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Self-consistency and Supersymmetry in a Many Fermion System

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Abstract

We show that, in the context of a specific simple model whose dynamical algebra is a Lie superalgebra, the thermodynamic self-consistent fermionic diagonalization condition is equivalent to supersymmetry.

In a recent paper [1] it was shown how the standard dynamical Lie algebraic approach to the solution of a many body problem can be extended to that of a dynamical superalgebra in the case of a many fermion system.

Unlike the conventional Lie algebraic formalism, which depends on pairing to reduce the system to one consisting entirely of bosonic operators, the superalgebra formalism allows us to consider additionally interactions involving odd numbers of fermions.

If we consider the general hamiltonian H of an interacting fermion system,

$$H = \sum_i \epsilon_i a_i^\dagger a_i + \frac{1}{2} \sum_{i,j,l,k} < ij|V|kl > a_i^\dagger a_j^\dagger a_l a_k, \tag{1}$$

with

$$\begin{gathered} \{a_k, a_{k'}\} = 0; \quad \{a_k, a_{k'}^\dagger\} = \delta_{k,k'}; \\ k \equiv (\mathrm{k}, \Uparrow), \quad -k \equiv (-\mathrm{k}, \Downarrow), \end{gathered} \tag{2}$$

*Permanent Address: Dipartimento di Fisica del Politecnico, Torino, Italy then the standard Lie algebraic procedure is as follows [2]:

Li) The pairing process linearises H to the form

$$H^{red} = \sum \epsilon_i a_i^\dagger a_i + \sum (\text{pairs of a's}), \tag{3}$$

which is now an element of a Lie algebra L.

Lii) The spectrum is obtained by means of a generalized Bogolubov transformation which is an automorphism $\Phi : \mathcal{L} \longleftarrow \mathcal{L}$ such that

$$\Phi(H^{red}) = \alpha_1 h_1 + \ldots + \alpha_l h_l, \tag{4}$$

where the set $\{h_1, \ldots, h_l; r_1, \ldots, e_{n-l}\}$ is a Cartan basis for the n-dimensional rank-l Lie algebra L.

Liii) The Cartan elements $\{h_1, \ldots, h_l\}$ represent observables which are conserved in the high temperature phase, but no longer conserved in some low temperature phase.

Liv) The remaining basis elements $\{e_1, \ldots, e_{n-l}\}$ represent order operators whose expectations $< e_i >$ give the relevant order parameters.

Lv) *Coherent states* [3] are obtained by the action of a unitary operator U which implements the automorphism Φ; e.g. the coherent state given by $|\Omega >= U^{-1}|\omega >$ corresponds to the cyclic vector $|\omega >$ which is the vacuum for the diagonalized H^{red}.

Lvi) Finally we impose self-consistency by demanding that the coefficients in the reduced hamiltonian obtained by linearisation of the original hamiltonian are equal to the expectations of the relevant operators with respect to equilibrium states induced by the reduced hamiltonian.

We can implement the linearization procedure *Li*) as follows. We consider the identity

$$\begin{aligned} AB &= (A- < A >)(B- < B >) \\ &+ < A > B + A < B > - < A >< B >, \end{aligned} \tag{5}$$

where $< \bullet >$ is the expectation in some state. If the first term on the r.h.s. can be considered "small" in some sense, this linearizes to

$$AB \approx < A > B + A < B > - < A >< B > . \tag{6}$$

This approximation is well-defined when A and B commute; for example, in the mean field reduction of hamiltonian (1), where $A = a_i^\dagger a_{-i}^\dagger \text{and} B = a_{-j} a_j$. In this case A and B are bosonic and their expectations are complex numbers.

However we can also implement the linearisation procedure in the case when A and B anticommute. Then $AB = -BA$ requires that $\vartheta_A =< A >$

and $\vartheta_B =< B >$ be anticommuting numbers which anticommute as well with the operators A and B.

An algebraic treatment of the hamiltonian problem may be implemented using the following steps, analogous to $Li)\ldots Lvi)$:

Si) The linearization procedure reduces H to:

$$H^{red} = \Sigma_i \epsilon_i a_i^\dagger a_i + \Sigma_i b_i B_i + \Sigma_i f_i F_i \tag{7}$$

where the B_i are products of even numbers of fermion operators. The coefficients ϵ_i are real numbers, the b_i are complex c-numbers whereas the f_is are anticommuting numbers. H^{red} is now an element of a Lie superalgebra A. Thus A is an algebra over an extended field which contains anticommuting numbers in a natural way.

