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String Motion on SDiff(M) and Hydrodynamics with Internal Degrees of Freedom

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Abstract

We consider classical string theory where the target space $SDiff(M)$ is the infinite dimensional group of volume preserving diffeomorphisms of a smooth manifold M. In analogy with the case of free point particle motion on $SDiff(M)$, where the geodesic equation is equivalent to the Euler equation for an incompressible fluid, we find a new set of coupled nonlinear equations. These can be interpreted as equations for a perfect fluid with some internal degrees of freedom. Some subtleties involved in quantizing this geometrical picture are discussed.

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1 Introduction

A geometrical understanding of the Euler equations for an incompressible fluid was given by Arnold in [1]. The basis of this modern perspective lies within the infinite dimensional group $SDiff(M)$ of volume preserving diffeomorphisms of the manifold M on which the fluid evolves. A fixed metric on M induces a natural metric on this infinite dimensional Lie group, and it was shown that the geodesic equation of the induced Riemannian connection on $SDiff(M)$ is equivalent to the Euler equations. A rigorous foundation for these ideas was later provided by Ebin and Marsden [2].

Another consequence of this geometrical construction, is the equivalence of a perfect fluid with the mechanics of a free point particle, where the evolution of the later is on the infinite dimensional space $SDiff(M)$. While one is accustomed in such a situation to linear equations of motion, the nonlinear nature of the Euler equation is nevertheless encoded in a subtle way.

In this paper we develop the variational approach to this picture and then extend it to the case of string motion on $SDiff(M)$. Once the correct variational techniques have been clearly identified, the equivalence of the Euler equation with geodesic motion on $SDiff(M)$ is easy to establish in a self-contained and economical fashion. This extends naturally to include a coupling of the fluid to an externally prescribed electromagnetic field. Such an understanding is required if one undertakes a path integral quantization of this theory where one has the intriguing possibility of relating single particle quantum mechanics on $SDiff(M)$ with a quantum field theory of fluids. It is also natural to consider the motion of strings on this space, and using the variational approach, we establish a new set of coupled nonlinear equations which may be interpreted as those of a fluid with specific internal degrees of freedom.

We begin in the next section with a review of the geometrical concepts required, and also include a self-contained derivation of the geodesic equation on $SDiff(M)$ similar to that in [3, 4]. The variational approach is then developed and this provides a simple alternative derivation of the equivalence between geodesic motion on $SDiff(M)$ and the Euler equations. It also has the advantage that the coupling of a fluid to an electromagnetic field also acquires a similar geometrical picture. These techniques naturally extend to string motion on $SDiff(M)$ and we treat that in detail. Finally, we consider some issues that arise in the quantization of these classical theories.

2 Point Particles on SDiff(M)

The geometrical understanding of the Euler equations for incompressible fluid flow on a manifold M is due to Arnold [1, 3], and was later developed by Ebin and Marsden [2]. In this approach, one can relate the Euler equations to geodesic motion on the infinite dimensional Lie group of volume preserving diffeomorphisms of M. We begin by reviewing some relevant mathematical concepts which will also serve to fix our notation.

Let M be a finite dimensional real manifold endowed with a fixed Riemannian metric which we denote either by $g(\cdot,\cdot)$ or by $\langle\cdot,\cdot\rangle$, and let ∇ denote the associated torsion free Riemannian connection.

Consider the group of smooth diffeomorphisms of M, denoted $Diff(M)$, where the group operation is simply composition of morphisms. Its structure as a (infinite dimensional) Riemannian manifold is central to our formulation, so let us begin by discussing its tangent space. Given any element $\eta \in Diff(M)$, we can formulate its action in the neighborhood U of some point p and its image point $\eta(p)$ using a single coordinate neighborhood $\varphi : U \rightarrow R^m$ since $\varphi \circ \eta^{-1}$ provides a coordinate system in a neighborhood of $\eta(p)$. Within R^m, if x^i denotes the coordinates of p and y^i the coordinates of $\eta(p)$, then an infinitesimal deformation of the identity transformation takes the form,

