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Dimensional Crossover and Finite Size Scaling Below T_c

F. Freire
The Blackett Laboratory,
Imperial College,
London SW7 2BZ, U.K.

Denjoe O'Connor
D.I.A.S.,
10 Burlington Rd.,
Dublin 4,
Ireland.

C.R. Stephens
Inst. for Theor. Physics,
Rijksuniversiteit,
Princetonplein 5,
P.O. Box 80006,
3508 TA Utrecht,
The Netherlands.

Abstract:
Using the formalism developed in [1] dimensional crossover in an Ising type system below $T_c(L)$ is considered, $T_c(L)$ being the critical temperature of the finite size system. The crossover considered is that on a d dimensional layered system satisfying periodic boundary conditions and of size L. Effective critical exponents δ_{eff} and β_{eff} are introduced and shown to crossover between their d and $d-1$ dimensional values for $\xi_L \to \infty$ in the limits $\frac{L}{\xi_L} \to \infty$, and $\frac{L}{\xi_L} \to 0$ respectively, ξ_L being the correlation length in the layers. The effective exponents are shown to satisfy natural generalizations of the standard scaling laws. L dependent, global, non-linear scaling fields which span the entire crossover are defined and a scaling form of the equation of state in terms of them derived from an L dependent renormalization group equation. A universal crossover equation of state and effective exponents are obtained to one loop and shown in the above asymptotic limits to reduce to known results.

1 Introduction

One of the most striking features of continuous phase transitions is the appearance of singularities, the singularities being associated with fixed points of the renormalization group (RG). The conventional RG formalism applied to critical phenomena is primarily concerned with the description of the non-analyticity at such singularities. The RG is well understood in its linearized form around a particular fixed point, and in a field theoretic context the well known Wilson-Fisher and Gaussian fixed points are also well understood. In all this one only requires knowledge of the RG in the neighbourhood of a fixed point. It is, however, very common that a system may exhibit different asymptotic behaviours, characteristic of two or more fixed points. In such cases one requires more global information about the RG. This is an altogether different proposition.

Systems that possess more than one fixed point can exhibit crossover behaviour between the various fixed points. This crossover behaviour is important both theoretically and experimentally but is difficult to treat. One can understand this intuitively in the following way: physical systems characteristically look very different at different "scales", exhibiting different effective degrees of freedom. For instance, for systems exhibiting a bicritical point one can see a crossover between degrees of freedom possessing an $O(N)$ symmetry to those possessing an $O(M)$ symmetry as one changes the scale of interest relative to the anisotropy scale. Systems with $O(N)$ and $O(M)$ symmetry are in different universality classes and hence possess different fixed points. Some other types of crossover are: crossover in uniaxial dipolar ferromagnets; crossover between critical, mean field and Gaussian behaviour; dimensional crossover (the subject of this paper) etc. They all exhibit the property of having different degrees of freedom at widely disparate scales. A general discussion of crossover behaviour in a RG context can be found in [2]

An extremely important feature of critical systems is the dominance of fluctuations. The RG was set up to describe such strongly fluctuating systems and does so in two ways, firstly by relating via a symmetry property, RG invariance, a system with strong fluctuations which is difficult to treat to one where fluctuations are not so important which is more tractable. In its most intuitive Kadanoff-Wilson formulation one thinks of course graining the degrees of freedom of the system. In light of the above comments about crossovers this immediately begs the question of what degrees of freedom one should be course graining. Obviously, the effective degrees of freedom which are a good description of the physics at a particular scale would be the correct ones to coarse grain. In other words one would wish to use a RG who's action as much as possible was a faithful representation of true scale changes in the system as generated by dilatations and more generally conformal transformations.

A crossover is typically induced by some "relevant" parameter e.g. dipole- dipole interactions, finite size effects etc. The effective degrees of freedom of the system will depend on these parameters, hence so will the action of the dilatation operator. A "good" RG should also depend on these parameters in order that it give a realistic representation of scale transformations in this environment. Depending on which set of parameters one chooses to include in the RG one can access different fixed point behaviour characteristic of the same physical system. A RG which is independent of some particular crossover parameter g will be, as we shall see, incapable of spanning the crossover, where g plays an essential role, being a good description only in the vicinity of the $g = 0$ fixed point. In principle there are a large number of inequivalent RG's, inequivalent in the sense that they give rise to different fixed points. If the RG's exhibit different asymptotic behaviour how can they represent the same physical system? The answer is that only a certain set, perhaps only one of the RG's will provide full global information on the scaling behaviour of the theory without extra input. For the other RG's extra non-perturbative information is required. Clearly if one can solve a problem exactly none of this matters.

Developing RG's that potentially offer full, global scaling information is not simple, one traditional method [3] has been to match RGs associated with different fixed points. It is most simply undertaken using field theoretic methods. Here though one encounters the commonly held prejudice that renormalization is entirely due to short distance singularities. If one holds to this view then it is not sensible to develop RGs that depend on relevant "infrared" scales. Implementing the point of view that renormalization can depend on important IR scales a small number of crossovers have been treated in a more appropriate manner e.g. crossover at a bicritical point [4], crossover in uniaxial dipolar ferromagnets [5] and dimensional crossover [1]. It is with the latter that we will be exculsively concerned and in particular with the extension of the techniques of [1] to systems below T_c.

Dimensional crossover has been chiefly addressed in the context of finite size scaling [6]. In most work on the RG applied in the context of finite size scaling, it has been a "bulk" RG, which is independent of the finite size scale L, that has been used. Such a RG has proved incapable of furnishing finite size scaling functions and dimensional crossover information except when supplemented by further non-perturbative information [7]. In [7] systems were considered that do not exhibit a true crossover in the sense that the finite system possesses only one fixed point — the "bulk" fixed point. In [1] a formalism was developed that can treat finite size systems that either do or do not possess more than one fixed point, though the emphasis was completely on the former. The essence of the methodology is an L dependent RG implemented in the spirit that the "true" effective degrees of freedom of the system are L dependent. Effective susceptibility and correlation length exponents were calculated perturbatively for an Ising type system on $S^1 \times R^{d-1}$ as were some scaling functions. All these quantities interpolated in a smooth, finite fashion between the forms and values expected of d and $d-1$ dimensional systems, as $\xi_L \to \infty$, in the limits $\frac{L}{\xi_L} \to \infty$ and $\frac{L}{\xi_L} \to 0$ respectively, ξ_L being the finite size correlation length.

The plan of this paper is as follows. In section 2, we present a summary of the problems linked to the normal approach to finite size crossover above T_c and their resolution. We discuss the question of choosing an expansion parameter and the concept of a floating fixed point introduced in [8]. The renormalization group equation - RGE - below T_c is deduced and the scaling form of vertex functions throughout the crossover are discussed in section 3. In section 4 the effective exponent laws involving δ_{eff} and β_{eff} are derived. This takes us to section 5, where the one-loop universal scaling form of the equation of state is calculated throughout the crossover. A discussion of universality in crossovers is given, elucidating how corrections to scaling are absorbed in the definition of the physical parameters at the one-loop level for the equation of state. In section 6, the asymptotic limits of the universal one-loop equation of state for the dimensional crossover are analysed, and shown to reduce to previous known results in appropriate limiting situations. Section 7 is reserved for conjectures and conclusions.

2 Crossover above T_c

In this section we briefly review crossover above T_c. The theory that we use as a prototypical example, throughout this paper, is an Ising-type system described by the Landau-Ginzburg-Wilson Hamiltonian

$$\mathcal{L} = \frac{1}{2}(\nabla\varphi_B)^2 + \frac{1}{2}m_B^2\varphi_B^2 + \frac{1}{2}t_B\varphi_B^2 + \frac{\lambda_B}{4!}\varphi_B^4 \tag{1}$$

on $S^1 \times \mathbb{R}^{d-1}$, i.e. a layered geometry with periodic boundary conditions with $d-1$ dimensional layers and of total thickness L. The system is renormalized so that $m^2 = 0$ and subject to the following conditions

$$\begin{aligned}
\Gamma^{(2)}(k=0, t=0, \lambda, L, \kappa) &= 0 &(2)\\
\frac{\partial\Gamma^{(2)}}{\partial \mathbf{k}^2}(k, t=\kappa^2, \lambda, L, \kappa)|_{k=0} &= 1 &(3)\\
\Gamma^{(4)}(k=0, t=\kappa^2, \lambda, L, \kappa) &= \lambda &(4)\\
\Gamma^{(2,1)}(k=0, t=\kappa^2, \lambda, L, \kappa) &= 1 &(5)
\end{aligned}$$

where our notation is that $\mathbf{k}$ is the momentum in the layers and k includes the discrete momentum perpendicular to the layers. Obviously as these normalization conditions are L dependent the consequent renormalized parameters are implicitly L dependent. In particular $t = T - T_c(L)$, $T_c(L)$ being the critical temperature of the finite system ($d \geq 2$).