Sii) The spectrum may be obtained by means of an automorphism Ψ of A $\quad \Psi = \exp(iAd \quad Z)$ where Z belongs to A, such that

$$\exp(iAd \quad Z)(H^{red}) = \sum_{i=1}^{l} \alpha_i h_i \tag{8}$$

where the set $\{h_1, \ldots, h_l, e_1^{(B)}, \ldots, e_r^{(B)}, e_1^{(F)}, \ldots, e_s^{(F)}\}$ is a Cartan basis for the Lie superalgebra.

Siii) The Cartan elements $\{h_1, \ldots, h_l\}$ again represent conserved observables in the high temperature phase, which are not conserved in the low temperature phase. These are bosonic in nature.

Siv) The remaining basis elements $\left\{e_1^{(B)}, \ldots, e_r^{(B)}, e_1^{(F)}, \ldots, e_s^{(F)}\right\}$ represent order operators, bosonic and fermionic, whose expectations give the relevant order parameters.

Sv) Supercoherent states are obtained by the action of the unitary operator U which implements the automorphism Φ. The existence of a unitary implementation of the automorphism is a consequence of A being an algebra over a field containing anti-commuting numbers; this enables one to consider the superalgebraic adjoint action of Z as a Lie algebraic one. Such supercoherent states generalise the supercoherent states of refs [4] and [5] in the same way as the generalised coherent states of ref [3] generalise the ordinary Glauber coherent states.

Svi) Self-consistency is imposed as in Liv) above. For example the fermionic coefficients f_i in Equation 7, which by the lineraization procedure can be written as $f_i = \Sigma_j c_{ij} < F_j >$, will be determined by an equation of the form:

$$f_i = \Sigma c_{ij} tr\{exp[-\beta H^{red}(f_i)]F_j\}/tr\{exp[-\beta H^{red}]\} \tag{9}$$

Eq (9) is a self-consistency condition because the coefficients f_i are determined in terms of H_{red} which itself depends on the $f_i^{\prime}s$.

We now illustrate the preceding steps Li) to Lvi) and Si) to Svi) by introducing a generalisation of the BCS model which includes umklapp processes.

First, from the interaction part of the hamiltonian (1) we retain only the following terms

(1) Cooper-pairing terms (BCS): $\frac{1}{2}\Sigma_{i,j} < i - i|V|j - j > a_i^\dagger a_{-i}^\dagger a_{-j} a_j$.

(2) Umklapp terms (U): $\frac{1}{2}\Sigma'_{ij} < ij|V| - j - i > a_i^\dagger a_{-i} a_{-j}$. These terms are permitted in a crystal, where momentum need only be conserved modulo a wave vector of the reciprocal lattice L (the prime indicates this restriction on the summation).

Then using the linearization procedure for commuting operators, our reduced hamiltonian is now of the form $H^1 = \Sigma_i H_i^{(1)}$, where

$$H_k^{(1)} = \epsilon_k(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + (\triangle_k a_{-k}^\dagger a_{-k}^\dagger + \nu_k a_k^\dagger a_{-k} + h.c.). \tag{10}$$

$$\triangle_k = \frac{1}{2}\Sigma_j < k - k|V|j - j >< a_j a_{-j} >; \tag{11}$$

$$\nu_k = \frac{1}{2}\Sigma'_j < kj|V| - j - k >< a_j^\dagger a_{-j} > . \tag{12}$$

The dynamical Lie algebra for this BCS-U model is $\bigoplus_k(\mathrm{su}(2)\bigoplus \mathrm{su}(2))_k$.

The spectrum $(\sqrt{\epsilon_k^2 + |\triangle_k|^2} \pm |\nu_k|)$ and the coherent states are obtained by means of a generalized Bogolubov transformation, as outlined above.

