covqmp.tex dias-stp-00-22

Covariant relativistic quantum theory

T. Garavaglia*

Institiúid Árd-léinn Bhaile Átha Cliath, Baile Átha Cliath 4, Éire†

Covariant classical particle dynamics is described, and the associated covariant relativistic particle quantum mechanics is derived. The invariant symmetric bracket is defined on the space of quantum amplitudes, and its relation to a generalized Hamiltonian dynamics and to a covariant Schrödinger type equation is shown. Examples for relativistic potential problems are solved. Mathematically and physically acceptable probability densities for the Klein-Gordon equation and for the Dirac equation are derived, and a new continuity equation for each case is given. The quantum distribution for mass is discussed, and unambiguous representations of four-velocity and four-acceleration operators are given.

PACS: 04.60, 03.65P, 03.65, 03.30 -

---

* E-mail: bronco@stp.dias.ie

† Also *Institiúid Teicneolaíochta Bhaile Átha Cliath.*

1. INTRODUCTION

The covariant form of classical dynamics [1] and [2] provides an elegant method for analyzing problems in relativistic particle dynamics. This is briefly described, and the corresponding covariant quantum formulation is presented. The method of analyzing potential problems in both the classical and the quantum case is given. Among the new results that follow from this analysis are the resolution of the ambiguity associated with the interpretation of probability density for the Klein-Gordon equation, a new equation of continuity for this case, and a similar equation for the Dirac equation. Also a representation of the covariant four-velocity operator is given. Since the covariant Hamiltonian operator is quadratic in four-momentum in the scalar case and linear in the spinor case, this four-velocity operator is unambiguously defined, and it is related to the corresponding classical quantities through the expectation values of the quantum operator. In addition, an exact quantum solution is found for the relativistic motion of a spin zero particle in a uniform field. The method can be easily extended to more complicated examples.

The Lorenz invariant parameter that is relevant for dynamical development in classical covariant mechanics is proper distance s, proper time times the speed of light, and the invariant parameter that characterizes the rest energy of a particle is its mass. These quantities have a fundamental quantum nature, and quantum distributions are associated with them. The relation of the c-number values of these quantities, which appear in the quantum and classical equations, to associated operators is described. The evolution of quantum states with respect to the parameters s is investigated.

The covariant free particle Green's function is derived, and it is shown how it provides a description of the evolution of initial quantum states. It is seen how an initial Gaussian state, in which the space-time coordinates are treated on an equal footing, spreads with respect to the proper time evolution parameter about the classical covariant path. Furthermore, the evolution of completely localized and completely delocalized initial states is discussed. It is interesting to observe how closely the covariant relativistic quantum results parallel the corresponding non-relativistic quantum results.

The covariant Schrödinger type equation describes the evolution of the quantum state for a massive scalar particle or spinor particle. The probability densities for observables are found from the quantum amplitudes associated with these states. The notion of the symmetric bracket defined on the space of quantum amplitudes is introduced, and it is shown how it is related to the covariant Schrödinger type equation. This results in the invariance of this bracket with respect to an one parameter transformation induced by a generator, which is bilinear in the quantum amplitudes. The coefficients of these amplitudes are the matrix elements of the covariant Hamiltonian that appears in either the scalar or the spinor covariant Schrödinger type equation. It is shown how this generator plays the role of an Hamiltonian on a generalized phase space formed from the real and imaginary parts of these amplitudes. The Hamiltonian flow generated on this space corresponds to the quantum state evolution provided by the covariant Schrödinger type equation.

The conventions used for four-vector, spinors, and Dirac gamma matrices are those of [3] with $\hbar = c = 1$. The space-time contravariant coordinate four-vector is defined as $q^\mu = (q^0, q^1, q^2, q^3) = (ct, x, y, z)$, and the dimensionless form is found by dividing the components with the fundamental units of length q_s. The Einstein summation convention is used throughout, and the covariant components are found from $q_\mu = g_{\mu\nu} q^\nu$, with the non-zero components of the metric tensor given by $(g_{00}, g_{11}, g_{22}, g_{33}) = (1, -1, -1, -1)$. The scalar product of two four-vectors is $q \cdot p = q_\mu p^\mu$, and the space-time measure is $ds = \sqrt{dq \cdot dq}$. The four-velocity is defined as $u^\mu = dq^\mu/ds$, and $u_\mu u^\mu = 1$. The four-momentum for a free particle is $p^\mu = m u^\mu = (p^0, \vec{p}) = (\gamma m, \gamma \vec{\beta} m)$, with $\vec{\beta} = d\vec{q}/dq^0$, $\gamma = 1/\sqrt{1 - \vec{\beta} \cdot \vec{\beta}}$, and $p \cdot p = m^2$.

The square root of the space-time measure can be found in another way, which leads to the linear representation of the relativistic wave equation found by Dirac. This method is based on the introduction of the four matrices γ^μ, $(\mu = 0, 1, 2, 3)$, which satisfy the relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} I. \tag{1.1}$$

In this way the space-time measure has the representation $d\not{s} = \gamma \cdot dq$, and $ds^2 = d\not{s} \cdot d\not{s} = dq \cdot dq$.

2. COVARIANT ACTION AND COVARIANT CLASSICAL DYNAMICS

The covariant classical methods, which are necessary for the development of covariant quantum theory, are described here. The covariant action is defined as

$$\mathcal{S} = \int \mathcal{L}\,(q, u)\, ds, \tag{2.1}$$

where $\mathcal{L}(q, u)$ is a Lorentz invariant. The requirement that the action be an extremum under a variation leads to the covariant Euler-Lagrange equation

$$\frac{d\frac{\partial \mathcal{L}}{\partial u^\mu}}{ds} - \frac{\partial \mathcal{L}}{\partial q^\mu} = 0. \tag{2.2}$$

The covariant Lagrangian for a particle of mass m and charge e interacting with a four-vector field $A^\mu(q)$ is

$$\mathcal{L} = \frac{m}{2}u^2 - eA \cdot u, \tag{2.3}$$

and the generalized four-momentum is

$$p_\mu = \frac{\partial \mathcal{L}}{\partial u^\mu} = mu_\mu - eA_\mu. \tag{2.4}$$

The equation of motion is found from Eq. (2.2) to be

$$\frac{dp_\mu}{ds} + \partial_\mu eA \cdot u = 0, \tag{2.5}$$

where $\partial_\nu = \partial/\partial q^\nu$. Since

$$\frac{dA_\mu\,(q)}{ds} = u^\nu \partial_\nu A_\mu\,(q), \tag{2.6}$$

the equation of motion becomes

$$ma_\mu = eu^\nu F_{\nu\mu}, \tag{2.7}$$

with

$$a_\mu = \frac{d^2 q_\mu}{ds^2}, \quad F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu. \tag{2.8}$$

From the covariant Lagrangian Eq. (2.3), the covariant Hamiltonian is defined as

$$\mathcal{H} = p \cdot u - \mathcal{L}. \tag{2.9}$$

Since $u_\mu = (p_\mu + eA_\mu)/m$, a simple calculation gives

$$\mathcal{H} = \frac{(p + eA)^2}{2m} = \frac{(mu)^2}{2m} = \frac{m}{2}. \tag{2.10}$$

From Eq. (2.9) and

$$\frac{d\mathcal{L}}{ds} = \frac{\partial \mathcal{L}}{\partial q} \cdot u + \frac{\partial \mathcal{L}}{\partial u} \cdot a, \tag{2.11}$$

it follows that

$$\frac{d\mathcal{H}}{ds} = \left(\frac{dp}{ds} - \frac{\partial \mathcal{L}}{\partial q}\right) \cdot u - \left(p - \frac{\partial \mathcal{L}}{\partial u}\right) \cdot a = 0, \tag{2.12}$$

which is a consequence of Eq. (2.2). In addition

$$\frac{d\mathcal{H}}{ds} = \frac{\partial \mathcal{H}}{\partial q} \cdot u + \frac{\partial \mathcal{H}}{\partial p} \cdot f = 0, \tag{2.13}$$

with $f^{\nu} = dp^{\nu}/ds$, and this implies the Hamilton equations

$$u_{\mu} = \frac{\partial \mathcal{H}}{\partial p^{\mu}} \qquad f_{\mu} = -\frac{\partial \mathcal{H}}{\partial q^{\mu}}. \tag{2.14}$$

These equations and the covariant Hamiltonian Eq. (2.10) give

$$u_{\mu} = \frac{\partial \mathcal{H}}{\partial p^{\mu}} = (p + eA)_{\mu}/m, \tag{2.15}$$

and

$$f_{\mu} = -\frac{\partial \mathcal{H}}{\partial q^{\mu}} = -u^{\nu}\partial_{\mu}eA_{\nu}. \tag{2.16}$$

From Eq. (2.15) it follows that

$$f_{\mu} = ma_{\mu} - e\frac{dA_{\mu}}{ds}, \tag{2.17}$$

which with Eq. (2.6) and Eq. (2.16) results in the equation of motion Eq. (2.7).

