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UNITARY IRREDUCIBLE REPRESENTATIONS OF LIE SUPERGROUPS

D. Williams† and J.F. Cornwell

Department of Physics,

University of St.Andrews,

St.Andrews,

Fife, Scotland.

ABSTRACT

Each Lie supergroup can be considered as equivalent to a certain family of Lie groups. The unitary irreducible representations of Lie supergroups are examined using this equivalence, with particular reference to the super Poincaré group.

† Present address: Dublin Institute for Advanced Study, 10 Burlington Road, Dublin 4, Ireland.

I. INTRODUCTION

The notions of supermanifold and supergroup have been refined over the years, the most useful definitions (in our opinion) being in the work of Rogers[1,2]. One important feature of this development is that a supermanifold M of even dimension m and odd dimension n over a Grassman algebra $E_L = E_{L0} + E_{L1}$ with L generators is topologically isomorphic to a real manifold $\mathcal{M}_L$ of dimension $(m+n)2^{L-1}$.

Many authors make use of this definition to build further structure onto supermanifolds[3] and to formulate physical theories in superspace[4]. All of these authors make the implicit assumption that any theory formulated on superspace corresponds to some theory formulated on space-time. However there are certain aspects of the relationship between these formulations which require closer investigation, and these will form the subject of a series of three papers, of which this is the first.

In an earlier paper[5] we were able to construct a theory of integration for Lie supergroups by using the topological isomorphism mentioned above and first investigated by Rogers.[1] The integral developed there was a Haar integral for the Lie group $\mathcal{G}_L$ equivalent to the Lie supergroup G which is unique up to a multiplicative constant. In this paper we examine the unitary irreducible representations of the equivalent Lie groups $\mathcal{G}_L$ of Lie supergroups, for which the existence of a Haar integral is an essential feature.

The method we are going to use is the theory of induced representations. This concept was discovered by Frobenius[6] over eighty years ago in his study of finite groups and was used by Wigner[7] to construct the unitary irreducible representations of the Poincaré group. Inducing was revealed as an indispensible tool for constructing representations of non-compact groups by the work of Mackey[8]. It has even been suggested that it is the only method that has been used to systematically obtain non-trivial representations of such groups[9]. We are able to use this theory because the equivalent Lie group can be expressed as a sequence of semidirect products each of which contribute representations.

The representations we construct act on a complex Hilbert space and have no Grassman analytic structure. In addition the representations given for the super Poincaré group contain only particles with a single spin value. These facts might, at first sight, suggest that they have nothing to do with the representations of the super Poincaré algebra, as described originally by Salam and Strathdee[10], but certain of the representations constructed here prove to be the required building blocks for the superfield representations of supersymmetry theories. This is covered in the last paper of this series[11].

The plan of this paper is as follows: in section II we discuss some necessary preliminaries. In section III we consider non-abelian Lie supergroups. Section IV is a review of super Poincaré groups. Section V is a preliminary discussion of the unitary irreducible representations of the four dimensional super Poincaré group. In section VI we consider the representations of the four dimensional super Poincaré group induced from a representation with $L=0$. In section VII we repeat this for $L=1$, and in section VIII for $L=2$. Section IX gives our concluding remarks. Our conventions for the Dirac Matrices are given in an Appendix. For all other undefined terminology and conventions we refer to our previous paper (ref.5).

II. PRELIMINARIES

First we need to define what we mean by a unitary representation of a group $\mathcal{G}$. We suppose that we have a Hilbert space $\mathcal{H}$ with scalar product $x.y$, $x,y\epsilon\mathcal{H}$, and that the action of the group elements on this Hilbert space is by the operators $U(g)$, $g\epsilon\mathcal{G}$, such that

$$(U(g)y).(U(g)x) = y.x \tag{1}$$

for each $g\epsilon\mathcal{G}$ and $x,y\epsilon\mathcal{H}$.

Now consider an arbitrary group $\mathcal{G}$, and the set of complex-valued square-integrable functions on G, $\mathcal{L}^2(\mathcal{G}, \mathbb{C})$. This set does not form a Hilbert space, but if we define the set of functions

$$T = \{f\epsilon\mathcal{L}^2(\mathcal{G}, \mathbb{C}), \int ff^* d\mu(\mathcal{G}) = 0\} \quad ,$$

then the set $\mathcal{H}(\mathcal{G}) = \mathcal{L}^2(\mathcal{G}, \mathbb{C})/T$ is a Hilbert space with inner product defined in terms of the left-invariant Haar integral by

$$f.h = \int f^* h \; d\mu(G) \quad .$$

For each $g\epsilon\mathcal{G}$ a unitary operator $U(g)$ acting on $\mathcal{H}(\mathcal{G})$ may be defined by

$$U(g)f(g') = f(g^{-1}g')$$

for all $g'\epsilon\mathcal{G}$. This, which is known as the 'left-regular' representation of $\mathcal{G}$, gives the archetype of a unitary representation of $\mathcal{G}$. It will be seen that the Haar integral on G plays an essential role.

Consider now a locally compact abelian group $\mathcal{G}$, which will then be isomorphic to $\mathbb{R}^n$ for some n, and consider the set of functions $\mathcal{H}(\mathcal{G})$ as defined above. Since all of the irreducible representations of an abelian group are one dimensional, we would expect $\mathcal{H}(\mathcal{G})$ to decompose into a sum of one dimensional subspaces. This does in fact happen and a convenient basis for these one dimensional subspaces is given by the characters of $\mathcal{G}$ defined as follows.

