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Critical Analysis of the Bogoliubov Theory of Superfluidity

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Abstract

The microscopic theory of superfluidity [1–3] was proposed by Bogoliubov in 1947 to explain the Landau-type excitation spectrum of ^{4}He. An analysis of the Bogoliubov theory has already been performed in the recent review [4]. Here we add some new critical analyses of this theory. This leads us to consider the superstable Bogoliubov model [5]. It gives rise to an improvement of the previous theory which will be explained with more details in a next paper [6]: coexistence in the superfluid liquid of particles inside and outside the Bose condensate (even at zero temperature), Bose/Bogoliubov statistics, "Cooper pairs" in the Bose condensate, Landau-type excitation spectrum...

Keywords : Bogoliubov, helium, superfluidity, Landau. excitation, spectrum, Cooper, Bose, condensation.

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1. Introduction

"Le monde progresse grâce aux choses impossibles qui ont t ralises."
André Maurois.

The first microscopic theory of superfluidity was originally proposed in 1947 by Bogoliubov in three revolutionary papers on the theory of interacting Bose gas [1–3]. His Weakly Imperfect Bose Gas (WIBG) arising from the truncation of a full interacting gas, was a starting point for this theory. However, only very few rigorous results concerning his WIBG and ansätze were known until 1998-2000. Then, the recent papers [7–11] expressed for the first time a rigorous analysis of this Bogoliubov model (WIBG) in the sense that the grand-canonical thermodynamic behavior is finally given at all temperatures and densities.

A deeper analysis of the Bogoliubov theory, including all recent studies [7–12] and some new critical analyses, has already been done from the point of view of rigorous results in the recent review [4]. However, the intention of our work is to check more carefully the Bogoliubov theory and its problematic ansätze in order to get solutions and explanations not included in [4].

Our detailed analysis gives rise to an improved *new* microscopic theory of superfluidity for liquid helium explained in [6]. Here we review and give a critical discussion of the standard microscopic theory of superfluidity, specially the Bogoliubov truncation and approximations corresponding to both canonical and grand-canonical Bogoliubov theories of superfluidity, see section 2. This will lead us in section 3 to our proposal for a new microscopic theory of superfluidity, for which we provide the physical arguments.
Notice that a recent historical overview of superfluidity is given in the paper [13]. Here we take into account *only homogeneous gases.* Concerning the inhomogeneous case, a rigorous proof that a 100% superfluid liquid occurs, corresponding also to a 100% Bose condensate, was performed for the first time with dilute trapped gases at zero-temperature whose the number of particles goes to infinity with an interacting potential converging to a Dirac function, see [14, 15].

2. The Bogoliubov theory of superfluidity

"Que celui qui n'a jamais péché jette au poisson la première pierre."
Francis Blanche, *Le Carnaval des animaux.*

We give here a detailed analysis of the Bogoliubov theory to emphasize on its different problems and questions.

2.1. Setup of the problem: the full interacting Bose gas

Let an interacting homogeneous gas of n spinless bosons with mass m be enclosed in a cubic box $\Lambda = \times_{\alpha=1}^{3} L \subset \mathbb{R}^3$. We denote by $\varphi(x) = \varphi(\|x\|)$ a (real) *two-body* interaction potential satisfying:

(A) $\varphi(x) \in L^1(\mathbb{R}^3)$.

(B) Its (real) Fourier transformation

$$\lambda_k = \int_{\mathbb{R}^3} d^3x \varphi(x) e^{-ikx}, \ k \in \mathbb{R}^3,$$

satisfies: $\lambda_0 > 0$ and $0 \leq \lambda_k \leq \lambda_0$ for $k \in \mathbb{R}^3$.

Using *periodic boundary conditions*, the corresponding Hamiltonian of the system acting on the boson Fock space $\mathcal{F}_\Lambda^B$ is equal to

$$H_{\Lambda,\lambda_0>0} = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + U_\Lambda^{MF} + \widetilde{U}_\Lambda, \tag{2.1}$$

with

$$\widetilde{U}_\Lambda \equiv \frac{1}{2V} \sum_{k_1,k_2,q\neq 0\in\Lambda^*} \lambda_q a_{k_1+q}^* a_{k_2-q}^* a_{k_1} a_{k_2}, \tag{2.2}$$

$$U_\Lambda^{MF} \equiv \frac{\lambda_0}{2V} \sum_{k_1,k_2\in\Lambda^*} a_{k_1}^* a_{k_2}^* a_{k_2} a_{k_1} = \frac{\lambda_0}{2V}\left(N_\Lambda^2 - N_\Lambda\right). \tag{2.3}$$

Here

$$N_\Lambda \equiv \sum_{k\in\Lambda^*} a_k^* a_k$$

is the particle number operator. $\varepsilon_k = \hbar^2 k^2/2m$ represents the one-particle energy spectrum and

$$\Lambda^* = \left\{ k \in \mathbb{R}^3 : \ k_\alpha = \frac{2\pi n_\alpha}{L}, \ n_\alpha = 0, \pm 1, \pm 2, ..., \ \alpha = 1,2,3 \right\}$$

is the set of wave vectors. Also, note that $a_k^\# = \{a_k^* \text{ or } a_k\}$ are the usual boson creation/annihilation operators in the one-particle state $\psi_k(x) = V^{-\frac{1}{2}} e^{ikx}$, $k \in \Lambda^*$, $x \in \Lambda$, acting on the boson Fock space

$$\mathcal{F}_\Lambda^B \equiv \bigoplus_{n=0}^{+\infty} \mathcal{H}_B^{(n)},$$

with $\mathcal{H}_B^{(n)}$ defined as the symmetrized n-particle Hilbert spaces

$$\mathcal{H}_B^{(n)} \equiv \left(L^2(\Lambda^n)\right)_{\text{symm}}, \ \mathcal{H}_B^{(0)} = \mathbb{C},$$

see [16, 17]. Under assumptions (A) and (B) on the interaction potential $\varphi(x)$, the full Hamiltonian $H_{\Lambda,\lambda_0>0}$ is superstable [16].