For real functions $F(q,p,s)$ and $G(q,p,s)$ of q_{ν}, p_{ν}, and s, the Poisson bracket is defined as

$$\{F, G\} = \partial F \cdot \bar{\partial} G - \partial G \cdot \bar{\partial} F, \tag{2.18}$$

with $\partial_{\mu} = \partial/\partial q^{\mu}$ and $\bar{\partial}_{\mu} = \partial/\partial p^{\mu}$. Since

$$\partial_{\mu} q^{\nu} = \bar{\partial}_{\mu} p^{\nu} = \delta^{\nu}_{\mu}, \tag{2.19}$$

one finds

$$\{q^{\mu}, p^{\nu}\} = g^{\mu\nu}. \tag{2.20}$$

From

$$\frac{dF}{ds} = \frac{\partial F}{\partial q} \cdot u + \frac{\partial F}{\partial p} \cdot f + \frac{\partial F}{\partial s}, \tag{2.21}$$

and Eq. (2.14), it follows that

$$\frac{dF}{ds} = \{F, \mathcal{H}\} + \frac{\partial F}{\partial s}. \tag{2.22}$$

The dynamical equations that are equivalent to Hamilton's equations Eq. (2.14) are

$$u_{\mu} = \{q_{\mu}, \mathcal{H}\} \qquad f_{\mu} = \{p_{\mu}, \mathcal{H}\} \tag{2.23}$$

The Hamiltonian is the generator for a one parameter transformation that leaves the Poisson bracket invariant. This is seen from

$$\frac{d}{ds}\{q(s)^{\mu}, p(s)^{\nu}\} = \{\frac{dq(s)^{\mu}}{ds}, p(s)^{\nu}\} + \{q(s)^{\mu}, \frac{dp(s)^{\nu}}{ds}\} = 0, \tag{2.24}$$

which follows from the dynamical equations Eq. (2.23) and the Jacobi identity for the Poisson bracket. This leads to the conclusion that

$$\{q(s)^{\mu}, p(s)^{\nu}\} = \{q(0)^{\mu}, p(0)^{\nu}\} = g^{\mu\nu}. \tag{2.25}$$

To illustrate the use of the covariant classical dynamical equations, examples are given that are to be compared with the corresponding covariant quantum solutions. These are associated with a vector potential of the form $A^{\mu} = (A^0, \vec{A}) = (-V(\vec{q}), 0)$. From the covariant Hamiltonian Eq. (2.10) and the dynamical equations Eq. (2.14) or Eq. (2.23), it follows that

$$u^0 = \left(p^0 + A^0\right)/m \qquad f^0 = \frac{(p+A)^0}{m}\partial_0 A_0 = 0; \tag{2.26}$$

hence, $p^0 = E$, a constant, and

$$E = m\gamma + V(\vec{q}), \tag{2.27}$$

since $u^0 = dq^0/ds = \gamma$. Also

$$f^i = \frac{(p+A)^0}{m}\partial_i A_0 = -u^0 \partial_i V(\vec{q}) \tag{2.28}$$

and

$$\frac{d\left(m\gamma\vec{\beta}\right)}{dq^0} = -\nabla V(\vec{q}). \tag{2.29}$$

For a free particle with $V(\vec{q}) = 0$, one finds $E = \gamma m$, and $\vec{p} = \gamma m \vec{\beta}$.

For a particle in a uniform field with potential $V(z) = -Kz$, the dynamical equations give

$$\frac{dq^0}{ds} = \left(p^0 - V(z)\right)/m, \quad \frac{dp^0}{ds} = 0, \quad p^0 = E, \tag{2.30}$$

and

$$\frac{dq^3}{ds} = \frac{p^3}{m}, \quad \frac{dp^3}{ds} = (E - V(z))\frac{\partial V(z)}{\partial z}. \tag{2.31}$$

Since $p^3 = m\dot{z}(dq^0/ds) = m\dot{z}(E - V(z))/m$, the invariant Hamiltonian becomes

$$(E - V(z))^2\left(1 - \dot{z}^2\right) = m^2. \tag{2.32}$$

For $z(0) = \dot{z}(0) = 0$, the solution is

$$z(t) = \frac{m}{K}\left(\sqrt{1 + \left(\frac{Kt}{m}\right)^2} - 1\right). \tag{2.33}$$

Additional related relativistic particle dynamics is found in [4], and applications of covariant dynamics in curvilinear coordinates to betatron physics may be found in [5] and [6]. Related concepts of covariant dynamics have been used also in the development of relativistic wave equations [7] and covariant quantum field theory [8].

3. COVARIANT QUANTUM MECHANICS AND SYMMETRIC BRACKET INVARIANCE

It is the objective of this paper to establish a quantum theory that corresponds to the covariant classical theory described above, where the dynamical parameter is proper time s. As a starting point, the scalar case is considered, and the extension to the spinor case is given later. This state is normalized to unity, a Lorentz invariant, and for the probability interpretation of quantum theory, it is to remain normalized to this value for changes in the value of the parameter s. This is represented as

$$\langle\Psi(s)|\Psi(s)\rangle = \langle\Psi(0)|\Psi(0)\rangle = 1, \tag{3.1}$$

and this requires the state $\Psi(s)$ to be related to the original state $\Psi(0)$ by either a unitary or antiunitary transformation [9] and [10]. It is well known that the antiunitary transformation is associated with the TCP theorem [3], and the norm preserving transformation associated with conservation of probability is unitary. The proper time developed state is induced by a one parameter unitary transformation $\hat{U}(s)$ such that

$$|\Psi(s)\rangle = \hat{U}(s)|\Psi(0)\rangle. \tag{3.2}$$

The product of two such transformations is of the form

$$\hat{U}(s_1)\,\hat{U}(s_2) = \hat{U}(s_1 + s_2)\,, \tag{3.3}$$

with

$$\hat{U}(s) = e^{-is\hat{\mathcal{H}}}, \tag{3.4}$$

where $\hat{\mathcal{H}}$ is an Hermitian operator. The differential representation of Eq. (3.1) is the covariant Schrödinger type equation

$$i\frac{\partial|\Psi(s)\rangle}{\partial s} = \hat{\mathcal{H}}|\Psi(s)\rangle. \tag{3.5}$$

Under a Lorentz transformation, the state $|\Psi(s)\rangle$ changes according to $|\Psi'(s)\rangle = \Lambda(\vec{\beta})|\Psi(s)\rangle$, where $\Lambda(\vec{\beta})$ is a scalar representation of the Lorentz group, and $\vec{\beta}$ is a dimensionless velocity vector. Under this transformation the norm is invariant and so is the form of Eq. (3.5).

