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Dimensional Reduction and the Non-triviality of $\lambda\phi^4$ in Four Dimensions at High Temperature.

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Abstract:
$\lambda\phi^4$ theory in four dimensions is shown to have a non-trivial fixed point at finite temperature, the relevant anomalous dimensions at the second order phase transition being the three dimensional ones. We emphasize the importance of having renormalization schemes and a renormalization group equation that can explicitly take into account the fact that the degrees of freedom of a theory may be qualitatively different at different scales. By applying such considerations to finite temperature $\lambda\phi^4$ where the low temperature degrees of freedom are effectively four dimensional and the high temperature ones three dimensional we are able to follow perturbatively the theory from zero to infinite temperature.

[1]Present Address

Finite temperature field theory is a subject of great popular interest (see [1] for a recent review) with many areas of application, primarily in condensed matter physics where it is implemented in a non-relativistic, low temperature setting. Nevertheless there are important physical systems where the thermal nature must be taken into account in a fully relativistic fashion, for example the early universe, and the quark-gluon plasma to name but two. It was in fact the discovery of Kirzhnits and Linde [2] that one should expect a symmetry restoring phase transition in four dimensional gauge theories and the subsequent more quantitative analysis of this discovery by Dolan and Jackiw [3] and Weinberg [4] that led to much interest in relativistic field theory at finite temperature.

Central to many investigations have been two main themes: i) that if one treats the thermality by using the imaginary time formalism then at "high temperature" one expects to see dimensional reduction; and ii) that near the phase transition one expects perturbation theory to break down due to the effects of infrared divergences [3, 4].The former notion seems to be used quite indiscriminantly and has been criticized, with some justification, by Landsman [5]. The latter has remained a sticking point though some large N work has ameliorated the problem. In this letter we wish to address both these matters in the simplified context of $\lambda\phi^4$ at finite temperature. The techniques implemented here have been used by us in the context of finite size effects in critical phenomena [6] (see also [7]) though the way temperature enters in the two cases is different.

Before embarking on a more detailed discussion of the problem we announce briefly our results. Firstly, using the renormalization group associated with a set of temperature dependent renormalizations we develop a formalism which describes the theory qualitatively and quantitatively as we continuously change from a high temperature regime $T \gg m_\beta$ to a low temperature regime $T \ll m_\beta$, m_β being the temperature dependent mass of the ϕ particles, not the zero temperature mass. Canonically what we will mean by a "high temperature" regime is that $T \gg$ any other mass scale in the problem. It is important to remember that temperature is a dimensionful quantity, therefore the terms high or low temperature have no intrinsic meaning except relative to some other mass scale. A renormalization group which as $\frac{T}{m_\beta} \to \infty$ describes three dimensional $\lambda\phi^4$ and as $\frac{T}{m_\beta} \to 0$ describes four dimensional $\lambda\phi^4$ is discovered by a careful examination of the four point vertex. From the latter we obtain a running coupling constant which has a fixed point characteristic of $\lambda\phi^4$ in three dimensions for the finite temperature theory and one characteristic of four dimensions for the zero temperature theory, thereby showing that $\lambda\phi^4$ theory in four dimensions at finite temperature is non-trivial.[2] Throughout we will work in the imaginary time formalism the real time argument being given at a later stage.

Consider the bare thermal Euclidean action given by

$$S_E = \int_0^\beta d\tau \int d^3x \left(\frac{1}{2}(\partial\phi_B)^2 + \frac{1}{2}m_B^2\phi_B^2 + \frac{\lambda_B}{4!}\phi_B^4 \right) \tag{1}$$

First we describe some of the pitfalls of treating this theory in the "standard" manner, with a calculation of the temperature dependent mass defined as $m_\beta^2 = \frac{\delta^2 V_\beta}{\delta\phi^2}\mid_{\phi=0}$ where V_β

[2]Note we do not necessarily imply the existence of a non-trivial continuum limit, though see later comments concerning the bare theory.

is the finite temperature effective potential derived using (1). To $O(\lambda)$ (anything without the subscript B will be taken to refer to a renormalized quantity) one finds

$$m_\beta^2 = m^2 + \frac{\lambda T}{2} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\mathbf{k}^2 + \frac{4\pi^2 n^2}{\beta^2} + m^2} + \delta m^2 \tag{2}$$

where δm^2 is an appropriate counterterm. The first question is: what is an appropriate counterterm? Standard lore follows the train of thought that δm^2 is there to remove the ultraviolet divergence from the loop integration. This ultraviolet divergence is temperature independent as it is associated with the short distance behaviour of the theory and therefore the counterterm may be chosen to be temperature independent. Put more succinctly, the counterterms one would use to renormalize the theory at zero temperature are sufficient to renormalize the theory at finite temperature. Let us illustrate that these are inadequate even when $m^2 > 0$, m^2 being the zero temperature mass parameter. If we choose minimal subtraction then

$$m_\beta^2 = m^2 + \frac{\bar{\lambda}}{32\pi^2} m^2 \ln\frac{m^2}{\kappa^2} + \frac{\bar{\lambda}}{4\pi^2} mT \sum_{n=1}^{\infty} \frac{1}{n} K_1\left(\frac{nm}{T}\right) \tag{3}$$

where κ is the renormalization scale and a dimensionless coupling constant $\bar{\lambda} = \lambda\kappa^\varepsilon$ has been introduced. For finite temperature theory in four dimensions $\bar{\lambda} = \lambda$. Note that the n in (3) refers to the winding number on the S^1 whereas n in (2) refers to the eigenvalue of the Laplacian on S^1. If one examines the low temperature limit of (3) the last term $\to 0$ and one is left with the standard four dimensional zero temperature result. In the high temperature limit $T \gg m$

$$m_\beta^2 \to m^2 \left(1 + \frac{\bar{\lambda}}{32\pi^2} \ln\frac{m^2}{\kappa^2} + \frac{\bar{\lambda} T^2}{24 m^2}\right) \tag{4}$$

