Spherically symmetric selfdual Yang-Mills instantons on curved backgrounds in all even dimensions

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Abstract

We discuss several different classes of selfdual Yang-Mills instantons in the backgrounds $d = 4p$ dimensional spacetimes with Euclidean signature. One class of solutions are time-independent and spherically symmetric in $d - 1$ dimensions, and the other class are spherically symmetric in all d dimensions. Some the solutions in the former class are evaluated numerically, all the rest being given in closed form. An analytic proof of existence covering all numerically evaluated solutions is given. All instantons studied have finite action and vanishing energy momentum tensor and do not disturb the geometry.

1 Introduction

The study of selfdual solutions of Yang-Mills (YM) theory on curved backgrounds has proven to be a fruitful field of research in physics and mathematics.

While most recent work on gravitating YM theory has been carried out in Lorentzian signature spacetimes, the earliest work on the subject, carried out by Charap and Duff [1], Chakrabarti and collaborators [2], was in four dimensional ($d = 4$) spacetimes of Euclidean signature. This was quite natural, as a sequel to the study of gravitational instantons [3]. In both [1] and [2], the YM connection A_μ is identified with the (gravitational) spin–connection ω_μ^{mn} as

$$A_\mu = -\frac{1}{2}\,\omega_\mu^{mn}\,\Sigma_{mn}^{(\pm)} \quad \Rightarrow \quad F_{\mu\nu} = -\frac{1}{2}\,R_{\mu\nu}^{mn}\,\Sigma_{mn}^{(\pm)}\,, \tag{1.1}$$

$F_{\mu\nu}$ and $R_{\mu\nu}^{mn}$ being the YM and the Riemann curvatures, and $\Sigma_{mn}^{(\pm)}$ one or other of the chiral representations of the algebra of $SO(4)$, *i.e.* left or right $SU(2)$. In both cases [1, 2], the instantons are *selfdual* in the YM curvature, and are evaluated in closed form. Selfduality of the YM curvature results in the vanishing of the stress tensor as a function of the non Abelian matter fields, so that the latter has no backreaction on gravity, *i.e.* these instantons are essentially given on a fixed curved background.

What is special about the $d = 4$ Charap–Duff (CD) instanton in [1], is that the Riemann curvature is also *double-selfdual*, which fixes the form metric of the metric background (*e.g.* the Euclideanised Schwarzschild background for the solution in [1]). However, we argue in this work that instanton configurations with rather similar properties exist for any spherically symmetric metric satisfying a suitable set of boundary conditions (this includes *e.g.* the Reissner-Nordstöm background). Although a closed form solution is found for a Schwarzschild metric only, we present an existence proof for the solutions we found numerically. In the present work, we will refer to this type of instantons (and it generalizations) as solutions of Type I.

Further to these (Euclidean time) static instantons [1, 2], a new type of $d = 4$ static YM instanton on a curved background was recently discovered in [4] to which we shall refer as Type II instantons. These are basically deformed Prasad–Sommerfield [5] (PS) monopoles. Like the Type I instantons, the solutions in [4] are also selfdual, but differ in an essential way from Type I instantons, in that they satisfy different boundary conditions and have a different action for the same background. As conjectured in [4], the Type II instantons exist for an arbitrary nonextremal $SO(3)$-spherically symmetric background, the PS solution being recovered in the $R^3 \times S^1$ flat space limit. The actions of both Types I and II instantons saturate the bound of the usual 2nd Chern–Pontryagin (CP) charge. They are both given on fixed Euclideanised black hole backgrounds.

The larger part of this paper is concerned with the generalization of the $d = 4$ solutions of both Types I and II to arbitrary even dimensions[1]. These instantons are static and spherically symmetric in $d-1$ dimensions and have a vanishing stress tensor. We argue that the form metric backgrounds are not crucial for the existence of these solutions, as long as the metric functions satisfy a rather week set of conditions. Here we will consider mainly Schwarzschild like backgrounds with and without a cosmological constant, and with a $U(1)$ field, presenting also an existence proof for a more general case.

In addition to these static solutions, we also study YM instantons which are spherically symmetric in the full d dimensional Euclidean spacetime. These are deformations of the BPST instanton [6], and are likewise selfdual, and hence are also solutions on a fixed curved background. In the case of AdS_4 background[2] this was given recently by Maldacena and Maoz [9] [3]. The deformed BPST instantons on AdS_4 and dS_4 are generalised to AdS_d and dS_d for all even d, the new solutions [4] in $d = 4p$ being deformations of the BPST hierarchy [15].

In general, gravitating YM instantons in higher dimensions are of physical relevance in the study of field theories arising from superstring theory [16, 17]. In particular, a special aspect of selfdual instantons is that they can be employed in supersymmetric gravity theories, for example in the analysis of branes in $4p+1$ dimensions generalising that of 5-Branes proposed in Ref. [18] (see also [19]). More recently, *six* dimensional instantons were employed in Cremmer–Scherck compactification over S^6 in Ref. [20], which can again be generalised to

[1]Restriction to even dimensions is because of our requirement of *selfduality*, without including Higgs or other scalar matter fields.

[2]Selfdual instantons on compact symmetric backgrounds, as opposed to ones on the noncompact space AdS_4, were known. For example YM instantons on S^4 were constructed by Jackiw and Rebbi [7], and those on $\mathbb{CP}^2$, by Gibbons and Pope [8] a long time ago.

[3]In [9] also wormhole solutions to the second order equations, where the matter field curves the geometry, are given, which in the dS_4 case were already known [10]. Here we have restricted to selfdual instantons.

[4]These are the instantons on noncompact symmetric spaces, corresponding to the already known ones on the compact spaces, namely on S^{4p} in [11], on S^{2n} in [12, 13], and on on $\mathbb{CP}^n$ in [14].

all *even* dimensions.

Instantons of non Abelian field systems in dimensions higher than 4 can be constructed for the hierarchy [21] of Yang–Mills models in all even dimensions

$$\mathcal{L}_{\rm YM}^{(P)} = \sum_{p=1}^{P} \frac{\tau_p^2}{2(2p)!}\ {\rm Tr}\, F(2p)^2\ , \tag{1.2}$$

in which the $2p$-form $F(2p)$ is the p-fold antisymmetrised product $F(2p) = F \wedge F \wedge ... \wedge F$ of the YM curvature 2-form F, and we choose the YM connections to take their values in the chiral representations $SO_{\pm}(d)$ of $SO(d)$. Here the maximum value of P in the superposition (1.2) is simply $P_{\rm max} \leq \frac{1}{2}d$. Such instantons are not necessarily selfdual. In particular, when all but the $p = \frac{1}{4}d$ term in (1.2) is retained, one finds a hierarchy of BPST instantons [15] in $d = 4p$ dimensions, which satisfy the corresponding selfduality equation

$$F(2p) = \pm\, {}^\star F(2p)\,, \tag{1.3}$$

${}^\star F(2p)$ being the Hodge dual of $F(2p)$, including the appropriate factor of $e = \sqrt{\det g}$.

The selfduality equations (1.3) feature higher orders of the YM $2-$form curvature. If one restricts to the usual ($p = 1$) YM model in higher dimensions, the action will be infinite. There are several other selfduality equations defined on higher *even* dimensions in the literature, which are linear in the YM curvature. But for none of these does the (usual) YM action saturate a topological lower bound and result in infinitely large action. Such selfduality equations are irrelevant for our purposes here. Also, the solution to the $p = 2$ member of (1.3) results in infinite action if it is not recognised [22] that the Lagrangian is the $p = 2$ member of the YM hierarchy in (1.2), and not the usual $p = 1$ member. It can be noted that there is a another hierarchy of, nonlinear in the YM curvature, selfduality equations [23] defined in all even dimensions. In $4p$ dimensions, this hierarchy coincides with the (1.3) of [15], saturating the action of the $p = \frac{1}{4}d$ in (1.2). In $4p + 2$ dimensions however, the Lagrangians of [23, 24, 25], whose field equations these nonlinear selfduality equations [23] solve, come in odd powers of the YM curvature and hence are not bounded from below. We therefore restrict our attention to the hierarchy (1.2) henceforth, both in $4p$ and $4p + 2$ dimensions.

In fact, since we are in effect concerned only with selfdual solutions, we will only ever consider two special cases of the YM hierarchy (1.2). In $d = 4p$ this is the system consisting of a single term with $p = \frac{1}{4}d$ saturated by (1.3). In $d = 2(p + q)$, with $q \neq p$, there are two terms in (1.2) labeled by p and q. The system in this case is saturated by the selfduality equations

$$\tau_p\, F(2p) = \pm\, \tau_q\, ({}^\star F(2q))(2p)\,, \tag{1.4}$$

where the Hodge dual of the $2q$ form on the right hand side of (1.4) is a $2p$ form, with an obvious relation between the dimensions of the constants τ_p and τ_q.

The choice of the hierarchy (1.2) consisting of higher order terms in the YM curvature can be justified in the light of the presence of such terms in the low energy string theory (see e.g. [26]-[28]) Lagrangian.

The imposition spherical symmetry in $d-1$ the spacelike dimensions for (Euclidean) time static fields is presented section **2**, while the more compact task of imposing spherical symmetry in all d Euclidean dimensions is deferred to section **5** where such instantons are constructed. The static solutions which are spherically symmetric in the $d-1$ dimensional subspace are presented in section **3**, while those in section **5** are spherically symmetric in all d dimensions. All instantons presented in section **5** are given in closed form. Section **3** is divided into two parts. In the first subsection, **3.1**, Type I *selfdual* instantons (generalising the $d=4$ CD instanton [1]) are evaluated in closed form for double-selfdual gravitational backgrounds. Also in subsection **3.1**, solutions satisfying Type I boundary conditions, but *not* on double-selfdual gravitational backgrounds, are constructed numerically. In subsection **3.2** Type II solutions, which satisfy boundary conditions that differ from those of Type I solutions, are presented. These are evaluated exclusively numerically. To underpin the numerically constructed solutions, analytic proofs for their existence are given in section **4**. All the solutions presented are selfdual satisfying the hierarchy of selfduality equations (1.3) and (1.4), respectively. In the case of Types I and II instantons the second order equations were integrated numerically in search of radial excitations, and none were found. A summary and discussion of our results is given in section **6**. Finally, an analysis of double–selfdual spaces is given in the Appendix, since these play an important role in the construction of the Charap–Duff hierarchy.

2 Symmetry imposition: spherical symmetry in $d-1$ dimensions

In this section, we impose spherical symmetry in $d-1$ dimensional subspace on the (Euclidean time) static gravitational and gauge fields.

2.1 General results

We consider a metric Ansatz with spherical symmetry in $d-1$ dimensional subspace,

$$ds^2 = N(r)\sigma^2(r)d\tau^2 \ + \ N(r)^{-1}dr^2 \ + \ r^2 d\Omega^2_{(d-2)} \,. \tag{2.5}$$

Here $d\Omega^2_{(d-2)}$ is the metric on a $(d-2)$-dimensional sphere, τ corresponds to the Euclidean time, while r is the radial coordinate. We shall be mainly interested in asymptotically flat background metrics whose fixed point set of the Euclidean time symmetry is of $d-2$ dimensions (a "bolt") and the range of the radial coordinate is restricted to $r_h \le r < \infty$, while

$$N(r) = N_1(r-r_h) + N_2(r-r_h)^2 + O(r-r_h)^3, \quad \sigma(r) = \sigma_h + \sigma_1(r-r_h) + O(r-r_h)^2, \tag{2.6}$$

where $N_1,\ N_2\ \sigma_h,\ \sigma_2$ are constants determined by the equations of motion[5] (with $N_1,\ \sigma_h$ positive quantities).

This type of metric usually corresponds to the analytical continuations of Lorentzian black hole solutions. The absence of conical singularities at $r = r_h$ fixes the periodicity of the coordinate τ

$$\beta = \frac{4\pi}{\sigma_h N_1}\ . \tag{2.7}$$

(Note that this holds for any gravity-matter model we consider.) As $r \to \infty$, the Euclideanised (thermal-)Minkowski background is approached, with $\sigma(r) \to 1$, $N(r) \to 1 - (r_0/r)^k$, with r_0 a positive constant, the value of k depending on the gravity model we are using (e.g. $k = d-3$ for the usual Einstein gravity).

Since some of the numerical work in section **3** is carried out for $p-$Einstein backgrounds defined for the system (A.1) with $p = q$, we state the reduced one dimensional gravitational Lagrangian in d spacetime subject to the static spherically symmetric metric Ansatz (2.5)

$$L^{(p,d)}_{(\mathrm{grav})} = \frac{\kappa_p}{2^{2p-1}}\,\frac{(d-2)!}{(d-2p-1)!}\ \sigma\,\frac{d}{dr}\left[r^{d-2p-1}(1-N)^p\right]\ . \tag{2.8}$$

Next, we impose spherical symmetry in $d-1$ dimensions on the static YM connection $A_\mu = (A_0, A_i)$, $i = 1, 2, ..., d-1$ and $\mu = 0, i$, resulting in the following Ansatz

$$A_0 = u(r)\,\hat{x}_j\,\Sigma^{(\pm)}_{j,d}\ ,\quad A_i = \left(\frac{1-w(r)}{r}\right)\Sigma^{(\pm)}_{ij}\hat{x}_j\ ,\quad \Sigma^{(\pm)}_{ij} = -\frac{1}{4}\left(\frac{1\pm\Gamma_{d+1}}{2}\right)[\Gamma_i,\Gamma_j]\ , \tag{2.9}$$

described by two functions $w(r)$ and $u(r)$ which we shall refer to as magnetic and electric potential, respectively. The Γ's denote the d-dimensional gamma matrices, and Γ_{d+1}, the chiral matrix in that dimension. The radial variable in (1.2) is $r = \sqrt{|x_i|^2}$ and $\hat{x}_i = x_i/r$ is the unit radius vector, while $x_0 = t$.

Inserting the YM Ansatz (2.9) in the p-th term in (1.2), we have the corresponding term in the resulting reduced one dimensional YM Lagrangian

$$\begin{aligned} L^{(p,d)}_{\mathrm{YM}} &= \frac{\tau_p}{2\cdot(2p)!}\frac{(d-2)!}{(d-2p-1)!}\,r^{d-4p}\Bigg\{\sigma\,(1-w^2)^{2(p-1)}\left[(2p)N\,w'^2 + \frac{(d-2p-1)}{r^2}(1-w^2)^2\right] \\ &+ \frac{2p}{2p-1}\,\frac{1}{\sigma}\left[\frac{\left([(1-w^2)^{p-1}u]'\right)^2}{d-2p}\,r^2 + \frac{2p-1}{N}\,\left[(1-w^2)^{p-1}u\right]^2\,w^2\right]\Bigg\}, \end{aligned} \tag{2.10}$$

where a prime denotes the derivative with respect to r.

We now adapt the expression (2.10) to the relevant models in $d = 4p$ and $d = 2(p+q)$, $(p \neq q)$.

[5] For most of this section we are not interested in the precise form of the functions N and σ, the considered YM instantons presenting some generic features for any choice of the background compatible with this behaviour.

2.2 $d = 4p$

In this case the action density corresponding to (1.2) with only one term $p = \frac{1}{4}d$ can be written as a sum of complete squares plus (or minus) a total derivative

$$\begin{aligned} L_{\rm YM}^{(p,d=4p)} &= \frac{\tau_p}{2\cdot(2p)!}\frac{(4p-2)!}{(2p-1)!}\Bigg\{\left[\frac{r}{\sqrt{\sigma(2p-1)}}\,\left[(1-w^2)^{p-1}\,u\right]'\mp\frac{\sqrt{\sigma(2p-1)}}{r}\,(1-w^2)^p\right]^2 \\ &+\ 2p\,(1-w^2)^{2(p-1)}\left[\sqrt{\sigma N}\,w'\pm\frac{1}{\sqrt{\sigma N}}\,u\,w\right]^2\pm\frac{d}{dr}\left[(1-w^2)^{2p-1}\,u\right]\Bigg\}, \end{aligned} \tag{2.11}$$

implying that the second order YM equations are solved by the following (anti-)selfduality equations

$$\sqrt{\sigma N}\,w' \ \pm\ \frac{1}{\sqrt{\sigma N}}\,u\,w = 0 \tag{2.12}$$

$$\frac{r}{\sqrt{\sigma(2p-1)}}\,\left[(1-w^2)^{p-1}\,u\right]' \ \mp\ \frac{\sqrt{\sigma(2p-1)}}{r}\,(1-w^2)^p = 0\,, \tag{2.13}$$

which arise directly from the imposition of spherical symmetry (2.9)-(2.5) on the (anti)selfduality equations (1.3).

