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Gauge invariance and background field formalism in the exact renormalisation group

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Abstract

We discuss gauge symmetry and Ward-Takahashi identities for Wilsonian flows in pure Yang-Mills theories. The background field formalism is used for the construction of a gauge invariant effective action. The symmetries of the effective action under gauge transformations for both the gauge field and the auxiliary background field are separately evaluated. We examine how the symmetry properties of the full theory are restored in the limit where the cut-off is removed.

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1. Introduction

The Wilsonian or exact renormalisation group (ERG) [1] has been successfully applied to both perturbative and non-perturbative phenomena in field theory. The main advantages of such an approach are its flexibility and the comparatively simple numerical implementation. Applications to gauge theories are much more involved because it is less obvious how a Wilsonian cut-off can be implemented for a (non-linear) gauge symmetry. Much work has been devoted to overcoming this intricacy [2–10] (see [10] for a review). We focus the discussion on pure Yang-Mills theory, since the inclusion of fermions is straightforward. The ERG equation for the corresponding effective action Γ_k describes how Γ_k changes under an infinitesimal variation of the infra-red scale k:

$$\partial_t\Gamma_k[A,c,c^*;\bar A]=\frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta A\delta A}+R_A\right)^{-1}\partial_t R_A\,-\,\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta c\delta c^*}+R_C\right)^{-1}\partial_t R_C\,. \tag{1}$$

Here, $t=\ln k$ is the logarithmic scale parameter and the trace Tr denotes a sum over momenta, Lorentz and gauge group indices. The functions R_A and R_C implement the Wilsonian infra-red cut-off for the gauge field A and the ghost fields c and c^* respectively. We also introduced an non-dynamical auxiliary field $\bar A$, the so-called background gauge field.

In the present Letter we discuss in detail the symmetry properties of coarse-grained effective actions for non-Abelian gauge theories and their flows (1) within the background field approach. A similar programme has been put forward for Abelian gauge theories in [6]. The most attractive feature of a background field formalism is that it provides a gauge invariant effective action [11], which is defined via an identification of the auxiliary background field with the original gauge field. This property can be maintained even within a Wilsonian approach [2].

The symmetry properties of a background field effective action are naturally encoded in Ward-Takahashi identities. These identities are derived by applying gauge transformations separately to dynamical fields and background field. Clearly, the investigation of the Ward-Takahashi identities play a pivotal rôle in the evaluation of the coarse grained effective action. We shall argue that it is crucial to separately discuss the action of gauge transformations on the dynamical fields and the gauge transformations on the background field. Solving the related Ward-Takahashi (or BRST) identities poses a fine-tuning problem which is known to be soluble in perturbation theory. For Wilsonian flows these identities have been discussed in [3,5,7]. In the present approach we deal with an additional Ward-Takahashi identity related to gauge transformations of the background field. Whether this imposes an additional fine-tuning condition is an important question which has not yet been addressed.

In the present contribution we close this conceptional gap in the formalism. The functional form of the Ward-Takahashi identities is established for both transformations. We argue that no additional fine tuning problem related to the existence of the new identity arises. Both Ward-Takahashi identities are shown to be compatible with the flow. This guarantees that both the usual Ward-Takahashi identity and the background field identity hold in the limit where the cut-off is removed. In this manner the gauge invariance displayed in the effective action is the physical gauge invariance rather than an auxiliary symmetry.

2. Background field formalism

We briefly summarise some important points about the background field formalism, in particular the rôle of the different gauge transformations present in this approach. In the background field formalism an auxiliary (non-dynamical) gauge field $\bar{A}$ is introduced. The formalism then hinges on the use of a gauge fixing condition which depends on this field in such a way that the condition is invariant under a simultaneous gauge transformation of $\bar{A}$ and of the dynamical fields A, c and c^*. This can be used to define an effective action which is invariant under the combined gauge transformation mentioned above. As the auxiliary field $\bar{A}$ is involved in this transformation it is clear that the invariance of the effective action is, *a priori*, an auxiliary symmetry. The essential point is that this symmetry for the special choice $\bar{A} = A$ becomes the inherent gauge symmetry of the theory.