Note that as $T \to \infty$, $m_\beta^2 \to \infty$. One may accept this at face value but one would also wish to investigate the validity of this result. If one considers the two loop contribution to m_β^2 the leading term is $O(\frac{\lambda^2 T^3}{m})$. Basically the theory is plagued by new divergences in the limit $\frac{T}{m} \to \infty$. Although these are not ultraviolet divergences they are unphysical and must be removed as in the case of the latter. In the analysis of Dolan and Jackiw [3] the summing of what they refer to as daisy diagrams converted m in the integral in (3) to m_β thereby ameliorating the infrared divergences. However, one must also investigate the vertex function, this they did not consider, but as we will see it is in fact vital to include these contributions.

Let us examine the latter question in more detail. One may assume a priori that λ is small enough so that one may investigate a large regime where perturbation theory has some validity. Eventually however, perturbation theory will inevitably break down, in particular as one approaches the physically interesting critical point. If one calculates $\Gamma^{(4)}$ at the symmetric point $\mathbf{p}_i \cdot \mathbf{p}_j = \frac{\kappa^2}{4}(4\delta_{ij} - 1)$ using minimal subtraction, one obtains

$$\Gamma^{(4)}(p_i, 0, \bar{\lambda}, T, \kappa)|_{SP} = \bar{\lambda}\kappa^\varepsilon \left(1 + \frac{3\bar{\lambda}\kappa^\varepsilon}{16\pi} F\left(\frac{T}{\kappa}\right)\right) \tag{5}$$

where $\bar{\lambda}$ is the minimally subtracted coupling constant and

$$F(\frac{T}{\kappa}) = T\kappa^{\varepsilon}\frac{3}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^{3-\varepsilon}k}{(2\pi)^{3-\varepsilon}}\frac{1}{(k^2+m^2+4\pi^2n^2T^2)((p-k)^2+m^2+4\pi^2n^2T^2)} \tag{6}$$

Then for $\frac{T}{\kappa} \gg 1$ one finds

$$\Gamma^{(4)}(p_i, 0, \bar{\lambda}, T, \kappa)|_{SP} \rightarrow \bar{\lambda}\kappa^{\varepsilon}\left(1+\frac{3\bar{\lambda}}{16\pi}\frac{T}{\kappa}\right) \tag{7}$$

which merely serves to emphasize the point that perturbation theory is breaking down in the "high" temperature regime.

It is enlightening to compare this situation with that encountered when considering bare perturbation theory at fixed cutoff in a zero temperature theory. Just as perturbation theory is untenable in terms of λ_B for scales $\ll \Lambda$ so perturbation theory in terms of $\lambda(T=0)$ is untenable for scales $m \sim T$. In the former case perturbation theory can be improved greatly by a reparametrization to the renormalized quantities m and λ. Correlation functions are then at least finite. In the latter, in the same fashion one should perform a renormalization of the parameters of the theory to remove any divergences which appear in some physical limit such as $\frac{T}{m_{\beta}} \rightarrow \infty$. To understand these remarks consider the relationship between renormalized and bare parameters which for large Λ is

$$\Gamma^{(2)} = m_B^2 + \frac{\bar{\lambda}_B}{32\pi^2}[\Lambda^2 - m_B^2\{ln(\frac{\Lambda^2}{m_B^2}) - \gamma\}] \qquad \Gamma^{(4)} = \bar{\lambda}_B - \frac{3\bar{\lambda}_B^2}{32\pi^2}\ln\frac{\Lambda^2}{m_B^2} \tag{8}$$

As $\frac{\Lambda}{m_B} \rightarrow \infty$ there is a breakdown in perturbation theory in terms of the bare coupling constant, similarly the theory is not massless when $m_B^2 = 0$. In the language of critical phenomena the mean field critical temperature has been shifted by the effects of fluctuations. To alleviate these problems one defines renormalized parameters by suitable subtractions of the above terms. For instance, if one uses minimal subtraction one finds

$$\Gamma^{(2)} = m^2 + \frac{\bar{\lambda}}{32\pi^2}m^2\ln\frac{m^2}{\kappa^2} \qquad \Gamma^{(4)} = \bar{\lambda} + \frac{3\bar{\lambda}^2}{32\pi^2}\ln\frac{m^2}{\kappa^2} \tag{9}$$

These functions are finite and manifestly independent of Λ.