Without any loss of generality, we will solve the selfduality equations by taking the upper sign in the above relations; the anti-instanton solutions are found by reversing the sign of the electric potential.

The action of the selfdual solutions is

$$S = \beta V_{d-2}\int_{r_h}^{\infty} dr\ L_{\rm YM}^{(p,d)} = \frac{\tau_p}{2\cdot(2p)!}\frac{(4p-2)!}{(2p-1)!}\left((1-w^2)^{2p-1}\,u\big|_{r=\infty}-(1-w^2)^{2p-1}\,u\big|_{r=r_h}\right) \tag{2.14}$$

where V_{d-2} is the area of the unit S^{d-2} sphere.

We consider in Section 3 two different sets of boundary conditions for the first order equations (2.12), (2.13), leading to different types of solutions and different values of the action (2.14).

2.3 $d = 2(p+q)$, $p \neq q$

The YM reduced one dimensional Lagrangian in this case is

$$L_{\rm YM} = L_{\rm YM}^{(p,d=2(p+q))} + L_{\rm YM}^{(q,d=2(p+q))}\,, \tag{2.15}$$

each of the two terms in which is readily read off (2.10), with coupling strengths τ_p^2 and τ_q^2 respectively. Just like (2.10) in $d = 4p$ was rewritten in the form (2.11), so can (2.15) be cast into the following useful form, consisting of sums of complete squares, plus (or minus) a total derivative.

$$
\begin{aligned}
L_{\rm YM} \ = \ & \left(\tau_p\sqrt{\frac{2p}{(2q-1)!}}r^{q-p}\sqrt{\sigma N}(1-w^2)^{p-1}w' \pm \tau_q\sqrt{\frac{2q}{(2p-1)!}}r^{p-q}\frac{1}{\sqrt{\sigma N}}(1-w^2)^{q-1}wu\right)^2 \\
+ & \left(\tau_q\sqrt{\frac{2q}{(2p-1)!}}r^{p-q}\sqrt{\sigma N}(1-w^2)^{q-1}w' \pm \tau_p\sqrt{\frac{2p}{(2q-1)!}}r^{q-p}\frac{1}{\sqrt{\sigma N}}(1-w^2)^{p-1}wu\right)^2 \\
+ & \left(\tau_p\sqrt{\frac{2p}{(2p-1)(2q)!}}r^{q-p+1}\frac{1}{\sqrt{\sigma}}\left[(1-w^2)^{p-1}u\right]' \mp \frac{\tau_q}{\sqrt{(2p-2)!}}r^{p-q-1}\sqrt{\sigma}(1-w^2)^{q}\right)^2 \\
+ & \left(\tau_q\sqrt{\frac{2q}{(2q-1)(2p)!}}r^{p-q+1}\frac{1}{\sqrt{\sigma}}\left[(1-w^2)^{q-1}u\right]' \mp \frac{\tau_p}{\sqrt{(2q-2)!}}r^{q-p-1}\sqrt{\sigma}(1-w^2)^{p}\right)^2 \\
& \pm\tau_p\tau_q\frac{4(p+q)}{\sqrt{(2p)!(2q)!}}\frac{d}{dr}\left[(1-w^2)^{p+q-1}\,u\right] \, . \qquad (2.16)
\end{aligned}
$$

(2.16) implies that the action of (2.15) is absolutely minimised by a set of (anti)selfduality equations. These can be expressed most simply by redefining the coupling strengths τ_p and τ_q in (2.15) and (2.16) according to

$$
\hat\tau_p = \tau_p\sqrt{(2p)!} \quad , \quad \hat\tau_q = \tau_q\sqrt{(2q)!}\,,
$$

resulting in

$$
\begin{aligned}
\hat\tau_p\, r^{q-p}\sqrt{\sigma N}\,(1-w^2)^{p-1}\,w' \ &= \mp \ \hat\tau_q\, r^{p-q}\frac{1}{\sqrt{\sigma N}}\,(1-w^2)^{q-1}\,wu \qquad (2.17) \\
\hat\tau_q\, r^{p-q}\sqrt{\sigma N}(1-w^2)^{q-1}\,w' \ &= \mp \ \hat\tau_p\, r^{q-p}\frac{1}{\sqrt{\sigma N}}\,(1-w^2)^{p-1}\,wu \qquad (2.18) \\
\hat\tau_p\, r^{q-p+1}\frac{1}{(2p-1)\sqrt{\sigma}}\,\left[(1-w^2)^{p-1}u\right]' \ &= \pm \ \hat\tau_q\, r^{p-q-1}\sqrt{\sigma}\,(1-w^2)^{q} \qquad (2.19) \\
\hat\tau_q\, r^{p-q+1}\frac{1}{(2q-1)\sqrt{\sigma}}\,\left[(1-w^2)^{q-1}u\right]' \ &= \pm \ \hat\tau_p\, r^{q-p-1}\sqrt{\sigma}\,(1-w^2)^{p}\,, \qquad (2.20)
\end{aligned}
$$

which also follow by directly imposing spherical symmetry (2.9)-(2.5) on the selfduality equation (1.4) Setting $p=q$, (2.17)-(2.20) and (1.4) revert to (2.12)-(2.13) and (1.3) respectively.

3 Solutions with spherical symmetry in $d-1$ dimensions

Here we will construct the Types I and II solutions in the following two subsections, respectively. Both these describe selfdual YM on black hole backgrounds, and differ from each other in the different boundary conditions they satisfy respectively.

3.1 Type I solutions: Extended Charap-Duff configurations and their deformations

This subsection is divided in three parts, the first two pertaining to solutions in $d = 4p$ and the third in $d = 2(p+q)$. In the first subsection we present closed form instantons on double–selfdual backgrounds in $d = 4p$, generalising the usual Schwarzschild black hole $d = 4$, to which we refer as $p-$Schwarzschild metrics. (These are not to be confused with the Schwarzschild-Tangherlini metrics in higher dimensions, which are *not* double–selfdual.) In the second subsection we construct numerical solutions on generic $4p$ dimensional backgrounds, which are not double–selfdual. The third subsection is concerned with solutions in $d = 2(p+q)$, which are given in fixed symmetric spaces only, and not on black holes.

3.1.1 Type I instantons in $d = 4p$ on double-selfdual backgrounds

For $p = 1$, the YM selfduality equations (2.12), (2.13) present a well known closed form solution, found a long time ago by Charap and Duff [1] (CD). This solution has been constructed for the case of double-selfdual $p-$Schwarzschild background [6].

The generalisation of the CD solution to $d = 4p$ case is given formally in [12], and here we construct these solutions concretely. This is straightforward and is effected by the replacement of the usual ($p = 1$) Schwarzschild background with the solution to the double-selfduality equation (A.6) corresponding to the $p-$Einstein gravity defined by (A.10). It should be emphasised here that, using the gravitational background of any other member of the gravitational hierarchy other than the $p-$Einstein gravity does not support a CD instanton solution. The YM instanton is found by embedding the gauge connection into the gravity spin connection according to (1.1). The resulting solution reads

$$w(r) = -\sqrt{N(r)}, \quad u(r) = -\frac{1}{2}N'(r), \quad \text{with} \quad \sigma(r) = 1\,, \tag{3.1}$$

where $N(r)$ is the metric function pertaining to the solution of the $p-$Einstein equations with cosmological term in $d = 4p$ dimensions. This is the $d = 4p$ special case of the solution given in [29], and can be found by solving

$$\left(\frac{1-N(r)}{r^2}\right)^p = c_1 + \left(\frac{r_0}{r}\right)^{d-1}\,, \tag{3.2}$$

which result from substitution of the metric Ansatz (2.5) in the double-selfduality equation (A.6). r_0 here is related to the mass of the solution, c_1 being fixed by the cosmological constant. This result, namely that the double-selfdual metric with Euclidean signature [7]

[6] It is worth noting that for the $p = 1$ case only, a generalisation of the CD instanton is obtained by replacing the Schwarzschild background employed in **3.1.1** above, by the Euclideanised Kerr background. Unfortunately, this more general solution cannot be extended to $d > 4$ since no higher dimensional counterparts of the Kerr solution are known in $p-$Einstein gravity for $p \geq 2$.

[7] That for Minkowskian signature double–selfduality of the metric does not lead to the $p-$Einstein equation is seen from (A.7) and (A.8) of the Appendix.

supports a YM instanton in the presence of a cosmological constant, agrees with that of Julia *et. al.* [30] in $d = 4$.

One can see that the gauge potentials diverge for solutions with AdS asymptotics which leads to a diverging action, according to (2.14.) For a vanishing cosmological constant $c_1 = 0$, these solutions have a finite action

$$S = \frac{\tau_p}{2 \cdot (2p)!} \frac{(4p-2)!}{(2p-1)!} 2\pi V_{d-2} \ . \tag{3.3}$$

(One can see that the background features do no enter here). Another interesting case is provided by dS instantons. Here the radial coordinate has a finite range and in the general case the spacetime presents a conical singularity at $r = r_h$ or $r = r_c$ (with $r_h < r_c$, $N(r_h) = N(r_c) = 0$). The action of these solutions is

$$S = \frac{\tau_p}{2 \cdot (2p)!} \frac{(4p-2)!}{(2p-1)!} \beta V_{d-2} \ (N'(r_h) - N'(r_c)). \tag{3.4}$$

3.1.2 Type I instantons in deformed $d = 4p$ $p-$Schwarzschild backgrounds

Interestingly, in addition to these solutions given in closed form, we have constructed numerical solutions with similar properties in other $d = 4p$ backgrounds with a vanishing cosmological constant. The only restriction we impose on these backgrounds is to present the expansion (2.6) as $r \to r_h$ and to approach asymptotically the $p-$Schwarzschild solution in p-Einstein gravity (*e.g.* $N(r) \to 1 - (r_0/r)^{(2p-1)/p}$ as $r \to \infty$).

These YM instanton solutions have the following expansion near the event horizon[8]

$$\begin{aligned} w(r) &= w_1\sqrt{r - r_h} + \frac{w_1}{2}\left(\frac{2(1-2p) - N_2 r_h^2}{N_1 r_h^2} - \frac{\sigma_1}{\sigma_0} + w_1^2(p-1)\right)(r - r_h)^{3/2} + o(r - r_h)^{5/2}, \\ u(r) &= -\frac{N_1\sigma_0}{2} + \sigma_0\left(\frac{2p-1}{r_h^2} - \frac{1}{2}N_1(p-1)w_1^2\right)(r - r_h) + o(r - r_h)^2 \end{aligned} \tag{3.5}$$

and at infinity,

$$w(r) = 1 - \frac{1}{2}\left(\frac{r_0}{r}\right)^{(2p-1)/p} + \dots, \quad u(r) = -\frac{2p-1}{2pr}\left(\frac{r_0}{r}\right)^{(2p-1)/p} + \dots, \tag{3.6}$$

Our numerical constructions of Type I $p-$YM selfdual solutions is limited here to those on Reissner-Nordström $p-$Einstein gravity backgrounds, as interesting examples of the generic case. This $p-$Reissner–Nordström metric is parametrised explicitly by the functions [29]

$$N(r) = 1 - \left[\left(\frac{r_0}{r}\right)^{2p-1} + \frac{c_2}{r^{2(3p-2)}}\right]^{1/p}, \qquad \sigma(r) = 1\,, \tag{3.7}$$

where $r_0 > 0$ and $c_2 \neq 0$ is an unspecified constant related to the electric charge, so that $N(r)$ has exactly one positive root at some $r = r_h$ and $N(r) > 0$ for all $r > r_h$. The

[8] Note that for the Schwarzschild-like coordinates we use, the slope of $w(r)$ diverges as $r \to r_h$. One can easily verify that this divergence disappears when using instead an isotropic coordinate system.

metric function (3.7) pertains to the $p-$Einstein(–Maxwell) system, which we here refer to as $p-$Reissner–Nordström. For small c_2, these can be viewed locally as deformations of the $p-$Schwarzschild double-selfdual backgrounds, which may give an heuristic explanation for the existence of these YM instantons.

Here we have excluded backgrounds of gravity with cosmological constant for purely practical reasons. Moreover, while in the presence of a cosmological constant the Einstein equations are satisfied by a double-selfdual metric, in the presence of a $U(1)$ field this is not the case. Thus, we learn something new by employing a $p-$Einstein Reissner-Nordström background, namely that even when the background Riemann curvature is not double-selfdual, the YM instantons remain selfdual. While this is consistent with the assertions in [1] and [12], namely that to construct (analytically or numerically) a single-selfdual YM solution it is *sufficient* to employ the embedding (1.1) of a double-selfdual Riemann curvature, it is not actually *necessary*.

For all considered solutions, the gauge functions $w(r)$ and $u(r)$ interpolate monotonically between the corresponding values at $r = r_h$ and the asymptotic values at infinity, without presenting any local extrema. Type I solutions exist for all values of the parameter r_h, in contrast to the Type II solutions presented in **3.2** below, which exist for r_h up to a maximal value. In Figure 1 we plot the $p = 2$ Charap-Duff solution (with $c_2 = 0$), known in closed form, together with the numerically numerically evaluated profiles of a typical selfdual YM solution in a $d = 8$ Euclideanised non–double-selfdual background. This last has been chosen to be the $p = 2$ Reissner-Nordström background (with $c_2 = 0.01$).

Another interesting property of numerically constructed solutions in $d = 4p$ with Type I boundary conditions concerns the solutions to the second order equations rather than the first order selfduality equations. In this case one might have expected that higher node radial excitations of the spherically symmetric selfdual solutions existed. Our numerical results indicate, quite definitely, that no such solutions exist. Had such non–selfdual solutions, describing the backreaction from gravity on the YM field been found, they would have been expected to be sphaleron-like configurations.

For $c_2 \neq 0$, the instanton solutions are evaluated numerically. For any choice of the metric functions $(N(r),\ \sigma(r))$ the action of the selfdual solutions satisfying (3.5), (3.6) is still given by (3.3). An existence proof for Type I solutions in a general metric background satisfying a suitable set of conditions is given in the next section. One can easily verify that the metric functions (3.7) satisfy the conditions there.

Somewhat surprisingly, it turns out that similar selfdual solutions to $p-$YM systems on *other* spherically symmetric background also exist. Therefore the condition for the metric background to approach asymptotically the $p-$Schwarzschild solution in p-Einstein gravity is not really crucial after all. These instanton configurations satisfy the same set of boundary conditions as the solutions above (*e.g.* $w(r_h) = 0,\ u(r_h) = 0$), with an expansion differs completely from (3.5), (3.6), however. We have tested this for the example of the $d = 8$ $2-$YM system on the $1-$Reissner–Nordström background (of the usual $p = 1$ Einstein-Maxwell gravity). These last differ from the former only quantitatively, the profiles of the functions asymptoting at least one order of magnitude longer, and exhibiting a similarly magnified steepness at the origin. Such solutions cannot be viewed as deformations of the

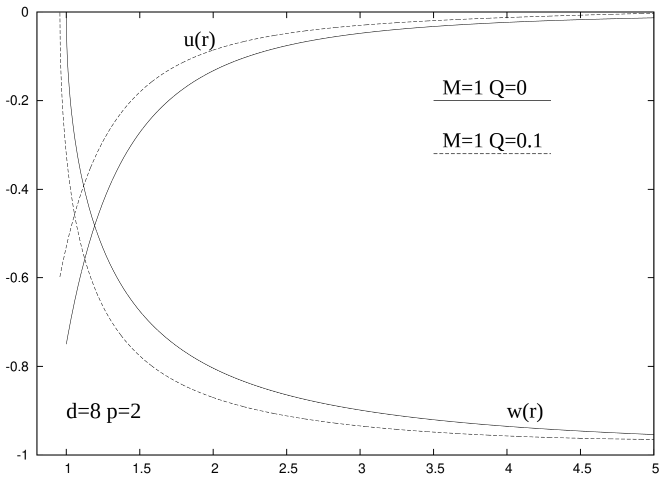

Figure 1: The YM gauge functions are shown as a function of the radial coordinate r for two Type I $p = 2$ YM selfdual solutions, showing the deformation of the explicit DC solution with $c_2 = 0$ by the $c_2 = 0.01$ 2−Reissner-Nordström metric

CD configurations. However, the numerical results are supported by the existence proof given in the next section. A discussion of these more general solutions will be presented elsewhere.