Intuitively we expect the critical behaviour of the system to be $d-1$ dimensional as the correlation length ξ goes to infinity for fixed L . However, if the limit $\frac{L}{\xi} \to \infty$, $\xi \to \infty$ is taken we expect it to be "bulk" i.e. d dimensional. The length L is the relevant quantity responsible for a crossover from the bulk theory to a dimensionally reduced one. As shown in [1] a smooth theoretical description of the crossover can be obtained if an appropriate L-dependent renormalization is carried out. This can be understood by thinking of the RG intuitively as a coarse graining procedure, then such an L dependent renormalization is akin to integrating out the physical degrees of freedom at the actual renormalization s

cale in question, and as one changes renormalization scale one is following the correct degrees of freedom, In contrast an L independent renormalization prescription is tantamount to "integrating out" only L independent degrees of freedom, however, one knows that for $\frac{L}{\xi} << 1$ the relevant degrees of freedom are $d-1$ dimensional. One is thus trying to describe a $d-1$ dimensional system via a perturbative expansion about a d dimensional system, i.e. to describe what are essentially $d-1$ dimensional degrees of freedom in terms of d dimensional degrees of freedom. The message is clear: the perturbative description of a finite size system in terms of bulk parameters is totally inadequate. What is more a standard non-perturbative approach such as using a bulk renormalization group is also inadequate, the fixed point of this group being the bulk one. Let us illustrate how the L dependent prescription works by proceeding to implement the L dependent renormalization conditions (2)-(5).

With these normalization conditions the RG equation for an N-point vertex function takes the form

$$\left(\kappa\frac{\partial}{\partial\kappa}+\beta(\bar{\lambda},L\kappa)\frac{\partial}{\partial\bar{\lambda}}+\gamma_{\varphi^2}(\bar{\lambda},L\kappa)t\frac{\partial}{\partial t}-\frac{N}{2}\gamma_{\varphi}(\bar{\lambda},L\kappa)\right)\Gamma^{(N)}(\mathbf{k}_i,t,\bar{\lambda},L,\kappa)=0 \qquad (6)$$

where $\bar{\lambda}=\lambda\kappa^{d-4}$ is the dimensionless coupling constant. Note that it is only in the $L\kappa$ dependence of the characteristic functions β, γ_{φ^2} and γ_φ that this equation differs from the conventional L-independent RGE. As L does not renormalize there is no $\frac{\partial}{\partial L}$ term and it scales with its canonical dimension. We have not included a dependence on t in the characteristic functions which therefore precludes observing the crossover to mean field theory. The crossover of interest here is between d and $d-1$ dimensional critical points. As an example let us consider the β function. With the normalization condition (4) one finds

$$\kappa\frac{d\bar{\lambda}}{d\kappa}=-(4-d)\bar{\lambda}+\frac{3\bar{\lambda}^2}{L\kappa}(4\pi)^{\frac{1-d}{2}}\Gamma(\frac{7-d}{2})\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2}\right)^{\frac{d-7}{2}}+O(\bar{\lambda}^3) \qquad (7)$$

As $L\kappa\to\infty$

$$\kappa\frac{d\bar{\lambda}}{d\kappa}\to-(4-d)\bar{\lambda}+\frac{3\bar{\lambda}^2}{(4\pi)^{\frac{d}{2}}}\Gamma(\frac{6-d}{2}) \qquad (8)$$

as expected for a d dimensional system. For $L\kappa\to 0$

$$\kappa\frac{d\bar{\lambda}}{d\kappa}\to-(4-d)\bar{\lambda}+\frac{3\bar{\lambda}^2}{L\kappa}(4\pi)^{\frac{1-d}{2}}\Gamma(\frac{7-d}{2}) \qquad (9)$$

The solution of $\beta = 0$ in (8) is $\bar{\lambda}^* = \frac{2(4\pi)^{\frac{d}{2}}}{3\Gamma(\frac{4-d}{2})}$, the Wilson-Fisher fixed point in d dimensions. For $d = 4$ (9) apparently displays only a Gaussian fixed point, however, one must be careful. The natural coupling constant in the $d-1$ dimensional limit is $\frac{\lambda}{L}$, the dimensionless version being $\frac{\bar{\lambda}}{L\kappa} = u$. Changing the dependent variable in (7) to u gives

$$\kappa\frac{du}{d\kappa} = -(5-d)u + 3u^2(4\pi)^{\frac{1-d}{2}}\Gamma(\frac{7-d}{2})\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2}\right)^{\frac{d-7}{2}} \tag{10}$$

As $L\kappa \to 0$

$$\kappa\frac{du}{d\kappa} = -(5-d)u + 3u^2(4\pi)^{\frac{1-d}{2}}\Gamma(\frac{7-d}{2}) \tag{11}$$

$\beta = 0$ gives $u^* = \frac{2(4\pi)^{\frac{1-d}{2}}}{3\Gamma(\frac{5-d}{2})}$, the Wilson-Fisher fixed point in $d-1$ dimensions. The differential equation (7) has the solution

$$\bar{\lambda}(\kappa)^{-1} = \bar{\lambda}^{-1}\kappa^{4-d} + \frac{3\kappa^{4-d}}{L}(4\pi)^{\frac{1-d}{2}}\Gamma(\frac{7-d}{2})\sum_{n=-\infty}^{\infty}\int_1^{\kappa}\frac{dx}{x^{6-d}}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2}\right)^{\frac{d-7}{2}} \tag{12}$$

where $\bar{\lambda} = \bar{\lambda}(1)$. One can take the solution (12) as the "small" parameter with respect to which perturbation theory is implemented. Equivalently one could also have solved (10) and used $u(\kappa)$ as a small parameter. What we mean by "small" here is a parameter which orders perturbation theory. It is very important to note that if one is computing a universal quantity, such as an effective critical exponent, it is irrelevant which expansion parameter is used, one still obtains the same answer. Going between $\bar{\lambda}$ and u is merely a change of variable in a differential equation which exhibits the $d-1$ dimensional fixed point in a more familiar way. The fact that $\bar{\lambda}(\kappa) \to 0$ as $L\kappa \to 0$ in no way means that interactions disappear, in fact quite the reverse. In the calculation of an universal quantity the contribution of a particular Feynman diagram in L dependent RG improved perturbation theory is composed of contributions from vertices and from loops. In the limit that $\bar{\lambda}(\kappa) \to 0$ one finds that the contribution from a loop $\to \infty$ in just such a way that the product yields the expected $d-1$ dimensional results.

In conventional critical phenomena one usually captures the dominant physics by expanding around a fixed point. In a crossover there is more than one. For example consider the solution of the characteristic equation for $t(\kappa)$ from (6)

$$t(\kappa) = te^{\int_1^{\kappa}\gamma_{\phi^2}(\bar{\lambda}(x,Lx),Lx)\frac{dx}{x}} = te^{\int_{\bar{\lambda}}^{\bar{\lambda}(\kappa)}\frac{\gamma_{\phi^2}}{\beta}\frac{d\bar{\lambda}'}{\bar{\lambda}'}} \tag{13}$$

where $t = t(\kappa = 1)$. One could expand around the d or $d-1$ dimensional fixed points giving $t(\kappa) = Xt\exp\int_1^{\kappa}\gamma^{\dagger}_{\phi^2}\frac{dx}{x}$ where $X = \exp\int\frac{(\gamma_{\phi^2}-\gamma^{\dagger}_{\phi^2})}{\beta}\frac{d\bar{\lambda}'}{\bar{\lambda}'}$ is a correction to scaling or metric factor. In conventional critical phenomena such corrections are treated as slowly varying and non-singular, as when the denominator of the integrand vanishes so does the numerator. However, for a crossover there is another fixed point where β can vanish, and where, more importantly, the numerator doesn't, having been expanded about a different fixed point. In this situation corrections to scaling are very large and in fact are what interpolate the crossover. It is important to note that these corrections to scaling can be explicitly calculated in the formalism herein.

It would be advantageous to mimic the standard formalism as much as possible by keeping corrections to scaling small. Consider then the change of variables $h = a_1\bar{\lambda}$ where a_1 the coefficient of the $O(\bar{\lambda}^2)$ term in (7) is

$$a_1 = \frac{3}{L\kappa}(4\pi)^{\frac{1-d}{2}}\Gamma(\frac{7-d}{2})\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2}\right)^{\frac{d-7}{2}} \tag{14}$$

One finds

$$\kappa\frac{dh}{d\kappa} = \beta(h) = -\varepsilon(L\kappa)h + h^2 + O(h^3) \tag{15}$$

where

$$\varepsilon(L\kappa) = 4 - d - \frac{d\ln a_1}{d\ln\kappa} = 5 - d - (7-d)\frac{\sum_{n=-\infty}^{\infty}\frac{4\pi^2n^2}{L^2\kappa^2}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2}\right)^{\frac{d-9}{2}}}{\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2}\right)^{\frac{d-7}{2}}} \tag{16}$$

Setting $\beta(h) = 0$ yields to lowest order

$$h^* = \varepsilon(L\kappa) + O(\varepsilon^2(L\kappa)) \tag{17}$$

We term h^* a floating fixed point. Its importance is two fold. Firstly, corrections to scaling around the floating fixed point are small as, for example, $\gamma_{\phi^2} = \gamma^*_{\phi^2}$ when $\beta(h) = 0$. Secondly, it is, like a conventional fixed point found from an algebraic property of the β function equation — its zeros. This is obviously computationally much simpler than having to solve a differential equation iteratively. The difference between using the solutions of (7) and (10) or the floating fixed point corresponds to slowly varying factors which are mere redefinitions of the L dependent crossover variables such as t. For the case at hand h^* varies between $4-d$ and $5-d$. It is clear that the floating fixed point is not necessarily numerically small. In order to achieve accurate estimates of physical quantities one would in principle wish to work to higher order and attempt some resummation procedure analogously to w

hat is done with the fixed dimension expansion [9]. Apart from lengthy calculation there is absolutely nothing to prevent this being done using the present techniques for $d < 4$. Although we will restrict attention to one loop results herein, two loop results for $T > T_c(L)$ have been calculated in [8]. One also knows by experience that one loop results are better than mean field theory and that two loop results are in fact quite often close to numerical and experimental results.