Returning to the T dependent system one sees that even after the removal of ultraviolet divergences there are new divergences appearing in the limit $\frac{T}{m} \rightarrow \infty$ as shown in equations (4) and (5). These play a quite analogous role to the $\frac{\Lambda}{m_B} \rightarrow \infty$ divergences in (8). They indicate that parameters defined at $T = 0$ are inadequate for the regime $T > m$. The main problem appears to be one of overambition. The inadequacy of the bare theory stems from the desire to take into account, albeit perturbatively here, all the degrees of freedom of the field between scales Λ and m_B. For $\frac{\Lambda}{m_B}$ very large this is more than we can do. Of course, if $m_B \sim \Lambda$ then the number of degrees of freedom we are explicitly taking into account is quite small, i.e. the dressing of the vertex is small. When we renormalize we effectively "hide" a high proportion of these degrees of freedom in the renormalized parameters. Subsequently we explicitly take into account only those degrees of freedom between scales m and κ. As long as $\frac{m}{\kappa}$ is not too large we should be on safe ground. One of the great virtues of the renormalization group is that it yields a relation between the theory at two different renormalization scales, κ and $\rho\kappa$ say. Renormalization group invariance tells us that the physics should be independent of ρ, therefore, even if $\frac{m}{\kappa}$ is large so that we are trying to take into account a large number of degrees of freedom, we can always choose ρ so that $\frac{m}{\rho\kappa}$ is not large and therefore only a relatively small number of degrees of freedom need be explicitly calculated. The above problem is relatively straightforward, because the degrees of freedom at high energy are qualitatively the same as at low energy. The finite temperature case is more difficult because as we shall see the degrees of freedom at high energy can be qualitatively different to those at low energy — in the case at hand four and three dimensional respectively. By using zero temperature renormalization schemes we are hiding only four dimensional degrees of freedom in the renormalized parameters. The ill behaved nature of expressions such as (7) is telling us that this is inadequate. The question is, can we circumvent this difficulty?

Related to the above problem of infrared divergences is that of dimensional reduction. The dimensional reduction is based on an extension of the Applequist-Carrazone decoupling theorem [8]. In the imaginary time formulation, the free propagator is equivalent to that of an infinite sum of three dimensional propagators with masses $M^2 = m^2 + 4\pi^2 n^2 T^2$. In the large temperature limit one naively believes there should be a decoupling of all but the $n = 0$ mode, leaving one with an effective three dimensional theory whose parameters get renormalised by the decoupling process. There is usually a tacit assumption that decoupling at the one loop level implies decoupling to all orders. As pointed out by Landsman [5] due to the presence of temperature induced mass terms m_β becomes of order T at some order in the loop expansion, this presents some difficulty since in order to see decoupling at all orders we require $\frac{m_\beta}{T} \rightarrow 0$.

Let us try to examine the situation from a different perspective. Consider the evaluation of the β function of the theory. Once again using the folklore that T independent counterterms are sufficient to renormalize the theory to one loop one would find the standard result

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} \tag{10}$$

Now, suppose we believe that there will be a dimensional reduction when $\frac{m_\beta}{T} \rightarrow 0$. Then surely for the sake of consistency we would expect there to be a non-trivial three dimensional fixed point not just a Gaussian one as shown by (10). This is not just something we can blame on inadequate perturbation theory either — if one uses a T independent renormalization scheme then $\beta(\lambda)$ will be T independent to all orders. Consequently, there is absolutely no way one can see a crossover to a non-trivial theory. Remember though in (4) and the subsequent discussion, it was shown that if a T independent subtraction scheme was used, such as minimal subtraction, then perturbation theory totally broke down in the regime $\frac{m}{T} \rightarrow 0$ i.e. just in the regime of interest where we might expect there to be a crossover to non-trivial behaviour. Clearly we are in a difficult situation, we are trying to describe the high temperature limit $T \gg m_\beta$ using a perturbation expansion about the zero temperature Gaussian fixed point, but this is the only fixed point seen if we insist on using a T independent renormalization scheme, whereas it is clear, at least intuitively, that in this limit the only possible fixed point will be a three dimensional one. The reader may be somewhat disturbed by this wishing to claim that the difference between a zero temperature renormalization scheme and a finite temperature renormalization scheme is a finite counterterm which cannot make any real difference. However, in the limit $\frac{T}{m_\beta} \to \infty$ this finite counterterm can become infinite thereby invalidating the previous statement, and it is just this regime we are interested in. In fact, distinction should be made not between infinite and finite counterterms but between large and small dressings of the vertices of the theory, i.e. the explicit taking into account of a large number of degrees of freedom or a small number. The point is that renormalization is not just about getting rid of ultraviolet divergences it is about defining a set of parameters with which to describe a theory and with which one can calculate. Just as bare parameters provide a good description of physics at scales $\sim \Lambda$ but a bad description at scales $\sim m$ so zero temperature counterterms lead to a set of parameters which are totally inadequate when it comes to a description of the $\frac{T}{m_\beta} \to \infty$ regime.