3.1.3 Selfdual Type I Yang-Mills solutions in $d = 2(p + q)$

The extension of the Charap-Duff solution in $d = 4$ to $d = 2(p + q)$ dimensions is also given formally in [12]. In this case however there exist no black hole solutions, and the only concrete instantons are those on the symmetric dS/AdS spaces given below.

On flat space the selfduality equations in $d = 2(p + q)$ dimensions (2.17)-(2.20) have no nontrivial solutions, but on a curved spacetime it is possible to find nontrivial solutions, albeit on maximally symmetric spaces. These solution minimise absolutely the action of the reduced one dimensional YM Lagrangian (2.15).

The dimensions of τ_p being different from the dimensions of τ_q, the system (2.15) is not scale invariant, and the selfduality equations (2.17)-(2.20) feature the dimensional constant $\frac{\tau_p}{\tau_q}$, as a result of which no asymptotically flat solutions to the latter exist.

Unfortunately, the only solution of these equations we could find is

$$w(r) = -\epsilon\sqrt{N(r)}, \quad u(r) = \epsilon\frac{1}{2}N'(r)\,, \tag{3.8}$$

where

$$N = 1 + \epsilon(\hat{\tau}_p/\hat{\tau}_q)^{1/(q-p)}r^2 \quad , \quad \sigma(r) = 1 \ \ , \tag{3.9}$$

and $q - p = 2n + 1$, with n an integer, while $\epsilon = \pm 1$. For $q - p = 2n$ one finds

$$w(r) = \sqrt{N(r)}, \quad u(r) = -\frac{1}{2}N'(r)\,, \tag{3.10}$$

with the metric functions given by (3.9) above. Restricting for simplicity to the case $\tau_p > 0$, $\tau_q > 0$, we see from (3.9) that these selfdual $(p,q)-$YM instantons are given on an Euclideanised dS ($\epsilon = -1$) or AdS ($\epsilon = 1$) background, the cosmological constant here being fixed by the coupling constants of the YM model[9].

3.2 Type II solutions: Deformed $p-$Prasad-Sommerfield configurations

This subsection deals only with $d = 4p$ solutions, and **not** $d = 2(p+q)$ ones with $p \neq q$. The reason for this is that the radial function $w(r)$ in (2.9) in this case vanishes asymptotically. The scaling properties consistent with finite action require that in $d = 2(p+q)$ both $F(2p)$ and $F(2q)$ terms be present in the Lagrangian. Then $w(r) \to 0$ for $r \to \infty$ causes the contribution of the $F(2p)$ (for $p < q$) to the action to diverge.

The hierarchy of Type II instantons basically consists of the deformed hierarchy of Prasad–Sommerfield [5] (PS) monopoles in $4p-1$ dimensions presented in [31], generalising the usual $3+1$ dimensional PS monopoles [5] to $(4p-1)+1$ dimensions.

These $p-$PS monopoles are deformed by the usual ($p = 1$) Einstein–Hilbert gravity. We shall refer to these as $p-$PS monopoles. Here we have used only $p = 1$ gravity in all $4p$ dimensions, since the background gravity here does not play a special role as it does in the Type I cases. It would have been equally valid to employ any $p-$Einstein gravity instead, but we chose to work with the simplest background. Type II instantons differ substantially from the Type I solutions given in **3.1**. In particular, they satisfy a different set of boundary conditions and have different actions.

These solutions are found for a set of boundary conditions familiar from previous studies on gravitating non Abelian solutions possesing an event horizon, where the YM connection A_μ has a nonvanishing electric component A_0 (see e.g. [32], [33]). Here the magnetic gauge potential w starts from a nonzero value at the horizon and vanishes at infinity, while the electric one u behaves in the opposite way. The YM potentials have the following expansion as $r \to r_h$

$$\begin{aligned} w(r) &= w_h + \frac{(2p-1)w_h(w_h^2-1)}{r_h^2 N_1}(r-r_h) + o(r-r_h)^2 , \\ u(r) &= \frac{(2p-1)\sigma_h(1-w_h^2)}{r_h^2}(r-r_h) + o(r-r_h)^2 , \end{aligned} \tag{3.11}$$

with $0 \leq w_h \leq 1$.

[9]In **3.1.1** above, where $p = q$, we studied numerically the second order equations to find out whether there existed any radial excitations, and the outcome was negative. Here too we inquire whether there might be non–selfdual solutions with the matter field deforming the geometry, and found that no such solutions can exist. We concluded this analytically, by noticing the impossibility to write for $d = 2(p+q)$ (with $p \neq q$) a consistent expansion near $r = r_h$ of the form (3.5).

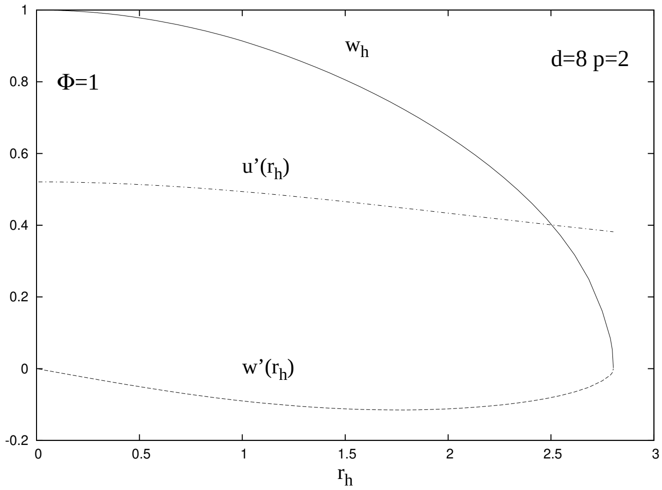

Figure 2: The parameters w_h, $w'(r_h)$ and $u'(r_h)$ of the $p = 2$ type II YM instantons in a $d = 8$ Schwarzschild-Tangherlini background are plotted as a function of r_h.

For large r, the solution reads

$$w(r) = \frac{e^{-\Phi r}}{r^{\Phi r_0 - 1}} + \ldots, \quad \text{for } p = 1, \quad w(r) = r^{2p-1} e^{-\Phi(r - (\frac{r_0}{r})^k/(k-1))} + \ldots, \quad \text{for } p \neq 1,$$
$$\text{and} \quad u(r) = \Phi - \frac{(2p-1)}{r} + \ldots, \quad \text{for any } p \ , \tag{3.12}$$

where Φ is an arbitrary nonzero constant. ¿From (2.14), we find the action of the Type II instanton solutions

$$S = \frac{\tau_p}{2 \cdot (2p)!} \frac{(4p-2)!}{(2p-1)!} \beta \, V_{d-2} \, \Phi \ . \tag{3.13}$$

One can see that the properties of the background metric enter here through the expression of β — the periodicity of the Euclidean time coordinate. Employing (3.11) and (3.12) to estimate the intergral of (2.13), implies the existence of a maximal allowed magnitude of the electric potential at infinity for a given r_h

$$\Phi < (2p-1) \int_{r_h}^{\infty} dr \, \frac{\sigma(r)}{r^2}. \tag{3.14}$$

In practice, we choose $\Phi = 1$ without any loss of generality, which sets the maximal value of the r_h for a given background. This is in contrast to the Type I solutions where the value of the horizon radius r_h is not constrained.

In Ref. [4], arguments for the existence of $p = 1$ type II selfdual Yang-Mills instantons for several $d = 4$ spherically symmetric backgrounds with Euclidean signature were presented. These solutions were evaluated numerically. The existence of similar solutions for any nonextremal $SO(3)$-spherically symmetric background approaching at infinity the $d = 4$ Euclideanised Minkowski spacetime was also conjectured. These solutions can be interpreted

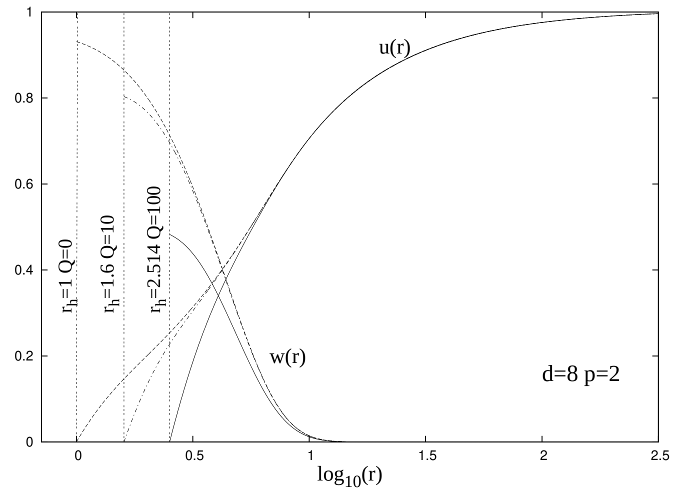

Figure 3: The YM gauge functions are shown as a function of the radial coordinate r for $p = 2$ type II YM selfdual solutions in $d = 8$ Euclideanised Reissner-Nordström-Tangherlini backgrounds.

as curved spacetime deformations of the well known Prasad-Sommerfield (PS) monopoles [5], viewed as instantons of the YM theory in a $R^3 \times S^1$ background. Here we extend some of these arguments to the $p-$PS monopoles [31] in $4p - 1$ spacelike dimensions. It is natural to suppose that these solutions will survive when the background has a nontrivial geometry, at least for a small curvature. Unlike in the previous case of Type I instantons where the gravitational background was specified to be the $p-$Einstein Schwarzschild like solution (with or without cosmological constant), here the curving of the background is not similarly constrained. Like in the 4 dimensional case [4], here we have found numerical arguments for the existence of an hierarchy of $d = 4p$ YM selfdual solutions, the $p = 1$ case in [4] being the first member only. For $p = 2,\ 3$, we have considered several different spherically symmetric backgrounds, the Schwarzschild-Tangherlini and Reissner-Nordström-Tangherlini solutions in Einstein-Maxwell theory being the simplest cases (the metric functions for the second situation are $\sigma = 1,\ N = 1 - ((\frac{r_0}{r})^{d-3} + \frac{c_2^2}{r^{2(d-3)}})$, r_0 being related to the mass and c_2 to the electric charge of the fixed backgrounds, respectively).

As in the $p = 1$ case, the $r_h \to 0$ limit (when physically possible), provides instanton solutions in a topologically trivial background. This is nicely illustrated by the case of $p-$YM selfdual instantons in the background of a Einstein-Yang-Mills purely magnetic hairy black holes discussed in [34], which solve also the field equations for an Euclidean signature. These solutions have a particle-like globally-regular limit with a nonvanishing curvature, the Killing vector $\partial/\partial\tau$ presenting in this case no fixed point sets (*i.e.* $g_{\tau\tau} > 0$ for any $r \geq 0$ and an arbitrary periodicity β). When taking instead the $r_h \to 0$ limit for a Schwarzschild background, the $p-$PS-type configurations in [31] are approached. In all these cases, the approximate expression of the YM instanton solutions as $r \to 0$ is

$$w(r) = 1 - br^2 + O(r^4), \quad u(r) = 2b\sigma_0 r + O(r^2), \tag{3.15}$$

(with $b > 0$ and $\sigma_0 = \sigma(r = 0)$), the asymptotic form (3.12) being valid in this case, too.

In all cases, the gauge functions $w(r)$ and $u(r)$ interpolate monotonically between the corresponding values at $r = r_h$ and the asymptotic values at infinity, without presenting any local extrema. For small enough values of r_h, the solutions look very similar to the flat space selfdual YM configuration. These solutions get deformed with the value of r_h increasing, while the value of the magnetic potential w at $r = r_h$ steadly decreases. As r_h approaches some maximal value implied by (3.14), we find that $w_h \to 0$ and the solution approaches the limiting configuration

$$w(r) = 0, \quad u(r) = \Phi + (2p-1) \int \frac{\sigma(r)}{r^2}\, dr. \tag{3.16}$$

In Figures 2 we plotted several relevant parameters of the YM instanton solutions as a function of r_h, for the case of $p = 2$ ($d = 8$) Schwarzschild-Tangherlini background. A typical selfdual YM solution in a Euclideanised Reissner-Nordström-Tangherlini black hole is plotted in Figure 3. These plots retain the generic features of the picture we found in other cases.

4 Analytic proofs of existence

To underpin the numerically constructed solutions with spherical symmetry in $d-1$ dimensions presented in the previous section, we present analytic existence proofs for these in the present section. As in section **3** above, we have split this section into two subsections, dealing with Types I and II solutions respectively.

4.1 Type I Solutions

4.1.1 The problem

In this subsection, we present an analytic proof for the existence of those Type I solutions which were evaluated numerically in the previous section.

Without loss of generality, we will consider only the case of upper signs in the system of selfduality equations (2.12) and (2.13) over (r_h, ∞), subject to the boundary conditions

$$\begin{aligned} w(r_h) &= 0, \quad u(r_h) = u_h, && (4.17)\\ w(\infty) &= 1, \quad u(\infty) = 0, && (4.18) \end{aligned}$$

where $u_h < 0$ is a constant. We will be interested in solutions such that u remains nonpositive and w remains nonnegative for all $r > r_h$.

¿From (2.12), we have

$$\sigma N w' + wu = 0, \quad r > r_h, \tag{4.19}$$

which implies that $w' \geq 0$ for all $r > r_h$. In fact, we also have $w > 0$ everywhere. Indeed, if there is an $r_0 > r_h$ such that $w(r_0) = 0$, then $w \equiv 0$ due to the uniqueness theorem for initial value problems of ordinary differential equations, which violates the boundary condition for w stated in (4.18). Similarly, $w < 1$ everywhere. Otherwise, if there is an $r_0 > r_h$ such that $w(r_0)=1$, then $w(r)\equiv 1$ for all $r\geq r_0$. Using the analyticity of solutions in the BPS system of equations, (2.12) and (2.13), we see that $w(r)\equiv 1$ for all $r>r_h$, which contradicts $w(r_h)=1$ in (4.17). These established facts now allow us to assert that $u(r)<0$ for all $r>r_h$. Suppose otherwise that there is an $r_0>r_h$ such that $u(r_0)=0$. Hence, r_0 is a maximum point for u and $u'(r_0)=0$. Inserting these into (2.13) evaluated at $r=r_0$, we obtain $w(r_0)=1$, which is false. A special consequence from the conclusion $w>0, u<0$ and (4.19) is that $w'>0$ for all $r>r_h$. Another is that the fact $w>0$ allows us to suppress (4.19) into

$$-\sigma N(\ln w)'=u,\quad r>r_h. \tag{4.20}$$

Inserting (4.20) into (2.13), we have

$$[(1-w^2)^{p-1}\sigma N(\ln w)']'+\frac{(2p-1)\sigma}{r^2}(1-w^2)^p=0,\quad r>r_h. \tag{4.21}$$

Furthermore, with $v=\ln w$ or $w=\mathrm{e}^v$, we rewrite (4.21) into the form

$$[(1-\mathrm{e}^{2v})^{p-1}\sigma Nv']'+\frac{(2p-1)\sigma}{r^2}(1-\mathrm{e}^{2v})^p=0,\quad r>r_h, \tag{4.22}$$

so that the boundary condition for w is converted to the boundary condition for v which says

$$v(r_h)=-\infty,\quad v(\infty)=0. \tag{4.23}$$

Recall that since $w(r)$ stays within the interval $(0,1)$ when $r>r_h$, the range of $v(r)$ for $r>r_h$ is $(-\infty,0)$. This property suggests that we may use the invertible transformation from $(-\infty,0]$ to itself defined by

$$f=P(v)=\int_0^v(1-\mathrm{e}^{2s})^{p-1}\,\mathrm{d}s. \tag{4.24}$$

to simplify (4.22) further into

$$f''+f'(\ln[\sigma N])'+\frac{(2p-1)}{r^2N}(1-\mathrm{e}^{2Q(f)})^p=0,\quad r_h<r<\infty. \tag{4.25}$$

Here and in the sequel, we use Q to denote the inverse of P over $(-\infty,0]$. It is clear that P,Q are increasing and $P(0)=Q(0)=0$.