We then extend the model by the inclusion of fermionic operators coming from the following umklapp terms:

$$(3) \qquad \frac{1}{2}\Sigma'_{i,k} < i - i|V|ki > a_i^\dagger a_{-i}^\dagger a_i a_k; (\mathrm{i} + \mathrm{k}) \in \mathrm{L},$$

$$(4) \qquad \frac{1}{2}\Sigma'_{i,k} < i - i|V|k - i > a_i^\dagger a_{-i} a_k; (\mathrm{i} - \mathrm{k}) \in \mathrm{L}.$$

Note that the inclusion of momentum non-conserving terms in such models is standard, as for example in charge density wave (CDW) models [6] where terms such as $a_k^\dagger a_{k+Q} (Q = 2k_F, k_F$ denoting the Fermi wave vector) occur. Just as in the CDW case this breaking of translational invariance gives rise to an order parameter below the relevant Peierls transition temperature, so in our case the mean-field description of the new umklapp terms gives rise to an order parameter, which, for processes (3) and (4) is described by an anticommuting number. In those latter processes, we employ the linearization procedure for anticommuting operators, so that for example

$$a_i^\dagger a_{-i}^\dagger a_i a_k \approx < a_i^\dagger a_{-i}^\dagger a_i > a_k + a_i^\dagger a_{-i}^\dagger a_i < a_k >$$

where the fermion averages $< \cdot >$ are anticommuting θ numbers.

The reduced Hamiltonian obtained in this way has the form $H^{(2)} = \Sigma_k H_k^{(2)}$ with

$$H_k^{(2)} = \Sigma_{i=1}^{6} b_i B_i + \Sigma_{j=0}^{8} f_j F_j \in su(2|2) \tag{13}$$

where we suppressed the k-dependence on the r.h.s. The operators $B_i, i = 1, \ldots, 6$ are the generators of the $(su(2) \bigoplus su(2))_k$ algebra introduced above:

$$\begin{aligned} J_+^{(k)} &= (J_-^{(k)})^\dagger = a_k^\dagger a_{-k}^\dagger, \overline{J}_3^{(k)} = \frac{1}{2}(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} - 1); \\ \overline{J}_+^{(k)} &= (\overline{J}_-^{(k)})^\dagger = a_k^\dagger a_{-k}, \\ \overline{J}_3^{(k)} &= \frac{1}{2}(a_k^\dagger a_k - a_{-k}^\dagger a_{-k}). \end{aligned} \tag{14}$$

while the $F_j, j = 1, \ldots, 8$ are the fermionic operators

$$\{a_k, a_{-k}, a_k^\dagger, a_{-k}^\dagger, n_k\ a_{-k}, n_{-k} a_k, a_{-k}^\dagger n_k, a_k^\dagger n_{-k}\}, \tag{15}$$

where $n_k \equiv a_k^\dagger a_k$. The set $B_1, \ldots, B_6; F_0, F_1, \ldots, F_8$ including$F_0 \equiv I$) forms a basis for the superalgebra $su(2|2)_k$. The coefficients b_i, f_i are elements of the extension ring $C[\theta_1 \theta_2, \ldots]$ generated by the *theta-* terms.
This model has been treated in ref [1], where the finite-temperature self-consistency equations (which are independent of θ were written down.

Within the context of the $su(2|2)$ superalgebra, it was shown in ref [1] that the hamiltonian $H^{(1)}$ is supersymmetric; that is we may define a charge $Q \in f(\bigoplus_k su(2|2)_k)$ (f denoting the fermionic sector) such that

$$H^{(1)} = \{Q, Q^\dagger\}, \quad Q^2 = 0, \quad [H^{(1)}, Q] = 0. \tag{16}$$

This is only possible when the coefficients in (10) satisfy the following condition

$$|\nu_k|^2 = |\triangle_k|^2 + \epsilon_k^2\ . \tag{17}$$

We may now treat $H^{(1)}$ (Eq 10) as an independently given hamiltonian by means of a self-consistent mean-field Fermi reduction, using the linearization scheme for anti-commuting operators on the interaction terms. This produces the following hamiltonian

$$\begin{aligned} H_k^F &= \epsilon_k(n_k + n_{-k}) + \{\triangle_k(< a_k^\dagger > a_{-k}^\dagger + a_k^\dagger < a_{-k}^\dagger >) \\ &+ \nu_k(< a_k^\dagger > a_{-k} + a_k^\dagger < a_{-k} >) + h.c.\} \end{aligned} \tag{18}$$

Define

$$\begin{aligned} \theta_-^{(0)}(k) &= -\overline{\triangle}_k < a_k > + \nu_k < a_k^\dagger >, \\ \theta_+^{(0)}(k) &= \overline{\triangle}_k < a_{-k} > + \overline{\nu}_k < a_{-k}^\dagger > \end{aligned} \tag{19}$$

so that

$$< a_k >= \frac{\triangle_k \theta_-^{(0)}(k) + \nu_k \overline{\theta}_-^{(0)}(k)}{|\nu_k|^2 - |\triangle_k|^2} \tag{20}$$