Some of these observations may come as no surprise to someone who has knowledge of decoupling, the latter is very much renormalization scheme dependent, there being certain schemes which lead to manifest decoupling [5]. Minimal subtraction is not one of these. Renormalization scheme dependence at first sight does not strike one as being a very attractive notion – after all the physics is supposed to be renormalization scheme independent. Using arguments gleaned from the theory of critical phenomena we shall argue that in some sense this renormalization scheme dependence is illusory and that the problems are in fact associated with temperature independent renormalization schemes, as these imply an expansion about the zero temperature Gaussian fixed point.

Led by the intuitive notion that there will be a dimensional reduction when $\frac{m_\beta}{T} \to 0$ we will rewrite the action in the form

$$\begin{aligned} S = \; & \int d^3x \big(\tfrac{1}{2}(\nabla\phi_0)^2 + \tfrac{1}{2}m^2\phi_0^2 + \tfrac{\lambda T}{4!}\phi_0^4 + \tfrac{1}{2}\sum_{n\neq 0}\nabla\phi_n\nabla\phi_{-n} + (m^2 + 4\pi^2 n^2 T^2)\phi_n\phi_{-n} \\ & + \tfrac{\lambda T}{4!}\sum_{n_1=n_2=n_3=n_4\neq 0} \delta(n_1+n_2+n_3+n_4)\phi_{n_1}\phi_{n_2}\phi_{n_3}\phi_{n_4} + C.T.\big) \end{aligned} \tag{11}$$

where: i) we have integrated over the imaginary time direction; ii) we have introduced the Fourier expanded field $\phi(\mathbf{x},\tau) = T^{\frac{1}{2}}\sum_n \phi_n(\mathbf{x})e^{i2\pi nT\tau}$; and iii) C.T. refers to some yet to specified counterterm. One might argue that only the $n = 0$ mode is relevant in the "high" temperature regime. With this assumption and using an ε expansion with minimal subtraction for $\Gamma^{(4)}$ one finds

$$\Gamma^{(4)} = \bar{\lambda} + \frac{3\bar{\lambda}^2}{32\pi^2}\frac{T}{\kappa}\ln\frac{m^2}{\kappa^2} \tag{12}$$

Defining a new coupling constant $u = \frac{\bar{\lambda}T}{\kappa}$ as is suggested by the term $\lambda\phi_0^4$ in (11), one finds a β-function

$$\beta(u) = -u + \frac{3u^2}{16\pi^2} \tag{13}$$

which indeed shows a non-trivial three dimensional fixed point. This has been at a price however. Apart from the implicit T dependence in u one has thrown all T dependence away, and in particular all the $n \neq 0$ modes which contain information about the low temperature limit of the theory. Here one has quite the reverse problem to that encountered using zero temperature counterterms. One is expanding about the fixed point associated with an infinite temperature system rather than a zero temperature system, hence the T independence of the β function. It is clear from (12) and (13) that in the high temperature limit the effective coupling constant is $\frac{\bar{\lambda}T}{\kappa}$, but of course in the low temperature limit it is $\bar{\lambda}$ hence there is a breakdown in perturbation theory at fixed $\bar{\lambda}$. We interpret this as telling us that if one wants to be able to interpolate sensibly between the low and high temperature regimes one must implement an explicitly T dependent renormalization scheme.

To implement such a renormalization scheme we choose the normalization conditions for the $\Gamma^{(N)}_{n\dots n}(p, t, \lambda, T, \kappa)$, n referring to the Euclidean momenta on the external legs

$$\Gamma^{(2)}_{00}(0, 0, \lambda, T, \kappa) = 0 \tag{14}$$

$$\Gamma^{(4)}_{0000}(k_i, 0, \lambda, T, \kappa)|_{S.P.} = \bar{\lambda}\kappa^{\varepsilon}T \tag{15}$$

where $S.P.$ denotes the symmetric point. With the normalization condition (15) the relationship between the bare and renormalized coupling constants to one loop is

$$\lambda_B = \bar{\lambda}\kappa^{\varepsilon} + \frac{3}{2}\bar{\lambda}^2 T\kappa^{2\varepsilon} \sum_{n=-\infty}^{\infty} \int_0^1 dx \int \frac{d^{3-\epsilon}k}{(2\pi)^{3-\epsilon}} \frac{1}{(\mathrm{k}^2 + 4\pi^2 n^2 T^2 + \kappa^2 x(1-x))^2} \tag{16}$$

Clearly in contradistinction to what one obtains via minimal subtraction the counterterms in this prescription are explicitly T dependent. One can calculate the β function, $\beta \equiv \kappa \frac{d\bar{\lambda}}{d\kappa}|_{T,\lambda_B}$, finding

$$\beta(\lambda) = -\varepsilon\bar{\lambda} + \frac{3\bar{\lambda}^2 T}{16\pi^2\kappa}\Gamma(\frac{3+\varepsilon}{2})(4\pi)^{\frac{1+\varepsilon}{2}} \sum_{n=-\infty}^{\infty} \int_0^1 \frac{x(1-x)dx}{(x(1-x) + \frac{4\pi^2 n^2 T^2}{\kappa^2})^{\frac{3+\varepsilon}{2}}} \tag{17}$$

Observing that the β-function (17) is finite as $\varepsilon \to 0$, by expanding the second term in the above for small ε as $\frac{T}{\kappa} \to 0$ it becomes

$$\beta(\lambda) = -\varepsilon\bar{\lambda} + \frac{3\bar{\lambda}^2}{16\pi^2} + O(\bar{\lambda}^2 e^{-\frac{\kappa}{T}}) \tag{18}$$