$$y^i(x) = x^i + \epsilon^i(x) \quad . \tag{1}$$

The infinitesimal deformation is encoded in a vector field ϵ on M, so we take the Lie algebra of this group, which is defined to be the tangent space at the identity diffeomorphism, to be the vector space of all vector fields on M. The Lie algebra product is the commutator of two vector fields. More generally, we can consider the tangent space to $Diff(M)$ at an arbitrary point η, and a deformation η' (in the same local coordinates) of η takes the form,

$$y'^i(x) = y^i(x) + \epsilon^i(y(x)) \quad . \tag{2}$$

The deformation $V(x) = \epsilon(y(x))$ is not a tangent vector to M at x, but is instead a vector at the image point of the diffeomorphism $y(x)$. The tangent space of $Diff(M)$ at a point η is defined to be the space of maps of M into its tangent bundle TM which are twisted with η; more precisely,

$$T_\eta Diff(M) = \{\ V : M \to TM | \pi \circ V(p) = \eta(p)\ \} \ , \tag{3}$$

where π is the projection map onto the base space of the tangent bundle.

We will be interested in a subgroup of $Diff(M)$ which preserves the Riemannian volume element

$$\varphi^{-1*}(\Omega_p) = d\mu_x = \sqrt{det[g_{ij}(x)]}\ dx^1 \wedge \cdots \wedge dx^m \tag{4}$$

at each point p,

$$\eta^*(\Omega_{\eta(p)}) = \Omega_p \ , \tag{5}$$

and we denote this subgroup by $SDiff(M)$. To obtain a local form of this constraint, just apply φ^{-1*} to this, and we find that (4) is equal to another local expression,

$$(\eta \circ \varphi^{-1})^*(\Omega_{\eta(p)}) = \sqrt{det[g_{ij}(y)]}\ dy^1 \wedge \cdots \wedge dy^m \ . \tag{6}$$

Now, y is a function of x via the diffeomorphism, so the final local form of the constraint is simply,

$$\sqrt{det[g_{ij}(x)]} = \sqrt{det[g_{ij}(y(x))]}\ det[\frac{\partial y}{\partial x}] \ . \tag{7}$$

By making a Taylor expansion of these quantities to lowest order in the ϵ which appears in (2), one quickly deduces the constraint on the deformation to be,

$$0 = \nabla_i^y\ \epsilon^i(y) \ . \tag{8}$$

The y superscript on the covariant derivative is just a reminder that we are differentiating with respect to that variable and not x.

The Lie algebra of the group of volume preserving diffeomorphisms is correspondingly the set of all divergence-free vector fields on M, and we have more generally,

$$T_\eta SDiff(M) = \{\ V : M \to TM | \pi \circ V(p) = \eta(p) \ , \quad \nabla_i (V \circ \eta^{-1})^i = 0 \ \} \ . \tag{9}$$

Sometimes it is convenient, as in the finite dimensional case, to denote a vector V at η by V_η.

In Appendix A we review the important aspects of finite dimensional Lie groups which we will here extend, in a formal way, to the infinite dimensional setting of $Diff(M)$ and its subgroup $SDiff(M)$; the reader may wish to consult that appendix at this point. We will need foremost a right invariant metric on the diffeomorphism group $SDiff(M)$. By a right action of $Diff(M)$ on itself, we mean the map $R_{\eta'} : Diff(M) \to Diff(M)$ defined by

$$R_{\eta'}(\eta) = \eta \circ \eta' \ , \tag{10}$$

where η' and η are elements of $Diff(M)$. One could also define a left action, but this will play no role in our presentation. The right action induces a map on the tangent space of $Diff(M)$, and this is given by:

$$R_{\eta' *} V = V \circ \eta' \ . \tag{11}$$

The meaning of the right hand side of this equation is as follows. If $V \in T_\eta Diff(M)$, then $R_{\eta' *} V \in T_{\eta \circ \eta'} Diff(M)$, and the quantity $\tilde{V} = V \circ \eta^{-1}$ is a vector field on M. By definition, $V \circ \eta'$ applied to a real valued function f on M, is the quantity $\tilde{V}(f)|_{\eta(\eta'(p))}$.