The starting point is the gauge-fixed classical action for a pure Yang-Mills theory including the ghost term

$$S = S_{\rm cl} + S_{\rm gf} + S_{\rm gh}\,. \tag{2}$$

The classical action $S_{\rm cl} = \frac{1}{4}\int_x F^a_{\mu\nu}F^a_{\mu\nu}$ contains the field strength tensor $F_{\mu\nu}(A) = \partial_\mu A_\mu - \partial_\nu A_\mu + g\,[A_\mu,\, A_\nu]$, where $F_{\mu\nu} \equiv F^a_{\mu\nu}t^a$ and $A_\mu = A^a_\mu t^a$ with the generators t^a satisfying $[t^a,\, t^b] = f^{abc}t^c$ and $\mathrm{tr}\, t^a t^b = -\frac{1}{2}\,\delta^{ab}$. We also employ the shorthand notation $\int_x \equiv \int d^dx$. In the adjoint representation, the covariant derivative is

$$D^{ab}_\mu(A) = \delta^{ab}\partial_\mu + g f^{acb} A^c_\mu\,. \tag{3}$$

The natural choice for the gauge fixing is the background field gauge

$$S_{\rm gf} = -\frac{1}{2\xi}\int_x\,(A-\bar{A})^a_\mu\,\bar{D}^{ab}_\mu \bar{D}^{bc}_\nu\,(A-\bar{A})^c_\nu \tag{4}$$

which involves the covariant derivative $\bar{D} \equiv D(\bar{A})$. The corresponding ghost action is given by

$$S_{\rm gh} = -\int_x c^*_a\, \bar{D}^{ac}_\mu D^{cd}_\mu\, c_d\ . \tag{5}$$

We now turn to the symmetries of the action in (2) and introduce two different gauge transformations. The first one, given by the infinitesimal generator $\mathcal{G}^a$, gauge transforms the original set of fields A, c, and c^*. It generates gauge transformations representing the underlying symmetry of the theory. The transformation is defined on arbitrary functionals of A, c, c^* and $\bar{A}$ as

$$\mathcal{G}^a = D^{ab}_\mu \frac{\delta}{\delta A^b_\mu} - g\, f^{abc}\left(c_c\,\frac{\delta}{\delta c_b} + c^*_c\,\frac{\delta}{\delta c^*_b}\right). \tag{6}$$

Finite gauge transformations with parameter ω^a are generated by $\exp[-i\int_x \omega^a \mathcal{G}^a]$. The action of (6) on the fields is given by

$$\begin{aligned} &\mathcal{G}^a(x) A^b_\mu(y) = D^{ab}_{\mu,x}(A)\delta(x-y), &&\mathcal{G}^a(x) c_b(y) = -g f^a{}_{bc}\, c_c(x)\delta(x-y),\\ &\mathcal{G}^a(x)\bar{A}^b_\mu(y) = 0, &&\mathcal{G}^a(x) c^*_b(y) = -g f^a{}_{bc}\, c^*_c(x)\delta(x-y). \end{aligned} \tag{7}$$

The gauge field A is transformed inhomogeneously, the ghosts transform as tensors according to their representation and the background field is invariant. The subscript x for the covariant derivative refers to the variable on which the derivative operates and it will be omitted whenever it is unambiguous. From (7) it follows that the covariant derivative transforms as a tensor,

$$\mathcal{G}^a(x) D^{bc}_{\mu,y} = g f^{bdc} D^{ad}_{\mu,x}\delta(x-y) \equiv g\,\left([t^a\,\delta(x-y),\, D_{\mu,y}]\,\right)^{bc}. \tag{8}$$

The second gauge transformation, given by the generator $\bar{\mathcal{G}}^a$, transforms only the background field $\bar{A}$

$$\bar{\mathcal{G}}^a = \bar{D}^{ab}_\mu \frac{\delta}{\delta \bar{A}^b_\mu}. \tag{9}$$