Setting $\varepsilon = 0$ gives the standard four dimensional Gaussian fixed point. As expected the temperature dependent corrections vanish in the zero temperature limit. In the limit $\frac{T}{\kappa} \to \infty$ only the $n = 0$ term above is important giving

$$\beta(\lambda) = -\varepsilon\bar{\lambda} + \frac{3\bar{\lambda}^2 T}{16\kappa} \tag{19}$$

If we set $\varepsilon = 0$ in the above one would be tempted to think once again that there was only a Gaussian fixed point and hence the theory was trivial at high temperature. Equation (17) though is a first order differential equation, the presence of the factor $\frac{T}{\kappa}$ in (19) is telling us that the point $\bar{\lambda} = 0$ is not a very good point around which to expand the solution of this differential equation when $\frac{T}{\kappa} \to \infty$. As the effective coupling constant there is not λ but λT, or in terms of dimensionless coupling constants $\frac{\bar{\lambda}T}{\kappa}$, when $\frac{T}{\kappa} \gg 1$ we are apparently in a strong coupling regime. Using the coupling constant u

$$\beta(u) = -(1+\varepsilon)u + \frac{3u^2}{16} + O(\frac{u^2\kappa^2}{T^2}) \tag{20}$$

Setting $\varepsilon = 0$ we now see that u has a non-trivial fixed point $u^* = \frac{16}{3}$, the temperature dependent corrections vanishing as $\frac{T}{\kappa} \to \infty$. As u^* is large it is not obvious that perturbation theory will be very good. Preferably one would wish to work to higher loop order and Borel sum. With the methods shown here there is nothing in principle to stop that from being done. Certainly low orders in perturbation theory will make qualitative sense at high temperature whereas for fixed $\bar{\lambda}$ one ends up with nonsense. The point is that the true fixed point of the theory is at $\bar{\lambda} \sim \frac{\kappa}{T}$. It is important to emphasize that we are not doing anything fishy by rewriting things in terms of u, all we are doing is choosing a coupling constant which makes the three dimensional limit look more familiar.[3] We see then that equation (17) interpolates between the trivial Gaussian theory in the zero temperature limit and a non-trivial theory in the high temperature limit.

One can in fact obtain the same results using a different renormalization scheme which is more amenable to higher loop computation, being a temperature dependent generalization of minimal subtraction. Standard minimal subtraction although regularizing ultraviolet divergences, i.e. the limit $\Lambda \to \infty$, by removing terms such as $\frac{\lambda\Lambda^2}{2}$ from $\Gamma^{(2)}$ leaves the $\frac{T}{m} \to \infty$ limit ill defined. So we will adopt the following philosophy — physical quantities should be regular in the $\frac{T}{m} \to \infty$ limit in the same way that they ought to be regular in the $\frac{\Lambda}{m} \to \infty$ limit therefore one should choose counterterms that not only remove ultraviolet divergences but also any other divergences that arise in some physically relevant limit such as $\frac{T}{m_\beta} \to \infty$ or $\frac{T}{\kappa} \to \infty$. With this in mind the relationship between λ_B and $\bar{\lambda}$ becomes

$$\lambda_B = \bar{\lambda}\kappa^\varepsilon + \frac{3\bar{\lambda}^2 T\kappa^\varepsilon}{32\pi^2\kappa(4\pi)^{\frac{1+\varepsilon}{2}}}\Gamma(\frac{1+\varepsilon}{2})\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2T^2}{\kappa^2}\right)^{-\frac{(1+\varepsilon)}{2}} \tag{21}$$

Thus

$$\beta(\bar{\lambda}) = -\varepsilon\bar{\lambda} + \frac{3\bar{\lambda}^2 T}{16\pi^2\kappa}(4\pi)^{\frac{1+\varepsilon}{2}}\Gamma(\frac{3+\varepsilon}{2})\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2T^2}{\kappa^2}\right)^{-\frac{(3+\varepsilon)}{2}} \tag{22}$$

Which yields essentially the same results for the fixed points as the normalization conditions. If we had used standard minimal subtraction the second term in (21) would have

[3]In fact one can show that for a universal (physical) quantity it does not matter whether one uses $\bar{\lambda}$ or u one gets the same answer

been $\frac{3\bar{\lambda}^2}{32\pi^2}\kappa^\varepsilon\Gamma(\frac{\varepsilon}{2})$ which exhibits a $\frac{1}{\varepsilon} \sim \frac{1}{(4-d)}$ pole. In the limit $\frac{T}{\kappa} \to 0$ the second term in (21) exhibits the same $\frac{1}{\varepsilon}$ pole. Because of the divergent nature of (21) confusion may arise as to which limit to take first when considering the high temperature regime: $\varepsilon \to 0$ then $\frac{T}{\kappa} \to \infty$ or visa versa. Our philosophy is always to work with finite expressions such as the β-function (22) where no such ambiguity can arise. Intuitively this is the correct procedure since (21) exhibits a large, non-linear dressing of the coupling constant; but the latter can only be calculated correctly by integrating an infinitessimal dressing. If one performs an expansion of the second term of (22) treating $\varepsilon + 1 = \varepsilon'$ as small one obtains a very simple expression for the β function

$$\beta(\bar{\lambda}) = -\varepsilon\bar{\lambda} + \frac{3\bar{\lambda}^2}{32\pi^2}\coth\frac{\kappa}{2T} \tag{23}$$

which clearly interpolates nicely between the four and three dimensional theories.