2.2. The Bogoliubov truncation and approximation

In order to get a microscopic explanation of the (phenomenological) Landau's theory of superfluidity [18,19], it is crucial to get a Landau-type excitation spectrum for the model (2.1). Indeed Landau [18,19] understood for the first time that the properties of quantum liquids like ^{4}He (or ^{3}He) can be entirely described by the spectrum of collective excitation, which for liquid ^{4}He, has two branches : "phonons" for long-wavelength excitations and "rotons" for a relatively short-wavelength collective excitations. This second assumption ensures the superfluidity of the Bose system. The computation of the spectrum of the full interacting model (2.1) is far from being solved for general two-body potential $\varphi(x)$. For a large class of interaction potentials $\varphi(x)$, the standard canonical or grand-canonical thermodynamic functions (free-energy density or pressure) of the Bose gas (2.1) in full interaction are unknown.
Two ways to extract some thermodynamic properties from the original model (2.1) would be either to use a very particular two-body potential $\varphi(x)$ (see [20–25]), or to truncate the full interaction of (2.1).
As an example, the Mean-Field Hamiltonian

$$H_\Lambda^{MF} \equiv T_\Lambda + U_\Lambda^{MF} = T_\Lambda + \frac{\lambda_0}{2V}\left(N_\Lambda^2 - N_\Lambda\right)$$

consists of either taking a constant two-body interaction potential $\varphi(x)$ in the box Λ , or cutting-off the terms with $q \neq 0$ in the full interaction of (2.1). This model has been analyzed exhaustively via the Imperfect Bose Gas

$$H_\Lambda^{IBG} \equiv T_\Lambda + \frac{\lambda_0}{2V} N_\Lambda^2$$

in [26–32] and is thermodynamically "very close" to the Perfect Bose Gas, see the discussions in [4,33,34].
The Bogoliubov WIBG, coming from the microscopic (Bogoliubov) theory of superfluidity [1–4,8,11,35,36], is also an example of such truncation procedures. Indeed, if one expects that the Bose-Einstein condensation, which occurs for the Perfect Bose-Gas in the mode $k = 0$, persists for a weak interaction $\varphi(x)$, then according to Bogoliubov [1–3,35,36] the most important terms in (2.1) should be those in which at least two operators a_0^*, a_0 appear. We are thus led to consider the following truncated Hamiltonian [1–3,35,36] :

$$H_{\Lambda,\lambda_0>0}^{B} \equiv T_\Lambda + U_\Lambda^D + U_\Lambda^{ND} + U_\Lambda^{BMF}, \tag{2.4}$$

where we recall that $\lambda_0 > 0$ (hypothesis (B)) and

$$\begin{aligned}
T_\Lambda &\equiv \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k, \\
U_\Lambda^D &\equiv \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k a_0^* a_0 \left(a_k^* a_k + a_{-k}^* a_{-k}\right), && (2.5)\\
U_\Lambda^{ND} &\equiv \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left(a_k^* a_{-k}^* a_0^2 + a_0^{*^2} a_k a_{-k}\right), && (2.6)\\
U_\Lambda^{BMF} &\equiv \frac{\lambda_0}{2V} a_0^{*^2} a_0^2 + \frac{\lambda_0}{V} a_0^* a_0 \sum_{k\in\Lambda^*\setminus\{0\}} a_k^* a_k. && (2.7)
\end{aligned}$$

This first step in the Bogoliubov theory, i.e. this truncation, is actually far from being exact. The Bogoliubov model $H^{B}_{\Lambda,\lambda_0>0}$ manifests, for high densities, a coexistence of two Bose condensations in the grand-canonical ensemble [4, 7, 8, 10, 11]. The first Bose condensation appears on the single mode $k = 0$ due to the nondiagonal interaction U^{ND}_{Λ} cf. [4, 7–10]. But it saturates for high densities and then coexists with a conventional Bose-Einstein condensation on modes next to the zero-mode ($\|k\| = 2\pi/L$), see [4, 11]. Then, for high densities, to be at least self-consistent in this procedure, the terms in (2.1) involving the 6 modes $\|k\| = 2\pi/L$ should not have been neglected in the truncation of the full interaction!

The Bogoliubov model (2.4) is "simpler" than the full Hamiltonian (2.1) but it is still *nondiagonal.* A very ingenious Bogoliubov treatment to solve this problem was to consider the two operators $a_0/\sqrt{V}$, $a_0^*/\sqrt{V}$ as complex numbers:

$$a_0/\sqrt{V} \to c,\ a_0^*/\sqrt{V} \to \bar{c},$$

since for large V, $a_0/\sqrt{V}$, $a_0^*/\sqrt{V}$ almost commute. This assumption is called the *Bogoliubov approximation.* Attempts of mathematical justification of this procedure and its intimate connection with representations of the Canonical Commutations Relations (CCR) was the subject of several papers, see e.g. [17, 37, 38]. A very interesting analysis was done by Ginibre [39] where he thermodynamically treated this problem for the full Hamiltonian (2.1).
This Bogoliubov treatment implies a new self-adjoint operator $H^{B}_{\Lambda,\lambda_0>0}(0,c)$ (cf. (A.1) with $\alpha = 0$) depending on operators $\{a_k\}_{k\in\Lambda^*\setminus\{0\}}$. This operator is well-defined on the Boson Fock space

$$\mathcal{F}'_B \equiv \bigoplus_{n=0}^{+\infty} \mathcal{H}^{(n)}_{B,k\neq 0},$$

constructed on the Hilbert space $\mathcal{H}'$ spanned by $\left\{\psi_k = e^{ikx}/\sqrt{V}\right\}_{k\in\Lambda^*\setminus\{0\}}$, where $\mathcal{H}^{(n)}_{B,k\neq 0}$ are the symmetrized n-particle Hilbert spaces appropriate for non-zero momentum bosons ($\mathcal{H}^{(0)}_B = \mathbb{C}$). However, the Hamiltonians $H_{\Lambda,\lambda_0>0}$ and $H^{B}_{\Lambda,\lambda_0>0}$ commute with the total particle-number N_Λ, whereas $H^{B}_{\Lambda,\lambda_0>0}(0,c)$ (A.1) *does not:*

$$\left[H^{B}_{\Lambda,\lambda_0>0}(0,c), N_\Lambda\right] \neq 0.$$

Then, in the canonical ensemble, Bogoliubov [3] suggests a different but similar way corresponding to a canonical Bogoliubov theory of superfluidity, whereas the intuitive Bogoliubov approximation takes place only in the grand-canonical Bogoliubov theory of superfluidity: see below.

