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Group Theoretical Approach To Squeezed States Using Generalized Bose Operators

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GROUP THEORETICAL APPROACH TO SQUEEZED STATES USING GENERALIZED BOSE OPERATORS.

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ABSTRACT

Generalized, k-boson Holstein-Primakoff realizations of SU(2) and SU(1,1) are introduced in terms of generalized bose operators. The corresponding group theoretical coherent states are studied with respect to their squeezing properties relative to k'-boson dynamical variables.

The group theoretical coherent state manifold [1] corresponds to the factor space G/H, where the group G has a representation acting on a vacuum state $|0\rangle$ whose stability subgroup is H. Any Lie algebra with a faithful nxn representation $\{X^i\}$,

$$[X^i,X^j] = \sum c^{ij}{}_k X^k \tag{1}$$

has an n-mode boson representation defined by

$$\hat{X}^i = \sum a_r^\dagger X^i{}_{rs} a_s \quad ; \tag{2}$$

that (2) is a representation of (1) follows from the boson commutation relations

$$[a_r, a_s^\dagger] = \delta_{rs} \qquad (r,s = 1,2,\cdots,n). \tag{3}$$

However, for such a representation the group theoretical coherent state manifold G/H, obtained by exponentiation of elements of the corresponding Lie algebra, is trivial; since in this case the stability subgroup H of the vacuum is G itself. This follows because, for each i,

$$\exp(\hat{X}^i)\,|0\rangle = \exp(\textstyle\sum a_r^\dagger X^i{}_{rs} a_s)\,|0\rangle = |0\rangle\,, \tag{4}$$

on using

$$a_r\,|0\rangle = 0. \tag{5}$$

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A way out of this impasse is to use a Holstein - Primakoff realization [2] of G, which does not have the simple bilinear form (2). For example, for SU(2) the standard one - boson H-P realization is

$$J_+ = (2\sigma + 1 - n)^{1/2}\; a^\dagger \quad (= J_-^\dagger)\ ,$$

$$J_3 = n - \sigma\ . \tag{6}$$

Clearly, the stability subgroup of the vacuum is now generated by $\{J_3, J_-\}$, strictly contained in su(2), and so a non-trivial coherent state manifold G/H results.

We may further refine this process by introducing the multi-boson operators [3],

$$A_{(k)}^\dagger = \{ [n/k](n-k)!/n! \}^{1/2} (a^\dagger)^k \tag{7}$$

which satisfy $[A_{(k)}, A_{(k)}^\dagger] = I$. These operators were originally introduced by Brandt and Greenberg [3]. They were investigated [4], generalized [5] and applied to the linearization of the quartic oscillator [6] as well as to the analysis of the time dependence of nonlinear optical phenomena such as second harmonic generation [5]. Generalized bose operators were implicit in Demkov's treatment of the dynamical algebra of the multidimensional anisotropic harmonic oscillator [7].

The canonical transformation

$$a^\dagger \dashrightarrow \phi_{(k)}(a^\dagger) = A_{(k)}^\dagger \tag{8}$$

forms an abelian semigroup. This follows from

$$\phi_{(k')}\ \phi_{(k)}(a^\dagger) = \phi_{(k')}(A_{(k)}^\dagger) = A_{(kk')}^\dagger = \phi_{(kk')}(a^\dagger)\,. \tag{9}$$

This semigroup can be extended to a group by postulating the existence of all the operators of the form $a_{(1/k)}$, for $k \in \mathbb{N}$. This group is isomorphic with $\mathbb{Q}^+$, the group of the positive rationals under multiplication.

Using the k-boson operators the H-P realization for SU(2) becomes

$$J_+^{(k)} = \left\{(2\sigma + 1 - [n/k])[n/k](n-k)!/n!\right\}^{1/2} (a^\dagger)^k = \left(J_-^{(k)}\right)^\dagger$$

$$J_3^{(k)} = [n/k] - \sigma. \tag{10}$$

[In (7) and (10) above, [x] means the greatest integer less than or equal to x; functions of the operator n are evaluated on the eigenstates of n as the corresponding functions of the eigenvalue]. The generalized harmonic oscillator (Weyl group), SU(2) and SU(1,1) coherent states are

$$|z;k\rangle = \exp(-|z|^2/2)\exp(zA_{(k)}^\dagger)|0\rangle \tag{11}$$

$$|z;k,\sigma\rangle = \exp[zJ_+^{(k)} - z^*J_-^{(k)}]\,|J_3=-\sigma; J=\sigma\rangle =$$

$$= (1+|\zeta|^2)^{-\sigma}\sum_{m=0}^{2\sigma}\zeta^m\sqrt{\binom{2\sigma}{m}}\,|km\rangle \qquad \text{where } \zeta = z\tan(|z|)/|z|$$

and

$$|z;k,\sigma\rangle = \exp[zJ_+^{(k)} - z^*J_-^{(k)}]\,|J_3=\sigma; J=\sigma\rangle =$$

$$= (1-|\zeta|^2)^{\sigma}\sum_{m=0}^{\infty}\zeta^m\sqrt{\binom{2\sigma+m-1}{m}}\,|km\rangle \qquad \text{where } \zeta = z\tanh(|z|)/|z|,$$

respectively. z will be referred to as the coherence parameter.