Further information can be gleaned from an examination of the renormalization group equation (a more detailed analysis of the RG equation will be given in another publication [9])

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta(\bar{\lambda},\frac{T}{\kappa})\frac{\partial}{\partial\bar{\lambda}} - \frac{N}{2}\gamma_\phi(\bar{\lambda},\frac{T}{\kappa}) + \gamma_{\phi^2}(\bar{\lambda},\frac{T}{\kappa})m_\beta^2\frac{\partial}{\partial m_\beta^2}\right)\Gamma^{(N)} = 0 \tag{24}$$

The solution of (24) is

$$\Gamma^{(N)}(\rho k_i, m_\beta, \bar{\lambda}, T, \kappa) = \rho^{N+d-\frac{Nd}{2}}e^{-\frac{N}{2}\int_1^\rho \frac{dx}{x}\gamma_\phi(\bar{\lambda}(x),\frac{T}{\kappa x})}\Gamma^{(N)}(k_i, \frac{m_\beta^2(\rho)}{\rho^2}, \bar{\lambda}(\rho), \frac{T}{\rho}, \kappa) \tag{25}$$

where $m_\beta(\rho)$ and $\bar{\lambda}(\rho)$ the running mass and running coupling constants are solutions of

$$\rho\frac{d\bar{\lambda}(\rho)}{d\rho} = \beta(\bar{\lambda}(\rho),\frac{T}{\kappa\rho}) = -\varepsilon\bar{\lambda} + \frac{3\bar{\lambda}^2}{32\pi^2}\coth\frac{\kappa\rho}{2T} \tag{26}$$

and

$$\rho\frac{dm_\beta^2(\rho)}{d\rho} = m_\beta^2(\rho)\gamma_{\phi^2}(\bar{\lambda}(\rho),\frac{T}{\kappa\rho}) \tag{27}$$

The solution of (26) is

$$\bar{\lambda}^{-1}(\rho) = \rho^{\varepsilon'}\bar{\lambda}^{-1} - \frac{3\kappa_0}{32\pi^2 T}\rho^{\varepsilon'}\int_1^\rho x^{-\varepsilon'}\coth\frac{\kappa_0 x}{2T}dx \tag{28}$$

As $\rho \to 0$ for fixed T, i.e. as we approach the phase transition from above, one can use a small argument expansion of $\coth x$ to find in terms of $u(\rho) = \frac{T}{\kappa\rho\bar{\lambda}(\rho)}$

$$u^{-1}(\rho) = \rho^{\varepsilon'}u^{-1} + \frac{3}{16\pi^2\varepsilon'}(1-\rho^{\varepsilon'}) - \frac{(\rho^2-\rho^{\varepsilon'})}{128\pi^2(2-\varepsilon')}(\frac{\kappa_0}{T})^2 + O(\frac{\kappa_0}{T})^4 \tag{29}$$

which gives $u(\rho) \to \frac{16\pi^2}{3}$ as $\rho \to 0$. For $\frac{\kappa_0}{T} \to \infty$, $\rho \to 0$ but $\frac{\kappa_0\rho}{T} \gg 1$ one uses a large argument expansion to find that

$$\bar{\lambda}^{-1}(\rho) = \bar{\lambda}^{-1}\rho^\varepsilon - \frac{3}{8\pi^2\varepsilon}(1-\rho^\varepsilon) + \frac{3\rho^\varepsilon}{8\pi^2}\int_1^\rho x^{-\varepsilon}e^{-\frac{\kappa_0 x}{T}}dx \tag{30}$$

which gives in the limit $\frac{\kappa_0\rho}{T} \to \infty$, $\rho \to 0$ that $\bar{\lambda} \to \frac{8\pi^2}{3|\ln\rho|} = 0$, i.e. the Gaussian theory. Whether we reach the trivial or non-trivial fixed point clearly depends on how we treat the $\rho \to 0$ limit. If, for fixed $\frac{\kappa_0}{T}$ we let $\rho \to 0$ then the three dimensional fixed point is reached; however, if $\rho \to 0$ but at the same time $\frac{\kappa_0}{T} \to \infty$ such that $\frac{\kappa_0\rho}{T} \to \infty$ the Gaussian fixed point is reached. The results here seem to be in qualitative agreement with the numerical work done by [10]. What this shows is that $\lambda\phi^4$ theory has non-trivial anomalous dimensions at high temperature. The existence of another mass scale makes the fixed point structure of the theory much richer than it would otherwise be.