One can think of $\varepsilon(L\kappa)$ as being a measure of the deviation from four of the effective dimensionality of the system. More generally one can define an effective dimensionality d_{eff}[1] via the relation

$$\frac{d\ln\Gamma^{(4)}}{d\ln t} = (4 - d_{eff} - 2\eta_{eff})\nu_{eff} \tag{18}$$

where $\nu_{eff} = (2 - \gamma_{\phi^2})^{-1}$ and $\eta_{eff} = \gamma_\phi$; γ_{ϕ^2} and γ_ϕ being the anomalous dimensions of ϕ^2 and ϕ across the crossover. These anomalous dimensions have been computed to two loops in [8]. From (18) and the solution of (6) for $N = 4$ we see that for $k = 0$

$$d_{eff} = d - \gamma_\lambda \tag{19}$$

γ_λ is the anomalous dimension of the dimensionful coupling constant and satisfies $\kappa\frac{d\lambda}{d\kappa} = \gamma_\lambda\lambda$. As $\gamma_\lambda = \varepsilon(L\kappa) + \frac{\beta(h)}{h}$ one finds $d_{eff} = d - \frac{\beta(\bar\lambda)}{\bar\lambda}$. d_{eff} clearly interpolates between d and $d-1$ as h varies from the bulk to the reduced fixed points. In line with the simpler notion of a floating fixed point one can define a floating d_{eff}, d^*_{eff} as

$$d^*_{eff} = 4 - \gamma^*_\lambda \tag{20}$$

d^*_{eff} also interpolates between d and $d-1$ and therefore captures the essence of the crossover, the difference between d_{eff} and d^*_{eff} being a slowly varying correction to scaling throughout the crossover. One can also define effective critical exponents ν^*_{eff} and η^*_{eff} with respect to the floating fixed point, i.e. $\nu^*_{eff} = \nu_{eff}(h = h^*)$ and $\eta^*_{eff} = \eta_{eff}(h = h^*)$. These concepts will be of great use in later sections. Having very briefly mentioned some pertinent facts for the $T > T_c$ case we go on now to consider below T_c.

3 Scaling below T_c

We know that in the broken phase of an Ising-type system the correlation functions are functions of the magnetisation density $M = <\varphi>$ of the system. We consider the magnetisation to be homogeneous. From multiplicative renormalizability the relation between the bare and renormalized vertex functions is

$$\Gamma^{(N)}(t, M, h, L, \kappa) = Z_\varphi^{\frac{N}{2}}(h, L\kappa, \frac{\Lambda}{\kappa})\Gamma_B^{(N)}(t_B, M_B, h_B, L, \Lambda) \tag{21}$$

where we assume the renormalization scheme to be L-dependent but t and M independent. This is a relevant scheme for the description of the dimensional crossover

[1] An analogous quantity was found by A. Bray in the context of the spherical model, private communication

as shown in the previous section. The multiplicative renormalization of t, M and h is defined by

$$
\begin{aligned}
t_B &= Z_{\varphi^2} t && (22)\\
M_B &= Z_\varphi^{\frac{1}{2}} M && (23)\\
h_B &= Z_h Z_\varphi^{-2} h && (24)
\end{aligned}
$$

From the κ-independence of $\Gamma_B^{(N)}$ we find the RGE for $\Gamma^{(N)}$

$$
\left\{\kappa\frac{\partial}{\partial\kappa} + \beta(h, L\kappa)\frac{\partial}{\partial h} + \gamma_{\varphi^2}(h, L\kappa)t\frac{\partial}{\partial t} - \frac{1}{2}\gamma_\varphi(h, L\kappa)\left[N + M\frac{\partial}{\partial M}\right]\right\}\Gamma^{(N)} = 0 \qquad (25)
$$

Its solution can be found by the method of characteristics, and using dimensional analysis it gives

$$
\begin{aligned}
&\Gamma^{(N)}(t, M, h, L, \kappa)\\
= \ &(\kappa\rho)^{N+d-\frac{1}{2}Nd}\exp\left[-\tfrac{N}{2}\int_\kappa^{\rho\kappa}\gamma_\varphi(h(x, Lx), Lx)\tfrac{dx}{x}\right]\Gamma^{(N)}\left(\tfrac{t(\rho)}{(\kappa\rho)^2}, \tfrac{M(\rho)}{(\kappa\rho)^{\frac{d}{2}-1}}, h(\rho), L\kappa\rho, 1\right) \qquad (26)
\end{aligned}
$$

where the running variables $t(\rho)$, $M(\rho)$ and $h(\rho)$ satisfy the characteristic equations

$$
\begin{aligned}
\rho\frac{dt(\rho)}{d\rho} &= \gamma_{\varphi^2}(h(\rho), L\kappa\rho)t(\rho) && (27)\\
\rho\frac{dM(\rho)}{d\rho} &= -\frac{1}{2}\gamma_\varphi(h(\rho), L\kappa\rho)M(\rho) && (28)\\
\rho\frac{dh(\rho)}{d\rho} &= \beta(h(\rho), L\kappa\rho) && (29)
\end{aligned}
$$

We can also rewrite (29) after a change of variable back to the dimensionful coupling λ as

$$
\rho\frac{d\lambda(\rho)}{d\rho} = \gamma_\lambda(\lambda(\rho), L\kappa\rho)\lambda(\rho) \qquad (30)
$$

The left-hand side of (26) cannot be evaluated perturbatively at $t = 0$ due to infrared divergences. This problem is surmounted by proceeding analogously to a system without crossover. The arbitrariness of ρ is utilized by trying to choose it so that the system is kept away from the infrared dangerous region for any value of L. Some possible conditions one might envision using to determine ρ are

$$
\begin{aligned}
t(\rho) &= (\kappa\rho)^2 && (31)\\
\tfrac{\lambda(\rho)}{2}M^2(\rho) &= \rho^2\kappa^2 && (32)\\
t(\rho) + \tfrac{\lambda(\rho)}{2}M^2(\rho) &= \rho^2\kappa^2 && (33)
\end{aligned}
$$

The usual condition $M(\rho) = (\rho\kappa)^{\frac{d}{2}-1}$ is inappropriate for the crossover problem and will be discussed further in section 7. In the next section we will discuss the relative merits of these conditions. Suffice it to say that if the problem could be solved exactly any condition would be as good as any other.

Consider the following sets of normalization conditions

$$\begin{aligned}&\Gamma^{(2)}(k=0, t(\rho), M(\rho)=0, h(\rho), L, \kappa\rho) = t(\rho)\\ &\frac{\partial}{\partial \mathbf{k}^2}\Gamma^{(2)}(k, t(\rho), M(\rho)=0, h(\rho), L, \kappa\rho)\Big|_{k=0} = 1\end{aligned} \tag{34}$$

$$\begin{aligned}&\Gamma^{(2)}(k=0, t(\rho)=0, M(\rho), h(\rho), L, \kappa\rho) = \frac{\lambda(\rho)}{2}M^2(\rho)\\ &\frac{\partial}{\partial \mathbf{k}^2}\Gamma^{(2)}(k, t(\rho)=0, M(\rho), h(\rho), L, \kappa\rho)\Big|_{k=0} = 1\end{aligned} \tag{35}$$

$$\begin{aligned}&\Gamma^{(2)}(k=0, t(\rho), M(\rho), h(\rho), L, \kappa\rho) = t(\rho) + \frac{\lambda(\rho)}{2}M^2(\rho)\\ &\frac{\partial}{\partial \mathbf{k}^2}\Gamma^{(2)}(k, t(\rho), M(\rho), h(\rho), L, \kappa\rho)\Big|_{k=0} = 1\end{aligned} \tag{36}$$

From the definition of the physical correlation length in the layers

$$\xi_L^2 = \frac{\int d^dx \mathbf{x}^2 G_L^{(2)}(x,0)}{2d\int d^dx G_L^{(2)}(r,0)} \tag{37}$$

where $\mathbf{x}^2$ is the distance squared in the layers, one sees that with the conditions (31) and (34) that $t(\rho) = \rho^2\kappa^2 = \xi_{Lt}^{-2}$ where ξ_{Lt} is the correlation length in the finite size system when $M = 0$. With (32) and (35) $\frac{\lambda(\rho)}{2}M^2(\rho) = \rho^2\kappa^2 = \xi_{LM}^{-2}$ where ξ_{LM} is the correlation length in the finite size system when $T = T_c(L)$. With (33) and (36), $t(\rho) + \frac{\lambda}{2}M^2(\rho) = \rho^2\kappa^2 = \xi_{LMt}^{-2}$ where ξ_{LMt} is the correlation length in the finite size system when $T < T_c(L)$. ξ_{Lt}, ξ_{LM} and ξ_{LMt} are all non-linear scaling fields which are capable of interpolating between the d and $d-1$ dimensional fixed points of the system for $\xi \to \infty$ in the limit $\frac{L}{\xi} \to \infty$ and $\frac{L}{\xi} \to 0$ respectively.