On the fields it acts as

$$\bar{\mathcal{G}}^a(x)\bar{A}^b_\mu(y) = \bar{D}^{ab}_\mu\delta(x-y), \qquad\qquad \bar{\mathcal{G}}^a A^b_\mu = \bar{\mathcal{G}}^a c_b = \bar{\mathcal{G}} c^*_b = 0\ . \tag{10}$$

Since $\bar{\mathcal{G}}^a$ acts on $\bar{A}$ as $\mathcal{G}^a$ on A it follows that the covariant derivative $\bar{D}$ transforms as a tensor as displayed in (8) replacing A with $\bar{A}$. $\bar{\mathcal{G}}^a$ transforms the background field inhomogeneously while leaving the dynamical fields unchanged. The background gauge transformation $\bar{\mathcal{G}}^a$ is introduced as an auxiliary transformation which, as it stands, does not carry any physical information. We remark that (10) implies $\bar{\mathcal{G}}^a(x)(A-\bar{A})^b_\mu(y) = -\bar{D}^{ab}_{\mu,x}\delta(x-y)$.

Let us now study the action of $\mathcal{G}^a$ and $\bar{\mathcal{G}}^a$ on the action S. The classical action is trivially invariant under both the gauge symmetry (6) and under the background gauge symmetry (9) since it does not depend on the background field. In turn, neither the gauge fixing term nor the ghost field action are invariant under (6) or (9). Their variation under (6) yields

$$\mathcal{G}^a(x)S_{\rm gf} = \frac{1}{\xi}D^{ab}_\mu \bar{D}^{bc}_\mu \bar{D}^{cd}_\nu (A-\bar{A})^d_\nu(x) \tag{11}$$

$$\mathcal{G}^a(x)S_{\rm gh} = f^{bdc}\bar{D}^{ad}_\mu \left(c^*_b D^{ce}_\mu c_e\right) \ . \tag{12}$$

However, making use of (4), (5) and (10) it is easy to see, that (11) and (12) are just $-\bar{\mathcal{G}}^a S_{\rm gf}$ and $-\bar{\mathcal{G}}^a S_{\rm gh}$ respectively. Thus, each term in the action $S[A,c,c^*;\bar{A}]$ is *separately* invariant under the combined transformation $\mathcal{G}+\bar{\mathcal{G}}$. This brings us to a key point of the background field formalism. The invariance of $S[A,c,c^*;\bar{A}]$ under $\mathcal{G}+\bar{\mathcal{G}}$ implies that the action $\hat{S}[A,c,c^*] \equiv S[A,c,c^*;\bar{A}=A]$ is invariant under the *physical* symmetry (6), $\mathcal{G}^a\hat{S}[A,c,c^*]=0$, with $S[A,c,c^*;\bar{A}]$ satisfying the classical 'Ward-Takahashi identity' $\mathcal{G}^a S = \mathcal{G}^a(S_{\rm gf}+S_{\rm gh})$.

At quantum level these statements turn into gauge invariance of the effective action $\Gamma[A,c,c^*;\bar{A}=A]$ with $\Gamma[A,c,c^*;\bar{A}]$ satisfying the Ward-Takahashi identity of a non-Abelian gauge theory. Note that only the combination of both statements gives a physical meaning to gauge invariance of $\Gamma[A,c,c^*;\bar{A}=A]$. In the quantised theory where the sources couple only to the fluctuation field

$$a^a_\mu = A^a_\mu - \bar{A}^a_\mu \ , \tag{13}$$

the resulting theory is evaluated for vanishing expectation value $\langle a\rangle = 0$ (Notice that the gauge fixing condition (4) only constrains a^a_μ).