DEFINITION II.1

(a) A character of an arbitrary locally compact abelian group is a continuous function

$$\chi : \mathcal{G} \dashrightarrow \mathbb{C}$$

such that

$$|\chi(g)| = 1$$

and

$$\chi(g)\chi(g') = \chi(gg') \quad \text{for all } g, g' \epsilon \mathcal{G} \quad .$$

(b) For $x = (x^1, x^2, \ldots, x^n) \epsilon \mathcal{G}$ and each $p \epsilon \mathbb{R}$ the function

$$\chi_p = exp(ip.x)$$

is a character of $\mathcal{G}$. The set $\{p\} = \mathbb{R}^n$ is called the dual group of $\mathcal{G}$ and is denoted $\hat{\mathcal{G}}$.

Each χ_p is not an element of $\mathcal{L}^2(\mathcal{G}, \mathbb{C})$ since

$$\int \chi_p \chi_p^* dx = \int e^{ip.x} e^{-ip.x} dx = \infty \quad .$$

But we can write any $f \epsilon \mathcal{L}^2(G, \mathbb{C})$ in the form

$$f(x) = (2\pi)^{-n/2} \int g(p) e^{ip.x} dp$$

with $$g(p) = (2\pi)^{-n/2} \int f(x) e^{-ip.x} dx \quad .$$

That is, each element of the regular representation can be expressed as an integral that can be thought of as being a weighted sum of the irreducible representations. This is called a direct integral decomposition. For a compact group $p \ \epsilon \ \{0, \pm½, \pm1, \ . \ . \ . \ \}$ and $0 \leq x^\mu \leq 2\pi$ so that $\int \chi_p \chi_p^* dx = 2\pi$ and we do have a decomposition into a direct sum of irreducible representations.

This decomposition into irreducible representations given by the characters is not unique, but whatever changes we make the representation remains equivalent to the one we have constructed. This typifies a

Type I representation. There are other types of representation called Type II and Type III which do not possess this property, but fortunately they do not seem to occur in particle physics. For an explanation of the various types of representation see Coleman[9].

Now $G = E_L^{m,n}$ is a locally compact Lie supergroup and is topologically isomorphic to $\mathbb{R}^{\mathcal{N}(m+n)}$, where $\mathcal{N} = 2^{L-1}$, so that the above arguments apply here. If we recall the definition of inner product as given in the appendix of ref.5 we can see that the characters of this supergroup G *can be labeled by* $(p,\phi)\epsilon E_L^{m,n}$ and written

$$\chi_{(p,\phi)} = exp\ i(p,\phi).(x,\theta) \quad \text{for} \quad (x,\theta)\epsilon G \quad .$$

These characters then serve as a basis for the one dimensional (complex not Grassman) representations of the supergroup G and any complex valued function f on superspace can be written

$$f(x,\theta) = \int d^m p\ d^n\phi\ \{g(p,\phi)e^{ip.x}e^{i\phi.\theta}\} \tag{2}$$

for some function $g: E_L^{m,n} \dashrightarrow \mathbb{C}$ and the integral as defined in ref.5.

Of course a function taking values in $\mathbb{C}E_L$ must be written as a sum of these integrals. Thus if $f': E_L^{m,n} \dashrightarrow \mathbb{C}E_L$ with component functions $f'_{\underline{e}_i}: E_L^{m,n} \dashrightarrow \mathbb{C}$ and $f'_{\underline{f}_j}: E_L^{m,n} \dashrightarrow \mathbb{C}$ then

$$f'(x,\theta) = \underline{e}_i f'_{\underline{e}_i}(x,\theta) + \underline{f}_j f'_{\underline{f}_j}(x,\theta) \quad , \tag{3}$$

with $f'(x,\theta)$

$$= \underline{e}_i \int d^m p\ d^n\phi \{g_{\underline{e}_i}(p,\phi)e^{ip.x}e^{i\phi.\theta}\} + \underline{f}_i \int d^m p\ d^n\phi \{g_{\underline{f}_i}(p,\phi)e^{ip.x}e^{i\phi.\theta}\} \ ,$$

for some choice of the functions $g_{\underline{e}_i}$ and $g_{\underline{f}_j}$. For a given function we determine the inverse functions $g_{\underline{e}_i}$ and $g_{\underline{f}_j}$ by the inverse Fourier transforms

$$g_{\underline{e}_i}(p,\phi) = (2\pi)^{N(m+n)/2}\int d^m x\ d^n\theta\ \{f'_{\underline{e}_i}(x,\theta)e^{-ip.x}e^{-i\phi.\theta}\} \qquad (4)$$

$$g_{\underline{f}_j}(p,\phi) = (2\pi)^{N(m+n)/2}\int d^m x\ d^n\theta\ \{f'_{\underline{f}_j}(x,\theta)e^{-ip.x}e^{-i\phi.\theta}\} \qquad (5)$$

Here $\underline{e}_i$ are a basis of E_{L0} *with* $i=0,1,2,\ldots,N-1$ and $\underline{f}_j$ are a basis of E_{L1} with $j=1,2,\ldots,$ as defined in ref.5 and the component functions are the projections in the corresponding directions. Also hereafter summation over repeated indices is implied.

Equations (2), (3), (4) and (5) give us the foundations of the theory of Fourier analysis on superspace, which we return to in the last paper in this series.