The spinor case requires some modifications of the above discussion. A general spinor state $|\Psi(0)\rangle$ has associated with it a conjugate state found from the adjoint state and represented as

$$\langle\bar{\Psi}(0)| = \langle\Psi(0)|\gamma^0, \tag{3.6}$$

where γ^0 is the time component Dirac matrix. In this case the scalar product is normalized to unity according to

$$\langle\bar{\Psi}(s)|\Psi(s)\rangle = \langle\bar{\Psi}(0)|\Psi(0)\rangle = 1. \tag{3.7}$$

This scalar product is a Lorentz invariant when the states transform according to

$$|\Psi'(s)\rangle = \Lambda\left(\vec{\beta}\right)|\Psi(s)\rangle \quad \langle\bar{\Psi}'(s)| = \langle\bar{\Psi}(s)|\Lambda^{-1}\left(\vec{\beta}\right), \tag{3.8}$$

where $\Lambda(\vec{\beta})$ is a spinor representation of the Lorentz group, and

$$\Lambda^{-1}\left(\vec{\beta}\right) = \gamma^0\Lambda\left(\vec{\beta}\right)\gamma^0. \tag{3.9}$$

The evolution of the spinor state $|\Psi(0)\rangle$ is generated by the transformation $\hat{U}(s)$, which has the inverse

$$\hat{U}^{-1}(s) = \gamma^0\hat{U}^{\dagger}(s)\,\gamma^0, \tag{3.10}$$

which preserves the scalar product Eq. (3.7).

For the scalar state case, an eigenstate in the rest system of a particle of mass m is represented by $|\lambda_i\rangle$, where λ represents a set of observables associated with measurements of the state. The general state of the particle is expanded at $s = 0$ in terms of these states, and the evolution of the state is generated by $\hat{U}(s)$. Introducing the identity operator, it is found that

$$|\Psi(s)\rangle = \hat{I}|\Psi(s)\rangle = \sum_i |\lambda_i\rangle\langle\lambda_i|\Psi(s)\rangle = \sum_i a_i(s)\,|\lambda_i\rangle, \tag{3.11}$$

where

$$a_i(s) = \langle\lambda_i|\Psi(s)\rangle = \langle\lambda_i|\hat{U}(s)\,|\Psi(0)\rangle. \tag{3.12}$$

Differentiating with respect to s, it is found that

$$\begin{aligned} i\frac{da_i(s)}{ds} &= \langle\lambda_i|\hat{\mathcal{H}}|\Psi(s)\rangle \\ &= \sum_j \langle\lambda_i|\hat{\mathcal{H}}|\lambda_j\rangle\langle\lambda_j|\Psi(s)\rangle \end{aligned}$$

$$= \sum_j \mathcal{H}_{ij} a_j (s) , \tag{3.13}$$

where $\mathcal{H}_{ij} = \langle \lambda_i | \hat{\mathcal{H}} | \lambda_j \rangle$.

It is now shown that this equation leads to invariance of the symmetric bracket defined on the space of complex quantum amplitudes a_i. For real functions $U(a_i, a_j^*)$ and $V(a_i, a_j^*)$, the symmetric bracket is defined as

$$\{U, V\}_+ = \sum_i \left(\frac{\partial U}{\partial a_i} \frac{\partial V}{\partial a_i^*} + \frac{\partial V}{\partial a_i} \frac{\partial U}{\partial a_i^*} \right) . \tag{3.14}$$

The complex coefficients a_i and a_j^* satisfy, in the discrete case, the conditions

$$\{a_i, a_j\}_+ = \{a_i^*, a_j^*\}_+ = 0, \qquad \{a_i, a_j^*\}_+ = \delta_{i,j}, \tag{3.15}$$

and in the continuum case these become

$$\{a_\lambda, a_{\lambda'}\}_+ = \{a_\lambda^*, a_{\lambda'}^*\}_+ = 0, \qquad \{a_\lambda, a_{\lambda'}^*\}_+ = \delta (\lambda - \lambda') , \tag{3.16}$$

with

$$\frac{\partial a_\lambda}{\partial a_{\lambda'}} = \delta (\lambda - \lambda') , \tag{3.17}$$

and its complex conjugate. Introducing the real function defined from the Hermitian matrix $\mathcal{H}_{ij}$

$$g (s) = \sum_i a_i^* (s) \mathcal{H}_{ij} a_j (s) , \tag{3.18}$$

it follows that

$$\begin{aligned} i \frac{d a_i (s)}{ds} &= \{a_i (s) , g (s)\}_+ = \frac{\partial g (s)}{\partial a_i^* (s)} \\ -i \frac{d a_i^* (s)}{ds} &= \{a_i^* (s) , g (s)\}_+ = \frac{\partial g (s)}{\partial a_i (s)} . \end{aligned} \tag{3.19}$$

Defining $a_i = a_i(0)$, the symmetric bracket invariance follows from

$$\{a_i (ds) , a_j^* (ds)\}_+ = \{a_i, a_j^*\}_+ + O \left(ds^2\right) \sim \delta_{ij} . \tag{3.20}$$

Following the methods of [11], the iteration of Eq. (3.19) shows that the development of $a(0)$ is generated by the operator $\hat{V}(s)$ defined in

$$a_i (s) = \hat{V} (s) a_i (0) = e^{is\delta_+} a_i (0) ,$$

which can be used to show that

$$\{a_i (s) , a_j^* (s)\}_+ = \{a_i, a_j^*\}_+ = \delta_{ij} . \tag{3.21}$$

In the definition of $\hat{V}(s)$, the operator $\delta_+ = \{g(0), \}_+$ has the properties

$$\delta_+ a_i = \{g (0) , a_i\}_+, \quad \delta_+^2 a_i = \{g (0) , \{g (0) , a_i\}_+\}_+, \quad etc. \tag{3.22}$$

The invariance of this bracket can be viewed as a principle, which leads to first and second quantization. This is discussed in detail in [11].

It is interesting to note that the equations for the development of $a_i(s)$, Eq. (3.19), are equivalent to generalized Hamilton's equations on a space where the real and imaginary parts of the amplitude $a_i(s) = \langle \lambda_i | \Psi(s) \rangle$ behave like phase space coordinates. To show this, the coordinates $Q_i(s)$ and $P_i(s)$ are defined as

$$Q_i (s) = \frac{a_i (s) + a_i^* (s)}{\sqrt{2}}, \quad P_i (s) = \frac{a_i (s) - a_i^* (s)}{i\sqrt{2}} . \tag{3.23}$$

From the Eq. (3.19), one can derive the Hamilton's equations for these coordinates,

$$\frac{dQ_i(s)}{ds} = \frac{\partial g(s)}{\partial P_i(s)}, \quad \frac{dP_i(s)}{ds} = -\frac{\partial g(s)}{\partial Q_i(s)}, \tag{3.24}$$

where $g(s)$ plays the role of the Hamiltonian on the space of these coordinates. This function takes a simple form if the matrix elements are diagonal, $\mathcal{H}_{ij} = (\lambda_i)\delta_{ij}$, and it becomes

$$g(s) = \sum_i \lambda_i |a_i(s)|^2 = \sum_i \lambda_i \frac{Q_i^2 + P_i^2}{2}. \tag{3.25}$$

It must be emphasized, that the Hamiltonian flow induced by this generator is not associated with the dynamics of ordinary phase space characterized by observable position coordinates q_i and observable conjugate momentum coordinates p_i, but it is the dynamical flow on the space of quantum amplitudes, $a_i(s) = \langle\lambda_i|\Psi(s)\rangle$, which are not observables. The Hamiltonian flow generated by Eq. (3.18) is equivalent to the covariant Schrödinger type equation Eq. (3.5), and it is the form of Eq. (3.18) and Eq. (3.19) that accounts for the linearity of this equation. Furthermore, the form of the generator in Eq. (3.25) that leaves the symmetric bracket invariant implies the probability interpretation of quantum theory, where $|a_i(s)|^2$ is the probability to observe the eigenvalue λ_i. The modification of the above discussion to include the spinor case is straightforward, and it follows from the properties associated with the conjugate state Eq. (3.6) and the scalar product Eq. (3.7).