3. Wilsonian flows

We now follow the strategy sketched above for the case of a coarse-grained effective action along the lines in [2]. Scale-dependent regulator terms for the gauge and the ghost fields, respectively, are added to the Yang-Mills action (2),

$$S_k = S + \Delta S_k \ , \qquad \Delta S_k = \Delta S_{k,A} + \Delta S_{k,C} \ . \tag{14}$$

The new terms are (non-local) operators quadratic in the fields and given by

$$\Delta S_{k,A} = \frac{1}{2}\int_x (A-\bar{A})^a_\mu \, R_{A\,\mu\nu}^{\,ab}(P_A^2)\,(A-\bar{A})^b_\nu \tag{15}$$

$$\Delta S_{k,C} = \int_x c^*_a \, R_C^{ab}(P_C^2)\, c_b \;, \tag{16}$$

where the arguments P_A^2 and P_C^2 of the regulator functions are appropriately defined Laplaceans. A suitable choice for them in the present context is $(P_A^2)^{ab}_{\mu\nu} = (-\bar{D}^2)^{ab}\delta_{\mu\nu}$ and $(P_C^2)^{ab} = (-\bar{D}^2)^{ab}$, which are operators with a positive spectrum of eigenvalues. The regulator functions R_A and R_C depend on both the coarse graining scale k, and on the background field $\bar{A}^a_\mu$ via P_A^2 and P_C^2. A typical example for an exponentially smooth regulator function is given by $R(P^2) = P^2/(\exp P^2/k^2 - 1)$.

In order to maintain the invariance of S_k under the combined transformation $\mathcal{G}^a + \bar{\mathcal{G}}^a$ we have to ensure that both (15) and (16) vanish under $\mathcal{G}^a + \bar{\mathcal{G}}^a$. For the action of $\mathcal{G}^a$ on the regulator terms we find

$$\mathcal{G}^a(x)\,\Delta S_k = -D^{ab}_\mu \, R_{A\,\mu\nu}^{\,bc}(P_A^2)\,(A-\bar{A})^c_\nu - g\int_y c^*_b(y)\left([t^a\delta(x-y),\, R_C(P^2_{C,y})]\right)^{bc} c_c(y)\;. \tag{17}$$

To compute how $\bar{\mathcal{G}}^a$ operates on (15) and (16), it is helpful to consider first the action of (9) on $\bar{D}^2$. From (8) (with $A=\bar{A}$) it follows immediately that $\bar{D}^2$ transforms as a tensor. Therefore P^2 and $R(P^2)$, which are both functions of $-\bar{D}^2$, also transform as tensors under (9)

$$\bar{\mathcal{G}}^a(x)\, R(P_y^2) = [t^a\delta(x-y),\, R(P_y^2)]. \tag{18}$$

Using also (10) it is straightforward to show that (17) equals $-\bar{\mathcal{G}}^a\,(\Delta S_{k,A} + \Delta S_{k,C})$ which establishes the desired property.[1]

So far, we have restricted the discussion to the classical action (2) with the regulator terms (14) added. The computation of the effective action Γ_k follows with the usual procedure. We introduce sources (J,η,η^*) for the fields $(A-\bar{A},c,c^*)$ to consider first the Schwinger functional $W_k \equiv W_k[J_\mu,\,\eta,\,\eta^*;\,\bar{A}_\mu]$, given by

$$\exp W_k = \int \prod_a \left\{\mathcal{D}A^a_\mu\,\mathcal{D}c_a\,\mathcal{D}c^*_a\right\}\,\exp\left[-S_k + \int (J^a_\mu (A-\bar{A})^a_\mu + \eta^*_a c_a - c^*_a\eta_a)\right]\;. \tag{19}$$

The effective action Γ_k is given by its Legendre transform

[1]More generally, P_A^2 and P_C^2 need not to be of the form as given in the text. The required symmetry properties remain unchanged as long as they transform as tensors under the background gauge transformation. Our choice above is one such example.

$$\Gamma_k[A,c,c^*;\bar{A}] = -W_k[J,\eta,\eta^*;\bar{A}] - \Delta S_k[A,c,c^*;\bar{A}] + \int_x \Big(J^a_\mu (A-\bar{A})^a_\mu + \eta^*_a \bar{c}_a - c^*_a \eta_a \Big). \quad (20)$$

Here, Γ_k is a functional of the expectation values of the fields (*e.g.* $A - \bar{A} \equiv -\delta W_k/\delta J$, etc).[2] The flow equation for Γ_k has already been given in (1).