Consider (26) with the conditions (31) and (34)

$$\Gamma^{(N)}(t, M, h, L, \kappa) = \xi_{Lt}^{\frac{1}{2}Nd-N-d} e^{-\frac{N}{2}\int_\kappa^{\xi_{Lt}^{-1}} \frac{dx}{x}\gamma_\varphi} \Gamma^{(N)}\left(1, \frac{\lambda(\rho)M^2(\rho)}{(\kappa\rho)^2}, h(\rho), L\kappa\rho, 1\right) \tag{38}$$

Solving the characteristic equations (28) and (29) gives

$$\frac{\lambda(\rho)M^2(\rho)}{\rho^2\kappa^2} = \frac{\lambda M^2}{\kappa^2}\exp\left[\int_\kappa^{\xi_{Lt}^{-1}}(\gamma_\lambda - \gamma_\phi - 2)\frac{dx}{x}\right] \tag{39}$$

Expanding around the floating fixed point $h = h^*$ then yields

$$\frac{\lambda(\rho)M^2(\rho)}{\rho^2\kappa^2} = \frac{\lambda M^2}{\kappa^2} e^{\int_\kappa^{\xi_{Lt}^{-1}}((\gamma_\lambda-\gamma_\lambda^*)-(\gamma_\phi-\gamma_\phi^*))\frac{dx}{x}} e^{\int_\kappa^{\xi_{Lt}^{-1}}(\gamma_\lambda^*-\gamma_\phi^*-2)\frac{dx}{x}} \tag{40}$$

γ_λ^* being the anomalous dimension of λ at the floating fixed point. The correction to scaling term above is slowly varying throughout the crossover and can be absorbed into a redefinition of M, as can λ. From section 2, we have $\gamma_\lambda^* = 4 - d_{eff}^*$ where d_{eff}^* is the "universal" part of d_{eff}. Hence

$$\frac{\lambda(\rho)M^2(\rho)}{\rho^2\kappa^2} = M^2 \exp\left[-\int_\kappa^{\xi_{Lt}^{-1}} (d_{eff}^* - 2 + \eta_{eff}^*)\frac{dx}{x}\right] \tag{41}$$

Substituting back into (38) gives

$$\Gamma^{(N)} = \xi_{Lt}^{\frac{Nd}{2}-N-d} e^{-\frac{N}{2}\int_\kappa^{\xi_{Lt}^{-1}} \eta_{eff}^* \frac{dx}{x}} \mathcal{F}_t^{(N)}\left(M^2 e^{-\int_\kappa^{\xi_{Lt}^{-1}}(d_{eff}^*-2+\eta_{eff}^*)\frac{dx}{x}}, \frac{L}{\xi_{Lt}}\right) \tag{42}$$

where $\mathcal{F}_t^{(N)}$ is a universal function. So, if $\Gamma^{(N)}$ is measured in units of ξ_{Lt} we see that the scaling functions are functions of two non-linear scaling variables $\frac{L}{\xi_{Lt}}$ and $M^2 e^{-\int_\kappa^{\xi_{Lt}^{-1}}(d_{eff}^*-2+\eta_{eff})\frac{dx}{x}}$.

With the conditions (32) and (35) instead of (31) and (34) one finds

$$\Gamma^{(N)}(t, M, h, L, \kappa) = \xi_{LM}^{\frac{Nd}{2}-N-d} e^{-\frac{N}{2}\int_\kappa^{\xi_{LM}^{-1}} \gamma_\phi \frac{dx}{x}} \Gamma^{(N)}\left(\frac{t(\rho)}{\rho^2\kappa^2}, 1, h(\rho), L\kappa\rho, 1\right) \tag{43}$$

The solution of the characteristic equation (27) is

$$\frac{t(\rho)}{\rho^2\kappa^2} = \frac{t}{\kappa^2}\exp\left[\int_\kappa^{\xi_{LM}^{-1}}(\gamma_{\phi^2} - 2)\frac{dx}{x}\right] \tag{44}$$

and expanding around the floating fixed point gives

$$\frac{t(\rho)}{\rho^2\kappa^2} = \frac{t}{\kappa^2} e^{\int_\kappa^{\xi_{LM}^{-1}}(\gamma_{\phi^2}-\gamma_{\phi^2}^*)\frac{dx}{x}} \exp\left[\int_\kappa^{\xi_{LM}^{-1}}(\gamma_{\phi^2}^* - 2)\frac{dx}{x}\right] \tag{45}$$

The correction to scaling factor can be absorbed into a redefinition of t and noting that $\gamma_{\phi^2}^* - 2 = -\frac{1}{\nu_{eff}^*}$ we have

$$\Gamma^{(N)} = \xi_{LM}^{\frac{Nd}{2}-N-d} e^{-\frac{N}{2}\int_\kappa^{\xi_{LM}^{-1}} \eta_{eff}^* \frac{dx}{x}} \mathcal{F}_M^{(N)}\left(t e^{-\int_\kappa^{\xi_{LM}^{-1}} \frac{1}{\nu_{eff}^*}\frac{dx}{x}}, \frac{L}{\xi_{LM}}\right) \tag{46}$$

where $\mathcal{F}_M^{(N)}$ is also a universal function. For $\Gamma^{(N)}$ measured in units of ξ_{LM} these scaling functions are functions of the two non-linear scaling fields $\frac{L}{\xi_{LM}}$ and $t e^{-\int_\kappa^{\xi_{LM}^{-1}} \frac{1}{\nu_{eff}^*}\frac{dx}{x}}$. ξ_{Lt} interpolates between $t^{-\nu_b}$ and $t^{-\nu_r}$, for $\xi_{LM} \to \infty$ in the limits $\frac{L}{\xi_{LM}} \to \infty$ and $\frac{L}{\xi_{Lt}} \to 0$ respectively, where ν_b and ν_r are the bulk and reduced correlation length exponents respectively. Note that all the above non-linear scaling fields are globally valid in the sense that they capture both the d and $d-1$ dimensional fixed points. We could also have written down scaling functions $\mathcal{F}_{Mt}$ which would be functions of ξ_{LMt} which is also a good non-linear scaling field for the crossover.

4 Scaling laws

In the previous section we investigated the scaling form of vertex functions below T_c in terms of two non-linear scaling fields ξ_{Lt} and ξ_{LM}. In this section we would like to proceed further with a general scaling formulation examining what happens to scaling laws for the crossover. In particular let us consider the crossover equation of state. From (26), as $H = \Gamma^{(1)}$ one finds the relation

$$H(t, M, h, L, \kappa) = (\rho\kappa)^{\frac{d}{2}+1} \exp\left[-\frac{1}{2}\int_{\kappa}^{\rho\kappa} \gamma_{\phi}\frac{dx}{x}\right] H(\frac{t(\rho)}{\rho^2\kappa^2}, \frac{\lambda(\rho)M^2(\rho)}{2\rho^2\kappa^2}, h(\rho), L\kappa\rho) \quad (47)$$

From the non-crossover equation of state

$$H = M^{\delta} f(tM^{-\frac{1}{\beta}}) \quad (48)$$

it is natural to define effective critical exponents for the crossover

$$\delta_{eff} = \left.\frac{d\ln H}{d\ln M}\right|_{t=0} \quad (49)$$

and

$$\beta_{eff} = \frac{d\ln M}{d\ln t} \quad (50)$$

the latter being defined on the crossover coexistence curve.

Consider (47) when $T = T_c(L)$, i.e. $t = t(\rho) = 0$. We impose the normalization condition

$$H\left(\frac{\lambda(\rho)M^2(\rho)}{2\rho^2\kappa^2} = 1, h(\rho), L\kappa\rho\right) = \frac{\lambda(\rho)M^3(\rho)}{6(\rho\kappa)^{\frac{d}{2}+1}} \quad (51)$$

on the dimensionless H on the right hand side of (47). This condition is consistent with the normalization condition on $\Gamma^{(2)}$ and motivated by the mean field theory case. Needless to say this renormalization is not necessary to remove UV divergences. We are, however, free to perform finite renormalizations of our variables. This particular one is designed for computational convenience and corresponds to just taking the tree level term of the equation of state when $t = 0$. It is analogous to the type of finite renormalization one would do in going between a minimally subtracted version of $\Gamma^{(2)}(t)$ and a normalization condition. With this normalization condition (47) expressed in terms of the dimensionless coupling $\bar{\lambda}$ gives

$$H = \frac{1}{3}\sqrt{\frac{2}{\bar{\lambda}(\rho)}}(\rho\kappa)^{\frac{d}{2}+1} \exp\left[-\frac{1}{2}\int_{\kappa}^{\rho\kappa} \gamma_{\phi}\frac{dx}{x}\right] \quad (52)$$

Using the characteristic equation for $\bar{\lambda}(\rho)$ one finds

$$H = \frac{1}{3}\sqrt{\frac{2}{\bar{\lambda}}}(\rho\kappa)^{\frac{d}{2}+1}\exp\left[-\frac{1}{2}\int_{\kappa}^{\rho\kappa}(\gamma_\phi + \frac{\beta(\bar{\lambda})}{\bar{\lambda}})\frac{dx}{x}\right] \tag{53}$$

where $\bar{\lambda} = \bar{\lambda}(1)$. Thus

$$\frac{d\ln H}{d\ln M} = \left(\frac{d}{2} + 1 - \frac{1}{2}\gamma_\phi - \frac{1}{2}\frac{\beta(\bar{\lambda}(\rho))}{\bar{\lambda}(\rho)}\right)\frac{d\ln\rho}{d\ln M} \tag{54}$$

From section 2, recall that $\frac{\beta(\bar{\lambda})}{\bar{\lambda}} = d - d_{eff}$, hence, with the condition $\frac{\lambda(\rho)M^2(\rho)}{2} = \rho^2\kappa^2$, one finds

$$\frac{d\ln\rho}{d\ln M} = \frac{2}{(d_{eff} - 2 + \gamma_\phi)} \tag{55}$$

Substituting back into (54) gives

$$\delta_{eff} = \frac{d_{eff} + 2 - \eta_{eff}}{d_{eff} - 2 + \eta_{eff}} \tag{56}$$