To motivate our general study, we start from the simplest (but instructive) situation in (2.12) and (2.13) for which $p=1$ in (3.7) and $\sigma\equiv 1$. Therefore $N(r)$ takes the form

$$N(r)=1-(\frac{r_0}{r}+\frac{c_2}{r^2}). \tag{4.26}$$

It is seen that the function $N(r)$ has a single positive root if and only if $c_2>0$ and the root is given by

$$r_h=\frac{1}{2}(r_0+\sqrt{r_0^2+4c_2}), \tag{4.27}$$

which allows us to rewrite (4.26) as

$$N(r) = \left(1 - \frac{r_h}{r}\right)\left(1 + \frac{R_h}{r}\right) = \frac{1}{r^2}(r - r_h)(r + R_h) \tag{4.28}$$

for some number $R_h > 0$ and consider the equations over $r_h < r < \infty$.

With $\sigma \equiv 1$ and N given in (4.28), the equation (4.25) becomes

$$f'' + f'\left(\frac{1}{\rho} + \frac{1}{\rho + r_h + R_h} - \frac{2}{\rho + r_h}\right) + \frac{1}{\rho(\rho + r_h + R_h)}(1 - \mathrm{e}^{2Q(f)}) = 0, \tag{4.29}$$

where we have used the translated radial variable $\rho = r - r_h > 0$. Using the Euler transformation $\rho = \mathrm{e}^t$, we obtain from (4.29) the equation

$$f_{tt} - f_t g(t) + h(t)(1 - \mathrm{e}^{2Q(f)}) = 0, \quad -\infty < t < \infty, \tag{4.30}$$

where the coefficient functions $g(t)$ and $h(t)$ are given by the expressions

$$g(t) = \frac{\mathrm{e}^t(\mathrm{e}^t + r_h + 2R_h)}{(\mathrm{e}^t + r_h)(\mathrm{e}^t + r_h + R_h)}, \tag{4.31}$$

$$h(t) = \frac{\mathrm{e}^t}{(\mathrm{e}^t + r_h + R_h)}. \tag{4.32}$$

4.1.2 The proof

We now consider the general situation when $p \geq 1$ and σ are arbitrary. With the same sequence of variable substitutions, we rewrite the governing equation in terms of the radial variable r as (4.25).

Similar to (4.28), we express $N(r)$ as

$$N(r) = \left(1 - \frac{r_h}{r}\right) M(r), \tag{4.33}$$

where $M > 0$ for all $r \geq r_h$ and $M(\infty) = 1$. With $\rho = r - r_h$, $t = \ln \rho$, and

$$g(t) = 1 - \left[\rho\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right)\right]_{\rho = \mathrm{e}^t}, \tag{4.34}$$

$$h(t) = \left[\frac{(2p-1)\rho}{rM(r)}\right]_{\rho = \mathrm{e}^t}, \tag{4.35}$$

the equation (4.25) becomes

$$f_{tt} - f_t g(t) + h(t)(1 - \mathrm{e}^{2Q(f)})^p = 0, \tag{4.36}$$

subject to the updated boundary condition

$$f(-\infty) = -\infty, \quad f(\infty) = 0. \tag{4.37}$$

It is seen that (4.36) generalizes (4.30). In view of (4.31) and (4.32), we impose the following conditions on the coefficient functions $g(t)$ and $h(t)$:

(i) $g(t) \geq 0, h(t) > 0$ for all t;

(ii) as $t \to -\infty$, there are the asymptotics

$$g(t) = \mathrm{O}(\mathrm{e}^{\delta t}), \quad h(t) = \mathrm{O}(\mathrm{e}^{\varepsilon t}) \tag{4.38}$$

for some constant $\delta, \varepsilon > 0$;

(iii) there hold

$$\lim_{t\to\infty} g(t) = g(\infty) > 0, \quad \lim_{t\to\infty} h(t) = h(\infty) > 0. \tag{4.39}$$

Note that in view of the definition (4.34)–(4.35) and the fact that $M(\infty) = 1$ we actually have $h(\infty) = 2p - 1$. However, this precise value is not important for our subsequent discussion.

In order to solve (4.36) subject to (4.37), we consider the solution of the equation (4.36) over the interval $-\infty < t < \infty$ subject to the initial value condition

$$f(t_0) = -a, \quad f_t(t_0) = b, \tag{4.40}$$

where $t_0 \in (-\infty, \infty)$ and $a > 0$. We shall show that for any $a > 0$, there exists a unique number $b(a) > 0$ so that when $b = b(a)$ the initial value problem consisted of (4.36) and (4.40) has a uniquely and globally defined solution $f(t)$ satisfies $f_t(t) > 0$ and $f(t) < 0$ for all t. Moreover, such a solution satisfies the desired boundary condition (4.37).

For technical reasons, we shall also consider the possibility that the solution f of (4.36) and (4.40) takes positive values under certain initial conditions. Consequently we need to modify (4.36) as

$$f_{tt} - f_t g(t) = h(t)R(f), \tag{4.41}$$

where we define

$$R(s) = \begin{cases} -(1 - \mathrm{e}^{2Q(s)})^p, & s \leq 0, \\ 2ps, & s > 0, \end{cases} \tag{4.42}$$

so that $R(s)$ is a differentiable increasing function.

For given t_0 and fixed $a > 0$, we use $f(t; b)$ to represent the unique solution of (4.41) satisfying (4.40) which is defined in its local or global interval of existence.

We will conduct a shooting analysis. To this end, we define our sets of shooting slopes as follows:

$$\begin{aligned} \mathcal{S}^- &= \{b \in \mathbb{R} \mid \exists t > t_0 \text{ so that } f_t(t; b) < 0\}, \\ \mathcal{S}^0 &= \{b \in \mathbb{R} \mid f_t(t; b) > 0 \text{ and } f(t; b) \leq 0 \text{ for all } t > t_0\}, \\ \mathcal{S}^+ &= \{b \in \mathbb{R} \mid f_t(t; b) > 0 \text{ for all } t \geq t_0 \text{ and } f(t; b) > 0 \text{ for some } t > t_0\}. \end{aligned}$$

Lemma 4.1. *The set of real numbers $\mathbb{R}$ may be expressed as the disjoint union $\mathbb{R} = \mathcal{S}^- \cup \mathcal{S}^0 \cup \mathcal{S}^+$.*

Proof. Let $t > t_0$ be any point in the interval of existence of the solution $f(b; t_0)$. Of course $(-\infty, 0) \subset \mathcal{S}^-$. For any $b \notin \mathcal{S}^-$, we have $f_t(t; b) \geq 0$ for all $t \geq t_0$. We claim that $f_t(t; b) > 0$ everywhere. In fact, if there is some point $t_1 > t_0$ such that $f_t(t_1; b) = 0$, then $f(t_1; b) \neq 0$ since $f = 0$ is an equilibrium of the equation (4.41) which cannot be attained by a solution trajectory originating from a non-equilibrium initial state. Using the fact that $f_t(t_1; b) = 0$ but $f(t_1; b) \neq 0$ in (4.41), we have

$$f_{tt}(t_1; b) = h(t_1)R(f(t_1; b)) \neq 0. \tag{4.43}$$

Hence, depending on the sign of $f_{tt}(t_1; b)$, we have either $f_t(t; b) < 0$ for $t < t_1$ but t is close to t_1 when $f_{tt}(t_1; b) > 0$ or $f_t(t_1; b) < 0$ for $t > t_1$ but t is close to t_1 when $f_{tt}(t_1; b) < 0$. Therefore, $b \in \mathcal{S}^-$, a contradiction. Hence $f_t(t; b) > 0$ for all $t > t_0$ which proves $b \in \mathcal{S}^- \cup \mathcal{S}^+$ as claimed.

Lemma 4.2. *The sets $\mathcal{S}^-$ and $\mathcal{S}^+$ are both open and nonempty.*

Proof. The set $\mathcal{S}^-$ is of course nonempty because $(-\infty, 0) \subset \mathcal{S}^-$ by the definition of $\mathcal{S}^-$. The openness of $\mathcal{S}^-$ follows immediately from the continuous dependence theorem of the solution of an ordinary differential equation on its initial values.

We now prove that $\mathcal{S}^+$ is also nonempty. To this end, we observe that, when $b > 0$, $f_t(t; b)$ remains positive for $t \in (t_0, t_0 + \varepsilon)$ when $\varepsilon > 0$ is small enough. Since $g(t) \geq 0$, we see that (4.41) gives us $f_{tt} \geq h(t)R(f)$. Integrating this inequality twice and using the initial condition (4.40), we have

$$f_t(t; b) \geq b + \int_{t_0}^{t} h(s_1)R(f(s_1; b))\,\mathrm{d}s_1, \quad t_0 < t < t_0 + \varepsilon, \tag{4.44}$$

$$\begin{aligned} f(t; b) \geq{}& -a + b(t - t_0) \\ &+ \int_{t_0}^{t}\int_{t_0}^{s_2} h(s_1)R(f(s_1; b))\,\mathrm{d}s_1\,\mathrm{d}s_2, \quad t_0 < t < t_0 + \varepsilon. \end{aligned} \tag{4.45}$$

Of course, (4.44) and (4.45) continue to hold wherever $f_t(t; b) \geq 0$ ($t > t_0$). We show that, when $b > 0$ is large enough, we have $b \in \mathcal{B}^+$. In fact, for any $t_1 > t_0$, the slope number $b > 0$ can be chosen so that

$$b + \int_{t_0}^{t_1} h(s_1)R(-a)\,\mathrm{d}s_1 > 0, \tag{4.46}$$

$$-a + b(t_1 - t_0) + \int_{t_0}^{t_1}\int_{t_0}^{s_2} h(s_1)R(-a)\,\mathrm{d}s_1\,\mathrm{d}s_2 > 0. \tag{4.47}$$

Initially, since $f_t(t; b) > 0$, we have $f(t; b) > f(t_0; b) = -a$ (for $t > t_0$). Hence $R(f(t; b)) > R(-a)$. In view of (4.44) and (4.46), we get

$$f_t(t; b) > b + \int_{t_0}^{t_1} h(s_1)R(-a)\,\mathrm{d}s_1 > 0, \quad t_0 < t \leq t_1, \tag{4.48}$$

which implies $f(t;b) > f(t_0;b) = -a$ and $R(f(t;b)) > R(-a)$ for all $t_0 < t \le t_1$. Using this fact in (4.45) and (4.47), we have $f(t_1;b) > 0$. Since $f(t;b)$ strictly increases in (t_0,t_1), there is a unique point $t_2 \in (t_0,t_1)$ such that $f(t_2;b) = 0$ but $f(t;b) < 0$ for all $t \in (t_0,t_2)$. However, the definition (4.42) says that $R(f) \ge 0$ whenever $f \ge 0$, we see that for all $t > t_2$, we have

$$
\begin{aligned}
f_t(t;b) &\ge b + \int_{t_0}^{t} h(s_1)R(f(s_1;b))\,\mathrm{d}s_1 \ge b + \int_{t_0}^{t_2} h(s_1)R(f(s_1;b))\,\mathrm{d}s_1 \\
&\ge b + \int_{t_0}^{t_1} h(s_1)R(-a)\,\mathrm{d}s_1 > 0, \qquad (4.49) \\
f(t;b) &> 0, \qquad (4.50)
\end{aligned}
$$

which establishes $b \in \mathcal{S}^+$ and the nonemptyness of $\mathcal{S}^+$ follows.

It is not hard to show that $\mathcal{S}^+$ is open. In fact, let $b_0 \in \mathcal{S}^+$. Then $f(t;b_0) > 0$ for all $t > t_0$ and there is a $t_3 > t_0$ so that $f(t_3;b_0) > 0$. By the continuous dependence theorem for the solution to the initial value problem of an ordinary differential equation, we see that when b is sufficiently close to b_0, we still have $f(t_3;b) > 0$ and $f_t(t;b) > 0$ for all $t \in [t_0,t_3]$. Applying the same argument as that for deriving (4.49), we conclude that $f_t(t;b) > 0$ for all $t > t_3$ as well. Therefore $b \in \mathcal{S}^+$ and $\mathcal{S}^+$ is indeed open.

Lemma 4.3. *The set $\mathcal{S}^0$ is nonempty and closed. Furthermore, for $b \in \mathcal{S}^0$, we have $f(t;b) < 0$ for all $t > t_0$ and $f(t;b) \to 0$ as $t \to \infty$.*

Proof. Since $\mathbb{R}$ is connected, it cannot be expressed as the disjoint union of two open sets, $\mathcal{S}^-$ and $\mathcal{S}^+$ established in Lemma 4.2. Hence $\mathcal{S}^0$ is nonempty and closed.

The definition of $\mathcal{S}^0$ gives us $f(t;b) \le 0$ for all $t > t_0$. If there is a point $t_1 > t_0$ such that $f(t_1;b) = 0$, then $f_t(t_1;b) = 0$ which is false.

Since $f(t;b)$ increases and stays negative-valued for all $t > t_0$, the limit

$$\eta \equiv \lim_{t\to\infty} f(t;b) \qquad (4.51)$$

exists and satisfies $-a < \eta \le 0$. The finiteness of the limit η in (4.51) implies that there is a sequence $\{t_j\}$ ($t_j \to \infty$ as $j \to \infty$) so that

$$f_t(t_j;b) \to 0 \quad \text{as } j \to \infty. \qquad (4.52)$$

As a consequence of (4.52) and (4.39), we see from (4.41) (or more precisely, (4.36)) that $f_{tt}(t;b)$ will stay below a negative number when t is sufficiently large if $\eta < 0$, which contradicts (4.52).

In fact, we know more about the 'good slope' set $\mathcal{S}^0$. We have

Lemma 4.4. *The set $\mathcal{S}^0$ consists of a single point.*

Proof. Let b_1 and b_2 be taken from $\mathcal{S}^0$ and $f(t;b_1)$ and $f(t;b_2)$ the corresponding solutions of (4.36) and (4.40). Then $z(t)=f(t;b_1)-f(t;b_2)$ $(t\geq t_0)$ satisfies

$$z_{tt}-g(t)z_t=h(t)R'(\xi(t))z,\quad t_0<t<\infty, \tag{4.53}$$

where $\xi(t)$ lies between the quantities $f(t;b_1)$ and $f(t;b_2)$. Using $h(t)R'(\xi(t))>0$, $z(t_0)=z(\infty)=0$, and the maximum principle in (4.53), we deduce $z\equiv 0$. In particular, $z_t(t_0)=b_1-b_2=0$ as claimed.

In view of Lemma 4.4, for given $a>0$ in (4.40), let the unique point in $\mathcal{S}^0$ be denoted by $b=b(a)$ and the corresponding solution of (4.36) and (4.40) be simply denoted by $f=f(t)$. We have

Lemma 4.5. *For $b=b(a)$, the solution $f(t)$ of (4.36) and (4.40) exists globally for all t. Furthermore, it satisfies $f_t(t)>0$ and $f(t)<0$ for all t and realizes the other expected boundary condition*

$$\lim_{t\to-\infty}f(t)=-\infty. \tag{4.54}$$

Proof. With the notation just mentioned, we consider the solution over the left-half line $t<t_0$. Multiplying (4.36) by $\mathrm{e}^{\int_t^{t_0}g(s)\,\mathrm{d}s}$ and integrating, we get

$$f_t(t)=\mathrm{e}^{-\int_t^{t_0}g(s)\mathrm{d}s}\left(b(a)+\int_t^{t_0}h(s)(1-\mathrm{e}^{2Q(f(s))})^p\mathrm{e}^{\int_s^{t_0}g(s_1)\mathrm{d}s_1}\,\mathrm{d}s\right),\quad t<t_0. \tag{4.55}$$

In particular,

$$f_t(t)>b(a)\mathrm{e}^{-\int_t^{t_0}g(s)\mathrm{d}s}>b(a)\mathrm{e}^{-\int_{-\infty}^{t_0}g(s)\mathrm{d}s}\equiv b_0>0 \tag{4.56}$$

for all $t<t_0$, where the convergence of the improper integral in (4.56) follows from (4.38). So $f(t)<f(t_0)-b_0(t_0-t)$ for $t<t_0$ and we obtain $f(-\infty)=-\infty$ as claimed.

We can now check the boundary conditions for the original field configuration pair w and u in terms of the radial variable r.

First, using the relations $v=\ln w$ and $v=Q(f)$, we may immediately deduce from Lemmas 4.3 and 4.5 that $w(r)\to 1$ as $r\to\infty$ and $w(r)\to 0$ as $r\to r_h$, respectively.