Finally we introduce the effective action $\hat{\Gamma}_k$ which corresponds to Γ_k evaluated at $\bar{A} = A$,

$$\hat{\Gamma}_k[A,c,c^*] \equiv \Gamma_k[A,c,c^*;\ \bar{A} = A]. \quad (21)$$

As we shall argue below, this action is gauge-invariant. Its flow equation is simply given by the one for Γ_k in (1), evaluated at $\bar{A} = A$. It is important to stress that the flow of $\hat{\Gamma}_k$, since it depends on the second functional derivatives of Γ_k w.r.t the dynamical fields (at $\bar{A} = A$), is a functional of Γ_k and *not* a functional of $\hat{\Gamma}_k$. This makes it mandatory to study not only the symmetries of $\hat{\Gamma}_k$ but also those of Γ_k.

4. Modified and background field Ward-Takahashi identities

We now turn to a detailed discussion of the Ward-Takahashi identities related to the transformations (6) and (9). Ward-Takahashi identities follow from an invariance of the Schwinger functional under gauge transformations. In the Wilsonian formalism, these identities are modified due to the presence of the regulator terms. The identity which follows from considering $\mathcal{G}^a\Gamma_k$ is denoted as the *modified* Ward-Takahashi identities (mWI). A second identity is derived from the background gauge transformations $\bar{\mathcal{G}}^a\Gamma_k$, leading to the *background field* Ward-Takahashi identities (bWI).

Let us first summarise some immediate consequences of the invariance of S_k under the action of $\mathcal{G}^a + \bar{\mathcal{G}}^a$. It can be read off from the definitions of the Schwinger functional (19) and the effective action (20), that the combination $\mathcal{G}^a + \bar{\mathcal{G}}^a$ leaves the functional Γ_k invariant for generic A and $\bar{A}$,

$$\Big(\mathcal{G}^a + \bar{\mathcal{G}}^a\Big)\, \Gamma_k = 0. \quad (22)$$

Some comments are in order. Within a Wilsonian approach, the physical Green's function are approached in the limit $k \to 0$, where Γ_k approaches the full quantum effective action. We have already pointed out that the statement of *physical* gauge invariance corresponds

[2]For simplicity, we do not introduce other names for these fields because we shall only be concerned with Γ_k in the remaining part of the letter.

to (22) at $k = 0$, with the fields A and $\bar{A}$ identified, only if $\Gamma_{k=0}$ satisfies the *usual* Ward-Takahashi identity connected to $\mathcal{G}^a$. Therefore it is necessary to keep track of the action of the transformations $\mathcal{G}^a$ and $\bar{\mathcal{G}}^a$ on Γ_k separately. Eq. (22) also implies that the effective action $\hat{\Gamma}[A, c, c^*]$ satisfies

$$\mathcal{G}^a \hat{\Gamma}_k = 0, \tag{23}$$

which for $k = 0$ expresses the desired physical gauge invariance. Consequently, for $k \neq 0$, physical gauge invariance is encoded in the behaviour of Γ_k under the transformation $\mathcal{G}^a$. This is also evident from the fact that the flow of $\hat{\Gamma}_k$ is a functional of Γ_k.

We now give a detailed derivation of the related modified Ward-Takahashi identity. We start by applying $\mathcal{G}^a$ to the Schwinger functional W_k (19). To be more precise, we apply $\mathcal{G}^a$ to the integration fields variables A, c, c^* which leaves W_k invariant since the path integral measure is invariant under the action of $\mathcal{G}^a$ and hence $\mathcal{G}^a W_k = 0$. Collecting all terms and making the Legendre transformation to Γ_k yields

$$\mathcal{G}^a \Gamma_k = \langle \mathcal{G}^a \left(S_{\rm gf} + S_{\rm gh} + \Delta S_k \right) \rangle_J, \tag{24}$$

where the expectation value $\langle \cdots \rangle_J$ stands for connected Green's functions in the external source $(J, \eta, \eta^*) = (\delta_A \Gamma_k, \delta_c \Gamma_k, \delta_{c^*} \Gamma_k)$. We evaluate the expectation values in (24) by using (11), (12) and (17). After some algebra we arrive at

$$\mathcal{G}^a(x)\, \Gamma_k = \mathcal{G}^a(x)\ (S_{\rm gf} + S_{\rm gh}) + L^a_k(x) + L^a_{R,k}(x)\ . \tag{25}$$