Now let us turn our attention to the relationship between M and t on the coexistence curve. Consider the solution of the RG equation for $G^{(1)} = M$

$$M(|t|, h, L, \kappa) = (\rho\kappa)^{\frac{d}{2}-1}\exp\left[\frac{1}{2}\int_{\kappa}^{\rho\kappa}\gamma_\phi\right] M(\frac{|t(\rho)|}{\rho^2\kappa^2}, h(\rho), L\kappa\rho) \tag{57}$$

Imposing the normalization condition

$$M(|t(\rho)| = \rho^2\kappa^2, \bar{\lambda}(\rho), L\kappa\rho) = \sqrt{\frac{6|t(\rho)|}{\bar{\lambda}(\rho)\rho^2\kappa^2}} \tag{58}$$

which is the equivalent of (51) and again corresponds to imposing the mean field condition at the normalization point, and requires only a finite renormalization of $G^{(1)}$. The right hand side of (58) is just the tree level term in the equation of state. Substituting (58) into (57) one finds

$$M(|t|, h, L, \kappa) = \sqrt{\frac{6}{\bar{\lambda}(\rho)}}(\rho\kappa)^{\frac{d}{2}-1}\exp\left[\frac{1}{2}\int_{1}^{\rho}\gamma_\phi\right] \tag{59}$$

Once again using the characteristic equation for $\bar{\lambda}(\rho)$ one finds

$$M(|t|, h, L, \kappa) = \sqrt{\frac{6}{\bar{\lambda}}}(\rho\kappa)^{\frac{d}{2}-1}\exp\left[\frac{1}{2}\int_{1}^{\rho}\gamma_\phi - \frac{\beta(\bar{\lambda})}{\bar{\lambda}}\right] \tag{60}$$

Hence

$$\frac{d\ln M}{d\ln|t|}=\frac{1}{2}\left(d-2+\gamma_\phi-\frac{\beta(\bar\lambda(\rho))}{\bar\lambda(\rho)}\right)\frac{d\ln\rho}{d\ln|t|} \tag{61}$$

With the condition $|t(\rho)|=\rho^2\kappa^2$ one has

$$\frac{d\ln\rho}{d\ln|t|}=\nu_{eff} \tag{62}$$

and substituting into (61) gives

$$\beta_{eff}=\frac{\nu_{eff}}{2}(d_{eff}-2+\eta_{eff}) \tag{63}$$

Thus we get the very interesting result that natural analogs of the conventional scaling laws are obeyed throughout the entire crossover. What this implies is that there is a generalization of universality which applies across the crossover in the sense that knowledge of γ_ϕ and γ_{ϕ^2} are sufficient to determine the entire crossover along with one more function d_{eff}. Knowledge of d_{eff} is equivalent to knowledge of γ_λ. In other words in contradistinction to the standard non-crossover problem where γ_λ merely represents slowly varying corrections to scaling here one requires γ_λ to obtain full knowledge of the crossover. It is also interesting that effective exponents defined with respect to the floating fixed point also obey scaling laws, explicitly

$$\delta^*_{eff}=\frac{(d^*_{eff}+2-\eta^*_{eff})}{(d^*_{eff}-2+\eta^*_{eff})} \tag{64}$$

and

$$\beta^*_{eff}=\frac{\nu^*_{eff}}{2}(d^*_{eff}-2+\eta^*_{eff}) \tag{65}$$

The difference between a floating fixed point and running coupling result amounts to no more than a redefinition of ones crossover variables by slowly varying non singular corrections to scaling across the crossover. In other words the floating fixed point captures the "universal" part of the crossover. We will return to this point later.

5 More on the Scaling Forms

Having introduced the effective exponents δ_{eff} and β_{eff} we can return to the considerations of section 3 and write the scaling forms in a slightly different way. Consider (42) and (46), first (42). The integrals in (42) are from an initial to a final inverse correlation length, having used the relation $\rho^2\kappa^2=\xi_{Lt}^{-1}$, hence we can change variables using the definition of ν_{eff}, i.e. $\frac{d\rho}{\rho}=-\frac{d\xi_{Lt}}{\xi_{Lt}}=\nu_{eff}\frac{dt}{t}$, to find

$$\Gamma^{(N)}=e^{\int_1^t(N+d-\frac{N}{2}(d+\eta_{eff}))\nu_{eff}\frac{dt'}{t'}}\mathcal{F}^{(N)}_t\left(Me^{-\int_1^t\beta_{eff}\frac{dt'}{t'}},Le^{\int_1^t\nu_{eff}\frac{dt'}{t'}}\right) \tag{66}$$

The two non-linear scaling fields entering the scaling function, in terms of $T-T_c(L)$, are $Me^{-\int_1^t \beta_{eff}\frac{dt'}{t'}}$ and $Le^{\int_1^t \nu_{eff}\frac{dt'}{t'}}$. For the equation of state

$$H = e^{\frac{1}{2}\int_1^t (d+2-\eta_{eff})\nu_{eff}\frac{dt'}{t'}}\mathcal{F}_t^{(1)}(Me^{-\int_1^t \beta_{eff}\frac{dt'}{t'}}, Le^{\int_1^t \nu_{eff}\frac{dt'}{t'}}) \tag{67}$$

Now consider (46). Using the condition fixing ρ in terms of M we can change variables via (55) i.e. $\frac{d\rho}{\rho} = -\frac{d\xi_{LM}}{\xi_{LM}} = (2/(d_{eff}-2+\eta_{eff}))\frac{dM}{M}$ to find

$$\Gamma^{(N)} = e^{\int_1^M (N+d-\frac{N}{2}(d+\eta_{eff}))\frac{\nu_{eff}}{\beta_{eff}}\frac{dM'}{M'}}\mathcal{F}_M^{(N)}(te^{-\int_1^M \frac{1}{\beta_{eff}}\frac{dM'}{M'}}, Le^{\int_1^M \frac{\nu_{eff}}{\beta_{eff}}\frac{dM'}{M'}}) \tag{68}$$

For the equation of state

$$H = e^{\frac{1}{2}\int_1^t (d+2-\eta_{eff})\frac{\nu_{eff}}{\beta_{eff}}\frac{dM'}{M'}}\mathcal{F}_M^{(1)} \tag{69}$$

Now, from the perturbative results, as we shall see in the next section, with the condition $\frac{\lambda(\rho)}{2}M^2(\rho) = \rho^2\kappa^2$ one can extract a factor $\sqrt{\frac{2}{\lambda(\rho)}}$ from $\mathcal{F}_M^{(1)}$, the remainder of $\mathcal{F}_M^{(1)}$ being a polynomial expansion in $\bar{\lambda}(\rho)$ (or $h(\rho)$). With

$$\bar{\lambda}(\rho) = \bar{\lambda}\exp\left(\int_1^M \left(\frac{2(d-d_{eff})}{d_{eff}-2+\eta_{eff}}\right)\frac{dM'}{M'}\right) \tag{70}$$

and using (56) and (63) one obtains

$$H = \exp\left(\int_1^M \delta_{eff}\frac{dM'}{M'}\right)\mathcal{G}(te^{-\int_1^M \frac{1}{\beta_{eff}}\frac{dM'}{M'}}, Le^{\int_1^M \frac{\nu_{eff}}{\beta_{eff}}\frac{dM'}{M'}}) \tag{71}$$

Thus we have the scaling form of the equation of state in terms of the two non-linear scaling fields $x = te^{-\int_1^M \frac{1}{\beta_{eff}}\frac{dM'}{M'}}$ and $y = Le^{\int_1^M \frac{\nu_{eff}}{\beta_{eff}}\frac{dM'}{M'}}$. For $t=0$, $\mathcal{G}=1$ and $H = \exp\int_1^M \delta_{eff}\frac{dM'}{M'}$ therefore yielding (56). For $H=0$ the equation of state is given by $\mathcal{G}(x,y)=0$, which yields a coexistence curve $x=g(y)$, hence

$$t = g(Le^{\int_1^M \frac{\nu_{eff}}{\beta_{eff}}\frac{dM'}{M'}})\exp\left(\int_1^M \frac{1}{\beta_{eff}}\frac{dM'}{M'}\right) \tag{72}$$

In order that we reproduce (63) we must have $g(y)=1$. In section 7, we show that in terms of apropriate variables $g(y)=1$. This is a self consistency condition for the effective exponent laws the variables x and y as written. We will now verify much of the above perturbatively.