Next, since (by (4.20))

$$\begin{aligned} u &= -\sigma N\frac{\mathrm{d}v}{\mathrm{d}r}=-\frac{\sigma(r)M(r)}{r}\rho\frac{\mathrm{d}v}{\mathrm{d}\rho}\\ &= -\frac{\sigma(\mathrm{e}^t+r_h)M(\mathrm{e}^t+r_h)}{(\mathrm{e}^t+r_h)}\frac{\mathrm{d}v}{\mathrm{d}t}\\ &= -\frac{\sigma(\mathrm{e}^t+r_h)M(\mathrm{e}^t+r_h)}{(\mathrm{e}^t+r_h)}(1-\mathrm{e}^{2Q(f(t))})^{-(p-1)}f_t(t), \end{aligned} \tag{4.57}$$

we can use the statement $f(-\infty)=-\infty$ in Lemma 4.5 to arrive at the expression

$$\lim_{r\to r_h}u(r)=-\frac{\sigma(r_h)M(r_h)}{r_h}\lim_{t\to-\infty}f_t(t)\equiv-\frac{\sigma(r_h)M(r_h)}{r_h}f_t(-\infty). \tag{4.58}$$

Note that, using (4.55), we have

$$f_t(-\infty) = \mathrm{e}^{-\int_{-\infty}^{t_0} g(s)\mathrm{d}s}\left(b(a) + \int_{-\infty}^{t_0} h(s)(1-\mathrm{e}^{2Q(f(s))})^p \mathrm{e}^{\int_s^{t_0} g(s_1)\mathrm{d}s_1}\,\mathrm{d}s\right) \tag{4.59}$$

and the uniform convergence of the right-hand side of (4.59) is a consequence of the assumption (4.38). In particular, the left-hand side of (4.59) is a well-defined positive number which gives rise to the negative limiting value of u at $r = r_h$.

In order to see what happens for u when $r \to \infty$, we can linearize (4.36) around $t = \infty$ to get

$$\theta_{tt} - g(\infty)\theta_t - 2[h(\infty)p]\theta = 0 \tag{4.60}$$

which has exactly one negative characteristic root, $-\lambda$ (say). Therefore f vanishes at $t = \infty$ exponentially fast like $\mathrm{e}^{-\lambda t}$. Using (4.107), we have $v = Q(f) = \mathrm{O}(\mathrm{e}^{-\lambda t/p})$ when t is large. Inserting these results into (4.57) and noting that $f_t(t) = \mathrm{O}(\mathrm{e}^{-\lambda t})$ for t large, we have

$$u = -\frac{\sigma(\mathrm{e}^t + r_h)M(\mathrm{e}^t + r_h)}{(\mathrm{e}^t + r_h)}\mathrm{O}(\mathrm{e}^{\lambda(p-1)t/p})f_t(t) = \mathrm{O}(\mathrm{e}^{-\lambda t/p}) \quad \text{as } t \to \infty. \tag{4.61}$$

Therefore we have shown that $u(r) \to 0$ as $r \to \infty$ as expected.

Returning to the original variables, we can summarize our study of the Type I solutions as follows.

Theorem 4.6. *Suppose that the background metric functions $N(r)$ and $\sigma(r)$ satisfy the conditions that $N(r)$ has exactly one positive root $r = r_h$ (say), $N(r) > 0$ when $r > r_h$,*

$$\lim_{r\to\infty} N(r) \equiv N(\infty) > 0, \tag{4.62}$$

$$\lim_{r\to\infty}\left[1 - (r - r_h)\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right)\right] \equiv g(\infty) > 0, \tag{4.63}$$

there are constants $\delta, \varepsilon > 0$ such that for r near r_h, there holds

$$1 - (r - r_h)\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right) = O((r - r_h)^\delta), \tag{4.64}$$

$$\frac{(r - r_h)^2}{r^2 N(r)} = O((r - r_h)^\varepsilon), \tag{4.65}$$

and for all $r > r_h$, there is the bound

$$(r - r_h)\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right) \leq 1. \tag{4.66}$$

Then the BPS system of equations (2.12) and (2.13) has a solution pair (w, u) over $r > r_h$ satisfying the boundary condition

$$w(r_h) = 0, \quad w(\infty) = 1; \quad u(r_h) = u_h, \quad u(\infty) = 0, \tag{4.67}$$

where $u_h < 0$ is a suitable constant, $w(r) > 0$, $w'(r) > 0$, and $u < 0$ for all $r > r_h$.

It can easily be seen that the conditions (4.62)-(4.65) are satisfied by any reasonable metric background and are in agreement with the asymptotics at the beginning of the section 2. The requirement (4.66) appears to be difficult to proof for an arbitrary metric. However, we have verified that this condition is satisfied in the concrete case we have considered in the numerics.

4.2 Type II solutions

4.2.1 The proof for a Schwarzschild background

We now consider type II solutions considered in section **3.2**. As before, we will start from a concrete situation.

We first set $\sigma \equiv 1$ in the system of selfduality equations (2.12)-(2.13), and we seek solutions with boundary conditions

$$w(r_h) = w_h, \qquad u(r_h) = 0; \tag{4.68}$$

$$w(\infty) = 0, \quad u(\infty) = \Phi, \tag{4.69}$$

where $w_h \in [0, 1]$ and $\Phi > 0$ are constants. For convenience, we shall now concentrate on nonnegative-valued solutions.

Like before, some elementary but useful properties of the solutions of the equation (2.12), (2.12) together with the boundary conditions (4.68), (4.69) may be deduced immediately. First, note that (2.12) implies that $w' \leq 0$. If $w_h = 0$ in (4.68), then it follows from (4.69) that $w \equiv 0$. Inserting this into (2.13) and using (4.68), we obtain $u(r) = (2p-1)(r_h^{-1} - r^{-1})$. Hence, in (4.69), we have

$$\Phi = \frac{(2p-1)}{r_h}. \tag{4.70}$$

In other words, the positive constant Φ in (4.69) in this trivial solution situation cannot be arbitrary. For the nontrivial solution situation, we have $0 < w_h \leq 1$. The uniqueness theorem for the initial value problem of ordinary differential equations implies that a nontrivial solution w of (2.12) cannot assume zero value at finite $r > r_h$. Hence $w(r) > 0$ for all $r > r_h$ which allows us to rewrite (2.12) as

$$-N(\ln w)' = u, \quad r > r_h. \tag{4.71}$$

Similarly, $w(r) < 1$ for all $r > r_h$. Otherwise, suppose that there is an $r_0 > r_h$ such that $w(r_0) = 1$. Hence $w_h = 1$ and $w(r) = 1$ for all $r_h < r < r_0$. Since the solution is necessarily analytic at r_0, we see that $w(r) = 1$ for r around r_0 which establishes $w(r) = 1$ for all $r > r_h$, contradicting $w(\infty) = 0$ in (4.69). We assert that $u(r) > 0$ for all $r > r_h$. Otherwise, suppose there is an $r_0 > r_h$ such that $u(r_0) = 0$. Then u attains its minimum at r_0. Therefore $u'(r_0) = 0$. Using these in (2.13), we arrive at a contradiction to the established fact $0 < w(r_0) < 1$. As a consequence of this fact and (2.12), we see that $w'(r) < 0$ for all $r > r_h$. These derived properties will serve as major clues for our resolution of the boundary value problem (2.12), (2.13), (4.68), (4.69) which is to follow in the sequel.

Let us now consider the concrete case where N is given as

$$N(r) = 1 - \left(\frac{r_h}{r}\right)^{d-3}, \quad r \geq r_h\ , \tag{4.72}$$

which corresponds to a Schwarzschild background. Our existence theorem for a nontrivial solution of (2.12), (2.13), (4.68), (4.69) may be stated as follows.

Theorem 4.7. *For the metric function N defined by (4.72) and $\sigma = 1$, the boundary value problem (2.12), (2.13), (4.68), (4.69) has a solution pair (w, u) for some constants $w_h \in (0, 1]$ and $\Phi > 0$ so that both w and u are positive-valued functions of the radial variable $r > r_h$ and w strictly increases.*

In order to get a proof of the theorem, we shall again pursue a suitable simplification of the system of equations (2.12) and (2.13). To this end, inserting (4.71) into (2.13), we obtain

$$[(1-w^2)^{p-1}N(\ln w)']' + \frac{(2p-1)}{r^2}(1-w^2)^p = 0. \tag{4.73}$$

Next set $v = \ln w$ or $w = \mathrm{e}^v$. We can rewrite (4.73) as

$$[(1-\mathrm{e}^{2v})^{p-1}Nv']' + \frac{(2p-1)}{r^2}(1-\mathrm{e}^{2v})^p = 0, \tag{4.74}$$

and arrive at the corresponding boundary condition

$$v(r_h) = v_h = \ln w_h \leq 0, \quad v(\infty) = -\infty. \tag{4.75}$$

Moreover, using (4.24) and its inverse, we can again rewrite (4.74) into a semilinear equation,

$$f'' + f'(\ln N)' + \frac{(2p-1)}{r^2 N}(1-\mathrm{e}^{2Q(f)})^p = 0, \quad r_h < r < \infty. \tag{4.76}$$

Set $r = \rho + r_h$. Then, in terms of the differentiation with respect to $\rho > 0$, we rewrite (4.76) as

$$\begin{aligned}\rho^2 f'' \quad &+ \quad \rho f' \frac{(d-3)r_h^{d-3}\rho}{([\rho + r_h]^{d-3} - r_h^{d-3})(\rho + r_h)} \\ &+ \quad \frac{(2p-1)(\rho+r_h)^{d-5}\rho^2}{(\rho+r_h)^{d-3} - r_h^{d-3}}(1-\mathrm{e}^{2Q(f)})^p = 0, \quad 0 < \rho < \infty.\end{aligned} \tag{4.77}$$

With $t = \ln \rho$, we convert (4.77) into

$$f_{tt} - f_t + g(t)f_t + h(t)(1-\mathrm{e}^{2Q(f)})^p = 0, \quad -\infty < t < \infty, \tag{4.78}$$

subject to the boundary conditions

$$f(-\infty) = -\alpha \quad (0 \leq \alpha < \infty), \quad f(\infty) = -\infty, \tag{4.79}$$

where the functions $g(t)$ and $h(t)$ in (4.78) are defined by

$$g(t) = \frac{(d-3)r_h^{d-3}\mathrm{e}^t}{([\mathrm{e}^t + r_h]^{d-3} - r_h^{d-3})(\mathrm{e}^t + r_h)}, \tag{4.80}$$

$$h(t) = \frac{(2p-1)(\mathrm{e}^t + r_h)^{d-5}\mathrm{e}^{2t}}{(\mathrm{e}^t + r_h)^{d-3} - r_h^{d-3}}, \tag{4.81}$$

and $\alpha = -P(v_h)$ (see (4.24)).

Recall that we are to solve (4.78) and (4.79) so that its solution $f(t)$ is a negative-valued decreasing function of t. For this purpose, we will use a shooting method and consider the initial value problem

$$f_{tt} - f_t + g(t)f_t + h(t)(1 - \mathrm{e}^{2Q(f)})^p = 0, \quad -\infty < t < \infty, \tag{4.82}$$

$$f(t_0) = -a, \quad f_t(t_0) = -b, \tag{4.83}$$

where $a, b > 0$ and t_0 is fixed. Of course, consistency requires

$$a > \alpha. \tag{4.84}$$

In order to realize the boundary condition $f(-\infty) = -\alpha$, we set $\tau = -t$, $\tau_0 = -t_0$, and convert (4.82) in the half interval $-\infty < t \leq t_0$ into the form

$$f_{\tau\tau} + f_\tau - G(\tau)f_\tau = H(\tau)R(f), \quad \tau_0 \leq \tau < \infty, \tag{4.85}$$

$$f(\tau_0) = -a, \quad f_\tau(\tau_0) = b, \tag{4.86}$$

where $G(\tau) = g(-\tau)$ and $H(\tau) = h(-\tau)$ are both positive-valued and $R(\cdot)$ is defined by (4.42) as before.

For fixed a satisfying (4.84), we use $f(\tau; b)$ to denote the unique solution of (4.85) and (4.86) which is defined in its interval of existence.

To engage in a shooting analysis for (4.85) and (4.86), we define

$$\begin{aligned}
\mathcal{B}^- &= \{b \in \mathbb{R} \,|\, \exists \tau > \tau_0 \text{ so that } f_\tau(\tau; b) < 0\}, \\
\mathcal{B}^0 &= \{b \in \mathbb{R} \,|\, f_\tau(\tau; b) > 0 \text{ and } f(\tau; b) \leq 0 \text{ for all } \tau > \tau_0\}, \\
\mathcal{B}^+ &= \{b \in \mathbb{R} \,|\, f_\tau(\tau; b) > 0 \text{ for all } \tau \geq \tau_0 \text{ and } f(\tau; b) > 0 \text{ for some } \tau > \tau_0\}.
\end{aligned}$$

Lemma 4.8. *We have the disjoint union* $\mathbb{R} = \mathcal{B}^- \cup \mathcal{B}^0 \cup \mathcal{B}^+$.

Proof. If $b \notin \mathcal{B}^-$, then $f_\tau(\tau; b) \geq 0$ for all $\tau > \tau_0$. If there is a point $\tau_1 > \tau_0$ so that $f_\tau(\tau_1; b) = 0$, then $f(\tau_1; b) \neq 0$ because $f = 0$ is an equilibrium point of the differential equation (4.85) which is not attainable in finite τ. Since $f(\tau_1; b) \neq 0$ but $f_\tau(\tau_1; b) = 0$, we see that either $f_{\tau\tau} > 0$ or $f_{\tau\tau} < 0$ at $\tau = \tau_1$. Hence, there is a $\tau < \tau_1$ or $\tau > \tau_1$ at which $f_\tau(\tau; b) < 0$ which implies $b \in \mathcal{B}^-$, a contradiction. Thus, $f_\tau(\tau; b) > 0$ for all $\tau > \tau_0$ and $b \in \mathcal{B}^0 \cup \mathcal{B}^+$.

Lemma 4.9. *The set $\mathcal{B}^-$ and $\mathcal{B}^+$ are both open and nonempty.*

Proof. The fact that $\mathcal{B}^- \neq \emptyset$ follows immediately from the fact that $(-\infty, 0) \subset \mathcal{B}^-$. The fact that $\mathcal{B}^-$ is open is self-evident. To see that $\mathcal{B}^+$ is nonempty, first note that $\mathcal{B}^+ \subset (0, \infty)$. Hence, for $\tau > \tau_0$ but τ is close to τ_0, we have $f_\tau > 0$ and (4.85) gives us

$$(\mathrm{e}^\tau f_\tau)_\tau > \mathrm{e}^\tau H(\tau) R(f). \tag{4.87}$$

Integrating (4.87) near τ_0 where $f_\tau > 0$, we have

$$\begin{aligned} f_\tau(\tau; b) &> \left(b\mathrm{e}^{\tau_0} + \int_{\tau_0}^{\tau} H(s_1) R(f(s_1; b)) \mathrm{e}^{s_1}\, \mathrm{d}s_1 \right) \mathrm{e}^{-\tau}, &(4.88)\\ f(\tau; b) &> -a + b(1 - \mathrm{e}^{-(\tau - \tau_0)}) \\ &\quad + \int_{\tau_0}^{\tau} \mathrm{e}^{-s_2} \left(\int_{\tau_0}^{s_2} H(s_1) R(f(s_1; b)) \mathrm{e}^{s_1}\, \mathrm{d}s_1 \right) \mathrm{d}s_2. &(4.89) \end{aligned}$$

For any fixed $\tau_1 > \tau_0$, we can choose $b > 0$ sufficiently large so that

$$b\mathrm{e}^{\tau_0} + \int_{\tau_0}^{\tau_1} H(s_1) R(-a) \mathrm{e}^{s_1}\, \mathrm{d}s_1 > 0, \tag{4.90}$$

$$-a + b(1 - \mathrm{e}^{-(\tau_1 - \tau_0)}) + \int_{\tau_0}^{\tau_1} \mathrm{e}^{-s_2} \left(\int_{\tau_0}^{s_2} H(s_1) R(-a) \mathrm{e}^{s_1}\, \mathrm{d}s_1 \right) \mathrm{d}s_2 > 0. \tag{4.91}$$

In view of (4.88)–(4.91), we see that there is a $\tau_2 \in (\tau_0, \tau_1)$ so that $f_\tau(\tau; b) > 0$ for $\tau \in [\tau_0, \tau_2]$, $f(\tau; b) < 0$ for $\tau \in [\tau_0, \tau_2)$, but $f(\tau_2; b) = 0$. Hence, for any $\tau > \tau_2$, there holds

$$\begin{aligned} f_\tau(\tau; b) &\geq \left(b\mathrm{e}^{\tau_0} + \int_{\tau_0}^{\tau_2} H(s_1) R(f(s_1; b)) \mathrm{e}^{s_1}\, \mathrm{d}s_1 \right) \mathrm{e}^{-\tau} \\ &\geq \left(b\mathrm{e}^{\tau_0} + \int_{\tau_0}^{\tau_1} H(s_1) R(-a) \mathrm{e}^{s_1}\, \mathrm{d}s_1 \right) \mathrm{e}^{-\tau} > 0, &(4.92)\\ f(\tau; b) &> 0. &(4.93) \end{aligned}$$

Therefore, $b \in \mathcal{B}^+$ and the nonemptyness of $\mathcal{B}^+$ is established.