4. COVARIANT SCHRÖDINGER TYPE WAVE EQUATION

The Schrödinger type equation in covariant form for the scalar case with wave function

$$\Psi(s,q) = \langle q|\Psi(s)\rangle, \tag{4.1}$$

is

$$i\frac{\partial\Psi(s,q)}{\partial s} = \hat{\mathcal{H}}\Psi(s,q). \tag{4.2}$$

Using a separable solution $\Psi(s,q) = \psi(s)\Phi(q)$, this equation becomes

$$\frac{i\frac{\partial\psi(s)}{\partial s}}{\psi(s)} = \lambda = \frac{\hat{\mathcal{H}}\Phi(q)}{\Phi(q)}. \tag{4.3}$$

For $\psi(s) \propto exp(-ims/2)$, one finds

$$\hat{\mathcal{H}}\Phi(q) = \frac{(\hat{p} + eA)^2}{2m}\Phi(q) = \frac{m}{2}\Phi(q), \tag{4.4}$$

with $\hat{p}^\mu = i\partial^\mu$. For the free particle case, $A^\mu = 0$,

$$\hat{\mathcal{H}} = \frac{(\hat{p})^2}{2m} = -\frac{1}{2m}\Box^2, \tag{4.5}$$

with

$$\Box^2 = \frac{\partial^2}{\partial^2 q^0} - \nabla^2 = \frac{\partial^2}{\partial^2 q^0} - \frac{\partial^2}{\partial^2 q^1} - \frac{\partial^2}{\partial^2 q^2} - \frac{\partial^2}{\partial^2 q^3}. \tag{4.6}$$

Potential problems can be included using the vector potential $A^\mu = (-V(q), 0, 0, 0)$, and the equation for $\Phi(q)$ becomes

$$\left(\left(\hat{p}^0 - V(q)\right)^2 + \nabla^2\right)\Phi(q) = m^2\Phi(q). \tag{4.7}$$

The time dependence is this equation is separable, and the solution $\Phi(q)$ takes the form

$$\Phi\left(q\right) \propto e^{-iEt}\phi\left(x,y,z\right). \tag{4.8}$$

This gives

$$\left(\left(E-V\left(x,y,z\right)\right)^2+\nabla^2\right)\phi\left(x,y,z\right)=m^2\phi\left(x,y,z\right). \tag{4.9}$$

In the one dimensional case, this becomes

$$\left(E-V\left(z\right)\right)^2\phi\left(z\right)=m^2\phi\left(z\right)-\frac{\partial^2\phi\left(z\right)}{\partial^2 z}. \tag{4.10}$$

The non-relativistic limit is found from

$$\left(E-V\left(z\right)\right)\sqrt{\phi\left(z\right)}=m\sqrt{\phi\left(z\right)}\sqrt{1-\frac{1}{m^2\phi\left(z\right)}\frac{d^2\phi\left(z\right)}{dz^2}}, \tag{4.11}$$

and this becomes

$$\left(E-V\left(z\right)\right)\sqrt{\phi\left(z\right)}\approx m\sqrt{\phi\left(z\right)}\left(1-\frac{1}{2m^2\phi\left(z\right)}\frac{d^2\phi\left(z\right)}{dz^2}+\ldots\right), \tag{4.12}$$

which is equivalent to

$$\frac{d^2\phi\left(z\right)}{dz^2}+2m\left(E-m-V\left(z\right)\right)\phi\left(z\right). \tag{4.13}$$

This is the usual time-independent Schrödinger equation; however, the eigenvalues $E-m$ contain the rest mass of the particle. The rest mass should be retained in the non-relativistic limit because in the free particle case the phase of the wave function is a Lorentz invariant, $Et-\vec{p}\cdot\vec{q}=ms$. Furthermore, the operator $\hat{p}^0$ is a generator for displacement of the time coordinate q^0; however, in this theory, it is not associated with a phase space Hamiltonian operator through a time dependent Schrödinger equation. In this way, time and energy are not conjugate coordinates expressible in terms of phase space operators, and an uncertainty relation of the type described in Appendix A is not possible. A related comment can be found in [12].

The equation of continuity for a free scalar particle follows from the covariant Schrödinger type equation. From Eq. (4.2) and the fact that $\hat{\mathcal{H}}$ is Hermitian, one can form

$$\psi^*\left(q,s\right)i\frac{\partial\psi\left(q,s\right)}{\partial s}+\psi\left(q,s\right)i\frac{\partial\psi^*\left(q,s\right)}{\partial s}=\psi^*\left(q,s\right)\mathcal{H}\psi\left(q,s\right)-\psi\left(q,s\right)\mathcal{H}\psi^*\left(q,s\right), \tag{4.14}$$

and this can be written as

$$i\frac{\partial\rho\left(q,s\right)}{\partial s}-\hat{p}_\mu j^\mu\left(q,s\right)=0, \tag{4.15}$$

with

$$\rho\left(q,s\right)=\psi^*\left(q,s\right)\psi\left(q,s\right), \tag{4.16}$$

and

$$j^\mu\left(q,s\right)=\frac{1}{2m}\left(\psi^*\left(q,s\right)\hat{p}^\mu\psi\left(q,s\right)-\psi\left(q,s\right)\hat{p}^\mu\psi^*\left(q,s\right)\right). \tag{4.17}$$

Here $\rho(q,s)$ represents the probability density in the four-vector q^μ representation, and $j^\mu(q,s)$ is associated with the charge density current [13]. The normalization condition $\langle\Psi(s)|\Psi(s)\rangle=1$ requires

$$\int d^4q\,\rho\left(q,s\right)=1. \tag{4.18}$$

For this theory, the probability density is positive definite, and this removes the historical problem associated with finding a suitable probability density for the Klein-Gordon equation.

5. FREE PARTICLE GREEN'S FUNCTION

The evolution of a free particle quantum state is determined from the the free particle Green's function. The Green's function for the covariant scalar quantum state $|\Psi(s)\rangle$ is defined as

$$g\left(q,q',s\right)=\theta\left(s\right)G\left(q,q',s\right)=\theta\left(s\right)\langle q|U^{-s\hat{\mathcal{H}}}|q'\rangle, \tag{5.1}$$

where $\theta(s)$ is the Heaviside function, and it satisfies the equation

$$\left(i\frac{\partial}{\partial s}-\hat{\mathcal{H}}\right)g\left(q,q',s\right)=\delta\left(s\right)\delta^4\left(q-q'\right), \tag{5.2}$$

and the condition

$$\lim_{s\to 0}g\left(q,q',s\right)=\delta^4\left(q-q'\right). \tag{5.3}$$

Following the method of Dirac [14], it can be obtained from the classical covariant action as

$$g\left(q,q',s\right)=\theta\left(s\right)A\left(s\right)e^{i\mathcal{S}\left(q,q',s\right)}. \tag{5.4}$$

For the free particle covariant Hamiltonian $\hat{\mathcal{H}}=\hat{p}\cdot\hat{p}/(2m)$, the solutions to the classical equations of motion are

$$q^\mu\left(s\right)-q'^\mu\left(s\right)=sp^\mu\left(s\right)/m,\ \text{and}\ p^\mu\left(s\right)=\text{constant}, \tag{5.5}$$

and the covariant classical action becomes

$$\mathcal{S}\left(q,q',s\right)=\int\frac{m}{2}\frac{dq}{ds}\cdot\frac{dq}{ds}ds=\frac{m\left(q-q'\right)\cdot\left(q-q'\right)}{2s}. \tag{5.6}$$

The Green's function that satisfies Eq. (5.2) and Eq. (5.3) is

$$g\left(q,q',s\right)=\theta\left(s\right)\left(\sqrt{\frac{m}{i2\pi s}}\right)^4e^{i\frac{m}{2s}\left[\left(t-t'\right)^2-\left(x-x'\right)^2-\left(y-y'\right)^2-\left(z-z'\right)^2\right]}. \tag{5.7}$$