2.3. Canonical Bogoliubov theory of superfluidity [3, 35, 40–43]

Let us consider the Bogoliubov Hamiltonian $H^{B}_{\Lambda,\lambda_0>0}$ in the canonical ensemble. Since $N_0 = a_0^* a_0$ is a non-negative self-adjoint operator, the operator $(N_0 + I)^{-1/2}$ is correctly defined and bounded. Let

$$\zeta_k = a_0^*\left(N_0 + \mathrm{I}\right)^{-1/2} a_k,\ \zeta_k^* = a_k^*\left(N_0 + \mathrm{I}\right)^{-1/2} a_0,\ k \in \Lambda^*. \tag{2.8}$$

Then for $k \neq 0$ these operators *satisfy* the canonical commutation relations (CCR), and via $(N_0 + \mathrm{I})^{-1/2} a_0 = a_0 N_0^{-1/2}$ the Hamiltonian $H^B_{\Lambda,\lambda_0>0}$ (2.4) can be written as

$$
\begin{aligned}
H^B_{\Lambda,\lambda_0>0} &= \sum_{k\in\Lambda^*} \varepsilon_k \zeta_k^* \zeta_k + \frac{N_0}{V}\lambda_0 \sum_{k\in\Lambda^*\setminus\{0\}} \zeta_k^*\zeta_k + \frac{\lambda_0}{2V} N_0 (N_0 - \mathrm{I}) \\
&+ \frac{N_0}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left(\zeta_k^*\zeta_k + \zeta_{-k}^*\zeta_{-k}\right) \\
&+ \frac{1}{2} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left(\zeta_k^*\zeta_{-k}^* \frac{N_0^{1/2}(N_0 - \mathrm{I})^{1/2}}{V} + \frac{N_0^{1/2}(N_0 - \mathrm{I})^{1/2}}{V} \zeta_k\zeta_{-k} \right). \qquad (2.9)
\end{aligned}
$$

The *canonical Bogoliubov approximation* for the Hamiltonian (2.9) corresponds to

$$
\frac{N_0}{V} \to |c|^2, \; \frac{N_0^{1/2}(N_0 - \mathrm{I})^{1/2}}{V} \to |c|^2. \qquad (2.10)
$$

The procedure implies the model $H^B_{\Lambda,\lambda_0>0}(0,c)$ (A.1) where the operators $\{a_k\}_{k\in\Lambda^*\setminus\{0\}}$ are replaced by $\{\zeta_k\}_{k\in\Lambda^*\setminus\{0\}}$. By (2.8), this model now conserves the number of particles, i.e. we can treat it in the canonical ensemble with some parameter c.
To exclude this uncertainty Bogoliubov proposed to eliminate the operator N_0 from (2.9) at the cost of further approximations, see [3, 35] and discussion in [40]. Since

$$
N_{\Lambda,k\neq 0} \equiv \sum_{k\in\Lambda^*\setminus\{0\}} a_k^* a_k = \sum_{k\in\Lambda^*\setminus\{0\}} \zeta_k^*\zeta_k, \; N_0 = N_\Lambda - N_{\Lambda,k\neq 0},
$$

he used the following approximation

$$
\frac{N_0^2}{2V} + \frac{N_0}{V} \sum_{k\in\Lambda^*\setminus\{0\}} \zeta_k^*\zeta_k \simeq \frac{N_\Lambda^2}{2V} \qquad (2.11)
$$

in the sum of the second and the third terms of (2.9) and

$$
\frac{N_0}{V} \simeq \frac{N_\Lambda}{V} \to \rho \qquad (2.12)
$$

in the third, the fourth and the fifth terms to arrive to an approximating Hamiltonian for the canonical ensemble with density ρ:

$$
\begin{aligned}
\widetilde{H}^B_{\Lambda,\lambda_0>0}(\rho) &= \sum_{k\in\Lambda^*\setminus\{0\}} (\varepsilon_k + \rho\lambda_k)\zeta_k^*\zeta_k + \frac{\rho}{2} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left(\zeta_k^*\zeta_{-k}^* + \zeta_k\zeta_{-k}\right) \\
&+ \lambda_0 \frac{N_\Lambda^2}{2V} - \frac{1}{2}\rho\lambda_0. \qquad (2.13)
\end{aligned}
$$

Since the last two terms are constants in the canonical ensemble we get a bilinear form in Bose-operators $\{\zeta_k\}_{k\in\Lambda^*\setminus\{0\}}$. Therefore, using the Bogoliubov canonical u-v transformation (see Appendix A with $a_k \to \zeta_k$), we finally get the well-known Bogoliubov gapless spectrum (cf.

(A.4) with $\lambda_0 = 0$ and $\alpha = 0$).

This canonical approach involves two main assumptions: (2.10) and (2.11)-(2.12). By analogy with some examples in the grand-canonical ensemble [7, 8, 10, 39], the canonical Bogoliubov approximation (2.10) should be true, but we should be very doubtful concerning (2.11)-(2.12). The approximation (2.11), taken in terms of operators, change the original Bogoliubov Hamiltonian $H^B_{\Lambda,\lambda_0>0}$ drastically, whereas (2.12) imposes a completely condensed particles density by fixing $|c|^2 = \rho$. This last assumption is not true for the original Bogoliubov Hamiltonian $H^B_{\Lambda,\lambda_0>0}$ (at least not in the grand-canonical ensemble, see [7–11]). Experimentally (cf. [44,45]), an estimate of the fraction of condensate in liquid ^4He at zero-temperature is only 9% !

2.4. Grand-canonical Bogoliubov theory of superfluidity [1, 2, 46–53]

Now, let us consider the more well-known approach, i.e. the grand-canonical Bogoliubov theory of superfluidity. Originally proposed by Bogoliubov [1, 2], and essentially advocated by Beliaev [46, 47], Hugenholtz and Pines [48–51], Tserkovnikov [52], and Tolmachev [53], to remove the problem of non-conservation of the particle number, they suggested to use the grand-canonical ensemble from the very beginning, i.e. they introduced a chemical potential α:

$$H^B_{\Lambda,\lambda_0>0}(\alpha) = H^B_{\Lambda,\lambda_0>0} - \alpha N_\Lambda. \tag{2.14}$$

Then the Bogoliubov approximation gives the Hamiltonian $H^B_{\Lambda,\lambda_0>0}(\alpha, c)$ (see Appendix A).

Remark 2.1. *The models, $\widetilde{H}^B_{\Lambda,\lambda_0>0}(\rho)$ (2.13) in the canonical ensemble and $H^B_{\Lambda,\lambda_0>0}(\alpha, c)$ (A.1) in the grand-canonical ensemble, represent two different Bose systems even for $|c|^2 = \rho$.*

After the gauge transformation (A.2), the Hamiltonian (A.1) depends only on $x = |c|^2$. Then, in [46, 48, 50, 53] the authors proposed to fix $x = |c|^2$ using the *variation principle* combined with the ground state ψ_0 of the Perfect Bose Gas:

$$\frac{\partial}{\partial |c|^2} \left(\psi_0, H^B_{\Lambda,\lambda_0>0}\psi_0\right)_{\mathcal{F}^B_\Lambda} = \frac{\partial}{\partial |c|^2}\left(\frac{1}{2}\lambda_0 |c|^4 V - \alpha |c|^2 V\right) = 0, \tag{2.15}$$

i.e.

$$\alpha = \lambda_0 |c|^2. \tag{2.16}$$

For a given total particle density ρ the chemical potential should be excluded from (2.16) by the subsidiary condition, which defines $|c|^2$ as a function of ρ. If one does this in the *first* approximation [46, 48, 50], then one gets $|c|^2 = \rho$, which returns us to (2.13). Therefore using again the Bogoliubov canonical *u-v* transformation (see Appendix A), we again get the Bogoliubov gapless spectrum (cf. (A.4) with $\lambda_0 = 0$ and $\alpha = 0$).