Moreover, for $b_0 \in \mathcal{B}^+$, there is a $\tau_1 > 0$ so that $f(\tau_1; b_0) > 0$. By the continuous dependence of f on the parameter b we see that when b_1 is close to b_0 we have $f_\tau(\tau; b_1) > 0$ for all $\tau \in [\tau_0, \tau_1]$ and $f(\tau_1, b_1) > 0$. Using (4.92) again, we see that $f_\tau(\tau; b_1) > 0$ for all $\tau > \tau_0$ as well, which proves $b_1 \in \mathcal{B}^+$. So $\mathcal{B}^+$ is open.

The fact that $\mathcal{B}^-$ is open is self-evident.

Lemma 4.10. *The set $\mathcal{B}^0$ is nonempty and closed. Furthermore, if $b \in \mathcal{B}^0$, then $f(\tau; b) < 0$ for all $\tau > \tau_0$.*

Proof. The first part of the lemma follows from the connectedness of $\mathbb{R}$, Lemma 4.8, and Lemma 4.9. To prove the second part, we assume otherwise that there is a $\tau_1 > \tau_0$ so that $f(\tau_1; b) = 0$. Since $f(\tau; b) \leq 0$ for all $\tau > \tau_0$, f attains its local maximum at τ_1. In particular, $f_\tau(\tau_1; b) = 0$, which contradicts the definition of $\mathcal{B}^0$.

Lemma 4.11. *For $b \in \mathcal{B}^0$, there is a number α satisfying $0 \leq \alpha < a$ such that $f(\tau; b) \to -\alpha$ as $\tau \to \infty$.*

Proof. Since f increases as a function of $\tau \geq \tau_0$ and $f < 0$ for all $\tau \geq \tau_0$, we see that the limit $\lim_{\tau\to\infty} f(\tau; b)$ exists and satisfies $-a < lim_{\tau\to\infty} f(\tau; b) \leq 0$.

Returning to the original variable $t = -\tau$, we see that we have obtained a solution of (4.78) over the left-half line $-\infty < t \leq t_0$ satisfying the boundary condition at $t = -\infty$ stated in (4.79).

We next consider the problem over the right-half line $t_0 \leq t < \infty$. For this purpose, let f be a local solution of (4.82) and (4.83) in a neighborhood of t_0. Since $1 - g(t) > 0$ and $h(t) > 0$, we deduce from (4.82) that f, f_t, f_{tt} all remain negative-valued for all $t \geq t_0$ in view of $f(t_0) < 0$ and $f_t(t_0) < 0$. In particular, the solution is defined globally for all $t_0 \leq t < \infty$. Using $h(\infty) = 2p - 1 > 0$, we see that $f_{tt}(t) \leq -c$ for some constant $c > 0$ for all $t \geq t_0$. Consequently, we must have $f(\infty) = -\infty$ which realizes the boundary condition at $t = \infty$ for f stated in (4.79). In other words, we have proved the existence of a solution of the two-point boundary value problem (4.78) and (4.79).

We are now ready to prove the existence theorem. To do so, we need only to examine the boundary conditions for the original field functions w and u in terms of the radial variable r. Using the relations among various variables, we obtain

$$w_h \equiv \lim_{r\to r_h} w = \lim_{\rho\to 0} w = \lim_{\rho\to 0} \mathrm{e}^v = \lim_{t\to-\infty} \mathrm{e}^{Q(f)} = \mathrm{e}^{Q(-\alpha)} \in (0, 1], \tag{4.94}$$

as desired because $\alpha \geq 0$ in view of Lemma 4.11.

In order to realize the boundary condition for $u = -N(\ln w)'$ (see (4.71)) at $r = r_h$, we insert the definition of the function N (see (4.72)) to get

$$\begin{aligned} u &= -N\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{([\rho + r_h]^{d-3} - r_h^{d-3})}{(\rho + r_h)^{d-3}}\frac{\mathrm{d}v}{\mathrm{d}\rho} \\ &= -\frac{(d-3)r_h^{d-4}(1 + \mathrm{O}(\rho))}{(\rho + r_h)^{d-3}}\rho\frac{\mathrm{d}v}{\mathrm{d}\rho}. \end{aligned} \tag{4.95}$$

Using (4.95), we obtain

$$\lim_{r\to r_h} u = -(d-3)r_h^{-1} \lim_{t\to-\infty} \frac{\mathrm{d}v}{\mathrm{d}t}. \tag{4.96}$$

To evaluate the right-hand side of (4.96), recall that (4.24) gives us

$$\frac{\mathrm{d}v}{\mathrm{d}t} = (1 - \mathrm{e}^{2v})^{-(p-1)}\frac{\mathrm{d}f}{\mathrm{d}t}. \tag{4.97}$$

The easier case is when $w_h < 1$. In view of (4.96) and (4.97), we have

$$\lim_{r\to r_h} u = -(d-3)r_h^{-1}(1 - w_h^2)^{-(p-1)} \lim_{t\to-\infty} \frac{\mathrm{d}f}{\mathrm{d}t}. \tag{4.98}$$

We again use the variable $\tau = -t$. Since f satisfies

$$f_{\tau\tau} + f_\tau(1 - G(\tau)) = -H(\tau)(1 - \mathrm{e}^{2Q(f)})^p, \quad \tau \geq \tau_0, \tag{4.99}$$

we may integrate (4.99) to get

$$f_\tau(\tau) = \mathrm{e}^{-\int_{\tau_0}^{\tau}(1-G(s))\,\mathrm{d}s}\bigg(b - \int_{\tau_0}^{\tau} H(s)(1 - \mathrm{e}^{2Q(f(s))})^p \mathrm{e}^{\int_{\tau_0}^{s}(1-G(s_1))\,\mathrm{d}s_1}\,\mathrm{d}s\bigg). \tag{4.100}$$

The definitions of $G(s)$ and $H(s)$ give us the asymptotics

$$1 - G(s) = \left(1 + \frac{d-4}{2}\right)\mathrm{e}^{-s} + \mathrm{O}(\mathrm{e}^{-2s}), \tag{4.101}$$

$$H(s) = \left(\frac{2p-1}{d-3}\right) r_h^{-1}\mathrm{e}^{-s} + \mathrm{O}(\mathrm{e}^{-2s}), \tag{4.102}$$

for s large. Using (4.101), (4.102), and the fact that $Q(f) < 0$ in (4.100), we see that $f_\tau(\tau)$ is bounded for $\tau \geq \tau_0$. As a consequence, (4.99) leads us to the estimate

$$f_{\tau\tau}(\tau) = \mathrm{O}(\mathrm{e}^{-\tau}) \quad \text{for large } \tau. \tag{4.103}$$

Since $f(\infty) = -\alpha$, we infer that there is a sequence $\{\tau_j\}$, $\tau_j \to \infty$ when $j \to \infty$, such that $f_\tau(\tau_j) \to 0$ when $j \to \infty$. From this fact and (4.103), we find

$$f_\tau(\tau) = \mathrm{O}(\mathrm{e}^{-\tau}) \quad \text{for large } \tau. \tag{4.104}$$

Inserting (4.104) into (4.98), we arrive at the expected result

$$\lim_{r \to r_h} u = 0. \tag{4.105}$$

We now consider the case when $w_h = 1$. Note that (4.100)–(4.104) are all valid. Using (4.104) and $f(\infty) = 0$, we have

$$f(\tau) = \mathrm{O}(\mathrm{e}^{-\tau}) \quad \text{for large } \tau. \tag{4.106}$$

Note also that (4.24) gives us the relation

$$f = \int_0^v \left(-2s - \frac{(2s)^2}{2!} - \cdots\right)^{p-1} \mathrm{d}s = (-2)^{p-1}\frac{v^p}{p} + \mathrm{O}(v^{p+1}). \tag{4.107}$$

Combining (4.106) and (4.107), we obtain

$$v(\tau) = \mathrm{O}(\mathrm{e}^{-\tau/p}) \quad \text{when } \tau \text{ is large.} \tag{4.108}$$

Inserting (4.104) and (4.108) into (4.97), we get

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v^{-(p-1)}(-2 + \mathrm{O}(v))^{-(p-1)}\frac{\mathrm{d}f}{\mathrm{d}t} = \mathrm{O}(\mathrm{e}^{t/p}) \quad \text{as } t \to -\infty, \tag{4.109}$$

which establishes (4.105) again in view of (4.96).

We finally examine the behavior of the solution pair (w, u) at $r = \infty$. Since we have derived the limit $f(t) \to -\infty$ as $t \to \infty$, we have

$$\lim_{r\to\infty} w = \lim_{\rho\to\infty} \mathrm{e}^{v} = \lim_{t\to\infty} \mathrm{e}^{Q(f)} = 0. \tag{4.110}$$

Besides, using the definition (4.72) for the background function N, the relation $u = -N(\ln w)_r$, and $v(\infty) = -\infty$, we have

$$\lim_{r\to\infty} u = -\lim_{\rho\to\infty} \frac{\mathrm{d}}{\mathrm{d}\rho}(\ln w) = -\lim_{t\to\infty} \mathrm{e}^{-t}\frac{\mathrm{d}v}{\mathrm{d}t} = -\lim_{t\to\infty} \mathrm{e}^{-t}\frac{\mathrm{d}f}{\mathrm{d}t}. \tag{4.111}$$

On the other hand, multiplying (4.78) by $\mathrm{e}^{-\int_{t_0}^{t}(1-g(s))\,\mathrm{d}s}$ and integrating over (t_0, t), we get

$$f_t(t) = \mathrm{e}^{\int_{t_0}^{t}(1-g(s))\,\mathrm{d}s}\left(-b - \int_{t_0}^{t} h(s)(1-\mathrm{e}^{2Q(f(s))})^p \mathrm{e}^{-\int_{t_0}^{s}(1-g(s_1))\,\mathrm{d}s_1}\,\mathrm{d}s\right), \tag{4.112}$$

which results in the expression

$$\begin{aligned} -\lim_{t\to\infty} \mathrm{e}^{-t} f_t(t) &= \mathrm{e}^{-t_0-\int_{t_0}^{\infty} g(s)\,\mathrm{d}s}\left(b + \int_{t_0}^{\infty} h(s)(1-\mathrm{e}^{2Q(f(s))})^p \mathrm{e}^{-\int_{t_0}^{s}(1-g(s_1))\,\mathrm{d}s_1}\,\mathrm{d}s\right) \\ &\equiv \Phi > 0, \end{aligned} \tag{4.113}$$

where we have used the properties

$$g(t) = \mathrm{O}(\mathrm{e}^{-(d-3)t}), \quad h(t) = \mathrm{O}(1) \quad \text{as } t \to \infty, \tag{4.114}$$

for the positive-valued functions $g(t)$ and $h(t)$ given in (4.80) and (4.81) to deduce the convergence of the two improper integrals in (4.113). In other words, in view of (4.111), the positive number Φ defined in (4.113) gives us the expected limit,

$$\lim_{r\to\infty} u = \Phi. \tag{4.115}$$

The proof of the existence theorem is now complete.

4.2.2 The proof for a general black hole background

We now turn our attention to the case where the blackhole metric is defined by general functions, $N > 0$ and $\sigma > 0$ (whenever $r > r_h$). Focusing again on the selfdual equations with the choice of upper signs, we have (2.12) and (2.13) subject to the same boundary conditions, (4.68) and (4.69). Note that we are interested again in nonnegative-valued solutions. As discussed earlier, when $w_h = 0$, the solution is unique and explicitly given by

$$w \equiv 0, \quad u(r) = (2p-1)\int_{r_h}^{r} \frac{\sigma(\rho)}{\rho^2}\,\mathrm{d}\rho, \quad \Phi = (2p-1)\int_{r_h}^{\infty} \frac{\sigma(\rho)}{\rho^2}\,\mathrm{d}\rho, \tag{4.116}$$

which may be called a trivial solution; when $w_h \in (0,1]$, however, the solution will not be trivial. Furthermore, we can see that if (w,u) is a solution with $w_h > 0$, then $w(r) > 0, u(r) > 0$ for $r > r_h$ and w strictly increases. In particular, (2.12) allows us to represent u by (4.20) which of course generalizes (4.71). Therefore, using the same substitution of variables, $v = \ln w$ and $f = P(v)$ as given in (4.24), we can transform (2.12) and (2.13) into the scalar equation (4.25).

Like before, we write $r = \rho + r_h$ ($\rho > 0$). Therefore, in terms of $t = \ln\rho$, we convert (4.25) into the familiar form (4.78) in which the coefficient functions $g(t)$ and $f(t)$ are now defined by the updated expressions

$$g(t) = \left[\rho\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right)\right]_{\rho=\mathrm{e}^t}, \tag{4.117}$$

$$h(t) = \left[(2p-1)\frac{\rho^2}{r^2 N(r)}\right]_{\rho=\mathrm{e}^t}. \tag{4.118}$$

Recall that the key properties of the functions $g(t)$ and $h(t)$ used in the proof of Theorem 4.7 are

(i) $g(t) \geq 0$ and $h(t) > 0$ for all t;

(ii) $1 - g(t) > 0$ for all t;

(iii) $h(\infty) > 0$;

(iv) as $t \to -\infty$, we have the asymptotics

$$1 - g(t) = \mathrm{O}(\mathrm{e}^{\varepsilon t}), \quad h(t) = \mathrm{O}(\mathrm{e}^{\delta t}), \tag{4.119}$$

for some $\varepsilon, \delta > 0$;

(v) as $t \to \infty$, we have the asymptotics

$$g(t) = \mathrm{O}(\mathrm{e}^{-\gamma t}), \quad h(t) = \mathrm{O}(1), \tag{4.120}$$

for some $\gamma > 0$.

It is direct to check that the proof of Theorem 4.7 is intact for the general N, σ case when the above properties are valid. Consequently, switching back to the original radial variable r, we arrive at the following general existence theorem.

Theorem 4.12. *For the general system of BPS equations (2.12) and (2.13) subject to the boundary condition given by (4.68) and (4.69), the same conclusion for existence of a solution as stated in Theorem 4.7 holds provided that the positive-valued metric functions N and σ satisfy the assumptions*

$$0 \leq (r - r_h)\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right) < 1, \quad \forall r > r_h, \tag{4.121}$$

$$\lim_{r\to\infty} N(r) = N_\infty > 0, \tag{4.122}$$

$$1 - (r - r_h)\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right) = O((r - r_h)^\varepsilon), \quad r \approx r_h, \tag{4.123}$$

$$\frac{(r-r_h)^2}{N(r)} = O((r-r_h)^\delta), \quad r \approx r_h, \tag{4.124}$$

$$(r-r_h)\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right) = O((r-r_h)^{-\gamma}), \quad r \approx \infty, \tag{4.125}$$

where $\varepsilon, \delta, \gamma$ are some positive constants.

The remarks at the end of the subsection 4.1.2 on the generality of these conditions apply in this case, too. Again, the only condition which seems to requires a knowledge of N, σ is (4.121). However, (4.121) is satisfied by the various backgrounds we have considered.