III. NON ABELIAN LIE SUPERGROUPS

Consider any $(m|n)$ dimensional Lie supergroup G with Lie superalgebra generators $\{\alpha_\mu, \beta_\sigma\}$. The equivalent Lie group $\mathcal{G}_A$ then has generators $\{\underline{e}_i\alpha_\mu, \underline{f}_j\beta_\sigma\}, i=0,1,\ldots,2^{A-1}-1;\ j=1,2,\ldots,2^{A-1}$ and the equivalent Lie group $\mathcal{G}_{A+1}$ has generators $\{\underline{e}_i\alpha_\mu,\ \underline{\varepsilon}_{A+1}\underline{f}_j\alpha_\mu,\ \underline{f}_j\beta_\sigma,\ \underline{\varepsilon}_{A+1}\underline{e}_i\beta_\sigma\}$. The additional generators in the step $A \to A+1$ ie. $\{\underline{\varepsilon}_{A+1}\underline{f}_j\alpha_\mu,\ \underline{\varepsilon}_{A+1}\underline{e}_i\beta_\sigma\}$ span an abelian invariant Lie subalgebra of $\mathcal{G}_{A+1}$ which we will denote by $\mathcal{G}'_{A+1}$. We then have the semi-direct product structure

$$\mathcal{G}_{A+1} = \mathcal{G}'_{A+1} \circledS \mathcal{G}_A \qquad (6)$$

and we can construct $\mathcal{G}_L$ for any L as a sequence of semidirect products as follows

$$\mathcal{G}_L = \mathcal{G}'_L \circledS (\mathcal{G}'_{L-1} \circledS (\mathcal{G}'_{L-2} \circledS (\ .\ .\ .\ (\mathcal{G}'_1 \circledS \mathcal{G}_0)))) \qquad (7)$$

with $\mathcal{G}_0$ the Lie group with generators $\{\underline{\varepsilon}_0\alpha_\mu\}$, so that it is the group corresponding to the even subalgebra of the supergroup G.

Now consider any Lie group $\mathcal{G}$ that admits the semidirect product structure $\mathcal{G} = \mathcal{N} \circledS \mathcal{H}$ with $\mathcal{N}$ an abelian invariant subgroup of $\mathcal{G}$. Let $a_h(n)$ denote the automorphism of $\mathcal{N}$ by $\mathcal{H}$ given by

$$a_h(n) = hnh^{-1} \tag{8}$$

for each $n\epsilon\mathcal{N}$ and fixed $h\epsilon\mathcal{H}$. For each $y\epsilon\mathcal{N}$ we define the transform of the character χ_y by h by

$$\chi_{h(y)}(n) = \chi_y(hnh^{-1}) = \chi_{y'}(n) \quad , \tag{9}$$

and define the orbit of χ to be the set of distinct elements *$h(y)$ for all $h\epsilon\mathcal{H}$.*

The group

$$\mathcal{H}(\chi_y) = \{h\epsilon\mathcal{H},\ \chi_{h(y)} = \chi_y\ \} \tag{10}$$

is called the stability group of χ_y. With these definitions we can state the main two theorems on induced representations that we will be using. These are as given by Mackey[8].

THEOREM III.1

For each χ, $y\epsilon\mathcal{N}$, choose an irreducible representation $\Delta_{\mathcal{H}(\chi_y)}$ of $\mathcal{H}(\chi_y)$ and consider the subgroup $S_y = \mathcal{N} \circledS \mathcal{H}(\chi_y)$ consisting of all $hn \epsilon \mathcal{N} \circledS \mathcal{H}$ with $h \epsilon \mathcal{H}(\chi_y)$. Then $\Delta_{S_y} = \Delta_{\mathcal{H}(\chi_y)}\chi(n)$ is an irreducible representation of S_y and the *induced representation* $(\Delta_{S_y})\uparrow\mathcal{G}$ is an irreducible representation of $\mathcal{G}$. If χ_y and $\chi_{y'}$ lie in the same orbit then every $(\Delta_{\mathcal{H}(\chi_y)}\chi_y(n))\uparrow\mathcal{G}$ is equivalent to some $(\Delta_{\mathcal{H}(\chi_{y'})}\chi_{y'}(n))\uparrow\mathcal{G}$. *Thus it is* sufficient to choose just one χ_y from each orbit.

Let C be the set of characters which includes just one member of each orbit. Then as χ_y varies over C and Δ varies over the set of irreducible representations of $\mathcal{H}(\chi_y)$ the irreducible representations $\Delta_{S_y}\uparrow\mathcal{G}$ are mutually inequivalent.

THEOREM III.2

Suppose it is possible to choose the set C of N such that it is a Borel set. Then every irreducible representation of $\mathcal{G}$ is equivalent to some $\Delta_{S_y}\uparrow\mathcal{G}$.

Of course we can replace 'irreducible representation' by 'unitary irreducible representation' in the statement of these theorems. Now consider equation (7) in the light of these theorems. We have the following results:

THEOREM III.3

Let G be a linear Lie supergroup (ie. a Lie supergroup admitting a faithful supermatrix representation). Then, given a value for L, we can proceed through the sequence $\mathcal{G}_0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow \ldots \rightarrow \mathcal{G}_L$ and obtain all the unitary irreducible representations of $\mathcal{G}_L$ provided we can ascertain the representations of $\mathcal{G}_0$ and its subgroups.