The clever Bogoliubov approximation on the model $H^B_{\Lambda,\lambda_0>0}$ is in fact true in terms of the thermodynamic behavior, see [4, 7, 8, 10]. However, the main assumption (2.16) which is crucial to get a gapless spectrum is false, in the sense that the theory is not rigorously consistent. Actually, in [5], the authors show for the first time that the condition (2.16) for $|c|^2 > 0$ involves a positive chemical potential where the pressure of the original Hamiltonian $H^B_{\Lambda,\lambda_0>0}$

does not exist. Then, in [4, 7, 8, 10], it is shown that the thermodynamically relevant spectrum of the original Hamiltonian $H^{B}_{\Lambda,\lambda_0>0}$ always has a gap for any chemical potential α in the existence domain of the pressure.

2.5. Remark on the spectrum of excitations

Before we embark on a strong revision of the Bogoliubov theory, we want to make precise the definition of the excitation spectrum of a system of particles. In particular, which is the relevant ensemble between the canonical and grand-canonical one, in terms of physical excitation spectrum?

It is clear that the spectrum of excitations should be understood as the spectrum of the corresponding Hamiltonian. Considering for example the Perfect Bose Gas, this spectrum is given by $\{\varepsilon_k\}_{k\in\mathbb{R}}$ in the canonical ensemble whereas in the grand-canonical ensemble it equals $\{\varepsilon_k-\alpha\}_{k\in\mathbb{R}}$, i.e. the spectrum has a gap for $\alpha<0$. Of course, the presence of this gap comes only from the *Lagrange multiplier* α associated with the operator N_Λ/V [54]. The excitation spectrum of the Perfect Bose Gas is then $\{\varepsilon_k\}_{k\in\mathbb{R}}$. The chemical potential α has no physical relevance in terms of spectrum of excitations, i.e. the *physical* spectrum of excitations should be seen *only* in the canonical ensemble.

An absence of gaps in the grand-canonical ensemble is *only a specific case.* For example, it is only in the presence of the conventional Bose-Einstein condensation that this property holds for the Perfect Bose Gas and then for the Mean-Field Bose Gas or the Imperfect Bose Gas, see [26–33]. This fact can also not be generalized to any Bose system having a Bose condensation, i.e. a gap on the spectrum in the grand-canonical ensemble may appear even if no gap exists in the canonical ensemble. For the Bogoliubov microscopic theory of superfluidity, the spectrum in the two ensembles gives the same result . However, it is only because of the *drastic* Bogoliubov assumption (2.16), that all effects of the chemical potential on the spectrum are removed in the grand-canonical ensemble (β,α).

Consequently, in terms of the spectrum of excitations, a Bose system should be thermodynamically analyzed *only in the canonical ensemble.*

3. A new microscopic theory of superfluidity?

To correct the Bogoliubov microscopic theory of superfluidity, the main guiding principle should be to get a gapless Hamiltonian, or at least a Hamiltonian whose spectrum seems to be gapless. Considering the complex Bose system (2.1), we should also truncate the Hamiltonian but, of course, in a different way than Bogoliubov did, see the above discussion in subsection 2.2.

Regarding the last subsection 2.5, this truncation should be understood in the framework of the canonical ensemble. In this ensemble (β,ρ) and in terms of thermodynamic properties, the full Hamiltonian (2.1) is completely equivalent to the model

$$H_{\Lambda,0}=\sum_{k\in\Lambda^*}\varepsilon_k a_k^*a_k+\widetilde{U}_\Lambda=\sum_{k\in\Lambda^*}\varepsilon_k a_k^*a_k+\frac{1}{2V}\sum_{k_1,k_2,q\neq0\in\Lambda^*}\lambda_q a_{k_1+q}^*a_{k_2-q}^*a_{k_1}a_{k_2}, \tag{3.1}$$

see (2.1)-(2.2), since the Mean-Field interaction U_Λ^{MF} (2.3) is simply a constant on the Hilbert space $\mathcal{H}_B^{(n=[\rho V])}$. Here β is the inverse temperature, and ρ the fixed full particle density, whereas $n=[\rho V]$, defined as the integer part of $V\rho$, is the number of particles.

Formally, the Mean-Field interaction U_Λ^{MF} does not change the "physical properties" of a Bose system (cf. [33, 34]) and the "physical" effect of the interaction potential should express itself by means of interaction $\widetilde{U}_\Lambda$. This nondiagonal interaction $\widetilde{U}_\Lambda$ (2.2) is then the *only interaction to truncate* following the first Bogoliubov procedure. We then get the following (*nondiagonal)* superstable Bogoliubov Hamiltonian in the canonical ensemble:

$$H_\Lambda^{SB} \equiv H_{\Lambda,\lambda_0=0}^{B} + U_\Lambda^{MF}, \tag{3.2}$$

see (2.3) and (2.4) for $\lambda_0 = 0$. In the canonical ensemble, the superstable Bogoliubov Hamiltonian H_Λ^{SB} is also equivalent to the model $H_{\Lambda,0}^{B}$. Now, considering that the canonical Bogoliubov approximation is true, we directly get

$$H_{\Lambda,\lambda_0}^{SB}(c) = \widehat{H}_{\Lambda,0}^{SB}(c) + \frac{\lambda_0}{2}\left(\rho^2 V - \rho\right), \tag{3.3}$$

in the canonical ensemble with

$$\begin{aligned}\widehat{H}_{\Lambda,0}^{SB}(c) &= \sum_{k\in\Lambda^*\setminus\{0\}} \varepsilon_k \zeta_k^* \zeta_k + \frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k |c|^2 \left[\zeta_k^*\zeta_k + \zeta_{-k}^*\zeta_{-k}\right] \\ &+ \frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left[c^2 \zeta_k^* \zeta_{-k}^* + \bar{c}^2 \zeta_k \zeta_{-k}\right].\end{aligned}$$

The Hamiltonian $\widehat{H}_{\Lambda,0}^{SB}(c)$ is again a bilinear form in Bose-operators $\{\zeta_k\}_{k\in\Lambda^*\setminus\{0\}}$ (2.8) diagonalizable with the Bogoliubov canonical u-v transformation (Appendix A with $a_k \to \zeta_k$). We then get the well-known Bogoliubov gapless spectrum for any $x = |c|^2 \geq 0$ (cf. (A.4) with $\lambda_0 = 0$ and $\alpha = 0$).