4.2.3 The proof for a topologically trivial background

We now consider type II YM instantons on a 'soliton' background for which the functions N and σ are everywhere positive and regular up to $r = 0$:

$$N(0) = 1, \quad \sigma(0) = \sigma_0 \quad \text{with } 0 < \sigma_0 < 1. \tag{4.126}$$

The governing equations are still (2.12) and (2.13) subject to the boundary conditions (4.68) and (4.69). In this situation, we may simply restate our sufficient conditions given in Theorem 4.12 by setting $r_h = 0$ in order to guarantee the existence of a solution with the only exception that (4.123) is no longer valid when $r_h = 0$ and the metric functions N and σ are regular at $r = 0$ which gives rise to the new property

$$\lim_{r \to 0} r\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right) = 0. \tag{4.127}$$

In fact, we note that, with $r_h = 0$, $r = \rho = \mathrm{e}^t$, and

$$g(t) = \left[r\left(\frac{\sigma'(r)}{\sigma(r)} + \frac{N'(r)}{N(r)}\right)\right]_{r=\mathrm{e}^t}, \tag{4.128}$$

$$h(t) = \left[\frac{(2p-1)}{N(r)}\right]_{r=\mathrm{e}^t}, \tag{4.129}$$

the equations (2.12) and (2.13) are condensed into the equation (4.78) as before. Thus, in view of (4.126)–(4.129), we have

$$\lim_{t \to -\infty} g(t) = 0, \quad \lim_{t \to -\infty} h(t) = 2p - 1, \tag{4.130}$$

which violates the condition (4.119). Therefore, our existence theorem may not be applicable directly to this 'regularised' problem. Below we shall show that (4.130) actually allows us to strengthen our existence theorem.

First, it is straightforward to see that (4.130) renders no barrier to the existence of a solution of (4.78) subject to the boundary condition (4.79) which may be obtained as before through solving the initial value problem (4.82)–(4.83) for given t_0, $a > 0$, and a suitable $b > 0$ (recall that the set $\mathcal{B}^0$ contains all such suitable b's).

Lemma 4.13. *Let f be the solution of (4.82)–(4.83) with $b \in \mathcal{B}^0$. Then $\alpha = 0$ or $f(-\infty) = 0$. In particular, $\mathcal{B}^0$ contains exactly one point $b = b(a, t_0)$ (say) and*

$$\mathcal{B}^- = (-\infty, b(a, t_0)), \quad \mathcal{B}^+ = (b(a, t_0), \infty). \tag{4.131}$$

Proof. Use the variable $t = -\tau$ and consider instead the problem (4.85)–(4.86) for $\tau \geq \tau_0$. We then arrive at (4.100) with $b \in \mathcal{B}^0$. Recall that (4.130) implies that

$$\lim_{s\to\infty} G(s) = 0, \quad \lim_{s\to\infty} H(s) = 2p - 1. \tag{4.132}$$

If $\alpha > 0$, then $f(\tau) < -\alpha$ for all $\tau \geq \tau_0$ and we conclude from (4.100) that $f_\tau(\tau) < 0$ when τ is sufficiently large, which contradicts the definition of $\mathcal{B}^0$.

If $b_1, b_2 \in \mathcal{B}^0$, then $z(\tau) = f(\tau; b_1) - f(\tau; b_2)$ satisfies the boundary condition $z(\tau_0) = z(\infty) = 0$. Inserting this into (4.85), we have

$$z_{\tau\tau} + (1 - G(\tau))z_\tau = H(\tau)R'(\xi)z, \quad \tau \geq \tau_0, \tag{4.133}$$

where ξ lies between $f(\tau; b_1)$ and $f(\tau; b_2)$. Applying the fact that $H(\tau)R'(\xi) > 0$ and the maximum principle in (4.133), we get $z \equiv 0$. In particular, $b_1 = b_2$. So $\mathcal{B}^0$ contains exactly one point as claimed.

To simplify notation, we set $t_0 = 0$ in (4.113). For any $a > 0$ in the initial value problem (4.82)–(4.83), let $b = b(a)$ be the unique point in $\mathcal{B}^0$ ensured by the above lemma. Hence we can rewrite the boundary value Φ given in (4.113) as a well-defined function of a, say $\Phi(a)$, as follows,

$$\Phi(a) = \mathrm{e}^{-\int_0^\infty g(s)\,\mathrm{d}s}\left(b(a) + \int_0^\infty h(s)(1 - \mathrm{e}^{2Q(f(s))})^p \mathrm{e}^{-\int_0^s (1-g(s_1))\,\mathrm{d}s_1}\,\mathrm{d}s\right). \tag{4.134}$$

Lemma 4.14. *The function $\Phi(\cdot)$ depends on $a > 0$ continuously so that $\Phi(a) \to 0^+$ as $a \to 0^+$ and $\Phi(a) \to \infty$ as $a \to \infty$. In particular, the range of $\Phi(\cdot)$ is the entire half interval $(0, \infty)$.*

Proof. First, we show that $b(a)$ is continuous with respect to the parameter $a > 0$. Otherwise there is a point $a_0 > 0$ and a sequence $\{a_j\} \subset (0, \infty)$ so that $a_j \to a_0$ as $j \to \infty$ but $|b(a_j) - b(a_0)| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$ and $j = 1, 2, \cdots$. From the proof of Lemma 4.9 (cf. (4.88)–(4.91)), we see that $\{b(a_j)\}$ is a bounded sequence. In fact, with $\tau_0 = 0$ and $\tau_1 = 1$ in (4.88)–(4.91), we see that $b \in \mathcal{B}^+$ when (4.90) and (4.91) or

$$b > \Gamma_1(a) \equiv \int_0^1 H(s)(1 - \mathrm{e}^{2Q(-a)})^p \mathrm{e}^s\,\mathrm{d}s \tag{4.135}$$

and

$$b(1 - \mathrm{e}^{-1}) > \Gamma_2(a) \equiv a + \int_0^1 \mathrm{e}^{-s_2}\left(\int_0^{s_2} H(s_1)(1 - \mathrm{e}^{2Q(-a)})^p \mathrm{e}^{s_1}\,\mathrm{d}s_1\right)\mathrm{d}s_2 \tag{4.136}$$

hold. In other words, (4.135) and (4.136) give us the upper bound

$$b(a) \leq \max\{\Gamma_1(a), (1-\mathrm{e}^{-1})^{-1}\Gamma_2(a)\}. \tag{4.137}$$

In particular, the boundedness of $\{b(a_j)\}$ follows. Hence, passing to a subsequence if necessary, we may assume $b(a_j) \to$ some b_0 as $j \to \infty$. Of course, $b_0 \neq b(a_0)$. It is clear that for (4.85)–(4.86) with $a = a_0$, both $b(a)$ and b_0 lie in $\mathcal{B}^0$, which contradicts Lemma 4.13 which states that $\mathcal{B}^0$ contains exactly one point.

The continuous dependence of $b(a)$ on a implies that the solution f of (4.85)–(4.86) with $b = b(a)$ depends on a continuously as well. Using this fact, we easily obtain the continuous dependence of $\Phi(a)$ on $a > 0$ because the improper integral containing f on the right-hand side of (4.134) is uniformly convergent with respect to the parameter $a > 0$ in view of (4.120).

We claim that $b(a) \to 0^+$ as $a \to 0^+$. Otherwise there is a sequence $\{a_j\}$ in $(0, \infty)$ and an ε_0 so that $a_j \to 0$ as $j \to \infty$ but $b(a_j) \geq \varepsilon_0$ $(j = 1, 2, \cdots)$. Using these in the initial value problem (4.85)–(4.86) with $a = a_j$ and $b = b(a_j)$, we observe that the solution will assume a positive value for a slightly positive τ when j is sufficiently large, which contradicts the definition of $b(a_j)$.

We can also claim that $b(a) \to \infty$ as $a \to \infty$. To see this, we insert the property $f_\tau \geq 0$ in (4.100) (with $\tau_0 = 0$) to get

$$b(a) \geq \int_0^\tau H(s)(1 - \mathrm{e}^{2Q(f(s))})^p \mathrm{e}^{\int_0^s (1-G(s_1))\mathrm{d}s_1}\,\mathrm{d}s, \quad \forall \tau \geq 0. \tag{4.138}$$

Since we also know that $f_{\tau\tau} \leq 0$ for $\tau \geq 0$, we have

$$f(\tau) \leq -a + b(a)\tau \quad \text{for } \tau \geq 0. \tag{4.139}$$

In view of (4.139), we see that $f(\tau) \leq -1$ (say) whenever τ satisfies

$$\tau \leq \frac{(a-1)}{b(a)}, \quad a > 1. \tag{4.140}$$

Combining (4.138) and (4.140), we get the lower bound

$$b(a) \geq \int_0^{(a-1)/b(a)} H(s)(1 - \mathrm{e}^{2Q(-1)})^p \mathrm{e}^{\int_0^s (1-G(s_1))\mathrm{d}s_1}\,\mathrm{d}s, \tag{4.141}$$

which implies that $b(a) \to \infty$ as $a \to \infty$ because the integral

$$\int_0^\infty H(s)(1 - \mathrm{e}^{2Q(-1)})^p \mathrm{e}^{\int_0^s (1-G(s_1))\mathrm{d}s_1}\,\mathrm{d}s \tag{4.142}$$

is divergent which allows us to argue by contradiction if $b(a) \not\to \infty$ when $a \to \infty$.

Hence $\Phi(a)$ defined in (4.134) is a continuous function with respect to $a \in (0, \infty)$ so that $\Phi(a) \to 0^+$ when $a \to 0^+$ and $\Phi(a) \to \infty$ when $a \to \infty$.

The proof of the lemma is now complete.

In summary, we can state our results for the existence of a solution in the regular case as follows.

Theorem 4.15. *Suppose that the background metric functions N and σ are regularly defined for all $r > 0$ and satisfy (4.126) at $r = 0$ and the conditions (4.121), (4.122), (4.124), (4.125) (all with $r_h = 0$), and (4.127). Then the system of equations (2.12) and (2.13) subject to the boundary conditions (4.68) and (4.69) governing a BPS monopole has a nontrivial solution if and only if $w_h = 1$. Moreover, the positive constant Φ in (4.69) may be taken to be any prescribed number and for any given $\Phi > 0$, the solution pair (w, u) satisfies $w > 0, u > 0$, and w strictly increases for $r > 0$.*

5 Solutions with spherical symmetry in (even) d-dimensions

Here we start by imposing spherical symmetry in the whole d dimensional (Euclidean) space, treating all coordinates on the same footing. The corresponding metric Ansatz in this case is

$$ds^2 = d\rho^2 + f^2(\rho) d\Omega^2_{(d-1)}, \tag{5.1}$$

where $f(\rho)$ is a function fixed by the gravity-matter field equations, ρ being the radial coordinate (with $\rho_a \leq \rho \leq \rho_b$).

The YM Ansatz compatible with the symmetries of the above line element is expressed as

$$A_\mu = \left(\frac{1 - w(\rho)}{\rho}\right) \Sigma^{(\pm)}_{\mu\nu} \hat{x}_\nu \,, \tag{5.2}$$

where the spin matrices are precisely those used in (1.2), the radial variable in is $\rho_\mu = \sqrt{|x_\mu|^2}$ and $\hat{x}_\mu = x_\mu/\rho$ is the unit radius vector.

The resulting reduced one dimensional YM Lagrangian for the p-th term in the YM ierarchy is

$$L^{(p,d)}_{\rm YM} \;=\; \frac{\tau_p}{2\cdot(2p)!}\frac{(d-1)!}{(d-2p)!} f^{d-4p+1}(w^2-1)^{2p-2}\left(w'^2 + \frac{d-2p}{2p}\frac{(w^2-1)^2}{f^2}\right) \;, \tag{5.3}$$

the corresponding $d = 4p$ YM selfduality equations taking the simple form

$$w' \pm \frac{w^2 - 1}{f(\rho)} = 0. \tag{5.4}$$

For any choice of the metric function $f(\rho)$, the solution of the above equation reads

$$w = \frac{1 - c_0 e^{\mp 2\int \frac{d\rho}{f(\rho)}}}{1 + c_0 e^{\mp 2\int \frac{d\rho}{f(\rho)}}} \tag{5.5}$$

where c_0 is an arbitrary positive constant.

The action of the selfdual solutions can be written as

$$S = \pm 2 \frac{\tau_p (4p-1)!}{2\cdot((2p)!)^2} V_{d-1} w \;{}_2F_1(\frac{1}{2}, 1-2p, \frac{3}{2}, w^2)\Big|^{\rho_1}_{\rho_0} \tag{5.6}$$

($_2F_1(a,b,c,z)$ being the hypergeometric function). For $f(\rho)=\rho$ one recovers the $d=4p$ generalization of the BPST instanton first found in [15], with $w=(\rho^2-c)/(\rho^2+c)$. An AdS background $f(\rho)=\rho_0\sinh\rho/\rho_0$ leads to a $d=4p$ generalization of the $d=4$ AdS selfdual instantons in [9], with $w=(\tanh^2(\rho/2\rho_0)-c)/(\tanh^2(\rho/2\rho_0)+c)$. The $d=4p$ selfdual instantons on a sphere (euclideanised dS space) are found by taking $\rho_0\to i\rho_0$ in the corresponding AdS relations.

Again, one can consider as well the superposition of two members of the YM hierarchy, say those labeled by p and q, with $d=2(p+q)$. The generic selfduality equations (1.4) reduce here to

$$w'\pm\sqrt{\frac{\kappa p}{q}}(1-w^2)^{q-p+1}f^{2p-2q-1}=0,\quad w'\pm\sqrt{\frac{q}{\kappa p}}(1-w^2)^{p-q+1}f^{2q-2p-1}=0\ , \tag{5.7}$$

where we note $\kappa=\hat{\tau}_p/\hat{\tau}_q$, which is supposed to be a positive quantity. The above are overdetermined and have the unique solutions

$$f=r_0\sin\rho/\rho_0,\ w=\cos\rho/\rho_0\ \text{ or }\ f=r_0\sinh\rho/\rho_0,\ w=\cosh\rho/\rho_0,\ \text{ with }\ r_0=(\kappa p/q)^{\frac{1}{4(q-p)}}\ , \tag{5.8}$$

accommodating fixed AdS and dS spaces, depending if κ is negative or positive. Note that when κ is negative, r_0 in (5.8) cannot be real, so we have really only a YM field on dS. Here, unlike in the $d=4p$ case featuring the single $F(2p)$, the expression of the metric function $f(\rho)$ is fixed to describe a curved maximally symmetric background. The dS case with $p=1,\ q=2$ solution of (5.8) was recently found in [13].

6 Summary and discussion

We have considered the problem of constructing instantons of gravitating Yang–Mills field systems in all even dimensions. Our constructions are limited to two highly symmetric kinds of solutions for which the effective field equations are one dimensional. The larger part of the work concerns (Euclidean time) static fields that are spherically symmetric in the $d-1$ space dimensions, and a smaller part deals with fields that are spherically symmetric in the whole d dimensional (Euclidean) spacetime, treating all coordinates on the same footing. The main task here was the extension of the known results [1, 4, 9] in $d=4$ to arbitrary even d.

The static, and spherically symmetric (in $d-1$ dimensions) solutions are interesting because in that case instantons on (Euclideanised) black holes can be described. The fully spherically symmetric (in d dimensions) solutions are also interesting because in that case the metric is a conformal deformation of the metric on S^{4p}, relevant to the $p-$BPST instantons. All cases studied are restricted to Yang–Mills selfdual solutions, which means that the gravitational background is a fixed one offering no backreaction to the YM field, since the stress tensor of selfdual YM fields vanishes. We have nontheless searched numerically for solutions to the second order field equations that might describe radial excitations of the selfdual ones we construct, and have found no such solutions.

The static spherically symmetric solutions we have studied come in two types. Type I are the d dimensional generalisations of the Charap–Duff [1] (CD) solution in 4 dimensions, while Type II generalise the deformed Prasad–Sommerfield (PS) monopole in [4]. For $d = 4p$ we give the exact Type I analogues of the CD solution on double-selfdual $p-$Schwarzschild backgrounds, including a cosmological constant, in closed form. On backgrounds that are not double-selfdual, the solutions are constructed numerically. In our numerical constructions we have mostly employed backgrounds arising from $p-$Einstein gravity, notably the $p-$Reissner–Nordström. In addition, we have verified that $p-$YM instantons of Type I satisfying CD like boundary conditions can be constructed numerically on $q-$Einstein backgrounds ($p \neq q$), but these are much less robust that when $p = q$. In $d = 4p + 2$ the only Type I selfdual solutions are those on fixed AdS_{2p} and dS_{2p} backgrounds, evaluated in closed form, and, in the AdS case the solution is not real. Type II solutions are evaluated only numerically, and only in $4p$ dimensions. These are deformations of $p-$Prasad-Sommerfield monopoles. Both Type I an II solutions describe the YM field on a fixed black hole, but while the radius of the horizon r_h for Type I is unconstrained, r_h for a Type II solution has a maximal value. The numerical results presented in section 3 are supported by the existence proofs given in section 4. In evaluating the numerical instantons of both Types, we solved the second order field equations, but found no radial excitations above the selfdual solutions. The existence of all numerically constructed solutions were proved analytically using a **Yisong, please supply a name, e.g. variational, shooting, etc, method** method.