PROOF

Suppose we have all the unitary irreducible representations of $\mathcal{G}_0$ and its subgroups. The stability group for each character of $\mathcal{G}_1'$ is then one of these subgroups and using Theorem III.1 we can obtain all the representations of $\mathcal{G}_1$. Now consider $\mathcal{G}_2$, the stability group of each character of $\mathcal{G}_2'$ is a subgroup of $\mathcal{G}_1$ which will admit the decomposition $\mathcal{G}_0'' \circledS \mathcal{G}_1''$ with $\mathcal{G}_0''$ a subgroup of $\mathcal{G}_0$ and $\mathcal{G}_1''$ a subgroup of $\mathcal{G}_1'$. Then since we have (by supposition) all the irreducible representations of $\mathcal{G}_0''$ and all the representations of $\mathcal{G}_1''$ are given by characters we can, by inducing twice, obtain all the representations of $\mathcal{G}_2$. We can repeat this procedure for each $A = 2, 3, \ . \ . \ . \ L$.

Now since $\mathcal{G}_L$ is a linear Lie supergroup the the orbit of each χ_y, $y \in \mathcal{G}'_{A+1}$ is a closed subset of $\mathcal{G}_{A+1}$ and each χ_y *lies in one and only one orbit.* The set C is then obtained by choosing one element from each of these closed subsets and must clearly be a Borel set. We thus satisfy the requirements of Theorem III.2

THEOREM III.4

If $\mathcal{G}_0$ has only Type I representations then $\mathcal{G}_L$ has only Type I representations.

PROOF

If $\mathcal{G}_0$ has only Type I representations then every subgroup of $\mathcal{G}_0$ has only Type I representations, since if a subgroup of $\mathcal{G}_0$ had Type II or III representations they would induce to give Type II or III representations for $\mathcal{G}_0$. Now the stability group of $n \epsilon \mathcal{G}'_1$ is one of these subgroups so that $\mathcal{G}_1$ has only Type I representations. Then proceeding through the *sequence* $A = 2, 3, \ ... \ , L$ we can see that $\mathcal{G}$ can have only Type I representations.

THEOREM III.5

At each step $\mathcal{G}_A \to \mathcal{G}_{A+1}$ the stability group of the character system $\chi(g) = expi(0, 0, . . .).(x, \theta)$, ie. $\chi(g) \equiv 1$ for all $g \epsilon \mathcal{G}'_{A+1}$ is $\mathcal{G}_A$, so that we retain all the representations so far obtained.

PROOF

This is an immediate consequence of Theorem III.3.

Thus it is clear that to obtain all the representations of a linear Lie supergroup we construct the representations obtained in each step $\mathcal{G}_A \dashrightarrow \mathcal{G}_{A+1}$ for $A=0,1,2,\ldots,L-1$. To construct the representations at each step we need to determine the set C and the stability group for each $y\epsilon\mathcal{G}'_A$. This analysis then tells us which representations we need to induce to representations of $\mathcal{G}_L$. To do this we need the definition of an induced representation. The one we give is based on Mackey[12]. It is not the most general definition and is applicable only if $\mathcal{G}$ is unimodular and if $\mathcal{G}/S_y$ possesses an invariant measure, which is true if the modular functions for $\mathcal{G}$ and S_y are equal so that we require that S_y is also unimodular. This is sufficient for all the cases we consider here.

DEFINITION III.6

Let V_Δ be a carrier space for a representation Φ_Δ of $S_y = \mathcal{H}_{X_y}\eta$ and and consider the linear space V of mappings of the right cosets $\mathcal{G}/S_y$ into V_Δ. For any $\phi\epsilon V$ and $g\epsilon\mathcal{G}$ define the operator Φ by

$$\Phi(g)\phi(S_yJ) = \Phi_\Delta(JgJ'^{-1})\phi(S_yJ') \tag{10}$$

with J' the coset representative such that

$$JgJ'^{-1}\epsilon\ S_y \quad . \tag{11}$$

THEOREM III.7

(a) For any $g,g'\ \epsilon\ \mathcal{G}$

$$\Phi(gg') = \Phi(g)\Phi(g') \qquad (120$$

so that Φ provides a representation of $\mathcal{G}$.

(b) If $\mathcal{G}/S$ possesses an invariant measure and Φ_Δ is a unitary representation with inner product $(\ ,\ )_V$ then Φ is a unitary operator with inner product defined by

$$(\phi^1, \phi^2) = \int (\phi^1(S_y J), \phi^2(S_y J))\; d\mu_{G/S_y} \quad . \tag{13}$$

Now suppose we choose $\hat{\chi}$ as a representative character for a given orbit, then the set of characters in the orbit are given by

$$\{\chi = \hat{\chi}g,\; g \in G\} \quad .$$

but g admits the decomposition $g = sJ$ for some coset J and $s \in$ *{Stability group of* $\hat{\chi}$*}*. that is $\chi = (\hat{\chi})sJ$. Thus $\chi = (\hat{\chi})J$ and this relationship must be one to one, so that we can put

$$J = J(\chi, \hat{\chi}) \quad . \tag{14}$$

Thus the cosets are labeled by our representative character and the characters in the corresponding orbit. Equation (10) can now be rewritten as

$$\Phi(g)\phi(S_y J(\chi,\hat{\chi})) = \Phi_\Delta(J(\chi,\hat{\chi})gJ^{-1}(\chi',\hat{\chi}))\phi(S_y J(\chi',\hat{\chi})) \quad . \tag{15}$$

Now equation (11) implies that χ' must satisfy

$$\chi' = \chi g \quad . \tag{16}$$

That is

$$\chi = \chi' g^{-1}, \tag{17}$$

and since the coset representatives are labeled by χ we can put $\phi(S_y J(\chi,\hat{\chi}) = \phi(\chi)$ and rewrite equation (15) as

$$\Phi(g)\phi(\chi) = \Phi_\Delta(J(\chi' g^{-1},\chi)gJ^{-1}(\chi',\chi)))\phi(\chi g) \quad . \tag{18}$$