Remark 3.1. *Under the assumption $|c|^2 = \rho$ we have the equality $H_\Lambda^{SB}(c) = \widetilde{H}_{\Lambda,\lambda_0=0}^{B}(\rho)$. The paper [6] shows that this assumption is exact only for $\rho \to \infty$.*

The second term of (3.3) should not be taken into account on the thermodynamic level, since it is a constant. In the canonical-ensemble, this means also that we can directly consider the Hamiltonian $H_{\Lambda,0}^{B}$ (see (2.4) with $\lambda_0 = 0$) instead of the superstable Hamiltonian H_Λ^{SB}. The Hamiltonian $H_{\Lambda,0}^{B}$ corresponds to the original Bogoliubov truncation done on $H_{\Lambda,0}$. But, unfortunately, in the grand-canonical ensemble, this Bose system is drastically instable at high densities, i.e. the terms of repulsion are not strong enough to prevent the system from collapse, see Appendix B. In order to analyze the thermodynamic properties of $H_{\Lambda,0}^{B}$ in the canonical ensemble, one should consider its supertabilized form [33, 34], i.e. the superstable Hamiltonian H_Λ^{SB} (cf. [6]). Then, we should concentrate our discussion only on the superstable gas H_Λ^{SB}, whose thermodynamic properties exist in the two ensembles (can./grand-can.) at all densities.

At this point, the reader may be very critical about this intuitive explanation of the gapless spectrum in the canonical ensemble, for the moment based on *only two* assumptions: a truncation of the full Hamiltonian [or $H_{\Lambda,0}$ (3.1)] implying H_Λ^{SB} [or $H_{\Lambda,0}^{B}$ (2.4)], and the canonical Bogoliubov approximation (2.10).

A first important question concerns the truncation. The Bogoliubov one was false: it was *not rigorously consistent* with the grand-canonical thermodynamic behavior of the Bogoliubov

Hamiltonian and the appearance of a second Bose condensation outside the zero-mode [4, 7–10], see subsection 2.2. We are going to explain why the truncation done here is also better from this point of view.
Actually, the paper [9] is very useful to point out the origin of this second Bose condensation for the Bogoliubov Hamiltonian $H^B_{\Lambda,\lambda_0>0}$ (2.4). Indeed, the apparition of the second (conventional) Bose-Einstein condensation for the Bogoliubov WIBG comes from the term of repulsion

$$\frac{\lambda_0}{2V} a_0^{*2} a_0^2 = \frac{\lambda_0}{2V} \left(N_0^2 - N_0 \right), \text{ with } N_0 \equiv a_0^* a_0, \tag{3.4}$$

which implies the saturation of the first (non-conventional) Bose condensation by excluding particles in the zero mode, since for any $k \neq 0$ the similar terms of repulsion

$$\left\{ \frac{\lambda_0}{2V} a_k^{*2} a_k^2 = \frac{\lambda_0}{2V} \left(N_k^2 - N_k \right), \text{ with } N_k \equiv a_k^* a_k \right\}_{k \in \Lambda^* \setminus \{0\}} \tag{3.5}$$

in the full Hamiltonian (2.1) are neglected in the Bogoliubov truncation. All terms (3.4)-(3.5) come from the Mean-Field (also called the "forward scattering") interaction U_Λ^{MF} (2.3). Consequently, keeping the interaction (2.3) in the superstable Bogoliubov Hamiltonian (3.2) allows us to avoid the appearance of a second Bose condensation, which would be inconsistent with this truncation.

A second remark concerns the canonical Bogoliubov approximation. In the grand-canonical ensemble (β, α), it is proven by Ginibre [39] that the Bogoliubov approximation is exact for any superstable Hamiltonian, including H_Λ^{SB}. We believe that this procedure works also in the canonical ensemble, and we are going to prove it for this specific model in [6].

Before going further, let us add an important remark. The canonical Bogoliubov approximation (2.10) done here for the superstable model H_Λ^{SB} may be interpreted, in the grand-canonical ensemble, as using the Bogoliubov approximation only on $H^B_{\Lambda,\lambda_0=0}$, i.e. not in U_Λ^{MF} (2.3). In the grand-canonical ensemble, doing this incomplete Bogoliubov approximation seems to be inexact [39].
This last procedure *for the grand-canonical ensemble* was used in papers [5, 55, 56]. Indeed, the same truncation and so the corresponding Bogoliubov Hamiltonian H_Λ^{SB} *was previously proposed by N. Angelescu, A. Verbeure and V.A. Zagrebnov in 1992* [5]. The main object of this superstable model was, for the authors [5], to correct the instability for positive chemical potentials of the Bogoliubov Hamiltonian (2.4). At the same time, the aim in [5] was to find a gapless Bogoliubov spectrum. In [56] the authors use a "generalized" Bogoliubov approximation. This "generalized" Bogoliubov approximation corresponds to partially changing the operators $\left\{a_0/\sqrt{V}, a_0^*/\sqrt{V}\right\}$ by a suitable function $\left\{b(c), \overline{b(c)}\right\}$ in (3.2) *except* in the Mean-Field interaction U_Λ^{MF} (2.3). Then, they prove a Bose condensation in zero-mode via second-order phase transition and a linear asymptotic of the elementary excitation spectrum in condensed phase for $\|k\| \rightarrow 0$, see also discussions in Section 3.4 of [4].

In [6] we show that the first procedure (partial Bogoliubov approx.) done in [5] is true but the other one [56] (partial generalized Bogoluibov approx.) performed on the superstable model (3.2) in the grand-canonical ensemble is *inexact*, in the sense that it is equivalent to drastic modifications of the original Hamiltonian (3.2). However, as Bogoliubov did, they were forced in [5] to add some *additional assumptions* to find a gapless spectrum since they use the grand-canonical ensemble, see the discussion of subsection 2.5. From the beginning it was unlikely that the *exact solution* of H_Λ^{SB} had a gapless spectrum even in the presence of Bose-condensation. In fact, we prove in [6] that, on the thermodynamic level, there is a gapless spectrum in the canonical ensemble, but *not* in the grand-canonical ensemble at all chemical potentials.