A natural metric on $Diff(M)$ is given by combining the metric on M, with an integration over M,

$$(V, W)_\eta = \int_M \langle V, W \rangle_{\eta(p)} \, \Omega_p \ , \tag{12}$$

where V, $W \in T_\eta Diff(M)$. The meaning of the integrand in this expression is of course,

$$\langle V, W \rangle_{\eta(p)} = g_{ij}(\eta(p)) \, V^i_{\eta(p)} \, W^j_{\eta(p)} \ . \tag{13}$$

By restricting this metric to tangent vectors to $SDiff(M)$, which as we saw above were just right translations of divergence-free vector fields on M, one has also a metric on this submanifold. Moreover, this metric on $SDiff(M)$ has the additional property of being right invariant:

$$(V, W)_\eta = (R_{\eta' *} V, R_{\eta' *} W)_{\eta \circ \eta'} \ , \tag{14}$$

because of (5) and the usual formula for a change of variables, $\int \eta^* \omega = \int \omega$.

Following the treatment of the finite dimensional case in Appendix A, we will need the bilinear function B defined implicitly through the relation,

$$([X,Y],Z)_e = (B(Z,X),Y)_e \quad . \tag{15}$$

We will suppress the symbol denoting the evaluation at the identity e in the following equations. The left hand side of this equation is given by,

$$([X,Y],Z) = \int_M \langle [X,Y],Z\rangle_p \, \Omega_p \quad , \tag{16}$$

and in a coordinate basis where $X = X^i \partial_i$, $Y = Y^i \partial_i$, and $Z = Z^i \partial_i$, we have

$$([X,Y],Z) = \int_M g_{ij} \, Z^j \, [X^k \partial_k Y^i - Y^k \partial_k X^i] \, d\mu_x \quad . \tag{17}$$

Using the fact that the vectors here are divergence free, (17) becomes,

$$\int_M g_{ij} \, Z^j \, \nabla_k (X^k Y^i - X^i Y^k) \, d\mu_x \quad . \tag{18}$$

Integrating by parts, we find that,

$$([X,Y],Z) = \int_M g_{kl} \, [g^{lm} X^n \nabla_{[m} Z_{n]}] \, Y^k \, d\mu_x \quad . \tag{19}$$

The bilinear function B can then be read off from this equation; it is the projection onto the divergence free part of the expression in square brackets:

$$B(Z,X) = P \circ [g^{kl} X^m \nabla_{[l} Z_{m]} \, \partial_k] \quad . \tag{20}$$

It is crucial to bear in mind this projection. In this analysis we have been using parallel transport in the larger ambient space $Diff(M)$; the connection on the submanifold $SDiff(M)$ is obtained by projecting the connection on $Diff(M)$ onto the tangent space of the submanifold. The geodesic equation $\dot{u} = B(u,u)$, derived in Appendix A for a finite dimensional Lie group, takes the form:

$$\partial_t \, u = P \circ \{ [\frac{1}{2} \nabla^k \, u^2 - u^j \, \nabla_j \, u^k \,] \, \partial_k \} \quad . \tag{21}$$

It is important to emphasize in the current situation precisely how u is related to a vector tangent to the geodesic. Analogous to eqn. (55) in Appendix A, u is a right translation of the velocity vector $\dot{\gamma}_t$ which here takes the form,

$$u = \dot{\gamma}_t \circ \gamma_t^{-1} \quad . \tag{22}$$

As a geodesic on $SDiff(M)$, γ_t is a volume preserving diffeomorphism for each parameter value t, and u is an element of the tangent space at the identity; i.e. a vector field on M.

It is worth elaborating a bit on the precise meaning of (22) as it will play a more subtle role in the next section. Explicitly, since u is a vector field on M, we can apply it to any function $f : M \to R$ and evaluate it at a point p and we have,

$$\begin{aligned} u(f)(p) &= \gamma_{t*}(\frac{d}{dt})(f) \circ \gamma_t^{-1}(p) \qquad (23) \\ &= \Big[\frac{d}{dt} f \circ \gamma_t(q) \Big] \Big|_{q=\gamma_t^{-1}(p)} \quad . \end{aligned}$$