The last type of instantons considered in this paper, namley those deforming the $p-$BPST instanton on $\mathbb{R}^{4p}$, are evaluated in closed form both on AdS and dS backgrounds. In $4p + 2$ dimensions, the selfduality equations yield the same fixed backgrounds as in the case of Type I solutions in these dimensions.

Perhaps one of the most remarkable qualitative features of Types I and II instantons is, that these are not instantons at all but rather are monopole like lumps. This is because for a genuine instanton, the radial function w appearing in the Ansatz (2.9) must change sign over the full range r_h to infinity of the radial coordinate, while in what we have here, the sign of w does not change in this range. This behaviour is typical of a monopole. This aspect of the CD solutions is consistent with the conclusion of Tekin [35], who has allowed a time dependent YM field on the $1-$Schwarzschild background in 4 dimensions, and found that the resulting solution remains static, namely the CD solution itself. This conclusion is clearly true also in the $4p$ dimensional $p-$CD instantons here. We intend to carry this line of investigation further.

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A Double selfdual spaces

The considered gravitational system in $d = 2(p+q)$ spacetime dimensions is the superposition of all possible scalars $R_{(p,q)}$

$$\mathcal{L}_{\rm grav} = \sum_{p=1}^{P} \frac{\kappa_p}{2p}\, e\, R_{(p,q)} \ , \tag{A.1}$$

where $R_{(p,q)}$ are constructed from the $2p$-form $R(2p) = R \wedge R \wedge ... \wedge R$ resulting from the totally antisymmetrised p-fold products of the Riemann curvature 2-forms R. We express $R_{(p,q)}$ in the notation of [12] as

$$e\, R_{(p,q)} = \varepsilon^{\mu_1\mu_2...\mu_{2p}\nu_1\nu_2...\nu_{2q}} e_{\nu_1}^{n_1} e_{\nu_2}^{n_{2q}} ... e_{\nu_{2q}}^{n_{2q}}\ \varepsilon_{m_1 m_2 ... m_{2p} n_1 n_2 ... n_{2q}}\, R_{\mu_1\mu_2....\mu_{2p}}^{m_1 m_2..m_{2p}} \ , \tag{A.2}$$

where e_ν^n are the *Vielbein* fields, $e = \det(e_\nu^n)$ in (A.1), and $R_{\mu_1\mu_2....\mu_{2p}}^{m_1 m_2..m_{2p}} = R(2p)$ is the p-fold totally antisymmetrised product of the Riemann curvature, in component notation. It is clear from the definition (A.2) that the spacetime dimensionality is $d = 2(p+q)$, and that the maximum value p in the sum (A.1) is $P = \frac{1}{2}(d-2)$, with the term $e\, R_{(p=\frac{d}{2},q=0)}$ being the (total divergence) Euler-Hirzebruch density. Subjecting (A.2) to the variational principle one arrives at the p-th order Einstein equation

$$G_{(p)}{}_\mu^m = R_{(p)}{}_\mu^m - \frac{1}{2p}\, R_{(p)}\, e_\mu^m \ , \tag{A.3}$$

in terms of the p-th order Einstein tensor $G_{(p)}{}_\mu^m$, with $R_{(p)}$ and $R_{(p)}{}_\mu^m$ being the p-th order Ricci scalar and the p-th order Ricci tensor defined respectively by

$$R_{(p)} = R_{\mu_1\,\mu_2....\mu_{2p}}^{m_1\,m_2..m_{2p}}\, e_{m_1}^{\mu_1}\, e_{m_2}^{\mu_1} ... e_{m_{2p}}^{\mu_{2p}} \tag{A.4}$$

$$R_{(p)}{}_\mu^m = R_{\mu\,\mu_2....\mu_{2p}}^{m\,m_2..m_{2p}}\, e_{m_2}^{\mu_2} ... e_{m_{2p}}^{\mu_{2p}} \ . \tag{A.5}$$

Let us first consider the special case of $p = q$, namely of a $4p$ dimensional spacetime. The double selfduality condition in that case is

$$R_{\mu_1\mu_2....\mu_{2p}}^{m_1 m_2..m_{2p}} = \pm \frac{e}{[(2p)!]^2}\, \varepsilon_{\mu_1\mu_2...\mu_{2p}\nu_1\nu_2...\nu_{2p}} R_{n_1 n_2..n_{2p}}^{\nu_1\nu_2....\nu_{2p}}\ \varepsilon^{m_1 m_2...m_{2p} n_1 n_2...n_{2p}} \ , \tag{A.6}$$

The $\pm$ sign in (A.6) pertains to Euclidean and Minkowskian signatures, respectively, which is in order to impose in this case unlike in the case of single selfduality in which case the Hodge dual for Minkowskian signature would introduce an undesirable factor of $i = \sqrt{-1}$. We shall soon see that it is gainful to keep only to Euclidean signature.

Contracting the l.h.s. of (A.6) with $e_{m_2}^{\mu_2}\, e_{m_2}^{\mu_2} ... e_{m_{2p}}^{\mu_{2p}}$, and relabeling the free indices (μ_1, m_1) as (μ, m) we get the p-th order Ricci tensor defined by (A.5). After applying the usual tensor identities this results in the constraint

$$R_{(p)}{}_\mu^m = \mp \left(R_{(p)}{}_\mu^m - \frac{1}{2p}\, R_{(p)}\, e_\mu^m \right) . \tag{A.7}$$

It is now obvious that in the case of Minkowskian signature (the lower sign) (A.7) leads to

$$R_{(p)} = 0 \tag{A.8}$$

which is too weak a constraint to satisfy any $p-$Einstein equation arising from the variation with respect to e^{μ}_{m}. Accordingly, we restrict to Euclidean spaces, whence (A.7) reads

$$G_{(p)}{}_{\mu}^{m} \;=\; -\frac{1}{4p}\, R_{(p)}\, e^{m}_{\mu}\,. \tag{A.9}$$

The most general Lagrangian whose field equations are solved by the constraint (A.9) is the following special case of (A.1) augmented with a cosmological constant Λ,

$$\mathcal{L}_{\rm grav} \;=\; e\, \left(\kappa^{2p}\, R_{(p,p)} \;+\; d!\, \Lambda\right)\,, \tag{A.10}$$

where κ is a constant with the dimension of a length. The Einstein equations of (A.10) are

$$\kappa^{2p}\, G_{(p)}{}_{\mu}^{m} = \frac{(4p)!}{(2p)((2p)!)^2}\, \Lambda\, e^{m}_{\mu}\,. \tag{A.11}$$

The consistency condition of the double selfduality condition (A.6) and the field equation (A.11) is

$$\kappa^{2p}\, R_{(p)} \;=\; -2\,\frac{(4p)!}{((2p)!)^2}\, \Lambda\,, \tag{A.12}$$

implying that if the p-th order Ricci scalar is in this way related to the cosmological constant Λ, then such a solution of the double selfduality equation satisfies also the Einstein equation. This is independent of the sign of Λ, and is of course true also for the particular case of vanishing cosmological constant $\Lambda = 0$.

Next we consider a spacetime of dimension $d = 2(p+q)$, with $q > p$. The double selfduality condition in this case is

$$R^{m_1 m_2 .. m_{2p}}_{\mu_1\mu_2....\mu_{2p}} = \pm\kappa^{2(q-p)}\frac{e}{[(2q)!]^2}\, \varepsilon_{\mu_1\mu_2...\mu_{2p}\nu_1\nu_2...\nu_{2q}}\, R^{\nu_1\nu_2....\nu_{2p}}_{n_1 n_2 .. n_{2q}}\;\; \varepsilon^{m_1 m_2 ... m_{2p} n_1 n_2 ... n_{2q}}\,, \tag{A.13}$$

where κ is a constant with dimensions of a length.

Let us contract the l.h.s. of (A.13) with $e^{\mu_2}_{m_2}\, e^{\mu_2}_{m_2} \ldots e^{\mu_{2p}}_{m_{2p}}$, and let us relabel the free indices (μ_1, m_1) as (μ, m). This yields the p-th order Ricci tensor defined by (A.5), which after applying the usual tensor identities results in the constraint

$$R_{(p)}{}_{\mu}^{m} \;=\; \mp\kappa^{2(q-p)}\,\frac{(2p-1)!}{(2q-1)!}\; G_{(q)}{}_{\mu}^{m}\,. \tag{A.14}$$

Before comparing this constraint with the Einstein equations of the appropriate gravitational system (a subsystem of (A.1) plus a cosmological constant), it is convenient to state the corresponding constraint arising from the inverse of the double selfduality constraint (A.13), namely of

$$R^{m_1 m_2 .. m_{2q}}_{\mu_1\mu_2....\mu_{2q}} = \pm\kappa^{2(p-q)}\frac{e}{[(2p)!]^2}\varepsilon_{\mu_1\mu_2...\mu_{2q}\nu_1\nu_2...\nu_{2p}} R^{\nu_1\nu_2....\nu_{2q}}_{n_1 n_2 .. n_{2p}}\;\varepsilon^{m_1 m_2 ... m_{2q} n_1 n_2 ... n_{2p}}\,. \tag{A.15}$$

This is

$$R_{(q)\mu}{}^{m} = \mp\kappa^{2(p-q)}\frac{(2q-1)!}{(2p-1)!}\,G_{(p)\mu}{}^{m}\,. \tag{A.16}$$

Again we consider the case of Minkowskian signature (the lower sign) first, to dispose of it as above. In this case (A.14) and (A.16) simply yield

$$(2q)!\,\kappa^{2p}\,R_{(p)} \;+\; (2p)!\,\kappa^{2q}\,R_{(q)} \;=\; 0\,, \tag{A.17}$$

which is too weak to solve an Einstein equations. Hence again we restrict to the Euclidean signature case.

Now the gravitational system in Euclidean space appropriate to (A.14) and (A.16) is

$$\mathcal{L}_{\rm grav} \;=\; e\,\left(\frac{1}{2}\kappa_1^{2p}\;R_{(p,q)} \;+\; \frac{1}{2}\kappa_2^{2q}\;R_{(q,p)} \;+\; d!\;\Lambda\right)\,, \tag{A.18}$$

whose Einstein equations are

$$\kappa_1^{2p}\;(2p)\;G_{(p)\mu}{}^{m} \;+\; \kappa_2^{2q}\;(2q)\;G_{(q)\mu}{}^{m} \;=\; \frac{(2(p+q))!}{(2p)!\,(2q)!}\,\Lambda\;e_\mu^m\,. \tag{A.19}$$

The consistency conditions arising from the identification of (A.14) with (A.19), and (A.16) with (A.19), are

$$\kappa_1^{2p}\,R_{(p)} \;=\; \kappa_2^{2q}\,R_{(q)} \;=\; -2\,\frac{(2(p+q))!}{(2p)!\,(2q)!}\;\Lambda \tag{A.20}$$

and

$$\kappa^{2(q-p)} = \frac{(2q)!}{(2p)!}\,\frac{\kappa_2^{2q}}{\kappa_1^{2p}}\,. \tag{A.21}$$

Condition (A.20) is the analogue of (A.12), to which it reduces when one sets $p=q$, while (A.21) is an additional condition in this case constraining the relative values of the three constants κ_1, κ_2 and κ, with dimensions of length.

Again, provided (A.21) is satisfied, the two conditions (A.20) imply that the Einstein equation is satisfied by the solutions of the double selfduality equations. This completes our discussion of double-selfdual spaces.

References

[1] J. M. Charap and M. J. Duff, Phys. Lett. B **69** (1977) 445.

[2] A. Chakrabarti, Fortsch. Phys. **35** (1987) 1.

[3] T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Rept. **66** (1980) 213.

[4] Y. Brihaye and E. Radu, Europhys. Lett. **75** (2006) 730 [arXiv:hep-th/0605111].

[5] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35** (1975) 760.

[6] A. A. Belavin, A. M. Polyakov, A. S. Shvarts and Y. S. Tyupkin, Phys. Lett. B **59** (1975) 85.

[7] R. Jackiw and C. Rebbi, Phys. Rev. D **14** (1976) 517.

[8] G. W. Gibbons and C. N. Pope, Commun. Math. Phys. **61** (1978) 239.

[9] J. M. Maldacena and L. Maoz, JHEP **0402** (2004) 053 [arXiv:hep-th/0401024].

[10] A. Hosoya and W. Ogura, Phys. Lett. B **225** (1989) 117.

[11] D. O'Se and D. H. Tchrakian, Lett. Math. Phys. **13** (1987) 211.

[12] G. M. O'Brien and D. H. Tchrakian, J. Math. Phys. **29** (1988) 1212.

[13] H. Kihara and M. Nitta, arXiv:hep-th/0703166.

[14] Z. Ma and D. H. Tchrakian, J. Math. Phys. **31** (1990) 1506.

[15] D. H. Tchrakian, Phys. Lett. B **150** (1985) 360.

[16] M. B. Green, J. H. Schwarz and E. Witten, *"Superstring Theory "* (1987), Cambridge, Cambridge University Press.

[17] J. Polchinski, *"String theory "* (1998), Cambridge, Cambridge University Press.

[18] A. Strominger, Nucl. Phys. B **343** (1990) 167 [Erratum-ibid. B **353** (1991) 565].

[19] G. W. Gibbons, D. Kastor, L. A. J. London, P. K. Townsend and J. H. Traschen, Nucl. Phys. B **416** (1994) 850 [arXiv:hep-th/9310118].

[20] H. Kihara and M. Nitta, arXiv:0704.0505 [hep-th].

[21] D. H. Tchrakian, in *"Tianjin 1992: Proceedings. Differential geometric methods in theoretical physics*, 584-587.

[22] B. Grossman, T. W. Kephart and J. D. Stasheff, Commun. Math. Phys. **96** (1984) 431.

[23] C. Saclioglu, Nucl. Phys. B **277** (1986) 487.

[24] K. Fujii, Lett. Math. Phys. **12** (1986) 363.

[25] K. Fujii, Lett. Math. Phys. **12** (1986) 371.

[26] A.A. Tseytlin, *Born–Infeld action, supersymmetry and string theory*, in *Yuri Golfand memorial volume*, ed. M. Shifman, World Scientific (2000).

[27] E. Bergshoeff, M. de Roo and A. Sevrin, Fortsch.Phys. 49 (2001) 433-440; Nucl.Phys.Proc.Suppl. 102 (2001) 50-55. E. A. Bergshoeff, M. de Roo and A. Sevrin, Fortsch. Phys. **49** (2001) 433 [Nucl. Phys. Proc. Suppl. **102** (2001) 50] [arXiv:hep-th/0011264].

[28] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, JHEP **0106** (2001) 034 [arXiv:hep-th/0102009].

[29] A. Chakrabarti and D. H. Tchrakian, Phys. Rev. D **65** (2002) 024029 [arXiv:hep-th/0101160].

[30] B. Julia, J. Levie and S. Ray, JHEP **0511** (2005) 025 [arXiv:hep-th/0507262].

[31] E. Radu and D. H. Tchrakian, Phys. Rev. D **71** (2005) 125013 [arXiv:hep-th/0502025].

[32] Y. Brihaye, B. Hartmann and J. Kunz, Phys. Lett. B **441** (1998) 77 [arXiv:hep-th/9807169].

[33] J. Bjoraker and Y. Hosotani, Phys. Rev. D **62** (2000) 043513 [arXiv:hep-th/0002098].

[34] Y. Brihaye, A. Chakrabarti and D. H. Tchrakian, Class. Quant. Grav. **20** (2003) 2765 [arXiv:hep-th/0202141].

[35] B. Tekin, Phys. Rev. D **65** (2002) 084035 [arXiv:hep-th/0201050].