Actually, the main problem of their methods (Bogoliubov *et al*) is to assume, à priori, the Bose condensation by directly doing the Bogoliubov approximation with an arbitrary choice of $|c|^2$, without exactly solving it in terms of the thermodynamic behavior. In particular, the explanation given after the truncation leading to (3.3) has to be proven. For example, Bogoliubov made the (shown to be wrong in [6]) assumption that $|c|^2 = \rho$ but what is our value of $|c|^2$ in (3.3)?

As the review [4] explained in the "outline" section, we should be discouraged *"from performing sloppy manipulations with Bose condensations, quantum fluctuations and different kinds of ansätze"*. The example of the exact solution of the WIBG-model (on the thermodynamic level) in relation with the Bogoliubov ansätze, provide a strong warning in doing it. Actually, the rigorous thermodynamic behavior of the *nondiagonal* superstable Bogoliubov Hamiltonian H_Λ^{SB} is performed in [6]. In particular, we rigorously prove that this model H_Λ^{SB} is "equivalent" in thermodynamic limit to the model $H_{\Lambda,0}^{B}(\widehat{c})$ (3.3) in the canonical ensemble. The value of $|\widehat{c}| = |\widehat{c}(\beta,\rho)| < \rho$ satisfies a variational principle, different from (2.15) in the canonical ensemble (β,ρ). This provides a new theory of superfluidity with a gapless spectrum at any particle densities and temperatures, leading to a deeper understanding of the Bose condensation phenomenon in liquid helium [6]: coexistence in the superfluid liquid of particles inside and outside the Bose condensate (even at zero temperature), Bose/Bogoliubov statistics, "Cooper pairs" in the Bose condensate.
This theory is based only on a weaker truncation (see (3.2)) than the Bogoliubov truncation (see (2.4) with $\lambda_0 > 0$). This *unique* hypothesis is not proven in this paper or [6] and it may not be exact even in thermodynamic limit (cf. discusions in [4, 12]. However, the paper [6] shows that the theory is, at least, self-consistent as intuitively explained. Moreover this implies the exact solution of a *nondiagonal continuous model*, i.e., $H_{\Lambda,0}^{B}$, far from the Perfect Bose Gas in the *canonical ensemble* at all temperatures and densities [6]. This is the first time for such a rigorous thermodynamic analysis to be performed on a non-trivial continuous gas.

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Appendix A. : the Bogoliubov u-v transformation

"L'ennemi se déguise parfois en géranium, mais on ne peut s'y tromper, car tandis que le géranium est à nos fenêtres, l'ennemi est à nos portes."
Pierre Desproges, *Manuel du savoir-vivre.*

In this subsection we recall the Bogoliubov canonical u-v transformation by applying it on the Bogoliubov approximation [39]

$$\begin{aligned} H^B_{\Lambda,\lambda_0}(\alpha,c) &= \sum_{k\in\Lambda^*\setminus\{0\}} \left[\varepsilon_k - \alpha + \lambda_0 |c|^2\right] a^*_k a_k + \frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k |c|^2 \left[a^*_k a_k + a^*_{-k}a_{-k}\right] \\ &+\frac{1}{2}\sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left[c^2 a^*_k a^*_{-k} + \bar{c}^2 a_k a_{-k}\right] - \alpha |c|^2 V + \frac{\lambda_0}{2}\left(|c|^4 V - |c|^2\right) \quad (A.1) \end{aligned}$$

of $H^B_\Lambda(\alpha) \equiv H^B_\Lambda - \alpha N_\Lambda$ (2.4) for any $\lambda_0 \geq 0$. After the canonical *gauge transformation* to boson operators

$$a_k e^{-i\arg c},\ k \in \Lambda^* \setminus \{0\}. \quad (A.2)$$

the Hamiltonian (A.1) depends only on $x = |c|^2$. Then, we compute the corresponding pressure

$$p^B_{\Lambda,\lambda_0}(\beta,\alpha,c) = \frac{1}{\beta V}\ln Tr_{\mathcal{F}'_B} e^{-\beta H^B_{\Lambda,\lambda_0}(\alpha,c)}. \quad (A.3)$$

Since $H^B_{\Lambda,\lambda_0}(\alpha,c)$ (A.1) is a bilinear form in boson operators $\left\{a^\#_k\right\}_{k\in\Lambda^*\setminus\{0\}}$, the Bogoliubov canonical u-v transformation diagonalizes it by using a new set of boson operators $\left\{b^\#_k\right\}_{k\in\Lambda^*\setminus\{0\}}$ defined by

$$a_k = \mathbf{u}_k b_k - \mathbf{v}_k b^*_{-k},\ a^*_k = \mathbf{u}_k b^*_k - \mathbf{v}_k b_{-k},$$

with real coefficients $\{\mathbf{u}_k = \mathbf{u}_{-k}\}_{k\in\Lambda^*\setminus\{0\}}$ and $\{\mathbf{v}_k = \mathbf{v}_{-k}\}_{k\in\Lambda^*\setminus\{0\}}$ satisfying:

$$\mathbf{u}^2_k - \mathbf{v}^2_k = 1,\ 2\mathbf{u}_k\mathbf{v}_k = \frac{x\lambda_k}{E^B_{k,\lambda_0}},\ \mathbf{u}^2_k + \mathbf{v}^2_k = \frac{\varepsilon_k}{E^B_{k,\lambda_0}}.$$

Here

$$\begin{aligned} f_{k,\lambda_0} &= \varepsilon_k - \alpha + x(\lambda_0 + \lambda_k), \\ E^B_{k,\lambda_0} &= \sqrt{f^2_{k,\lambda_0} - x^2\lambda^2_k} = \sqrt{(\varepsilon_k - \alpha + x\lambda_0)(\varepsilon_k - \alpha + x(\lambda_0 + 2\lambda_k))}, \end{aligned} \quad (A.4)$$

for $x \equiv |c|^2$. Thus

$$\mathbf{u}^2_k = \frac{1}{2}\left(\frac{f_{k,\lambda_0}}{E^B_{k,\lambda_0}} + 1\right),\ \mathbf{v}^2_k = \frac{1}{2}\left(\frac{f_{k,\lambda_0}}{E^B_{k,\lambda_0}} - 1\right).$$

Notice that $f_{k,\lambda_0} \geq x\lambda_k$ and, $|c|^2$ and α satisfy the inequality:

$$\alpha \leq |c|^2 \lambda_0 + \min_{k\in\Lambda^*\backslash\{0\}} \varepsilon_k. \tag{A.5}$$