When $f = x^i$, a coordinate function near p, this then gives a local formula for the classical object $u^i(t,x)$. Taking $p \to \gamma_t(p)$ gives an equivalent, and perhaps simpler looking, expression,

$$u^i(t, \gamma_t(x)) = \frac{d}{dt} \gamma_t^i(x) \quad . \tag{24}$$

Returning now to eqn. (21), the projection P can be achieved by putting in a term to soak-up the divergence part of the right hand side of (21),

$$\frac{1}{2} \nabla^k u^2 - u^j \nabla_j u^k = -\nabla^k p' \quad . \tag{25}$$

Redefining the variable $p = p' + \frac{1}{2}u^2$, we have the Euler equation,

$$\partial_t u^k + u^j \nabla_j u^k = -\nabla^k p \quad , \tag{26}$$

and the incompressibility condition $\nabla_k u^k = 0$ is naturally encoded in the constraint that we are working on $SDiff(M)$. The Euler equations therefore correspond to the geodesic equation on the group of volume preserving diffeomorphisms.

3 Variational Approach

In this section we will rederive the Euler equations from a variational principle. Our derivation is self-contained, and does not rely on the main results of the previous section or Appendix A. This approach also enables us to couple the fluid to an external electromagnetic potential in a natural way.

An action principle for the hydrodynamic Euler equations is easily given in terms of an "energy" functional [5],

$$S_{point} = \frac{1}{2} \int_L (\dot{\gamma}_t, \dot{\gamma}_t)\, dt \ , \tag{27}$$

where S_{point} is to be interpreted as a functional of the map $\gamma_t : L \rightarrow SDiff(M)$. In this context one would normally take L to be the real line, but the circle will also be interesting when one undertakes a quantization of this theory. As $\dot{\gamma}_t$ is a tangent vector on $SDiff(M)$ at the point γ_t, we have explicitly,

$$(\dot{\gamma}_t, \dot{\gamma}_t)_{\gamma_t} = \int_M d\mu_{\gamma_t(x)}\, g_{ij}(\gamma_t(x))\, \dot{\gamma}_t^i(x)\, \dot{\gamma}_t^j(x) \ . \tag{28}$$

Note that the volume preserving nature of γ_t (i.e. $d\mu_{\gamma_t(x)} = d\mu_x$) allows one to write the point particle action S_{point} equivalently as $\frac{1}{2} \int_L (u, u)\, dt$.

On formal grounds, we expect that the variation of the above action will give,

$$\delta\, S_{point} = \int_L \left(\delta\gamma_t, -\frac{D\dot{\gamma}_t}{Dt}\right) dt \ , \tag{29}$$

where we are making an arbitrary variation of γ_t subject to the constraint that we always remain on the submanifold of volume preserving diffeomorphisms, and that variations vanish on the boundary if present. We see then that the object $\delta\gamma_t$ is an element of $T_{\gamma_t} SDiff(M)$, defined in (9). This is another way of appreciating the need for the projection P we encountered in the previous section, as the covariant derivative D we computed is in the ambient space $Diff(M)$. Let us now show that eqn. (29) actually follows from first principles.

We have already remarked that the measure $d\mu_x = d\mu_{\gamma_t(x)}$ is independent of γ_t, so we need not vary that factor. As in the previous section, we let

x denote coordinates at $p \in M$ while $y(x)$ denotes the coordinates of $\gamma_t(p)$ within the same coordinate system. A variation of the action now gives,

$$
\begin{aligned}
\delta S_{point} &= \int_L dt \int_M d\mu_x \{\frac{1}{2}\delta g_{ij}(y)\,\dot{y}^i\,\dot{y}^j + g_{ij}(y)\,\dot{y}^i\,\delta\,\dot{y}^j\} \\
&= \int_L dt \int_M d\mu_x \{\frac{1}{2}\delta y^k\,\partial^y_k g_{ij}(y)\,\dot{y}^i\,\dot{y}^j - \delta y^i\,\dot{y}^j\,\dot{y}^k\,\partial^y_k g_{ij}(y) \\
&\qquad -\delta y^i\,g_{ij}(y)\,\frac{d\dot{y}^j}{dt}\}\ ,
\end{aligned} \tag{30}
$$

where we have integrated by parts in going to the second line of this expression. Now take note of the definition (24) which relates u to $\dot{y}$; this implies

$$
\frac{d\dot{y}^i}{dt} = \partial_t u^i(t,y) + u^j(t,y)\,\partial^y_j\,u^i(t,y)\ . \tag{31}
$$