Now suppose that the Φ_Δ admits a matrix representation Δ with basis $\{\psi_n\}$ ie.

$$\Phi(s)\psi_n = \Delta(s)_{n'n}\psi_{n'} \quad \text{for } s \in S_y \tag{19}$$

and define

$$\phi_{\chi,n}(\chi') = \begin{cases} \psi_n & \text{if } \chi = \chi' \quad , \\ 0 & \text{otherwise} \quad , \end{cases} \tag{20}$$

so that

$$\phi_{\chi' g^{-1}, n}(\chi) = \begin{cases} \psi_n & \text{if } \chi' = \chi g \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{21}$$

and $$\phi_{\chi', n}(\chi g) = \begin{cases} \psi_n & \text{if } \chi' = \chi g \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{22}$$

If we consider the action of $\Phi(g)$ on these mappings we obtain, from equation (19) using equation (21),

$$\begin{aligned} \Phi(g)\phi_{\chi', n}(\chi) &= \Delta(J(\chi' g^{-1}, \hat{\chi}) g J^{-1}(\chi', \hat{\chi}))\phi_{\chi', n}(\chi g) \\ &= \begin{cases} \Delta(J(\chi' g^{-1}, \hat{\chi}) g J^{-1}(\chi', \hat{\chi}))_{mn}\psi_m & \text{if } \chi' = \chi \ , \\ 0 \quad \text{otherwise} \ , \end{cases} \\ &= \Delta(J(\chi' g^{-1}, \hat{\chi}) g J^{-1}(\chi', \hat{\chi}))_{mn}\phi_{\chi' g^{-1}, m}(\chi) \ , \end{aligned} \tag{23}$$

where we have used equation (22). Then replacing χ' by χ and deleting the redundant argument of the mappings ϕ we obtain

$$\Phi(g)\phi_{\chi, n} = \Delta(J(\chi' g^{-1}, \hat{\chi}) g J^{-1}(\chi', \hat{\chi}))_{mn} \ \phi_{\chi g^{-1}, m} \ . \tag{24}$$

Now the characters χ are in one-to-one correspondence to the elements $y \in \hat{\eta}$ so that the above equation could equally well have been written in terms of y. In fact this is the most convenient way to express it. To do this we first have to determine how the group $\mathcal{H}$ acts on $\hat{\eta}$ in some convenient way. In our case $\eta = \mathcal{G}'_{A+1}$, and $\mathcal{H} = \mathcal{G}_A$. Let $g \in \mathcal{G}_A$ and $g_n \in \mathcal{G}'_{A+1}$. Now all elements of $\mathcal{G}'_{A+1}/\mathcal{G}_A$ can be written in the form

$$g_n = I + \alpha_\mu x^\mu + \beta_\sigma \theta^\sigma \ , \tag{25}$$

with $x^\mu = x^\mu_j \underline{f}_j \underline{\epsilon}_{A+1}$ and $\theta^\sigma = \theta^\sigma_i \underline{e}_i \underline{\epsilon}_{A+1}$.

The action of the automorphism $a_h(n)$ defined by equation (8) is then given by

$$\begin{aligned} a_g(g_n) &= g g_n g^{-1} \\ &= I + g(\alpha, \beta)_a g \begin{bmatrix} x \\ \theta \end{bmatrix}_a \ , \end{aligned} \tag{26}$$

here $(\alpha, \beta)_a = (\alpha_1, \alpha_2, \ . \ . \ . \ . \ , \beta_1, \beta_2 \ . \ . \ . \ )_a$ and is an even row supervector, and $\begin{bmatrix} x \\ \theta \end{bmatrix}_a$ is the even column supervector $(x^1, x^2, \ . \ . \ . \ . \ , \theta^1, \theta^2, \ . \ . \ . \ . \ )^t$. This is more succinctly written in terms of the adjoint representation of G. This and the adjoint representation of a Lie superalgebra are defined as follows.

DEFINITION III.8

Let G be a linear Lie supergroup of dimension $(n|m)$, let $\{\alpha_\mu, \beta_\sigma\}$ be a basis of its Lie superalgebra and let $\mathcal{L} = E_L \otimes L$ be the Lie supermodule constructed from L. Then:

(a) For each $g \in G$ let $Ad(g)$ be the supermatrix defined by

$$g(\alpha,\beta)_a g^{-1} = (\alpha,\beta)_b (Ad(g))_{ba} \quad \text{for } a = 1, 2, \ldots, m+n.$$

(b) For every $\gamma \in \mathcal{L}$ define the supermatrix $ad(\gamma)$ by

$$[\gamma, (\alpha,\beta)_b] = (\alpha,\beta)_a (ad(\gamma))_{ab} \quad .$$

These representations are linked in the same manner as for Lie groups, that is $ad(\gamma)$ is the associated representation of $\mathcal{L}_0$, the Lie module of G corresponding to $Ad(g)$. Now equation (26) can be rewritten

$$a_g(g_n) = I + (\alpha,\beta)_b (Ad(g))_{ba} \begin{bmatrix} x \\ \theta \end{bmatrix}_a \quad . \tag{27}$$

and since the characters of η are given by

$$\chi_{(y,\phi)}(g_n) = exp\ i\{(y,\phi).(x,\theta)\}$$

we have

$$(\chi_{(y,\phi)}(g_n))g = exp\ i\{(y,\phi).(Ad(g))_{ba} \begin{bmatrix} x \\ \theta \end{bmatrix}_a\}$$

$$= exp\ i\{(y,\phi)_b (Ad(g))_{ba}).(x,\theta)\} \quad . \tag{28}$$

Thus rewriting equation (24) in terms of (y,ϕ) we obtain:

$$\Phi^{\hat{y},\hat{\phi},\Delta_{\mathcal{H}}}(g)\phi_{(y,\phi),n} \tag{29}$$

$$= \Delta^{y,\phi,\Delta_{\mathcal{H}}}(J((y,\phi)Ad(g^{-1}),(\hat{y},\hat{\phi}))gJ^{-1}((y,\phi),(\hat{y},\hat{\phi})))_{mn}\phi_{(y.\phi)Ad(g^{-1}),m} \; .$$