We have shown that both normalization conditions at temperature T and GMS both lead to a sensible description of the crossover from low temperature to high temperature. So will any old temperature dependent scheme do? Obviously not, the key criterion a sensible scheme must satisfy is that it remove any $\frac{T}{m} \to \infty$ divergences as well as any ultraviolet divergences. In other words if we try to explicitly evaluate too many degrees of freedom we will run into trouble. The difference between GMS and normalization conditions at fixed T is a finite counterterm throughout the entire crossover. Any renormalization scheme which differs from these by a finite counterterm throughout the crossover will lead to essentially the same physics. The set of all renormalization schemes S for our model system can be partitioned into distinct sets S_L and S_∞. In the former the limit $\frac{T}{m} \to \infty$ is regular in the latter it is not. Within each set of schemes physical quantities will be renormalization scheme independent. This will not be true between the two sets though. For instance effective scaling dimensions of operators are different if calculated using a zero temperature scheme versus a scheme such as GMS whereas GMS or normalization conditions at finite temperature will yield the same answer.

Further insight can be gained by considering the crossover from the point of view of the bare theory. Consider

$$\Gamma_B^{(4)} = \bar{\lambda}_B - \frac{3\bar{\lambda}_B^2}{2} T \int_0^1 dx \sum_n \int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + 4\pi^2 n^2 T^2 + m_B^2 + x(1-x)\kappa^2)^2} \tag{31}$$

where Λ is an ultraviolet cutoff. Performing the sum explicitly yields

$$\Gamma_B^{(4)} = \bar{\lambda}_B - \frac{3\bar{\lambda}_B^2}{16\pi^2} \int_0^1 dx \int^\Lambda \frac{d^3k}{(k^2+m_B^2+x(1-x)\kappa^2)^{\frac{1}{2}}}$$
$$\times\left(\coth\left(\frac{(k^2+m_B^2+x(1-x)\kappa^2)^{\frac{1}{2}}}{2T}\right) + \frac{(k^2+m_B^2+x(1-x)\kappa^2)^{\frac{1}{2}}}{2T\sinh^2\left(\frac{(k^2+m_B^2+x(1-x)\kappa^2)^{\frac{1}{2}}}{2T}\right)}\right) \tag{32}$$

The β function at fixed λ, κ, T is

$$\beta(\bar{\lambda}_B, T, m_B, \Lambda) = \Lambda\frac{\partial\bar{\lambda}_B}{\partial\Lambda}\Big|_{\lambda,\kappa,T} = \frac{3\bar{\lambda}_B^2}{16\pi^2}\int_0^1 dx \frac{\Lambda^3}{(\Lambda^2+m_B^2+x(1-x)\kappa^2)^{\frac{3}{2}}}$$
$$\times\left(\coth\left(\frac{(\Lambda^2+m_B^2+x(1-x)\kappa^2)^{\frac{1}{2}}}{2T}\right) + \frac{(\Lambda^2+m_B^2+x(1-x)\kappa^2)^{\frac{1}{2}}}{2T\sinh\left(\frac{(\Lambda^2+m_B^2+x(1-x)\kappa^2)^{\frac{1}{2}}}{2T}\right)}\right) \tag{33}$$

If one takes the limit $\Lambda \longrightarrow \infty$ one finds the bare version of (12) which is just what we expect — to recover the four dimensional fixed point in this limit, the argument being that such a regime is microscopic. Does this mean that the bulk fixed point is the appropriate one? Can we see a crossover in this formalism? First, let us ask what β is actually telling us here. It tells us the effect of including degrees of freedom between Λ and $\Lambda + d\Lambda$. For $\frac{\Lambda}{T} \longrightarrow \infty$ (33) tells us that at the fixed point $\bar{\lambda}_B^*$ the Λ dependence disappears — this is the standard argument for universality. In a theory where there are only two length scales, m and Λ say, β is a function of $\frac{\Lambda}{m}$ only, so the limit $\Lambda \longrightarrow \infty$ is essentially the same as the limit $m \longrightarrow 0$. This is the intuitive reasoning behind why the continuum limit is supposed to be equivalent to a second order phase transition. The existence of an extra scale T makes the present case more subtle. If one took the $T \longrightarrow \infty$ limit in (33) one would find

$$\beta(\bar{\lambda}_B) = \frac{3\bar{\lambda}_B^2 T}{8\pi^2 \Lambda} \int_0^1 dx \frac{\Lambda^4}{(\Lambda^2 + m_B^2 + x(1-x)\kappa^2)^2} \tag{34}$$

Defining a new bare coupling $u_B = \frac{\bar{\lambda}_B T}{\Lambda}$ gives

$$\beta(u_B) = -u_B + \frac{3u_B^2}{8\pi^2} \int_0^1 dx \frac{\Lambda^4}{(\Lambda^2 + m_B^2 + x(1-x)\kappa^2)^2} \tag{35}$$