Combining terms to form the covariant derivative ∇_j on M, we have simply that,

$$
\begin{aligned}
\delta S_{point} &= -\int_L dt \int_M d\mu_x\,g_{il}(y)\,\delta y^l\ \cdot \\
&\qquad \{\partial_t u^i(t,y) + u^j(t,y)\,\nabla^y_j\,u^i(t,y)\}\ .
\end{aligned} \tag{32}
$$

If we shift back to x, remembering to insert the projection operator P, eqn. (32) then gives the usual Euler equation as the condition for an extremum of the functional S_{point}.

In order to couple a point particle to an electromagnetic field we can add to S_{point} the interaction,

$$
S_{em} = \int_L dt \int_M d\mu_x\,\dot{\gamma}^i_t\,A_i(t,\gamma_t(x))\ , \tag{33}
$$

where A is an externally prescribed potential field. This is of course in complete analogy to what one considers for a point particle on a finite dimensional manifold. We can easily vary this quantity as we did for the free particle, and one finds,

$$
\begin{aligned}
\delta(S_{point} + S_{em}) &= -\int_L dt \int_M d\mu_x\,g_{il}(y)\,\delta y^l\ \cdot \\
&\qquad \{\partial_t u^i(t,y) + u^j(t,y)\,\nabla^y_j\,u^i(t,y) + \\
&\qquad g^{ik}(y)\,[\partial_t A_k(t,y) + u^j(t,y)\,F_{jk}(t,y)]\}\ .
\end{aligned} \tag{34}
$$

Here, $F_{ij}(t,y) = \partial_i^y A_j(t,y) - \partial_j^y A_i(t,y)$ is the field strength of the potential. As in the free particle case, we can again shift back to x by a change of variables, and one finds the classical equation,

$$\partial_t u^i + u^j \nabla_j u^i + g^{ik} (\partial_t A_k + u^j F_{jk}) = -\nabla^i p \ , \tag{35}$$

for the coupled system; here all variables are at the point (t,x).

4 Strings on SDiff(M)

Given a simple point particle action which encodes the usual Euler equations for an incompressible fluid, it is geometrically natural to consider a similar action for nonrelativistic strings. We should emphasize from the outset that the string picture is in the infinite dimensional context of strings moving on $SDiff(M)$. When we look at the extremal surface conditions - the two dimensional analog of the geodesic equation - we will have a point particle fluid coupled to some internal degrees of freedom.

Let us begin the discussion from the action,

$$S_{string} = \int_\Sigma d\tau\, d\sigma\, \sqrt{h}\, h^{\alpha\beta}\, (\partial_\alpha \gamma, \partial_\beta \gamma) \ , \tag{36}$$

where $h_{\alpha\beta}$ is a metric on some Riemann surface Σ with coordinates τ and σ. As usual, h denotes $|det[h_{\alpha\beta}]|$, and γ is the embedding of the surface in $SDiff(M)$. To be perfectly uniform with the preceding section (eg. eqn. (22)), we should write $\gamma_{\tau,\sigma}$ here instead of γ, but the subscripts become too cumbersome and so we will usually leave the parameter dependence of γ understood. The signature of h need not be Riemannian; in the semi-Riemannian (Minkowski) case we will see that the conserved energy is manifestly positive definite. For the remainder of this section, let us take $\Sigma = R \times S^1$ to be a cylinder where the "time" coordinate τ ranges over the real line and σ is the coordinate for S^1. Furthermore, h is taken to be diagonal with $h_{\tau\tau} = -1$ and $h_{\sigma\sigma} = +1$ (Minkowski signature).