6 The universal one-loop equation of state

To start with, we write the running equation of state in its non-universal form

$$H = \frac{t(\rho)M(\rho)}{(\rho\kappa)^{\frac{d}{2}+1}} + \frac{1}{6}\frac{\bar{\lambda}(\rho)M(\rho)^3}{(\rho\kappa)^{\frac{3d}{2}-3}} +$$

$$\frac{\bar\lambda(\rho)M(\rho)}{2L(\kappa\rho)^{\frac{d}{2}}}(4\pi)^{\frac{(1-d)}{2}}\Gamma(\tfrac{(5-d)}{2})\left(\tfrac{t(\rho)}{\rho^2\kappa^2}+\tfrac{\bar\lambda(\rho)}{2}\tfrac{M(\rho)^2}{(\rho\kappa)^{d-2}}\right)\sum_{n=-\infty}^{+\infty}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-5}{2}} \tag{73}$$

$$+\frac{\bar\lambda(\rho)M(\rho)}{2L(\kappa\rho)^{\frac{d}{2}}}(4\pi)^{\frac{1-d}{2}}\Gamma(\tfrac{3-d}{2})\sum_{n=-\infty}^{\infty}\left[\left(\frac{t(\rho)}{\rho^2\kappa^2}+\frac{\bar\lambda(\rho)}{2}\frac{M(\rho)^2}{(\rho\kappa)^{d-2}}+\frac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-3}{2}}-\left(\frac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-3}{2}}\right]$$

Using the normalization condition on H, remember this corresponds to a finite renormalization,

$$H(t(\rho)=0,\frac{\lambda(\rho)}{2}M(\rho)^2=\rho^2\kappa^2,L,\lambda(\rho),\kappa\rho)=\frac{\lambda(\rho)}{6}M(\rho)^3 \tag{74}$$

one finds that the dimensionless magnetic field is

$$H(\frac{t(\rho)}{\rho^2\kappa^2}=0)=\frac{1}{3}\sqrt{\frac{2}{\bar\lambda(\rho)}} \tag{75}$$

and hence

$$\begin{aligned}H=&\frac{t(\rho)M(\rho)}{(\rho\kappa)^{\frac{d}{2}+1}}+\frac{1}{6}\frac{\bar\lambda(\rho)M(\rho)^3}{(\rho\kappa)^{\frac{3d}{2}-3}}+\\
&\frac{\bar\lambda(\rho)M(\rho)}{2L(\kappa\rho)^{\frac{d}{2}}}(4\pi)^{\frac{(1-d)}{2}}\Gamma(\tfrac{(5-d)}{2})\left((\tfrac{t(\rho)}{\rho^2\kappa^2}+\tfrac{\bar\lambda(\rho)}{2}\tfrac{M(\rho)^2}{(\rho\kappa)^{d-2}}\right)\sum_{n=-\infty}^{+\infty}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-5}{2}}\\
&+\frac{\bar\lambda(\rho)M(\rho)}{2L(\kappa\rho)^{\frac{d}{2}}}(4\pi)^{\frac{1-d}{2}}\Gamma(\tfrac{3-d}{2})\sum_{n=-\infty}^{+\infty}\left[\left(\frac{t(\rho)}{\rho^2\kappa^2}+\frac{\bar\lambda(\rho)}{2}\frac{M(\rho)^2}{(\rho\kappa)^{d-2}}+\frac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-3}{2}}-\left(\frac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-3}{2}}\right]\\
&-\frac{\bar\lambda(\rho)}{2L\kappa\rho}\sqrt{\frac{2}{\bar\lambda(\rho)}}(4\pi)^{\frac{1-d}{2}}\Gamma(\tfrac{3-d}{2})\sum_{n=-\infty}^{+\infty}[\left(1+\frac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-3}{2}}-\left(\frac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-3}{2}}\\
&+\tfrac{(3-d)}{2}\left(1+\tfrac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-5}{2}}]\end{aligned} \tag{76}$$

Fixing ρ by condition (32) gives

$$\begin{aligned}H(\rho)=&(\rho\kappa)^{\frac{d}{2}+1}\sqrt{\tfrac{2}{\bar\lambda(\rho)}}((\tfrac{1}{3}+\tfrac{t(\rho)}{\rho^2\kappa^2}+\tfrac{\bar\lambda(\rho)}{2L\kappa\rho}(4\pi)^{\frac{1-d}{2}}\Gamma(\tfrac{3-d}{2})\sum_{n=-\infty}^{+\infty}[\left(1+\frac{t(\rho)}{\rho^2\kappa^2}+\frac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-3}{2}}\\
&-\left(1+\tfrac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-3}{2}}+\tfrac{d-3}{2}\tfrac{t(\rho)}{\rho^2\kappa^2}\left(1+\tfrac{4\pi^2n^2}{L^2\kappa^2\rho^2}\right)^{\frac{d-5}{2}}])\end{aligned} \tag{77}$$

When $t(\rho)=0$

$$H(\rho)=\frac{1}{3}(\rho\kappa)^{\frac{d}{2}+1}\sqrt{\frac{2}{\bar\lambda(\rho)}} \tag{78}$$

To obtain a universal equation of state we need to make two demands. Firstly that $H=\exp(\int_1^M\delta_{eff}\frac{dM'}{M'})$ and secondly that for $H=0$ the equation of state has a zero at $x=-1$ where x is the non-linear scaling field introduced in section 5. These demands are obviously implementations of the effective exponent laws (56) and (63). Using $\bar{\lambda}(\rho)=\exp(\int_1^\rho(d-d_{eff})\frac{dx}{x})$, in a one loop approximation and setting $\kappa=1$ for convenience, one finds for $t=0$

$$H=\frac{\sqrt{2}}{3}e^{\int_1^\rho(\frac{d_{eff}}{2}+1)\frac{dx}{x}} \tag{79}$$

As $\frac{d\rho}{\rho}=\frac{2}{(d_{eff}-2)}\frac{dM}{M}$ one gets

$$H=\frac{\sqrt{2}}{3}e^{\int_1^M\left(\frac{d_{eff}+2}{d_{eff}-2}\right)\frac{dM'}{M'}} \tag{80}$$

We can absorb the $\sqrt{2}/3$ into a redefinition of κ, of course equivalently one can define a new magnetic field

$$H=\frac{\sqrt{2}}{3}H' \tag{81}$$

then

$$H'=e^{\int_1^M\left(\frac{d_{eff}+2}{d_{eff}-2}\right)\frac{dM'}{M'}} \tag{82}$$

Now consider the case when $t\neq 0$, our task is to get our expressions into the universal form (71). Now observing the form of (77) we see the prefactor is the same as we have identified in (80). We therefore can rewrite (77) as

$$H'=e^{\int_1^M\left(\frac{d_{eff}+2}{d_{eff}-2}\right)\frac{dM'}{M'}}\mathcal{G} \tag{83}$$

where

$$\begin{aligned}\mathcal{G}= \;& 1+3\frac{t(\rho)}{\rho^2\kappa^2}+\frac{3\bar{\lambda}(\rho)}{2L\kappa\rho}\frac{\Gamma(\frac{3-d}{2})}{(4\pi)^{\frac{d-1}{2}}}\sum_{n=-\infty}^{+\infty}\left[\left(1+\frac{t(\rho)}{\rho^2\kappa^2}+(\frac{2\pi n}{\rho\kappa L})^2\right)^{\frac{d-3}{2}}\right.\\ &\left.-\left(1+(\frac{2\pi n}{\rho\kappa L})^2\right)^{\frac{d-3}{2}}+\frac{(d-3)}{2}\frac{t(\rho)}{\rho^2\kappa^2}\left(1+(\frac{2\pi n}{\rho\kappa L})^2\right)^{\frac{d-5}{2}}\right]\end{aligned} \tag{84}$$

With the condition (32), we can express (84) in terms of the non-linear scaling fields x and y, which of course must be evaluated up to the order we're working in. First of all we define a variable $x'=\frac{t(\rho)}{\rho^2\kappa^2}$ to obtain

$$\begin{aligned}\mathcal{G}= \;& 1+3x'+3\frac{\bar{\lambda}(y)}{2y}(4\pi)^{\frac{1-d}{2}}\Gamma(\tfrac{3-d}{2})\sum_{n=-\infty}^{+\infty}\left[\left(1+x'+((\frac{2\pi n}{y}))^2\right)^{\frac{d-3}{2}}\right.\\ &\left.-\left(1+(\frac{2\pi n}{y})^2\right)^{\frac{d-3}{2}}+\frac{d-3}{2}x'\left(1+(\frac{2\pi n}{y})^2\right)^{\frac{d-5}{2}}\right]\end{aligned} \tag{85}$$

Now, we wish to redefine $x' = (a + b(y)\bar{\lambda}(y))x$ such that for $H' = 0$, $x = -1$ is a zero of the equation of state thereby ensuring the validity of (71). Comparing powers of $\bar{\lambda}$ gives $a = \frac{1}{3}$ and

$$b(y) = \frac{\Gamma(\frac{3-d}{2})}{2y(4\pi)^{\frac{d-1}{2}}} \sum_{n=-\infty}^{+\infty} \left[\left(\frac{2}{3} + (\frac{2\pi n}{y})^2 \right)^{\frac{d-3}{2}} - \left(1 + (\frac{2\pi n}{y})^2 \right)^{\frac{d-3}{2}} + \frac{d-3}{2} \left(1 + (\frac{2\pi n}{y})^2 \right)^{\frac{d-5}{2}} \right] \quad (86)$$

Substituting back into (85) gives

$$\begin{aligned} \mathcal{G} = \ & 1 + x - \tfrac{3\bar{\lambda}(y)}{2y}(4\pi)^{\frac{1-d}{2}}\Gamma(\tfrac{3-d}{2}) \sum_{n=-\infty}^{+\infty} \left[(1+x)\left(1 + (\frac{2\pi n}{y})^2\right)^{\frac{d-3}{2}} \right. \\ & \left. -x\left(\tfrac{2}{3} + (\tfrac{2\pi n}{y})^2\right)^{\frac{d-3}{2}} - \left(1 + \tfrac{x}{3} + (\tfrac{2\pi n}{y})^2\right)^{\frac{d-3}{2}} \right] \end{aligned} \quad (87)$$

We will now write things in terms of the coupling $h(y)$ which to one loop is given by $h = a_1\bar{\lambda}$. Thus

$$\begin{aligned} H = \ & e^{\int_1^M \left(\frac{d_{eff}+2}{d_{eff}-2}\right)\frac{dM'}{M'}} \left(1 + x - \frac{\frac{2h(y)}{(5-d)(3-d)}}{\sum_{-\infty}^{\infty}\left(1 + (\frac{2\pi n}{y})^2\right)^{\frac{d-7}{2}}} \sum_{-\infty}^{\infty} \left[(1+x)\left(1 + (\frac{2\pi n}{y})^2\right)^{\frac{d-3}{2}} \right.\right. \\ & \left.\left. -x\left(\tfrac{2}{3} + (\tfrac{2\pi n}{y})^2\right)^{\frac{d-3}{2}} - \left(1 + \tfrac{x}{3} + (\tfrac{2\pi n}{y})^2\right)^{\frac{d-3}{2}} \right] \right) \end{aligned} \quad (88)$$

where we have dropped the $'$ on H for convenience. An essentially equivalent expression is obtained in expanding about the floating fixed point, the quantities in (89) being replaced by their floating fixed point values to this order. Equation (89) is the universal one loop equation of state in terms of the two non-linear scaling fields x and y.