Here (y,ϕ) are the parameters corresponding to the choice of a representative character $\hat{\chi}$ and we have written $\Delta^{\hat{y},\hat{\phi},\Delta_{\mathcal{H}}}$ and $\Phi^{\hat{y},\hat{\phi},\Delta_{\mathcal{H}}}$ to indicate the dependence of the representation on the choice of representative character and the choice of irreducible representation of $\mathcal{H}_{\hat{y},\hat{\phi}} = \mathcal{H}_{\chi(\hat{y},\hat{\phi})}$.

We note that equation (28) defines the orbit of a character. This can be used to obtain the stability group, but in practice it is easier to use the Lie algebra of $\mathcal{G}_A$ acting on the Lie algebra of $\mathcal{G}'_{A+1}$. To see this we observe that there is a one to one correspondence between the elements of $\hat{\eta}$ and the elements of the Lie algebra of $\mathcal{G}'_{A+1}$, so that requiring that h be an element of the stability group of an element (y,ϕ) is equivalent to demanding that h leaves invariant the corresponding element of the Lie algebra of $\mathcal{G}'_{A+1}$. Now h must lie in some one parameter subgroup of $\mathcal{G}_A$ so that we can put $h = exp\ a\gamma$, $a \in \mathbb{R}$, $\gamma \in \mathcal{L}(\mathcal{G}_A)$. Then our requirement becomes $h\delta h^{-1} = \delta$ where δ is the Lie algebra element corresponding to the character ie. $exp(a\gamma)\delta exp(-a\gamma) = \delta$ so that we require $[\gamma,\delta] = 0$.

IV. A REVIEW OF N=1 SUPER POINCARE GROUPS

N=1 Super Poincaré groups can be constructed in most (but not all) dimensions. In addition they are real supergroups only if it is possible to define Majorana spinors in the chosen dimension. The generators of the Lie superalgebra of the Poincaré group are:

(i) $L_{\lambda\mu} = -L_{\mu\lambda}$, the Lorentz generators,

(ii) K_σ the translation generators

and (iii) Q_α the supersymmetry generators.

Their commutation relations are:

$$\begin{aligned}
[L_{\lambda\mu}, L_{\sigma\rho}] &= g_{\lambda\sigma}L_{\mu\rho} - g_{\lambda\rho}L_{\mu\sigma} - g_{\mu\sigma}L_{\lambda\rho} + g_{\mu\rho}L_{\lambda\sigma} \quad ,\\
[L_{\lambda\mu}, K_\sigma] &= g_{\lambda\sigma}K_\mu - g_{\mu\sigma}K_\lambda \quad ,\\
[L_{\lambda\mu}, Q_\alpha] &= \tfrac{1}{2}(\gamma_\lambda\gamma_\mu)_{\alpha\beta}Q_\beta \quad ,\\
[K_\sigma , K_\rho] &= 0 \quad ,
\end{aligned} \qquad (30)$$

$$[K_\sigma\ ,\ Q_\alpha] = 0 \qquad .$$

and $$[Q_\alpha\ ,\ Q_\beta] = (\gamma^\sigma C)_{\alpha\beta} K_\sigma \qquad .$$

Here, if d is the number of dimensions, $\lambda, \mu, \sigma, \rho = 1, 2, \ldots, d$ and $\alpha, \beta = 1, 2, \ldots, 2^\omega$ with ω given by $\omega = ½(d-1)$ for d odd and $\omega = ½d$ for d even; $g_{\lambda\lambda} = -1$ for a spacelike dimension and $g_{\lambda\lambda} = 1$ for a timelike dimension, with $g_{\lambda\mu} = 0$ if $\lambda \neq \mu$. The Dirac matrices are as appropriate to the number of dimensions. Our conventions for $d = 4$ and one time dimension are given in the appendix. Note that we choose $g_{\lambda\mu} = diag(-1,\ -1,\ -1,\ 1)$.

A supermatrix representation of the superalgebra can be constructed in the block form

$$\begin{bmatrix} L_{\lambda\mu} & K_\sigma & U(Q_\alpha) \\ 0 & 0 & 0 \\ 0 & V(Q_\alpha) & \Gamma(L_{\lambda\mu}) \end{bmatrix} \tag{31}$$

with
$$\begin{aligned} (L_{\lambda\mu})_{ba} &= \delta_{b\mu} g_{a\lambda} - \delta_{b\lambda} g_{a\mu} \qquad , \\ (K_\sigma)_a &= \delta_{a\sigma} \qquad , \\ \Gamma(L_{\lambda\mu})_{ba} &= ½((\gamma_\lambda \gamma_\mu)^t)_{ba} \qquad , \\ (V(Q_\alpha))_a &= \delta_{\alpha a} \qquad , \end{aligned} \tag{32}$$

and $$(U(Q_\alpha))_{ab} = ½(\gamma^a C)_{\alpha b} \qquad .$$

This representation has been written in the standard block form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and acts on the space $(x^\mu,\ 1,\ \theta^\sigma)^t$ with $\mu = 1, 2, \ldots, d$ and $\sigma = 1, 2, \ldots, 2^\omega$, ie. the extended Minkowski space (superspace) of supersymmetry theories.