As $\partial_\tau \gamma$ and $\partial_\sigma \gamma$ are tangent vectors to $SDiff(M)$, their right translations to the identity, $u_\tau = \partial_\tau\gamma \circ \gamma^{-1}$ and $u_\sigma = \partial_\sigma\gamma \circ \gamma^{-1}$, are vector fields on M and are divergence-free,

$$0 = \nabla_k u_\tau^k = \nabla_k u_\sigma^k \ . \tag{37}$$

The extremal conditions are obtained by varying γ in the usual way and integrating by parts to obtain,

$$P \circ \left[\frac{D\,\partial_\tau \gamma}{D\tau} - \frac{D\,\partial_\sigma \gamma}{D\sigma} \right] = 0 \ . \tag{38}$$

The covariant derivatives here are in the ambient space $Diff(M)$, so the projection P is needed to remove the divergence part. Let us simply note that the relative minus sign on the left hand side is just a consequence of the Minkowski signature metric h. The structure of each of those terms above is analogous to the point particle case, and we have,

$$\dot{u}^i_\tau + u^j_\tau \, \nabla_j \, u^i_\tau - u'^i_\sigma - u^j_\sigma \, \nabla_j \, u^i_\sigma = -\nabla^i \, p \ , \tag{39}$$

where the notation $\dot{u} = \partial_\tau \, u$ and $u' = \partial_\sigma \, u$ has been used.

At first sight, the minimal surface equation appears to be just a sum of two copies of the geodesic equation we encountered in the previous section. However, one has a crucial constraint which is a consequence of the equality of mixed partial derivatives. Let us explicitly write out the analog of (24),

$$\begin{aligned} u^i_\tau(\tau, \sigma, \gamma_{\tau,\sigma}(x)) &= \partial_\tau \, \gamma^i_{\tau,\sigma}(x) \\ u^i_\sigma(\tau, \sigma, \gamma_{\tau,\sigma}(x)) &= \partial_\sigma \, \gamma^i_{\tau,\sigma}(x) \ , \end{aligned} \tag{40}$$

where for once we have deliberately emphasized the parameter dependence implicit in γ. If we differentiate the first of these with respect to σ and the second with respect to τ, then those expressions are equal, and one quickly shows that,

$$\dot{u}^i_\sigma - u'^i_\tau + u^j_\tau \, \nabla_j \, u^i_\sigma - u^j_\sigma \, \nabla_j \, u^i_\tau = 0 \ . \tag{41}$$

Note that the Christoffel connection term in ∇_j actually drops out of the last two terms in (41), and it could equivalently be written with ∂_j. All functions in (37,39,41) have dependence (τ, σ, x); this set of equations embodies the "stringy" generalization of the hydrodynamic equations of Euler.

The conserved quantity of perhaps greatest importance is the energy which is here given by,

$$E = \int_{S^1} (u_\tau, u_\tau) + (u_\sigma, u_\sigma) \, d\sigma \ . \tag{42}$$

As always $(\cdot, \cdot)$ denotes the right invariant metric which includes an integral over M. The conservation of E is easily established by computing $\partial_\tau E$ and using eqns. (37,39,41). We see here the significance of choosing the Minkowski metric on Σ; the action is then of the form $T - V$ (kinetic minus potential energies) and the total energy E is $T + V$, which in this model is manifestly non-negative. Similarly, one easily establishes that the quantity,

$$\int_{S^1} (u_\tau, u_\sigma) \, d\sigma \quad , \tag{43}$$

is a constant of motion. Taken together with the energy, this suggests that the variables $A^i = u^i_\tau + u^i_\sigma$ and $B^i = u^i_\tau - u^i_\sigma$ might be more appropriate to the analysis of this set of equations, since the above integrals are equivalent to the conservation of $\int A^2$ and $\int B^2$ separately. In terms of these variables, the complete set of classical equations takes the form,

$$\begin{aligned} &\dot{A}^i - A'^i + B^j \nabla_j A^i = -\nabla^i p \\ &\dot{B}^i + B'^i + A^j \nabla_j B^i = -\nabla^i p \\ &\nabla_i A^i = \nabla_i B^i = 0 \quad . \end{aligned} \tag{44}$$

One possible interpretation of the "stringy fluid" equations is that of a fluid of point particles with internal structure. The situation where $A' = B' = 0$ - essentially rigid internal structure - might be the simplest to first investigate. In this case, the equations can be interpreted as describing a point particle fluid where each particle has a vector degree of freedom; the usual Euler equation describes scalar constituent entities. We wish to emphasize that this is not just any arbitrary set of equations, but a system with as much geometrical significance as the Euler equation itself. We conjecture that it will be possible to establish existence and uniqueness of classical solutions which are equally strong [2].