The Hamiltonian (A.1) becomes:

$$\mathrm{H}^B_{\Lambda,\lambda_0}(\alpha, c) = \sum_{k\in\Lambda^*\backslash\{0\}} E^B_{k,\lambda_0} b^*_k b_k + \frac{1}{2} \sum_{k\in\Lambda^*\backslash\{0\}} \left(E^B_{k,\lambda_0} - f_{k,\lambda_0}\right) - \alpha x + \frac{\lambda_0}{2}\left(x^2 - \frac{x}{V}\right). \tag{A.6}$$

Therefore, the pressure $p^B_{\Lambda,\lambda_0}(\beta, \alpha, c)$ (A.3) equals

$$\begin{aligned}
p^B_{\Lambda,\lambda_0}(\beta,\alpha,c) &= \xi_{\Lambda,\lambda_0}\left(\beta,\alpha,x\equiv|c|^2\right) + \eta_{\Lambda,\lambda_0}\left(\alpha, x\equiv|c|^2\right),\\
\xi_{\Lambda,\lambda_0}(\beta,\alpha,x) &= \frac{1}{\beta V}\sum_{k\in\Lambda^*\backslash\{0\}} \ln\left(1-e^{-\beta E^B_{k,\lambda_0}}\right)^{-1},\\
\eta_{\Lambda,\lambda_0}(\alpha,x) &= \frac{1}{2V}\sum_{k\in\Lambda^*\backslash\{0\}} \left(f_{k,\lambda_0} - E^B_{k,\lambda_0}\right) + \alpha x - \frac{\lambda_0}{2}\left(x^2-\frac{x}{V}\right),
\end{aligned} \tag{A.7}$$

and has the following thermodynamic limit:

$$\begin{aligned}
p^B_{\lambda_0}\left(\beta,\alpha,x\equiv|c|^2\right) &\equiv \lim_{\Lambda} p^B_{\Lambda,\lambda_0}(\beta,\alpha,c) = \xi_{\lambda_0}(\beta,\alpha,x) + \eta_{\lambda_0}(\alpha,x),\\
\xi_{\lambda_0}(\beta,\alpha,x) &\equiv \lim_{\Lambda}\xi_{\Lambda,\lambda_0}(\beta,\alpha,x) = \frac{1}{(2\pi)^3\beta}\int_{\mathbb{R}^3}\ln\left(1-e^{-\beta E^B_{k,\lambda_0}}\right)^{-1} d^3k,\\
\eta_{\lambda_0}(\alpha,x) &\equiv \lim_{\Lambda}\eta_{\Lambda,\lambda_0}(\alpha,x) = \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\left(f_{k,\lambda_0} - E^B_{k,\lambda_0}\right)d^3k + \alpha x - \frac{\lambda_0}{2}x^2,
\end{aligned} \tag{A.8}$$

with $E^B_{k,\lambda_0\geq 0}$, $f_{k,\lambda_0\geq 0}$ defined by (A.4), and $\alpha \leq x\lambda_0$ by (A.5).

Appendix B. : The grand-canonical Bogoliubov Hamiltonian

"C'est encore plus beau lorsque c'est inutile."
Edmond Rostand, *Cyrano de Bergerac.*

We are exploring the thermodynamic behavior of the Hamiltonian $H^B_{\Lambda,0}$ (2.4), because the results in [4,8,11] are not useful anymore to deduce the thermodynamic properties since they are valid only for $\lambda_0 > 0$.

The pressure in the grand-canonical ensemble for a chemical potential α and an inverse temperature $\beta \geq 0$, is given by

$$p^B_{\Lambda,0}(\beta,\alpha) \equiv \frac{1}{\beta V}\ln Tr_{\mathcal{F}^B_\Lambda}\left(e^{-\beta\left(H^B_{\Lambda,0}-\alpha N_\Lambda\right)}\right),$$

and the grand-canonical particle density by

$$\rho^B_{\Lambda,0}(\beta,\alpha) \equiv \left\langle \frac{N_\Lambda}{V}\right\rangle_{H^B_{\Lambda,0}}(\beta,\alpha) = \partial_\alpha p^B_{\Lambda,0}(\beta,\alpha).$$

B.1. An upper bound for the grand-canonical pressure

Regrouping terms in (2.4) one has

$$H^B_{\Lambda,0} = H^I_\Lambda + \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left(a_0^* a_k + a_{-k}^* a_0\right)^* \left(a_0^* a_k + a_{-k}^* a_0\right) \geq H^I_\Lambda,$$

where

$$H^I_\Lambda = \sum_{k\in\Lambda^*\setminus\{0\}} \left(\varepsilon_k - \frac{\lambda_k}{2V}\right) N_k - \left(\frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k\right) N_0.$$

Hence we obtain

$$\begin{aligned} p^B_{\Lambda,0}(\beta,\alpha) \;&\leq\; p^I_\Lambda(\beta,\alpha) \equiv \frac{1}{\beta V} \sum_{k\in\Lambda^*\setminus\{0\}} \ln\left(1 - e^{-\beta\left(\varepsilon_k - \frac{\lambda_k}{2V} - \alpha\right)}\right)^{-1} + \\ &+ \frac{1}{\beta V} \ln\left(1 - e^{\beta\left(\alpha - \alpha_{\sup,\Lambda}\right)}\right)^{-1}, \end{aligned}$$

for

$$\alpha < \alpha_{\sup,\Lambda} \equiv -\frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k < 0. \tag{B.1}$$

B.2. A lower bound for the grand-canonical pressure using the Bogoliubov approximation

The corresponding lower bound for the Bogoliubov Hamiltonian $H^B_{\Lambda,\lambda_0>0}$ (2.4) found in [5] remains valid even for $\lambda_0 = 0$ and one gets

$$p^B_{\Lambda,0}(\beta,\alpha) \geq \sup_{c\in\mathbb{C}} p^B_{\Lambda,0}(\beta,\alpha,c), \tag{B.2}$$

where $p^B_{\Lambda,0}(\beta,\alpha,c)$ is defined by (A.7) in Appendix A. Therefore one has to analyze the lower bound $\sup_{c\in\mathbb{C}} p^B_{\Lambda,0}(\beta,\alpha.c)$.

Lemma B.1. *Let $p^B_{\Lambda,0}(\beta,\alpha,c)$ be given as in (A.7). Then*

$$\sup_{c\in\mathbb{C}} p^B_{\Lambda,0}(\beta,\alpha,c) = \begin{cases} p^B_{\Lambda,0}(\beta,\alpha,0) = p^{PBG}_\Lambda(\beta,\alpha)\ ;\ \text{for } \alpha \leq \alpha_{\sup,\Lambda} < 0 \\ \qquad\qquad +\infty\ ;\ \text{for } \alpha > \alpha_{\sup,\Lambda}, \end{cases}$$

where $p^{PBG}_\Lambda(\beta,\alpha)$ is the grand-canonical pressure for the Perfect Bose Gas.