Both L_k and $L_{R,k}$ display loop terms. The first term L_k stands for the well-known loop contributions to Ward-Takahashi identities in non-Abelian gauge theories originating from $\langle \mathcal{G}^a (S_{\rm gf} + S_{\rm gh}) \rangle_J$. L_k is given by

$$\begin{aligned} L^a_k(x) = g &\left[\frac{1}{\xi}\ f^{adb} \left(\left(\bar{D} \otimes \bar{D} \right)^{bc}_{\mu\nu} G_A{}^{cd}_{\nu\mu} \right)(x,x) - f^{bdc} \bar{D}^{ad}_{\mu,x} \left(D^{ce}_\mu G_C{}^{eb} \right)(x,x) \right] \\ &- g^2 \bar{D}^{ad}_{\mu,x} \left(f^{bdc} f^{che} \Big[G_A{}^{hg}_{\mu\nu} \frac{\delta}{\delta A^g_\nu} + G_{AC}{}^{hg}_\mu \frac{\delta}{\delta c^g} + G_{AC^*}{}^{hg}_\mu \frac{\delta}{\delta c^{*g}} \Big] G_C{}^{eb} \right)(x,x), \end{aligned} \tag{26}$$

where $(\bar{D} \otimes \bar{D})^{ab}_{\mu\nu} = \bar{D}^{ac}_\mu \bar{D}^{cb}_\nu$. We have also introduced the propagators G_A, G_C, G_{AC} and G_{AC^*}, whose inverses are given by

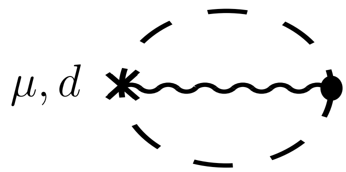
$$G^{-1}_{k,A}{}^{ab}_{\mu\nu} \equiv \frac{\delta^2 \Gamma_k}{\delta A^\mu_a \delta A^\nu_b} + R^{ab}_{A,\mu\nu}\ , \qquad G^{-1}_{k,C}{}^{ab} \equiv \frac{\delta^2 \Gamma_k}{\delta c_a \delta c^*_b} + R^{ab}_C\ , \tag{27}$$

$$G^{-1}_{AC}{}^{ab}_{\mu} \equiv \frac{\delta^2 \Gamma_k}{\delta A^\mu_a \delta c_b}\ , \qquad G^{-1}_{AC^*}{}^{ab}_{\mu} \equiv \frac{\delta^2 \Gamma_k}{\delta A^\mu_a \delta c^*_b}\ . \tag{28}$$

The term in the second line in (26) is a two-loop contribution to the Ward-Takahashi identity. To see this more explicitly, let us write out the first of its contributions:

$$\bar{D}^{ad}_{\mu,x}\left(f^{bdc}f^{che}\int_y {G_A}^{hg}_{\mu\nu}(x,y)\frac{\delta}{\delta A^g_\nu(y)}{G_C}^{eb}(x,x)\right)\ . \tag{29}$$

Note that $\frac{\delta}{\delta A(y)}G_C(x,x)$ is a loop closing at x with a gauge field vertex (at y) and $G_A(x,y)$ is a line emanating at x and connecting to the vertex at y (see figure).



Feynman-diagram for the expression in braces in (29)

The term $L_{R,k}$ in (25) comprises the loop contribution of $\langle \mathcal{G}^a \Delta S_k\rangle_J$ coming directly from the coarse graining and it is given by

$$L^a_{R,k}(x) = g\mathrm{tr}_{\mathrm{ad}}\, t^a\, \left(\left[G_A^{\mu\nu},\, R_A^{\nu\mu}(P_A^2)\right](x,x) - \left[G_C,\, R_C(P_C^2)\right](x,x)\right)\ . \tag{30}$$

This term clearly disappears in the limit $k\to 0$, $L_{R,0}\equiv 0$.