5 Aspects of Quantization

Given a Lagrangian formulation for the classical theories that we have been discussing, it is natural to consider their path integral quantization. In the point particle fluid, the Lagrangian is that of a free particle moving in the space $SDiff(M)$. Simple quantum mechanics of this particle is therefore equivalent to a quantum field theory of the Euler equations for incompressible fluid flow. In the stringy generalization that we have formulated, one considers a free sigma model on the same space.

In a path integral quantization for a particle fluid, one integrates over the space of paths in $SDiff(M)$; if the paths have fixed boundaries one is just computing the propagator of a point particle. Without this Lagrangian perspective, the natural approach for quantizing the fluid equations would be to work in terms of the vector field u (see for example [6]). Interestingly, these approaches are not generally equivalent for topological reasons. Consider the case where $M = S^2$, the 2-sphere. There, a direct approach would be to work in terms of the function space of the globally defined stream function ψ (related to the velocity through $u^i = \epsilon^{ij}\,\partial_j\,\psi$); this space is homotopically trivial. However, it is known that the homotopy type of $SDiff(S^2)$ is the same as that of $SO(3)$ [7, 8], and the later has $\pi_1(SO(3)) = Z_2$. The topology of the configuration spaces in these approaches is therefore different. To construct the quantum field theory from the Lagrangian formulation, one might begin be reducing the path space of $SDiff(M)$ to more conventional function spaces, taking into account the topological features, by using coordinates on $SDiff(M)$. Similar remarks apply to the stringy generalization we considered in this paper, but we will not carry out this program here.

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Appendix A: Geodesics on Lie Groups

Let G be a finite dimensional Lie group [9, 10]. For each $g \in G$, we can define both a left action $L_g : G \to G$ and a right action $R_g : G \to G$ in the natural way,

$$\begin{aligned} L_g(h) &= g\,h \\ R_g(h) &= h\,g\ , \end{aligned} \tag{45}$$

where the group product is understood on the right hand side of these equations. Note that $L_g \circ L_{g'} = L_{g\,g'}$, whereas $R_g \circ R_{g'} = R_{g'\,g}$. The maps L_g and

R_g induce linear maps of the tangent space of G at each point $h \in G$,

$$\begin{aligned} L_{g*} : T_h G &\rightarrow T_{g\,h} G \\ R_{g*} : T_h G &\rightarrow T_{h\,g} G \ . \end{aligned} \tag{46}$$

Given any metric $(\cdot, \cdot)_e$ on the tangent space at the identity e of G, one can use these group actions to define a Riemannian metric on G. Let us focus on the right action with a metric defined by

$$(X, Y)_g = (R_{g^{-1}*} X, R_{g^{-1}*} Y)_e \ , \tag{47}$$

where X and Y are tangent vectors at the point g. The group actions are simply used to transport vectors located at some point g back to the identity e where a given metric has already been specified. Moreover, this metric is also right invariant in the sense that

$$(R_{h*} X, R_{h*} Y)_{g\,h} = (X, Y)_g \ , \tag{48}$$

since $R_{(g\,h)^{-1}*} \cdot R_{h*} = R_{g^{-1}*}$.

Given a Riemannian metric, one naturally has the associated torsion free Riemannian connection D defined implicitly through the two conditions,

$$\begin{aligned} [X, Y] &= D_X Y - D_Y X \\ X(Y, Z) &= (D_X Y, Z) + (Y, D_X Z) \ , \end{aligned} \tag{49}$$

where X, Y, and Z are any vector fields on G. We will be interested in a formula for the special case where both the metric $(\cdot, \cdot)$ and the vector fields are right invariant (a vector field X is right invariant if $R_{g'*}(X_g) = X_{g\,g'}$ for all $g' \in G$). Taking a sum of three combinations of (49) with permuted arguments, and using the fact that $X(Y, Z) = 0$ for right invariant quantities, one easily establishes,

$$(D_X Y, Z) = \frac{1}{2} \left\{ ([X, Y], Z) - ([Y, Z], X) + ([Z, X], Y) \right\} \ . \tag{50}$$