(Of course we have similar equations for $< a_{-k} >, < a^\dagger_{-k} >$ as well, and write, for generic anti-commuting variables $\theta_+(k)$ and $\theta_-(k)$,

$$\begin{aligned} & a(\theta_\pm(k)) \equiv \theta_\pm(k) a_{\pm k}; a^\dagger(\theta_\pm(k)) \\ & \equiv a^\dagger_{\pm k}\theta_\pm(k) = [a(\theta_\pm(k))]^\dagger \end{aligned} \tag{21}$$

With this notation the hamiltonian H_k^F becomes

$$H_k^F = \epsilon_k(n_k + n_{-k}) + a(\theta_-^{(0)}(k)) + a(\theta_+^{(0)}(k)) + h.c., \tag{22}$$

which is an element of a solvable SLA $A_k \subset su(2|2)_k$ To diagonalize H^F, according to $Sii)$ we consider the adjoint action exp(adiZ) of an element $Z\epsilon A,$ where $A = \bigoplus_k A_k, Z = \bigoplus_k Z_k$, and

$$Z_k = a(\theta_+(k)) + a(\theta_-(k)) + h.c. \tag{23}$$

The condition that $\exp(adi)(H^F) \equiv U(\theta) H^F U^{-1}(\theta)$ be free of non-diagonal terms is

$$\theta_\pm(k) = \frac{i}{\epsilon_k}\theta_\pm^{(0)}(k). \tag{24}$$

We now wish to set-up a self-consistant scheme as in Svi) above. We must therefore evaluate the thermodynamic averages of the fermion operators in the Gibbs ensemble determined by H^F. We readily find that, for example,

$$< a(\theta_+) >_\beta = \frac{tr\{e^{-\beta H^F} a(\theta_\pm)\}}{tr\{e^{-\beta H^F}\}} = i\overline{\theta}_\pm\theta_\pm \tag{25}$$

whereby

$$< a_{\pm k} >= -i\overline{\theta}_+(k). \tag{26}$$

It is worth noticing that we obtain the same average by evaluating the expectation of the operators $a(\theta_\pm)$ in the supercoherent state [7] $|\Omega >= U^{-1}(\theta)|\omega >$:

$$\begin{aligned} & < \Omega|a(\theta_\pm|\Omega >=< \omega|U(\theta)a(\theta_\pm) \\ & U^{-1}(\theta)|\omega >=< \omega|exp(iadZ)(\theta_\pm)|\omega > \end{aligned} \tag{27}$$

thus, using eq.(24), $< a_k >= -\overline{\theta}_+^{(0)}(k)\epsilon_k$. We thus obtain four linear equations homogeneous in $\theta_+^{(0)}(k)$, $\theta_-^{(0)}(k)$, $\overline{\theta}_+^{(0)}(k)$, $\overline{\theta}_-^{(0)}(k)$, leading to the determinantal condition

$$|\nu_k|^2 = |\triangle_k|^2 + \epsilon_k^2, \tag{28}$$

which is the same as eq. (17) for the hamiltonian $H^{(1)}$ to be supersymmetric.

The fact that the conditions for self-consistent fermionic diagonalization of H^F and the supersymmetricity of $H(1)$ coincide is somewhat puzzling. It is worth noting however that the common condition (17) is expressible solely in terms of the of the Casimir invariants of the Lie subalgebra $su(2) \bigoplus su(2)$ of $su(2|2)$.

This may indicate the possible generalization of the above result to more physically realistic models, such as those indicated in ref [7].

References

1. A Montorsi, M Rasetti and A I Solomon, Phys.Rev.Lett 59, 2243 (1987)

2. A I Solomon, J.Phys A: Math. Gen. 14, 2177 (1981)

3. M Rasetti, Inst. J. Theor. Phys. 13, 425 (1975)

4. J W F Valle, Inst. J. Theor. Phys. 18, 923 (1979)

5. C Aragone and F Zypman, J. Phys. A: Math. Gen. 19, 2267 (1986)

6. E Peierls, "Quantum Theory of Solids", Oxford University Press, 1977; ch.v.
   J Birman and A I Solomon, Phys. Lett. 88A, 413 (1982)

7. A Montorsi, M Rasetti and A I Solomon, Phys. Lett. submitted for publication