We note that the choice of representation $\Gamma(L_{\lambda\mu})$ given, this is the negative transpose of what one might expect, and is chosen to conform to standard usage in the physics literature.

The algebra, of course, admits the semi direct structure

$$\begin{bmatrix} L_{\lambda\mu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Gamma(L_{\lambda\mu}) \end{bmatrix} \begin{bmatrix} 0 & K_\sigma & U(Q_\alpha) \\ 0 & 0 & 0 \\ 0 & V(Q_\alpha) & 0 \end{bmatrix} \tag{33}$$

of the Lorentz algebra $so(d-t,t;\mathbb{R})$ generated by $\{L_{\lambda\mu}\}$ and the supersymmetry algebra generated by $\{K_\sigma, Q_\alpha\}$ and denoted by $st(d-t,t;\mathbb{R})$. The translations $t(d-t,t;\mathbb{R})$ generated by $\{K_\sigma\}$ also generate a subalgebra, but the supersymmetry generators on their own do not.

Corresponding to the decomposition of the algebra we can construct the supergroup as

$$g =$$

$$\begin{bmatrix} I & t & T(-\tau) \\ 0 & 1 & 0 \\ 0 & -\tau & I \end{bmatrix} \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma(\Lambda) \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda & t & T(-\tau)\Gamma(\Lambda) \\ 0 & 1 & 0 \\ 0 & -\tau & \Gamma(\Lambda) \end{bmatrix} . \tag{34}$$

We note that the inverse is

$$g^{-1} =$$

$$\begin{bmatrix} \Lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma(\Lambda^{-1}) \end{bmatrix} \begin{bmatrix} I & -t & T(\tau) \\ 0 & 1 & 0 \\ 0 & \tau & I \end{bmatrix}$$

$$= \begin{bmatrix} \Lambda^{-1} & -\Lambda^{-1}t & \Lambda^{-1}T(-\tau) \\ 0 & 1 & 0 \\ 0 & \Gamma(\Lambda^{-1})\tau & \Gamma(\Lambda^{-1}) \end{bmatrix} . \tag{35}$$

In these expressions Λ is a Grassman valued Lorentz transformation in the appropriate number of dimensions, $\Gamma(\Lambda)$ is the representation

corresponding to the representation $\Gamma(L_{\lambda\mu})$, $t = t^\sigma \epsilon\ E_L$ is a column vector corresponding to the translations, $\tau = \tau^\alpha \epsilon\ E_L$ a column vector corresponding to the supertranslations and $(T(-\tau))_{ab} = ½\tau^{\alpha}(\gamma^{a}C)_{\alpha b}$. Note the fact that we have $-\tau$, this is due to the rule for multiplying supermatrices by scalars from E_L. We observe that $(T(-\tau))(-\tau) = 0$.

Consider now the subgroup obtained by setting $t = 0$ and $\tau = 0$, this will in general be a four component group corresponding to the four components of the Lorentz group. These will be linked by the operators corresponding to space inversion, time inversion and total inversion. In this paper we consider only the one component supergroup, the proper orthochronous super Poincaré group, which we denote by $SO_0(d-t,t;E_L)$, or more precisely its covering group denoted by $\overline{SO}_0(d-t,t;E_L)$, which for $d = 4$ and $t = 1$ can be taken to be isomorphic to $ST(3,1;E_L) \circledS SL(2;\mathbb{C}E_L)$. It is convenient, in the sequel, to denote an element of this supergroup by $[\Lambda|t|\tau]$ where Λ, t, τ refer to the matrix blocks of equation (34). In this notation group multiplication is given by

$$[\Lambda|t|\tau][\Lambda'|t'|\tau'] = [\Lambda'|\ t' + t + T(\tau)\Gamma(\Lambda)\tau'|\Gamma(\Lambda)\tau' + \tau] \quad . \tag{36}$$

Also

$$[\Lambda|t|\tau]^{-1} = [\Lambda^{-1}|-\Lambda^{-1}t|-\Gamma(\Lambda^{-1})\tau] \quad . \tag{37}$$

In the sequel we will need the adjoint representation. If we choose the order of the generators to be such that the matrix $ad(\gamma)$ is given by

$$[\gamma,\ (L_{\phi\psi},\ K_{\rho},\ Q_{\beta}\ )_a] = (L_{\epsilon\delta},\ K_{\sigma},\ Q_{\gamma}\ )_b(ad(\gamma))_{ba} \tag{38}$$

for any γ in the Lie superalgebra of the super Poincaré group, then $ad(\gamma)$ can be determined from the commutation relations of equation (30) to be the $(½(d^2 + d)|\omega)$ dimensional supermatrix written in block form as