For $d = 4$ in terms of the floating fixed point we have

$$\begin{aligned} \mathcal{G}(x,y) = 1 + x + & \frac{\sum_{-\infty}^{\infty}\left(1 - 2(\frac{2\pi n}{y})^2\right)\left(1 + (\frac{2\pi n}{y})^2\right)^{-\frac{5}{2}}}{\left(\sum_{-\infty}^{\infty}\left(1 + (\frac{2\pi n}{y})^2\right)^{-\frac{3}{2}}\right)^2}) \\ & \sum_{-\infty}^{\infty} \left[(1+x')\left(1 + (\frac{2\pi n}{y})^2\right)^{\frac{1}{2}} - x\left(\frac{2}{3} + (\frac{2\pi n}{y})^2\right)^{\frac{1}{2}} - \left(1 + \frac{x}{3} + (\frac{2\pi n}{y})^2\right)^{\frac{1}{2}} \right] \end{aligned} \quad (89)$$

For $d = 3$ care should be taken in taking the limit, one finds again in terms of the floating fixed point

$$\mathcal{G}(x,y) = 1 + x + \frac{\sum_{-\infty}^{\infty} \left(1 - (\frac{2\pi n}{y})^2\right)\left(1 + (\frac{2\pi n}{y})^2\right)^{-3}}{\left(\sum_{-\infty}^{\infty} \left(1 + (\frac{2\pi n}{y})^2\right)^2\right)^2}$$

$$\sum_{-\infty}^{\infty} \left[\ln \left(\frac{(1 + (\frac{2\pi n}{y})^2)}{(1 + \frac{x}{3} + (\frac{2\pi n}{y})^2)} \right) + x \ln \left(\frac{(1 + (\frac{2\pi n}{y})^2)}{(\frac{2}{3} + (\frac{2\pi n}{y})^2)} \right) \right] \quad (90)$$

Now let us make contact with known results. There is one basic observation that assists in this. It is to note that at each stage of the computation taking the limit $\rho\kappa L \to \infty$ is equivalent to performing the integral over one additional momentum and therefore, in this limit, expressions involving d get mapped into the same expression but with d replaced by $d+1$ and n replaced by $n = 0$. The opposite limit $\rho\kappa L \to 0$ is achieved by merely retaining the $n = 0$ expressions. Therefore the as

ymptotic limit $y \to \infty$ can be read off from (89) by implementing the above prescription. The expression in the $y \to 0$ limit is

$$\mathcal{G}(x,0) = 1 + x + \frac{2}{(d-3)} \left[1 + x - x(\frac{2}{3})^{\frac{d-3}{2}} - (1 + \frac{x}{3})^{\frac{d-3}{2}} \right] \quad (91)$$

and in the limit $y \to \infty$ we have

$$\mathcal{G}(x,\infty) = 1 + x + \frac{2}{(d-2)} \left[1 + x - x(\frac{2}{3})^{\frac{d-2}{2}} - (1 + \frac{x}{3})^{\frac{d-2}{2}} \right] \quad (92)$$

where in the respective limits we used $\varepsilon(0) = 5 - d$ and $\varepsilon(\infty) = 4 - d$ Thus if we work in dimension $d = 4 - \varepsilon$ where ε is a fixed small constant we can write these expressions as

$$\mathcal{G}(x,0) = 1 + x + \frac{2}{(1-\varepsilon)} \left[1 + x - x(\frac{2}{3})^{\frac{1-\varepsilon}{2}} - (1 + \frac{x}{3})^{\frac{1-\varepsilon}{2}} \right] \quad (93)$$

and in the limit $y \to \infty$ we have

$$\mathcal{G}(x,\infty) = 1 + x - \frac{1}{(1-\frac{\varepsilon}{2})} \left[1 + x - x(\frac{2}{3})^{1-\frac{\varepsilon}{2}} - (1 + \frac{x}{3})^{1-\frac{\varepsilon}{2}} \right] \quad (94)$$

In an expansion in ε to first order we find

$$\mathcal{G}(x,0) = 1 + x + 2\left[1 + x - x(\tfrac{2}{3})^{\frac{1}{2}} - (1 + \tfrac{x}{3})^{\frac{1}{2}}\right] \quad (95)$$

$$-2\varepsilon \left[(1+x) + x(\tfrac{2}{3})^{\frac{1}{2}}(1 + \tfrac{1}{2}\ln(\tfrac{2}{3})) + (1 + \tfrac{x}{3})^{\frac{1}{2}}(1 + \tfrac{1}{2}\ln(1 + \tfrac{x}{3})\right] \quad (96)$$

and in the limit $y \to \infty$ we have

$$\mathcal{G}(x,\infty) = 1 + x + \frac{\varepsilon}{6}\left[(3+x)\ln(3+x) - 3(1+x)\ln 3) + 2x\ln 2\right] \tag{97}$$

It is important to note that these expressions are obtained from an expansion in the same ε. It is also important to note that ε is being considered as a small deviation from the floating fixed point. One cannot of course make use of a direct ε expansion around mean field theory. It is the global nature of our renormalization group that allows us to do this. Without both fixed points having been retained in the one scheme a direct comparison of the $4-\varepsilon$ theory with the dimensionally reduced $3-\varepsilon$ theory would be impossible. These results apply to a $4-\varepsilon$ dimensional layer geometry geometry with periodic boundary conditions on the layers, and ε is assumed small. Thus the one loop universal equation of state above has asymptotic limits which agree in an ε type expansion with known results [10]. It is important to emphasize though that the expression (89) is valid throughout the entire crossover.

7 Effective Exponents to One Loop

In this section we will derive expressions for δ_{eff} and β_{eff} to one loop. From (54) noting that $\gamma_\phi = 0$ to one loop we have

$$\frac{d\ln H}{d\ln M} = \frac{1}{2}\left(d + 2 - \frac{\beta(\bar{\lambda})}{\bar{\lambda}}\right)\frac{d\ln\rho\kappa}{d\ln M} \tag{98}$$

With (55) one finds

$$\frac{d\ln\rho\kappa}{d\ln M} = \frac{2}{\left(d - 2 - \frac{\beta(\bar{\lambda})}{\bar{\lambda}}\right)} \tag{99}$$

And we have to one loop

$$\delta_{eff} = \frac{d\ln H}{d\ln M} = \left(\frac{d_{eff}(\rho) + 2}{d_{eff}(\rho) - 2}\right) \tag{100}$$

Working in terms of the floating fixed point and absorbing correction to scaling factors into redefinitions of H and M, δ_{eff} becomes

$$\delta^*_{eff} = \left(\frac{d^*_{eff}(\rho) + 2}{d^*_{eff}(\rho) - 2}\right) = \left(\frac{6 - \varepsilon(L\kappa\rho)}{2 - \varepsilon(L\kappa\rho)}\right) = 3 + \varepsilon(L\kappa\rho) \tag{101}$$

where ρ is the solution of (32) and we have expanded the denominator in $\varepsilon(L\kappa\rho)$. This is necessary as we are implementing perturbation theory in terms of the floating fixed point. At the floating fixed point (32) gives

$$(\rho\kappa)^{d-2} = \frac{\varepsilon(L\kappa\rho)}{a_1(L\kappa\rho)}M^2(\rho) \tag{102}$$

Using (15) and (16) fro $a_1(\rho\kappa L)$ and $\varepsilon(\rho\kappa L)$

$$(\rho\kappa)^{d-3} = \frac{LM^2(\rho)}{3} \frac{\left[5-d-(7-d)\frac{\sum_{n=-\infty}^{\infty} \frac{4\pi^2 n^2}{L^2\kappa^2}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2}\right)^{\frac{d-9}{2}}}{\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2}\right)^{\frac{d-7}{2}}} \right]}{\left[\Gamma(\frac{7-d}{2})(4\pi)^{\frac{1-d}{2}} \sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{L^2\kappa^2}\right)^{\frac{d-7}{2}}\right]} \tag{103}$$

This transcendental equation must be solved for ρ and the solution substituted into

$$\delta_{eff} = 8-d-(7-d)\frac{\sum_{n=-\infty}^{\infty} \frac{4\pi^2 n^2}{\rho^2 L^2\kappa^2}\left(1+\frac{4\pi^2n^2}{\rho^2 L^2\kappa^2}\right)^{\frac{d-9}{2}}}{\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{\rho^2 L^2\kappa^2}\right)^{\frac{d-7}{2}}} \tag{104}$$

For $d=4$

$$\delta_{eff} = 4-3\frac{\sum_{n=-\infty}^{\infty} \frac{4\pi^2 n^2}{\rho^2 L^2\kappa^2}\left(1+\frac{4\pi^2n^2}{\rho^2 L^2\kappa^2}\right)^{\frac{-5}{2}}}{\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{\rho^2 L^2\kappa^2}\right)^{\frac{-3}{2}}} \tag{105}$$

For $d=3$

$$\delta_{eff} = 4+\frac{L\kappa\rho}{\sinh L\kappa\rho} \tag{106}$$

Going back to (102) $\varepsilon(L\kappa\rho)$ is just a function which varies between $5-d$ and $4-d$ and therefore can be treated as a correction to scaling and absorbed into a redefinition of M as indeed can the other purely numerical factors. Thus

$$(\rho\kappa)^{d-3}\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{\rho^2 L^2\kappa^2}\right)^{\frac{d-7}{2}} = LM^2(\rho) \tag{107}$$

Numerical solutions of (107) will be presented a later paper. For the moment denoting the solution as $L\kappa\rho = g(LM^{\frac{2}{(d-2)}})$ gives for $d=4$

$$\delta_{eff} = 4-3\frac{\sum_{n=-\infty}^{\infty} \frac{4\pi^2 n^2}{g^2}\left(1+\frac{4\pi^2n^2}{g^2}\right)^{\frac{-5}{2}}}{\sum_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{g^2}\right)^{\frac{-3}{2}}} \tag{108}$$

As $LM^{\frac{2}{(d-2)}} \to 0$, $\delta_{eff} \to 4$ and as $LM^{\frac{2}{(d-2)}} \to \infty$, $\delta_{eff} \to 3$.