This evolution function can be used to find the wave function $\psi(q,s)$ from the initial state $\psi(q,0)$ using

$$\psi\left(q,s\right)=\int G\left(q,q',s\right)\psi\left(q',0\right)d^4q'. \tag{5.8}$$

Three cases of free particle motion are considered:
case 1: The initial particle is localized in space-time, with wave function

$$\psi\left(q,0\right)=\delta\left(q^0\right)\delta\left(q^1\right)\delta\left(q^2\right)\delta\left(q^3\right), \tag{5.9}$$

and the evolved wave function is

$$\psi\left(q,s\right)=\left(\sqrt{\frac{m}{i2\pi s}}\right)^4e^{i\frac{m}{2s}\left[(t)^2-(x)^2-(y)^2-(z)^2\right]}. \tag{5.10}$$

case 2: The initial particle is delocalized in space-time, with wave function

$$\psi\left(q,0\right)=\frac{1}{4\pi^2}e^{-ip\cdot q}=\frac{1}{4\pi^2}e^{-ip^0q^0+i\vec{p}\cdot\vec{q}}, \tag{5.11}$$

and the evolved wave function is

$$\psi\left(q,s\right)=\frac{1}{4\pi^2}e^{-ip\cdot q}e^{i\left(-p^0p^0+\vec{p}\cdot\vec{p}\right)s/(2m)}. \tag{5.12}$$

case 3: The initial particle is described by Gaussian distributions, with initial state wave function

$$\begin{aligned}\psi(q,0) &= \frac{1}{2\pi\left(\sigma_0\sigma_1\sigma_2\sigma_3\right)^{1/2}}e^{-ip(0)\cdot q} \\ &\times e^{-\left(q^0\right)^2/\left(4\sigma_0^2\right)}e^{-\left(q^1\right)^2/\left(4\sigma_1^2\right)} \\ &\times e^{-\left(q^2\right)^2/\left(4\sigma_2^2\right)}e^{-\left(q^3\right)^2/\left(4\sigma_3^2\right)},\end{aligned} \tag{5.13}$$

and the evolved wave function is

$$\begin{aligned}\psi(q,s) = e^{i\tan^{-1}(s)}\psi_0\left(q^0,p^0(0),s\right)\psi_1\left(q^1,-p^1(0),-s\right) \\ \times\ \psi_2\left(q^2,-p^2(0),-s\right)\psi_3\left(q^3,-p^3(0),-s\right),\end{aligned} \tag{5.14}$$

with

$$\begin{aligned}\psi_\mu\left(q^\mu,p^\mu(0),s\right) &= \frac{1}{\left(2\pi\sigma_\mu^2(s)\right)^{1/4}}e^{-\frac{\left(q^\mu-p^\mu(0)s\right)^2}{2\sigma^2(s)}(1+is)} \\ &\times\ e^{i\frac{\left(p^\mu(0)\right)^2 s}{2}}e^{-iq^\mu p^\mu(0)},\end{aligned} \tag{5.15}$$

where

$$\sigma_\mu(s) = \sigma_\mu\sigma(s) = \sigma_\mu\sqrt{1+s^2},$$

and where

$$\begin{aligned}q^\mu &\to q^\mu/\delta\left(q^\mu\right) \\ p^\mu &\to p^\mu/\delta\left(p^\mu\right) \\ s &\to s/\delta(s) \\ \delta\left(q^\mu\right)\delta\left(p^\mu\right) &= \hbar \\ \delta(s)\,\delta\left(p^\mu\right) &= mc\delta\left(q^\mu\right) \\ \delta\left(q^\mu\right) &= \sqrt{2}\sigma_\mu.\end{aligned} \tag{5.16}$$

This is the free particle motion, which is to be compared with the classical motion resulting from Eq. (2.30) and Eq. (2.31).

Each of these solutions can be confirmed to be a solution of the covariant Schrödinger type equation Eq. (4.2). In addition each is a solution to the equation of continuity Eq. (4.15). The normalization condition Eq. (4.18) is satisfied by the wave function Eq. (5.14). The wave function Eq. (5.9) must be viewed as the limit as $\sigma_\mu \to 0$ in Eq. (5.13). However; Eq. (5.11) satisfies the condition for delta function normalization. The function $\psi^*(q,-s)$ are also solutions to the covariant Schrödinger type equation and the equation of continuity. They represent the solutions obtained from the initial functions above with $p \to -p$. This is seen from the complex conjugation of the covariant Schrödinger type equation followed by $s \to -s$. Also for the free particle Green's function

$$g\left(q,q',s\right) = g^*\left(q,q'-s\right), \tag{5.17}$$

and

$$\psi^*(q,-s) = \int g^*\left(q,q',-s\right)\psi^*\left(q',0\right)d^4q. \tag{5.18}$$

The appropriate wave functions to associate with the classical limits of a quantum theory are found from the minimum uncertainty states. For the spatial coordinates, these states are found from the eigenvalue equation

$$\hat{a}^j\left|a^j\right\rangle = \frac{\hat{q}^j + i\hat{p}^j}{\sqrt{2}}\left|a^j\right\rangle = a^j\left|a^j\right\rangle, \tag{5.19}$$

for $j = 1, 2$ or 3 with $a^j = \left(q^j(0) + ip^j(0)\right)/\sqrt{2}$ and $[\hat{q}^i, \hat{p}^j] = i\delta_{ij}$. The normalized wave functions that satisfy this equation are

$$\langle q^i|a^i\rangle = \frac{1}{\pi^{1/4}} e^{ip^i(0)q^i - \left(q^i - q^i(0)\right)^2/2}. \tag{5.20}$$

For the time component, the coordinate and momentum space commutation relations are

$$\left[q^0, i\frac{\partial}{\partial q^0}\right] = -i, \quad \left[p^0, -i\frac{\partial}{\partial p^0}\right] = i. \tag{5.21}$$

The momentum space minimum uncertainty state is found from

$$\hat{b}^0|b^0\rangle = \frac{\hat{p}^0 + i\hat{q}^0}{\sqrt{2}}|b^0\rangle = b^0|a^0\rangle, \tag{5.22}$$

with $b^0 = (p^0(0) + iq^0(0))/\sqrt{2}$, and the wave function is

$$\langle p^0|b^0\rangle = \frac{1}{\pi^{1/4}} e^{iq^0(0)p^0 - \left(p^0 - p^0(0)\right)^2/2}. \tag{5.23}$$

The coordinate space function is found from

$$\langle q^0|b^0\rangle = \int dp^0 \langle q^0|p^0\rangle\langle p^0|b^0\rangle, \tag{5.24}$$

with $\langle q^0|p^0\rangle = (1/\sqrt{2\pi})\exp(-ip^0q^0)$, and it becomes

$$\langle q^0|b^0\rangle = \frac{1}{\pi^{1/4}} e^{-ip^0(0)\left(q^0 - q^0(0)\right) - \left(q^0 - q^0(0)\right)^2/2}. \tag{5.25}$$

The momentum space functions associated with the coordinate space functions are found from the complex conjugate of Eq. (5.20) and the replacement $q \leftrightarrow p$. The uncertainty for an operator $\hat{Q}$ is defined with respect to the state $|\psi\rangle$ as

$$\sigma(Q) = \sqrt{\langle\psi|\left(\hat{Q} - \bar{Q}\right)^2|\psi\rangle}, \tag{5.26}$$

with $\bar{Q} = \langle\psi|\hat{Q}|\psi\rangle$, and, for the minimum uncertainty states, they have the value $\sigma(q^\mu) = \sigma(p^\mu) = \frac{1}{\sqrt{2}}$. It is these states that are used for the initial state wave functions that appear in Eq. (5.13), which are found with the coordinate replacements Eq. (5.16).