The identity expressed by (25) is the modified Ward-Takahashi identities where the contribution from the coarse graining is contained in $L_{R,k}$. It follows that the mWI (25) turns into the usual WI for $k=0$:

$$\mathcal{G}^a\Gamma = L^a_0. \tag{31}$$

It is left to cast $\bar{\mathcal{G}}^a\Gamma_k$ into an explicit form. Starting by applying $\bar{\mathcal{G}}^a$ to $W_k[J,\eta,\bar\eta;\bar A]$ leads to $\bar{\mathcal{G}}^a W_k = -\bar{D}^{ab}_\mu J^b_\mu - \langle\bar{\mathcal{G}}^a\left(S_{\mathrm{gf}}+S_{\mathrm{gh}}+\Delta S_k\right)\rangle_J$. It follows from (20) that

$$\bar{\mathcal{G}}^a(\Gamma_k+\Delta S_k) = \langle\bar{\mathcal{G}}^a\left(S_{\mathrm{gf}}+S_{\mathrm{gh}}+\Delta S_k\right)\rangle_J. \tag{32}$$

Using (11), (12), (17) and (18) this last equation takes the form

$$\bar{\mathcal{G}}^a\Gamma_k = \bar{\mathcal{G}}^a(S_{\mathrm{gf}}+S_{\mathrm{gh}}) - (L^a_k + L^a_{R,k}). \tag{33}$$

Eq. (22) follows immediately from this identity and the mWI (25).

We close this section with a comment on the finiteness of (25) and (33). The quantum corrections in L^a_k are familiar as they appear already in the usual WI. In perturbation theory, these terms require an additional UV regularisation and renormalisation. In the present Wilsonian framework, however, we are dealing with UV *finite* quantities. The flow of the effective action Γ_k starts with a finite initial condition at $k = \Lambda$, Γ_Λ. The finiteness of L_k^a then follows from the observation that the flow (which is IR *and* UV finite) cannot generate UV divergences. Therefore, in contrast to perturbation theory, no additional renormalisation is needed to make them finite, which is one of the key advantages of the ERG approach.

5. Symmetries of the flow and physical gauge invariance

The implementation of coarse graining modifies the gauge symmetry of the theory as we have discussed. At the formal level it is clear that the original symmetry is restored when the coarse graining scale is removed (see also (31)). A more delicate problem is to guarantee that this also happens at the level of the solution to the flow equation.

To understand how gauge invariance is encoded throughout the flow, it is pivotal to also study the action of the symmetry transformations on $\partial_t\Gamma_k$ (see (1)). Firstly we derive how the combined transformation $\mathcal{G}^a + \bar{\mathcal{G}}^a$ acts on $\partial_t\Gamma$ where we only want to argue at the level of the flow equation. The flow equation (1) functionally depends on second derivatives of Γ_k w.r.t. fields A, c, c^* and on $R, \partial_t R$. Hence, we are interested on the action of $\mathcal{G}^a + \bar{\mathcal{G}}^a$ on these quantities. We note that

$$
\begin{aligned}
(\mathcal{G}^a + \bar{\mathcal{G}}^a)(x)\frac{\delta^2\Gamma_k}{\delta A_b^\mu \delta A_c^\nu} &= \Big(\Big[\delta_x t^a,\, \frac{\delta^2\Gamma_k}{\delta A_\mu \delta A_\nu}\Big]\Big)^{bc},\\
(\mathcal{G}^a + \bar{\mathcal{G}}^a)(x)\,\frac{\delta^2\Gamma_k}{\delta c_b \delta c_c^*} &= \Big(\Big[\delta_x t^a,\, \frac{\delta^2\Gamma_k}{\delta c \delta c^*}\Big]\Big)^{bc}
\end{aligned} \tag{34}
$$