Turning now to β_{eff}, given by (65) which to one loop is

$$\frac{d\ln M}{d\ln|t|} = \frac{1}{2}\left(d - 2 - \frac{\beta(\bar{\lambda})}{\bar{\lambda}}\right)\frac{d\ln\rho\kappa}{d\ln|t|} \tag{109}$$

where the normalization condition (32) has been used. With the condition (31), $\frac{d\ln\rho\kappa}{d\ln|t|} = \nu_{eff}$, hence to one loop

$$\beta_{eff} = \frac{d\ln M}{d\ln|t|} = \frac{\nu_{eff}}{2}(d_{eff} - 2) \tag{110}$$

Once again working in terms of the floating fixed point and absorbing corrections to scaling into redefinitions of t and M (110) becomes

$$\beta^*_{eff} = \frac{1}{4}\left(1 + \frac{\varepsilon(L\kappa\rho)}{6}\right)(2 - \varepsilon(L\kappa\rho)) \tag{111}$$

where the result for ν^*_{eff} to $O(\varepsilon(L\kappa\rho))$ has been taken from ([1]). As $\varepsilon(L\kappa\rho)$ is our "small" parameter for generating perturbation theory we neglect the $O(\varepsilon^2(L\kappa\rho))$ term to find

$$\beta_{eff} = \frac{1}{2} - \frac{\varepsilon(L\kappa\rho)}{6} \tag{112}$$

To find ρ we need to solve (31). To lowest order it gives $\rho\kappa = t^{\frac{1}{2}}$ thus

$$\beta^*_{eff} = \frac{d-2}{6} + (7-d)\frac{\sum\limits_{n=-\infty}^{\infty}\frac{4\pi^2n^2}{L^2t}\left(1+\frac{4\pi^2n^2}{L^2t}\right)^{\frac{d-9}{2}}}{\sum\limits_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{L^2t}\right)^{\frac{d-7}{2}}} \tag{113}$$

For $d = 4$

$$\beta^*_{eff} = \frac{1}{3} + \frac{1}{2}\frac{\sum\limits_{n=-\infty}^{\infty}\frac{4\pi^2n^2}{L^2t}\left(1+\frac{4\pi^2n^2}{L^2t}\right)^{\frac{-5}{2}}}{\sum\limits_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{L^2t}\right)^{\frac{-3}{2}}} \tag{114}$$

As $L^2|t| \to 0$, $\beta^*_{eff} \to \frac{1}{3}$ and as $L^2|t| \to \infty$, $\beta^*_{eff} \to \frac{1}{2}$. For $d = 3$

$$\beta^*_{eff} = \frac{1}{6} + \frac{2}{3}\frac{\sum\limits_{n=-\infty}^{\infty}\frac{4\pi^2n^2}{L^2t}\left(1+\frac{4\pi^2n^2}{L^2t}\right)^{-3}}{\sum\limits_{n=-\infty}^{\infty}\left(1+\frac{4\pi^2n^2}{L^2t}\right)^{-2}} = \frac{1}{3} - \frac{1}{6}\frac{L|t|^{\frac{1}{2}}}{\sinh L|t|^{\frac{1}{2}}} \tag{115}$$

Obviously working with β^*_{eff} is much simpler than δ^*_{eff} because condition (31) is a much more amenable to a perturbative solution than (32).

One might enquire as to why the usual condition $M(\rho) = (\rho\kappa)^{\frac{d}{2}-1}$ was not used, it is after all the condition used in the non-crossover problem. The reason why it cannot be used is that it leads to an ill-defined perturbation theory in the limit $LM^{\frac{2}{(d-2)}} \to 0$. Terms which diverge in this limit appear in the equation of state. The reason for this is simple but subtle. In order to make perturbation theory work we wish to work away from a regime where the correlation length is large i.e. the effective mass is small. For $t = 0$ the effective mass is $\sim \frac{\lambda}{2}M^2$. Setting a condition on M does not keep away from the critical region if λ can become very small. This cannot happen in the non-crossover case but does happen here. In the limit $LM^{\frac{2}{(d-2)}} \to 0$ the running coupling $\bar{\lambda}(LM^{\frac{2}{(d-2)}}) \to 0$ and so the critical region where perturbation theory breaks down is entered.

8 Conclusion

Previously [1] we had set out a formulation of how to perturbatively treat the crossover above T_c for a finite size system, wherein the finite system itself could exhibit critical behaviour. The present paper is a natural extension of this formulation to below T_c. The canonical problem to a large extent from the crossover point of view is the same either above or below T_c in the sense that one would like a RG that "coarse grains" the effective degrees of freedom in an L dependent way as one knows that the physics, i.e. how the system looks at different scales, is very L dependent. The natural consequence of an L dependent RG is seen to be L dependent anomalous dimensions and the appearance of ξ_L as the most natural scaling field in the problem as opposed to the bulk correlation length. We identified three such scaling fields that were capable of spanning the crossover between d and $d-1$ dimensional fixed points representing second order phase transitions: ξ_{Lt}, ξ_{LM} and ξ_{LMt}. The first two represent physically the correlation length in finite size systems above $T_c(L)$ in zero magnetic field and at $T = T_c(L)$ respectively. ξ_{LMt} is the true correlation length in the real physical system. For the crossover in question however, all three are equally good non-linear scaling fields. The L dependent RG shows how correlation functions and particularly the equation of state can be written in a natural scaling form in terms of these scaling fields.

We defined natural analogs of the critical exponents δ and β for the crossover and showed that these exponents satisfy scaling laws which are the analogs of the standard relations for the non-crossover case. These were the natural extension of the scaling law $\gamma_{eff} = \nu_{eff}(2-\eta_{eff})$ derived in [8]. One subtlety was the appearance of an effective dimensionality d_{eff} in these relations. This object was seen to naturally appear as a representation of the fact that the scaling dimension of the operator ϕ^4 and hence the coupling constant λ changed across the crossover. In the non-crossover case γ_λ plays a rather minor role, for instance representing the slowly varying and non-singular corrections to scaling about the Wilson-Fisher fixed point. In the crossover case the change in degree of relevance of the ϕ^4 operator is very important and must be accounted for. d_{eff} does this in a very natural fashion. It also appears very naturally if one thinks of it in the context of universality. The universality class of systems is specified by space dimensionality and symmetry. Here we interpolate between two universality classes with different space dimensions, hence it is quite natural to have a generalized universality in the sense that only γ_{ϕ^2}, γ_ϕ and d_{eff} are required for a complete description. The effective exponents themselves are also universal quantities. Obviously only two of the effective exponents need be known, the rest follow automatically. The derivation of the scaling law $\alpha_{eff} = 2 - \nu_{eff} d_{eff}$ will be left for another publication where the crossover in the specific heat will be considered. The non-linear scaling fields for the crossover were shown to have a very natural representation in terms of the effective exponents and interpolated between just the ones one would expect in the asymptotic regimes.

Having determined a universal form for the equation of state we proceeded to determine it explicitly perturbatively. By implementing the effective exponent scaling laws one could determine the variable redefinitions necessary in order to make the equation universal. The equation of the crossover coexistence curve was determined. The equation of state was shown to reduce in it's asymptotic limits to known ϵ expansion results. One interesting technical point was the inadequacy of the condition $M(\rho) = (\rho\kappa)^{\frac{d}{2}-1}$ for determining a regime where perturbation theory could safetly be used. The reason for this was that the crossover behaviour of the coupling constant was sufficient to drive the system into a perturbatively ill-defined region in spite of the condition on $M(\rho)$.

There are several problems which are worth considering which stem directly from the considerations herein. First and foremost is the question of the discontinuity fixed point at $T = 0$ i.e. at the end of the coexistence curve. This fixed point cannot be seen in any of the expressions we derived here for basically the same reason that we mentioned earlier that precluded us from examining the crossover to mean field theory i.e. the parameter that induces the crossover has not been included in the renormalization prescription, therefore one's RG will be independent of it and hence the crossover will not be seen. The crossover to mean field theory could have been found by making the renormalization prescription explicitly t and momentum dependent. In the case of the strong coupling fixed point the natural thing to do is to implement a M dependent renormalization, hence one's anomalous dimensions etc would all be explicitly M dependent. We will return to this issue in a future publication. Related to this is the question of the behaviour below $T_c(L)$ of an $O(N)$ model i.e the non-linear σ model. Once again we will return to this issue in the future.

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