The classical representations of the parameters m and s are

$$m = \sqrt{p \cdot p}, \quad \text{and} \quad s = \sqrt{q \cdot q}. \tag{5.27}$$

The Poisson bracket of these quantities has the value

$$\{m, s\} = \frac{p.q}{sm} = 1, \tag{5.28}$$

and this suggest that there could be a corresponding commutation relation for the operators associated with $m(p)$ and $s(q)$. However, the spectrum of m must be positive, and as shown in Appendix A this is not possible if $[\hat{m}, \hat{s}] = i$.

6. PARTICLE IN AN EXTERNAL FIELD

To illustrate the method of solution for Eq. (4.7), I consider the relativistic motion of a particle in an external field with potential $A^0(z) = -V(z) = Kz$. This is the covariant quantum analogue to the classical example given in Eq. (2.32). The equation for $\Phi(z)$ becomes

$$(E + kz)^2 \phi(z) + \frac{\partial^2}{\partial z^2}\phi(z) = m^2 \phi(z) . \tag{6.1}$$

Using the substitutions

$$\xi = \frac{1}{\sqrt{k}}(E + kz), \quad \lambda = -\frac{m^2}{k}, \tag{6.2}$$

one finds the parabolic cylindrical equation [15]

$$\frac{d^2\phi(\xi)}{d\xi^2} + \left(\xi^2 + \lambda\right)\phi(\xi) = 0, \tag{6.3}$$

$$\phi(\xi) = D_{-\frac{1+i\lambda}{2}}\left[\pm(1+i)\xi\right], \tag{6.4}$$

$$D_p\left[(1+i)\xi\right] = \frac{2^{\frac{p+1}{2}}}{\Gamma\left(-\frac{p}{2}\right)}\int_1^\infty e^{-\frac{i}{2}\xi^2 x}\frac{(x+1)^{\frac{p-1}{2}}}{(x-1)^{1+\frac{p}{2}}}dx, \tag{6.5}$$

where for [RE $p<0$; RE $i\xi^2 \geq 0$],

$$\begin{aligned} D_p(\xi) &= 2^{\frac{1}{4}+\frac{p}{2}} W_{\frac{1}{4}+\frac{p}{2},-\frac{1}{4}}\left(\frac{\xi^2}{2}\right)\xi^{-\frac{1}{2}} \\ &= 2^{\frac{p}{2}} e^{-\frac{\xi^2}{4}}\left\{\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)}\Phi\left(-\frac{p}{2},\frac{1}{2};\frac{\xi^2}{2}\right) - \frac{\sqrt{2\pi}\xi}{\Gamma\left(-\frac{p}{2}\right)}\Phi\left(\frac{1-p}{2},\frac{3}{2};\frac{\xi^2}{2}\right)\right\}, \end{aligned} \tag{6.6}$$

and where $W_{\lambda,\mu}(z)$ are Whittaker functions, and $\Phi(\alpha,\gamma;x)$ are confluent hypergeometric functions. The eigenvalue spectrum for the energy is continuous, and in the non-relativistic limit the wave functions become Airy functions.

7. COVARIANT FERMION CASE

A similar analysis can be done for the case of spin 1/2 particles. The covariant Hamiltonian associated with the classical equations is

$$\mathcal{H} = \frac{\langle\bar{u}|\not{p} + e\not{A}(q)|u\rangle}{2m} = (p + eA)_\mu \frac{\langle\bar{u}|\gamma^\mu|u\rangle}{2m}, \tag{7.1}$$

where $|u\rangle$ is a spinor and $\langle\bar{u}| = \langle u|\gamma^0$ with normalization $\langle\bar{u}|u\rangle = 2\sqrt{p\cdot p}$. For scalar products and matrix elements with spinors, an average is made over helicity states. Hamilton's equations for this case give

$$\frac{dq^\mu}{ds} = \frac{\partial\mathcal{H}}{\partial p_\mu} = \frac{\langle\bar{u}|\gamma^\mu|u\rangle}{2m} \tag{7.2}$$

$$\frac{dp^\mu}{ds} = -\frac{\partial\mathcal{H}}{\partial q_\mu} = -\frac{\partial eA_\nu}{\partial q_\mu}\frac{\langle\bar{u}|\gamma^\nu|u\rangle}{2m} = -\frac{\partial eA\cdot u}{\partial q_\mu}, \tag{7.3}$$

and this gives Eq. (2.5).

The covariant wave equation associated with this case is

$$i\frac{\partial|\psi(s)\rangle}{\partial s} = (i\not{p} + e\not{A})\,|\psi(s)\rangle = \hat{\mathcal{H}}|\psi(s)\rangle, \tag{7.4}$$

and the Hamiltonian is

$$\hat{\mathcal{H}} = (\not{p} + e\not{A}) = \gamma^0\hat{\mathcal{H}}^\dagger\gamma^0. \tag{7.5}$$

The state $|\psi(s)\rangle$ is generated from $|\psi(0)\rangle$ using

$$|\psi(s)\rangle = \hat{U}(s)\,|\psi(0)\rangle = e^{-is\hat{\mathcal{H}}}|\psi(0)\rangle \tag{7.6}$$

$$\langle\bar{\psi}(s)| = \langle\bar{\psi}(0)|\hat{U}(-s) = \langle\bar{\psi}(0)|e^{is\hat{\mathcal{H}}}, \tag{7.7}$$

such that

$$\hat{U}(-s) = \gamma^0\hat{U}^\dagger(s)\,\gamma^0 \quad \text{and} \quad \langle\bar{\psi}(s)|\psi(s)\rangle = 1. \tag{7.8}$$

In the q-representation, the probability density for the four-vector q is given by

$$\rho(q,s) = \langle\bar{\psi}(s)|q\rangle\langle q|\psi(s)\rangle, \tag{7.9}$$

which is normalized such that

$$\begin{aligned}\int\rho(q,s)\,d^4q &= \langle\bar{\psi}(s)|\psi(s)\rangle = \langle\bar{\psi}(0)|\hat{U}^{-1}(s)\,\hat{U}(s)\,|\psi(0)\rangle \\ &= \langle\bar{\psi}(0)|\psi(0)\rangle = 1.\end{aligned} \tag{7.10}$$

Pre-multiplying the covariant equation Eq. (7.4) for $\langle q|\psi(s)\rangle$ by $\langle\bar{\psi}(s)|q\rangle$, and post-multiplying the matrix transformed complex conjugate of Eq. (7.4) by $\gamma^0\langle q|\psi(s)\rangle$ gives, when the second resulting equation is subtracted from the first, the equation of continuity

$$\frac{\partial\rho(q,s)}{\partial s} = \frac{\partial j^\mu(q,s)}{\partial q^\mu}, \tag{7.11}$$

where the current density is

$$j^\mu(q,s) = \langle\bar{\psi}(s)|q\rangle\gamma^\mu\langle q|\psi(s)\rangle. \tag{7.12}$$

Upon integration over the four-volume d^4q, one finds from Eq. (7.11)

$$\frac{\partial\int\rho(q,s)\,d^4q}{\partial s} = \int\frac{\partial j^\mu(q,s)}{\partial q^\mu}d^4q = \int j^\mu(q,s)\,dS_\mu = 0, \tag{7.13}$$

where dS_μ is an element of three dimensional hypersurface orthogonal to the direction μ. This is the statement of conservation of charge. As an example, I consider the free particle case when the spinor state is expanded in four-momentum p and helicity λ states such that

$$\begin{aligned}|\psi(s)\rangle &= \sum_\lambda\int\hat{U}(s)\,a_\lambda(p)\,\frac{|u(p,\lambda)\rangle\otimes|p\rangle}{\sqrt{\langle\bar{u}(p,\lambda)|u(p,\lambda)\rangle}}d^4p \\ &= \sum_\lambda\int e^{-is\sqrt{p\cdot p}}a_\lambda(p)\,\frac{|u(p,\lambda)\rangle\otimes|p\rangle}{\sqrt{\langle\bar{u}(p,\lambda)|u(p,\lambda)\rangle}}d^4p,\end{aligned} \tag{7.14}$$

and the wave function in the q representation is

$$\langle q|\psi(s)\rangle = \sum_\lambda\int e^{-is\sqrt{p\cdot p}}e^{-ip\cdot q}\frac{a_\lambda(p)\,|u(p,\lambda)\rangle}{\sqrt{\langle\bar{u}(p,\lambda)|u(p,\lambda)\rangle}}\frac{d^4p}{\left(\sqrt{2\pi}\right)^4}. \tag{7.15}$$