Proof. Through (A.4) and (A.7) in Appendix A, one gets that for $\alpha \leq 0$:
(i)

$$\partial_x \eta_{\Lambda,0}(\alpha,x) = \alpha + \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k - \frac{1}{2V} \sum_{k\in\Lambda^*\setminus\{0\}} \lambda_k \left(1 + x\frac{2\lambda_k}{\varepsilon_k - \alpha}\right)^{-1/2},$$

$$\partial_x \eta_{\Lambda,0}(\alpha,0) = \alpha < 0;$$

(ii)

$$\partial_x^2 \eta_{\Lambda,0}(\alpha,x) = \frac{1}{2V}\sum_{k\in\Lambda^*\backslash\{0\}}\left(\frac{\lambda_k^2\sqrt{\varepsilon_k-\alpha}}{(\varepsilon_k-\alpha+2x\lambda_k)^{3/2}}\right) > 0.$$

Since

$$\lim_{x\to+\infty}\frac{1}{2V}\sum_{k\in\Lambda^*\backslash\{0\}}\lambda_k\left(1+x\frac{2\lambda_k}{\varepsilon_k-\alpha}\right)^{-1/2} = 0,$$

even in the thermodynamic limit, (i) implies

$$\alpha \le \partial_x \eta_{\Lambda,0}(\alpha,x) \le \alpha - \alpha_{\sup,\Lambda} \text{ for all } x \ge 0$$

and

$$\lim_{x\to+\infty}\left\{\partial_x\eta_{\Lambda,0}(\alpha,x) - \alpha + \alpha_{\sup,\Lambda}\right\} = 0,$$

we get with (ii)

$$\sup_{x\ge 0}\left\{\eta_{\Lambda,0}(\alpha,x)\right\} = \begin{cases} \eta_{\Lambda,0}(\alpha,x=0)\ ;\ \text{for } \alpha\le\alpha_{\sup,\Lambda} \\ +\infty\ ;\ \text{for } \alpha>\alpha_{\sup,\Lambda}. \end{cases} \tag{B.3}$$

Therefore, for $\beta\to\infty$ (zero-temperature) the corresponding pressure $p_{\Lambda,0}^B(\beta,\alpha,c)$ (A.7) attains its supremum at $c=0$ if $\alpha\le\alpha_{\sup,\Lambda}$ whereas $\sup_{c\in\mathbb{C}} p_{\Lambda,0}^B(\beta,\alpha,c)$ does not exist for any $\alpha>\alpha_{\sup,\Lambda}$. By (A.4) and (A.7) note that

$$\begin{array}{l} (i)\ \partial_x\xi_{\lambda_0\ge0}(\beta,\alpha,x)<0 \text{ and } \lim\limits_{x\to+\infty}\xi_{\lambda_0\ge0}(\beta,\alpha,x)=0, \\ (ii)\ \partial_\beta\xi_{\lambda_0\ge0}(\beta,\alpha,x)<0 \text{ and } \lim\limits_{\beta\to+\infty}\xi_{\lambda_0\ge0}(\beta,\alpha,x)=0. \end{array} \tag{B.4}$$

Hence via (B.3) and (B.4) the lemma holds. ■

Consequently, combining (B.2) with Lemma B.1, we find

$$p_{\Lambda,0}^B(\beta,\alpha) \ge p_{\Lambda,0}^B(\beta,\alpha,0) = p_\Lambda^{PBG}(\beta,\alpha), \tag{B.5}$$

for any $\alpha\le\alpha_{\sup,\Lambda}$, whereas for $\alpha>\alpha_{\sup,\Lambda}$ the pressure $p_{\Lambda,0}^B(\beta,\alpha)$ does not exist.

B.3. Thermodynamic behavior of the model

Via the previous upper bound and (B.5) we get

$$p_\Lambda^{PBG}(\beta,\alpha) \le p_{\Lambda,0}^B(\beta,\alpha) \le p_\Lambda^I(\beta,\alpha),$$

for $\alpha<\alpha_{\sup,\Lambda}$, which gives

$$p_0^B(\beta,\alpha) = \lim_\Lambda p_{\Lambda,0}^B(\beta,\alpha) = p^{PBG}(\beta,\alpha) \tag{B.6}$$

in the thermodynamic limit for

$$\alpha<\alpha_{\sup}\equiv \lim_\Lambda \alpha_{\sup,\Lambda} = -\frac{1}{2}\varphi(0),$$

and which can be extended by continuity of the pressure to $\alpha \leq \alpha_{\text{sup}}$. Here $p^{PBG}(\beta, \alpha)$ is the infinite volume pressure for the Perfect Bose Gas. From (B.6) and Griffiths lemma [57,58] the infinite volume particle density $\rho_0^B(\beta, \alpha)$ equals

$$\rho_0^B(\beta, \alpha) = \partial_\alpha p^{PBG}(\beta, \alpha) = \rho^{PBG}(\beta, \alpha) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \left(e^{\beta(\varepsilon_k - \alpha)} - 1\right)^{-1}$$

for $\alpha < \alpha_{\text{sup}}$ and therefore

$$\lim_{\alpha \to \alpha_{\text{sup}}^-} \rho_0^B(\beta, \alpha) = \rho^{PBG}(\beta, \alpha_{\text{sup}}) < +\infty, \tag{B.7}$$

i.e. it is not possible to reach high densities regimes in the grand-canonical ensemble (β, α).

Hence the properties of the model $H_{\Lambda,0}^B$ are, in a way, trivial for rather negative chemical potential $\alpha \leq \alpha_{\text{sup},\Lambda}$: they are equivalent to the Perfect Bose Gas. The nondiagonal interaction U_Λ^{ND} (2.6) is not able to change the system for sufficiently negative chemical potential $\alpha \leq \alpha_{\text{sup}}$. This fact is not surprising since it is exactly the same for the Bogoliubov Hamiltonian H_Λ^B for $\alpha \leq \alpha_{\text{sup},\Lambda}$, see the corresponding lower and upper bounds in [5] and discussions in [4,8]. Actually, as soon as the nondiagonal interaction U_Λ^{ND} (2.6) beats the kinetic part for $\alpha > \alpha_{\text{sup},\Lambda}$ by attracting particles in the zero-mode [9], the system becomes unstable, i.e. all particles collapse in the zero-mode because of the absence of strong enough repulsion terms such as (3.4). Such terms as (3.4) are then crucial to induce the non-conventional Bose condensation mechanism without any instability.

The model $H_{\Lambda,0}^B$ turns out to be not sufficient for a microscopic theory of superfluidity in the grand-canonical ensemble.

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