which is the three dimensional result with some corrections to scaling that vanish as $\frac{\Lambda}{m_B} \longrightarrow \infty$. The reader may wonder as to why one would be so perverse as to take $\Lambda < T$. From a standard quantum field theory point of view it perhaps sounds strange. In the spirit of Wilson [11] regarding it as a floating cut off due to the integration of a range of momenta, it seems quite sensible. When one begins to integrate out momenta with wavelengths $\sim T^{-1}$ then one finds a qualitative change in the physics and (33) does indeed exhibit a crossover. Clearly one must be careful about the $\Lambda \longrightarrow \infty$ limit. One may take this limit for fixed L or consider $\Lambda \longrightarrow \infty$, $T \longrightarrow \infty$ such that $\frac{\Lambda}{T} \longrightarrow 0$. The former yields the bulk fixed point and the latter the reduced one. One can see that the $\Lambda \longrightarrow \infty$ for fixed T is naive as the integration of such a β function yields only the contribution of momenta that play an unimportant role when scales of interest are $\sim T^{-1}$. From (32) it is not difficult to see that for $m_B \sim 0$ the most important contributions to the integrand are from momenta $m_B < k < T$. If one took the naive continuum limit of (33) one would never know this. These important fluctuations ensure that an expansion about the bulk fixed point is badly behaved. In this sense the bare theory parallels very closely the problems encountered in the renormalized theory when T independent counterterms are used. So we see that adequately interpreted the bare theory yields a crossover totally in accord with that exhibited in the renormalized theory, thus verifying that $\lambda\phi^4$ at finite temperatures is non-trivial.

One other point concerns the evaluation of $\Gamma^{(2)}$, which we showed to be badly behaved in the $\frac{T}{m} \longrightarrow \infty$ limit when renormalized using minimal subtraction. Working with m_β as the mass parameter was deemed necessary. The non-trivial fixed point and consequent dimensional reduction are associated with the regime $\frac{T}{m_\beta} \longrightarrow \infty$ and this limit was well defined leading to an effective three dimensional theory. Parametrizing one's theory with parameters defined at temperature T is a very natural procedure from a critical phenomena or a lattice theory point of view. A particle physicist however, would prefer to parametrize the finite temperature results using zero temperature parameters as these are the ones he knows. One of the main thrusts of this article has been to show the inadequacy of such parameters for a direct perturbative analysis of the problem.

As we have seen, there is a severe problem with infrared divergences at high temperature. One may attempt to ameliorate these infrared problems in other ways than the present one, for example by summing up infinite sets of Feynman diagrams. For instance Dolan and Jackiw [3] showed that the sum over daisy diagrams was equivalent to the Schwinger-Dyson equation for the propagator in the large N limit. One knows however, that a divergent series can yield different results depending on how the terms in the series are grouped. For instance in this daisy sum if one of the loop propagators is removed one obtains a series of contributions to the four point vertex (not necessarily 1-particle irreducible). By iterating these contributions one would obtain a new coupling constant. We feel it is preferable to avoid such ambiguities and use a more systematic approach such as the present one. In terms of the Schwinger-Dyson equations themselves our prescription would require that one start with the Schwinger-Dyson equation for the four point function, find a good approximation for the coupling constant from this, then eliminate the bare coupling constant in the Schwinger-Dyson equation for the propagator in terms of this coupling. This amounts to a renormalization of the coupling constant to one that is more physical. It does not seem sufficient to us to look at the Schwinger-Dyson equation for $\Gamma^{(2)}$ in isolation. The important point to note is that the Schwinger-Dyson equations need to be renormalized and this amounts to a choice of appropriate parameters to summarize the physics. Our point of view is that this is best done by a study of the renormalization goup flow of these parameters, and if one finds a set of these parameters which do not get driven out of the domain of perturbation theory these are obvously the best parameters to choose. Once again it would seem to us that one would be drawn inevitably to the use of finite temperature renormalization schemes as only these schemes are capable of showing that the degrees of freedom are qualitatively changing.

So let us summarize. $\lambda\phi^4$ (and many other theories) at high temperature is plagued by infrared divergences. Rather than treat the problem in the somewhat ambiguous manner of summing up infinite sets of Feynman diagrams we have tried to discover what is the physical reasoning behind the breakdown of the theory. A field theory generally represents a system with a large (often infinite) number of degrees of freedom. Trying to explicitly calculate the effects of this large number is too difficult. Renormalization tries to help by "hiding" a large number of the degrees of freedom in renormalized parameters and renormalized fields. This is exactly what one wishes to do, however, if the degrees of freedom are qualitatively different at different energy scales one can run into problems. One may renormalize by combining certain degrees of freedom into the renormalized parameters only to find out that at the scales of interest these were not the appropriate ones. In the case at hand using zero temperature renormalization schemes ensured that only four dimensional degrees of freedom were absorbed in the renormalized parameters, whereas for scales $\ll T$ it was the case that one should have been absorbing three dimensional degrees of freedom. Fortunately, the theory told us that we were doing something stupid as was manifest in the breakdown in perturbation theory. What we have developed here is a renormalization formalism that correctly takes into account the interpolation of the degrees of freedom between four and three dimensional ones. When this is done one ends up with a finite theory that interpolates fully between the high and low temperature limits, as is manifest in the crossover in the β function between the four and three dimensional fixed points. This shows that $\lambda\phi^4$ in four dimensions at high temperature is not a trivial theory. We also indicated some pitfalls of looking at the continuum limit at finite temperature (at least perturbatively). It is clear that one can apply the formalism herein to other theories, gauge theories of course would be of great interest. It is important to note that nowhere do we assume what the asymptotic $\frac{T}{m_\beta} \to \infty$ limit of the theory will be, we actually derive it. This we believe is one of the strengths of the formalism. There might be cases where dimensional reduction does not come about, in which case our formalism would show this.

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