This leads to the density function

$$\rho(q,s) = \sum_{\lambda}\sum_{\lambda'}\int d^4p' \int d^4p\, a^*_{\lambda}(p')\, a_{\lambda}(p)\, e^{-is\left(\sqrt{p.p}-\sqrt{p'.p'}\right)} e^{-i(p-p')\cdot q} \times \frac{\langle \bar{u}(p',\lambda')|u(p,\lambda)\rangle}{\sqrt{\langle \bar{u}(p',\lambda')|u(p',\lambda')\rangle}\sqrt{\langle \bar{u}(p,\lambda)|u(p,\lambda)\rangle}} \frac{1}{\left(\sqrt{2\pi}\right)^8}, \tag{7.16}$$

and the charge current density becomes

$$j^{\mu}(q,s) = e\sum_{\lambda}\int |a_{\lambda}(p)|^2 \frac{\langle \bar{u}(p,\lambda)|\gamma^{\mu}|u(p,\lambda)\rangle}{\langle \bar{u}(p,\lambda)|u(p,\lambda)\rangle} d^4p, \tag{7.17}$$

or

$$j^{\mu}(q,s) \approx e\sum_{\lambda}\int |a_{\lambda}(p)|^2 \frac{p^{\mu}}{m} d^4p. \tag{7.18}$$

8. PARAMETER VALUES AND NORMALIZATION

The parameters m and s that appear in the covariant evolution equations Eq. (3.5), and in the classical equations are associated with quantum operators. This association is described for the mass parameter, and the corresponding relation for the proper time parameter s may be found is a similar manner. For the spin zero scalar case, the quantum state for a free particle may be expanded in terms of the four-momentum eigenstates as

$$|\Psi(s)\rangle = \int a_p(s)\; |p\rangle\, d^4p, \tag{8.1}$$

and the scalar product is

$$1 = \langle \Psi(s)|\Psi(s)\rangle = \int |a_p(0)\;|^2\, d^4p. \tag{8.2}$$

The mass parameter m is found from

$$m^2 = \langle \Psi(s)|\hat{p}\cdot\hat{p}|\Psi(s)\rangle = \int |a_p(0)\,|^2 p\cdot p d^4p. \tag{8.3}$$

For the spin 1/2 case, the spinor state is

$$|\Psi(s)\rangle = \sum_{\lambda}\int a_{\lambda}(p,s)\; \frac{|u(p,\lambda)\rangle \otimes |p\rangle}{\sqrt{\langle \bar{u}(p,\lambda)|u(p,\lambda)\rangle}}\, d^4p, \tag{8.4}$$

where $|u(p,\lambda)\rangle$ is a spinor that is normalized such that

$$\langle \bar{u}(\pm p,\lambda)|u(\pm p,\lambda)\rangle = \pm 2\sqrt{p\cdot p}, \tag{8.5}$$

and satisfies the condition

$$\langle \bar{u}(\pm p,\lambda)|\gamma^{\mu}|u(\pm p,\lambda)\rangle = 2p^{\mu}. \tag{8.6}$$

The scalar product in this case is

$$\langle \bar{\Psi}(s)|\Psi(s)\rangle = \sum_{\lambda}\int |a_{\lambda}(p)\;|^2\, d^4p = 1. \tag{8.7}$$

The mean value of the operator $\not{p}$ is

$$\frac{\langle \bar{\Psi}(s)|\not{p}|\Psi(s)\rangle}{\langle \bar{\Psi}|\Psi\rangle} = \sum_{\lambda} \int |a_{\lambda}(p)|^{2} \sqrt{p \cdot p}\, d^{4}p. \tag{8.8}$$

In general, the moments of the operators $\hat{\mathcal{H}}$ may be evaluated, and these can be used to construct the associated characteristic function. The distributions associated with this operator can be found from the Fourier transforms of the characteristic function. For example, the characteristic function for the operator $\hat{\mathcal{H}}$ is

$$\Phi(\alpha) = \langle \Psi(s)|e^{i\alpha\hat{\mathcal{H}}}|\Psi(s)\rangle, \tag{8.9}$$

and the distribution of $m/2$ is

$$\rho(m/2, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha m/2} \Phi(\alpha)\, d\alpha. \tag{8.10}$$

This shows the quantum nature of the mass distribution, which in principle can be measured. For example, measurements of the particle speed of an ensemble of identical free particles with identical initial conditions can be made. Since the magnitude of the three momentum and energy are related by the relation $|\vec{p}| = \beta E$, the value of m^2 can be determined with an independent measurement of either $|\vec{p}|$ or E. For a neutral particle, E can be found from an impact; however, for a charged particle, $|\vec{p}|$ can be found from its curvature in a magnetic field. For an ensemble of free particles, measurements of s^2 can be made from the observation of the distance traveled by a free particle in time t and the measurement of the distance $q^0 = ct$ traveled by light in the same time.

9. FOUR-VELOCITY AND FOUR-ACCELERATION OPERATORS

The observed values of four-velocity are associated with the four-velocity operator $\hat{v}^{\mu} = i[\hat{q}^{\mu}, \hat{\mathcal{H}}]$. These values are found in the scalar case, with $\hat{\mathcal{H}}$ given by Eq. (4.4), from the expectation value

$$\frac{dq^{\mu}}{ds} = u^{\mu} = \frac{\langle \Psi(s)|i\left[\hat{q}^{\mu}, \hat{\mathcal{H}}\right]|\Psi(s)\rangle}{\langle \Psi|\Psi\rangle}, \tag{9.1}$$

and in the spinor case from

$$\frac{dq^{\mu}}{ds} = u^{\mu} = \frac{\langle \bar{\Psi}(s)|i\left[\hat{q}^{\mu}, \hat{\mathcal{H}}\right]|\Psi(s)\rangle}{\langle \bar{\Psi}|\Psi\rangle}, \tag{9.2}$$

where $\hat{\mathcal{H}}$ is given by Eq. (7.5). In the free particle case, these both become

$$u^{\mu} = \frac{p^{\mu}}{m}, \tag{9.3}$$

when the phase space operators satisfy the quantum bracket condition

$$[\hat{q}^{\mu}, \hat{p}^{\nu}] = -ig^{\mu\nu}. \tag{9.4}$$

The usual velocity vectors are found from dq^i/dq^0. It is clear that the operators associated with four-velocity are unambiguously defined, and that the matrix elements in Eq. (9.1) and Eq. (9.2) give the predicted observed mean values for four-velocity. The operator $\hat{\mathcal{H}}$ is either linear or quadratic in $\hat{p}^{\mu}$, and it does not involve the square root of the scalar product of the three momentum operator. This is a clear advantage over the non-covariant forms in [16] and [17] proposed to represent the velocity operator.