For conventional uncertainty calculations of the electromagnetic field components $x=(a^\dagger + a)/\sqrt{2}$ and $p=i(a^\dagger-a)/\sqrt{2}$, only terms $\langle a^\lambda\rangle$, for $\lambda=1$ or 2, arise; as a consequence only the values k=1 and k=2 are significant in (7) and (9). Introducing the generalized dynamical variables

$$X_{(k')}=(A_{(k')}^\dagger+A_{(k')})/\sqrt{2}$$

and (12)

$$P_{(k')}=i(A_{(k')}^\dagger-A_{(k')})/\sqrt{2}$$

we find that matrix elements of k' boson operators with respect to k boson states depend only on the fractional boson index r=k'/k. This is closely related to the semigroup property expressed by eq. (9). Matrix elements corresponding to SU(1,1) operators can be obtained from those of SU(2) operators by Minkowski's unitary trick, i. e., replacing σ by $-\sigma$. When only the first and second moments of the position operators are discussed non-trivial results are only obtained for integral and half integral values of r. For the SU(2) representations, for which the carrying space is finite dimensional, both the integral and the half-integral position (or momentum) uncertainties have finite minima for finite values of the coherence parameter. For SU(1,1) the uncertainties corresponding to half-integral r have finite minima, whereas those corresponding to integral r values can be made arbitrarily close to zero, for any σ, by an appropriate choice of the coherence parameter. For the generalized harmonic oscillator coherent states the position uncertainties have finite minima for half-integral r and are monotonically decreasing to zero upon increase of the coherence parameter to infinity.

We summarize some of the results obtained:

1. G = Weyl Group

   $r=1$

   conventional harmonic oscillator coherent states [8]; $\Delta x=\Delta p=1/2$. No squeezing [9].

   r=integer > 1

   Δx is a monotonically decreasing function of the coherence parameter, vanishing in the limit $z \to \infty$.

   r=half-integer

   Δx (or Δp) is squeezed to some finite optimal value [10,11,13]. The most squeezed state is obtained for $r=1/2$ [10].

2. G = SU(2)

   For any value of σ Δx has some finite minimal value at some finite value of the coherence parameter.

   r=integer

   The minimal Δx is squeezed to lower and lower values upon increase of σ, the $\sigma \to \infty$ limit being zero.

   r=half-integer

   The minimal Δx keeps reducing upon increase of σ, the $\sigma \to \infty$ limit being that obtained for the generalized harmonic oscillator coherent state.

3. G = SU(1,1)

   r=integer

   the optimal value of Δx, for any σ, is zero.

   r=half-integer

   the optimal value of Δx increases with σ, reaching the same limit as the SU(2) state for $\sigma \to \infty$ [11].

4. G = SU(n)

   for the conventional Holstein-Primakoff realizations [12] which correspond to a restricted class of irreducible representations, the optimal values of Δx coincide with those for SU(2) [14].

Recent work on the Holstein-Primakoff realizations corresponding to all the unitary irreducible representations of the symplectic groups [15 and references therein] suggests some further possibilities for the generation of squeezed states.

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In response to a question of Professor C. Quesne we point out that the k- boson operator $A_{(k)}^\dagger$ generates k sets of generalized coherent states

$$|z;k;i\rangle = \exp(-|z|^2/2)\exp(zA_{(k)}^\dagger)|i\rangle, \qquad i=0,1,2,\cdots,k-1.$$

In the above we have only investigated the set corresponding to i=0, denoted by $|z;k\rangle$.

The resolution of the identity can be written in the form

$$(1/\pi)\sum_{i=0}^{k-1}\int |z;k;i\rangle\langle z;k;i|\, d^2z = 1.$$

To clarify the properties of $A_{(k)}^\dagger$ we note further that

$$A_{(k)}^\dagger|nk+i\rangle = \sqrt{(n+1)}\,|(n+1)k+i\rangle, \qquad i=0,1,2,\cdots,k-1.$$