For a Lie group, since the right action R_g is a diffeomorphism, one has a global frame of smooth vector fields on G; let such a basis of vector fields be given by $\{e_i\}$,

$$e_i|_g = R_{g*}(e_i|_e) \ . \tag{51}$$

We identify the Lie algebra T_eG of G with the right invariant vector fields on G. Given any curve γ_t on G, we can then decompose its tangent vector $\dot{\gamma}_t$ into components with respect to this basis. Let us just remind the reader that as a vector, $\dot{\gamma}_t$ has the action,

$$\dot{\gamma}_t(f)\Big|_{\gamma_t} = \frac{d}{dt} f(\gamma_t) = \frac{df}{dt} \quad , \tag{52}$$

for any smooth real valued function $f : G \to R$. Note that $\dot{\gamma}_t$ is a vector at the point γ_t in this equation.

We will need a formula for the covariant derivative of a vector field $\bar{A}$ along the curve γ_t, and we will proceed to first use the basis e_k. Recalling the defining properties of a connection [9], we have

$$\begin{aligned} \left(\frac{D\bar{A}}{Dt}, e_k\right) &= (D_{\dot{\gamma}_t^i e_i}(\bar{A}^j e_j), e_k) \\ &= \frac{d\bar{A}_k}{dt} + \dot{\gamma}_t^i \, \bar{A}^j \, (D_{e_i} e_j, e_k) \;, \end{aligned} \tag{53}$$

where the index k has been lowered through the use of the metric, $\bar{A}_k = \bar{A}^l(e_l, e_k)$. We should again emphasize that the the component functions $\bar{A}^i$ and $\dot{\gamma}_t^i$ are evaluated at the point γ_t in this expression. Using (50), the right hand side of the above equation becomes,

$$\frac{d\bar{A}_k}{dt} + \dot{\gamma}_t^i \, \bar{A}^j \, \frac{1}{2} \{ ([e_i, e_j], e_k) - ([e_j, e_k], e_i) + ([e_k, e_i], e_j) \} \; . \tag{54}$$

It will be useful to rewrite this in an invariant way without reference to any particular basis. Pushing $\bar{A}$ and $\dot{\gamma}_t$ back to the identity e, let us define the time dependent vectors

$$\begin{aligned} A|_e &= R_{\gamma_t^{-1}*}(\bar{A}|_{\gamma_t}) \;, \\ u|_e &= R_{\gamma_t^{-1}*}(\dot{\gamma}_t|_{\gamma_t}) \;. \end{aligned} \tag{55}$$

The time dependence here is clear in that we are pushing a vector $\bar{A}|_{\gamma_t}$ back to the identity e of G at different times. Since

$$R_{\gamma_t^{-1}*}(\bar{A}|_{\gamma_t}) = \bar{A}^k(\gamma_t) \, R_{\gamma_t^{-1}*}(e_k|_{\gamma_t}) = \bar{A}^k(\gamma_t) \, e_k|_e \;, \tag{56}$$

this means that $A^k(e) = \bar{A}^k(\gamma_t)$, where $A = A^k(e)\, e_k|_e$, and similarly we have $u^k(e) = \dot{\gamma}_t^k(\gamma_t)$. Let us further define a bilinear function $B : T_eG \times T_eG \to T_eG$ by,

$$([X,Y], Z) = (B(Z,X), Y) \ . \tag{57}$$

Taking all these relations together, we have,

$$R_{\gamma_t^{-1}*}\left(\frac{D\bar{A}}{Dt}\right) = \frac{dA}{dt} + \frac{1}{2}\left\{ [u,A] - B(u,A) - B(A,u) \right\} \ . \tag{58}$$

If γ_t is a geodesic, then $D_{\dot{\gamma}_t}\dot{\gamma}_t = 0$, and this relation reduces to,

$$\frac{du}{dt} - B(u,u) = 0 \ . \tag{59}$$

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