From the four-velocity operator, one can also obtain the quantum equations that correspond to the classical equations of motion for a point particle of mass m interacting with a four-vector potential. For the scalar case, the four-velocity operator $\hat{u}^{\mu} = (\hat{p}^{\mu} + eA^{\mu})/m$ satisfies the commutation relation

$$[\hat{u}^{\mu}, \hat{u}^{\nu}] = \frac{ie}{m^2} F^{\mu\nu}, \tag{9.5}$$

and the four-acceleration operator is found from

$$\hat{a}^{\mu} = i \left[\hat{u}^{\mu}, \hat{\mathcal{H}} \right], \tag{9.6}$$

which gives

$$ma^{\mu} + \frac{e}{2} \langle \Psi\,(s) |\, (F^{\mu\nu} \hat{u}_{\nu} + \hat{u}_{\nu} F^{\mu\nu})\, | \Psi\,(s) \rangle, \tag{9.7}$$

which is to be compared with Eq. (2.7). For the spinor case with interaction,

$$u^{\mu} = \frac{\langle \bar{\Psi}\,(s) | \gamma^{\mu} | \Psi\,(s) \rangle}{\langle \bar{\Psi} | \Psi \rangle}, \tag{9.8}$$

and

$$\frac{dp^{\mu}}{ds} = \frac{\langle \bar{\Psi}\,(s) | i\,[\hat{p}^{\mu}, \not{p} + e\not{A}]\, | \Psi\,(s) \rangle}{\langle \bar{\Psi} | \Psi \rangle}. \tag{9.9}$$

This becomes

$$\frac{dp^{\mu}}{ds} + e \frac{\langle \bar{\Psi}\,(s) | \partial^{\mu} \not{A}\, | \Psi\,(s) \rangle}{\langle \bar{\Psi} | \Psi \rangle}, \tag{9.10}$$

which is to be compared with Eq. (2.5). It is important to note that consistency of these equations with the classical results depends upon the commutation relation Eq. (9.4), and this justifies its introduction.

10. CONCLUSIONS

We have seen that the covariant formulation of quantum mechanics provides representations that include both the scalar and spinor description of particle states. It gives a clear connection of the covariant quantum description with classical covariant particle dynamics. The covariant free particle Green's function describes the evolution of initial states in terms of the proper time parameter s, and it is seen how the initial Gaussian state propagates with spreading width with respect to the classical covariant path. In addition, it gives an acceptable representation of the probability density in both the scalar and spinor representations, and it produces a new set of equations of continuity, which show the connection between probability density, charge density, and current density. The solution of potential problems also follows from this formulation. This permits the description of practical relativistic quantum effects, which are the relativistic generalization of standard non-relativistic quantum situations. Also in the covariant formulation, the representation of the four-velocity and the four-acceleration operators is unambiguous, and one can clearly see how these operators are associated with observed values.

The properties of quantum amplitudes are shown to be associated with an Hamiltonian formulation, which acts on the space of these amplitudes. This leads to the notion of invariance of the symmetric bracket, which is defined on this space. This invariance and the associated Hamiltonian formulation are a consequence of the covariant Schrödinger type equation. This provides a subtle connection between the probability interpretation of quantum theory and the evolution of quantum states. It is shown in [11] how the the non-covariant form of this symmetric bracket invariance leads to second quantization for fermion and boson states.

REFERENCES

1. G. Garrod, Phys. Rev. **167**, 1143 (1968).
2. H. Goldstein, *Classical Mechanics* (Addison Wesley, Reading Mass., 1959) p. 207.
3. V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics, Second Edition* (Pergamon Press, New York, 1982).
4. G. Stephenson and C. W. Kilmister, *Special Relativity for Physicists* (Longmans, Green and Co., London, 1958) p. 65.
5. T. Garavaglia, in *Proceedings of the 1991 Symposium on the Superconducting Super Collider, Corpus Christi, SSCL-SR-1213*, edited by V. Kelly and G. P. Yost (Superconducting Super Collider Laboratory, Dallas Texas, 1991) p. 669.
6. T. Garavaglia, in *Conference Record of the 1991 IEEE Particle Accelerator Conference, San Francisco*, edited by L. Lizama and J. Chew (IEEE, 1991) Vol. I, p. 231,
(See also http://WWW.hep/ftp/ssc/new/COLLIDER/colliderpapers, SSCL-416 Rev).
7. V. Fock, Physik Z. Sowjetunion **12**, 404 (1937).
8. J. Schwinger, Phys. Rev. **82**, 664 (1951).
9. E. P. Wigner, Nachrichten d. Gesell. d. Wissenshaften z. Göttingen Math.-Phys. Klass, 546 (1932).
10. E. P. Wigner, J. Math. Phys. **1**, 409 (1960); J. Math. Phys. **1**, 414 (1960).
11. T. Garavaglia and S. K. Kauffmann, *The principle of symmetric bracket invariance as the origin of first and second quantization*, (http://xxx.lanl.gov, hep-th/9907059), Contributed paper to XIX International Symposium on Lepton and Photon Interactions at High Energies, Stanford University, August 9-14, 1999.(http://lp99.slac.stanford.edu).
12. W. Pauli, *Prinzipien der Quantentheorie 1, Handbuch der Physik, Band V, Teil 1.* (Springer-Verlag, Berlin, 1958) p. 60.
13. W. Pauli, and V. Weisskopf, Helv. Phys. Acta **7**, 709(1934).
14. P. A. M. Dirac, Physikalische Zeitschrift der Sowjetunion, Band **3**, Heft **1**, 64 (1933).
15. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products, Fifth Edition* (Academic Press, New York, 1996).
16. S. M. W. Ahmad and E. P. Wigner, Il Nuovo Cimento **28A**, 1 (1975).
17. R. F. O'Connell and E. P. Wigner, Phys. Lett. **61A**, 353 (1977); Phys. Lett. **67A**, 319 (1978).

A. MINIMUM UNCERTAINTY PRINCIPLE

For the Hermitian operators $\hat{A}$ and $\hat{B}$ that satisfy the commutation relation $[\hat{A}, \hat{B}] = i$, one defines the deviation operators $\hat{\Delta}(A)$ and $\hat{\Delta}(B)$ where $\hat{\Delta}(Q) = \hat{Q} - \bar{Q}$, with $\bar{Q} = \langle\psi|\hat{Q}|\psi\rangle$. It follows that $[\hat{\Delta}(A), \hat{\Delta}(B)] = i$. Introducing the operator

$$\hat{a} = \frac{\hat{A} + i\lambda\hat{B}}{\sqrt{2}}, \tag{A.1}$$

for real λ, the inner product inequality

$$\langle\psi|\hat{\Delta}^{\dagger}(a)\,\hat{\Delta}(a)\,|\psi\rangle \geq 0 \tag{A.2}$$

becomes

$$\left(\lambda - \frac{1}{2\sigma^2(B)}\right)^2 + \frac{\sigma^2(A)}{\sigma^2(B)} \geq \frac{1}{4\sigma^4(B)}, \tag{A.3}$$

where $\sigma(Q)$ is defined in Eq. (5.26). This equation implies

$$\sigma(A)\,\sigma(B) \geq \frac{1}{2}. \tag{A.4}$$

If equality is satisfied in Eq. $(A.2)$ and Eq. $(A.4)$, then $\lambda = 1/(2\sigma^2(B)$. The quantum state closest to the classical result has $\sigma(A) = \sigma(B)$, and this implies $\lambda = 1$. These conditions imply

$$\hat{\Delta}(a)\,|a\rangle = 0, \tag{A.5}$$

where $|a\rangle$ is the well known coherent state, which is generated from the vacuum state $|0\rangle$ with the displacement operator

$$|a\rangle = \hat{D}(a)\,|0\rangle = e^{\left(a\hat{a}^{\dagger} - a^{*}\hat{a}\right)}|0\rangle. \tag{A.6}$$

If $\hat{A}$ and $\hat{B}$ are Hermitian and satisfy the commutation relation $[\hat{A}, \hat{B}] = i$, then the spectrum of $\hat{B}$ is in general both positive and negative and unbounded in the B representation where $\hat{B}|B\rangle = B|B\rangle$. This is seen from

$$\hat{U}(A)\,\hat{B}\hat{U}^{\dagger}(A)\,\hat{U}(A)\,|B\rangle = B\hat{U}(A)\,|B\rangle, \tag{A.7}$$

which become

$$\hat{B}\hat{U}(A)\,|B\rangle = (B + \phi)\,\hat{U}(A)\,|B\rangle, \tag{A.8}$$

where $\hat{U}(A) = \exp(i\phi\hat{A})$, with $-\infty < \phi < \infty$. A similar result is found for the A representation.