and similar identities for mixed derivatives. Here we have used (22) and the commutator of two derivatives w.r.t. the fields A, c, c^* and $\mathcal{G}^a$. For the sake of simplicity we have introduced the short hand notation $[\delta_x,\, \mathcal{O}](y,z) = \delta(y-x)\mathcal{O}(y,z) - \mathcal{O}(y,z)\delta(z-x)$. This facilitates the following conclusion. Eq. (34) states that second derivatives of Γ_k w.r.t. the fields A, c, c^* transform as tensors under $\mathcal{G}^a + \bar{\mathcal{G}}^a$. Together with (18) this implies that the propagators G_A and G_C transform as tensors:

$$
(\mathcal{G}^a + \bar{\mathcal{G}}^a)(x)G_A = [t^a\delta_x,\, G_A], \qquad (\mathcal{G}^a + \bar{\mathcal{G}}^a)G_C = [t^a\delta_x,\, G_C]. \tag{35}
$$

With (18) and (35) we conclude

$$
(\mathcal{G}^a + \bar{\mathcal{G}}^a)\partial_t\Gamma_k = 0. \tag{36}
$$

This implies that $\mathcal{G}^a\partial_t\hat{\Gamma}_k = 0$. Note that the only input for (36) was the invariance of Γ_k. Thus, if the initial effective action Γ_Λ is invariant under $\mathcal{G}^a + \bar{\mathcal{G}}^a$ it follows that the full effective action Γ_0 satisfies $(\mathcal{G}^a + \bar{\mathcal{G}}^a)\Gamma_0 = 0$. In other words, (22) and (36) are the proof that flow and $(\mathcal{G}^a + \bar{\mathcal{G}}^a)$ commute. Moreover, Γ_0 satisfies the usual WI (31). This means that we can follow the line of arguments of the background field formalism as explained in the second section. Thus we conclude that $\mathcal{G}^a\hat{\Gamma}_0 = 0$ displays physical gauge invariance.

Now we continue with a remark on the consistency of the mWI (25) with the flow. As for other formulations of Wilsonian flows in gauge theories [3,7,8,10], the flow of the modified Ward-Takahashi identity is proportional to the mWI itself. Very schematically this identity has the form

$$\Big(\partial_t - \mathcal{O}\Big)(\mathcal{G}^a\Gamma_k - \mathcal{G}^a\left(S_{\rm gf} + S_{\rm gh}\right) - L^a_k - L^a_{R,k}) = 0, \tag{37}$$

where $\mathcal{O}$ does only depend on Γ_k and derivatives of the field. An explicit expression for $\mathcal{O}$ in (37) has been given in [10]. Eq. (37) states that if the effective action Γ_k satisfies the mWI at some (initial) scale $k = \Lambda$, then Γ_k *automatically* satisfies the mWI for all scales k, provided it is obtained from integrating the flow equation. Hence, Γ_0 satisfies the usual Ward-Takahashi identity.

We also like to briefly discuss the connection between gauge invariance in the present approach and BRST-invariance in the BRST formalism. An analogous treatment within a BRST formulation has been given [3,5,7] where the information about the gauge symmetry is carried by a modified BRST-identity. This identity reduces to the mWI (25) by integrating out the ghosts and putting the BRST charges to zero. Its advantage in the standard perturbative approach is that BRST invariance leads to a bilinear equation in derivatives of the effective action (the well-known master equation). In the presence of a coarse graining term, however, this identity receives an additional term which contains the propagator derived from the effective action, thus spoiling the bilinear structure. This additional term has the same form as the loop terms already present in Ward-Takahashi identities for non-linear symmetries. Therefore, we see no advantage in studying BRST invariance rather than the usual Ward-Takahashi identities.

In summary, we have established the complete set of equations relevant for the control of gauge invariance within the present approach. In particular, it has been shown that there is no additional fine-tuning condition, despite the presence of a background field. These results can be used for further interesting applications in the Wilsonian approach to non-Abelian gauge theories. The formalism is well-suited for analytic calculations. For example a consistent calculation of 2-loop quantities can be done analytically [12]. Note that the introduction of a background field to the cut-off terms is not restricted to the background field gauge discussed here. In axial gauges a similar procedure can be used to obtain a gauge invariant effective action. Here analytic calculations of the full effective action (in some approximation) are accessible [13].

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