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \quad .$$

Here a is $½(d^2 - d)\times½(d^2 - d)$ with the double index $\phi\psi$ labeling the rows and $\epsilon\delta$ labeling the columns; e is $d\times d$ with ρ labeling its rows and π

labeling its columns and j is $\omega\times\omega$ with β labeling its rows and γ its columns. The dimensions of the other blocks follow from these. We find that the matrices are

$$ad(K_\sigma) = \begin{bmatrix} 0 & 0 & 0 \\ (g_{\psi\sigma}\delta^\pi_\phi - g_{\phi\sigma}\delta^\pi_\psi) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{38a}$$

$$ad(L_{\lambda\mu}) = \begin{bmatrix} \begin{matrix}(g_{\lambda\phi}\delta^\epsilon_\mu\delta^\delta_\psi - g_{\lambda\psi}\delta^\epsilon_\mu\delta^\delta_\phi \\ - g_{\mu\phi}\delta^\epsilon_\lambda\delta^\delta_\psi + g_{\mu\psi}\delta^\epsilon_\lambda\delta^\delta_\phi)\end{matrix} & 0 & 0 \\ 0 & (g_{\lambda\rho}\delta^\pi_\mu - g_{\mu\rho}\delta^\pi_\lambda) & 0 \\ 0 & 0 & (-\tfrac{1}{2}(\gamma_\lambda\gamma_\mu)^t)_{\gamma\beta} \end{bmatrix} \tag{38b}$$

and

$$ad(Q_\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (\gamma^\pi C)_{\alpha\beta} \\ (\tfrac{1}{2}(\gamma_\phi\gamma_\psi)^t)_{\gamma\alpha} & 0 & 0 \end{bmatrix} . \tag{38c}$$

As before we can obtain a matrix representation of the supergroup by exponentiating the subgroups generated by $\{L_{\lambda\mu}\}$ and $\{K_\sigma, Q_\alpha\}$. This representation is then the adjoint representation of the supergroup. We obtain

$$Ad(g) = Ad([\Lambda|t|\tau])$$

$$= \begin{bmatrix} I & 0 & 0 \\ V+TU & I & 2T \\ U & 0 & I \end{bmatrix} \begin{bmatrix} Ad(\Lambda) & 0 & 0 \\ 0 & \boldsymbol{\Lambda} & 0 \\ 0 & 0 & \Gamma(\Lambda) \end{bmatrix} ,$$

$$= \begin{bmatrix} Ad(\Lambda) & 0 & 0 \\ (V+TU)Ad(\Lambda) & \boldsymbol{\Lambda} & 2T\Gamma(\Lambda) \\ UAd(\Lambda) & 0 & \Gamma(\Lambda) \end{bmatrix} . \tag{39}$$

Here Λ, $\Gamma(\Lambda)$ are the matrix representations of the covering group of the Lorentz group as given before in this section, T is as previously defined, $Ad(\Lambda)$ is the adjoint representation of the super Lorentz group and the matrices U and V are given by

$$U_{\phi\psi,\gamma} = \tfrac{1}{2}\tau^{\alpha}((\gamma_{\phi}\gamma_{\psi})^{t})_{\gamma\alpha} \tag{40}$$

and

$$V_{\phi\psi}{}^{\pi} = t^{\sigma}(g_{\psi\sigma}\delta^{\pi}_{\phi} - g_{\phi\sigma}\delta^{\pi}_{\psi}) \quad . \tag{41}$$

V. THE UNITARY REPRESENTATIONS OF THE SUPER POINCARE GROUP: PRELIMINARIES

In this section we make a start in the study of the irreducible representations of the super Poincaré group. We consider only the case with $d = 4$ and $t = 1$. We are looking for representations that we can associate with the well known representations of the Poincaré Lie superalgebra, as originally described by Salam and Strathdee[10]. These are known to be labeled by a 'superspin' index $j = 0, \tfrac{1}{2}, 1, \ldots$ and a 'mass parameter' M. They consist of the direct sum of four Poincaré type representations, in the massive case, (except for $j = 0$ which has only three such representations) with spins of j, $j+\tfrac{1}{2}$, $j-\tfrac{1}{2}$, j (the $j-\tfrac{1}{2}$ representation does not exist for $j = 0$) together with the supersymmetry generators Q_{α} which link the representations.

We will show by a series of examples that the unitary irreducible representations we can construct act on state vectors which have at most one index that can be associated with spin, and that this is such that an irreducible representation acts on state vectors with a single fixed spin. The connection between the representations we construct here and the Salam-Strathdee representations is the subject of the last paper in this series[11].

We consider group elements of the form $g = [\Lambda | t | \tau]$ for $g \epsilon G_L$ parametrized by $t^{\sigma} \epsilon E_{L0}$, $\sigma = 1, 2, 3, 4$ for the translations, $\tau^{\alpha} \epsilon E_{L1}$, $\alpha = 1, 2, 3, 4$ for the supertranslations and for the Lorentz transformations we parametrize by $y^{\mu\lambda}\epsilon E_L$, μ, $\lambda = 1, 2, 3, 4$; $\mu\neq\lambda$ and $y^{\mu\lambda} = -y^{\lambda\mu}$ which we take to correspond to the Lie algebra generators $L_{\mu\lambda}$ ie. $\Lambda = exp(y^{\mu\lambda}L_{\mu\lambda})$. We will need to write these as a vector, in which case we choose their order to be $y = (y^{12}, y^{13}, y^{14}, y^{23}, y^{24}, y^{34})$. The elements of $\Lambda = \mathcal{G}'_{A+1}$ are then parametrized by

$$(y^{\mu\lambda}_{j,(A+1)}\underline{f}_j\underline{\epsilon}_{A+1},\ t^{\sigma}_{j,(A+1)}\underline{f}_j\underline{\epsilon}_{A+1},\ \tau^{\alpha}_{i,(A+1)}\underline{e}_i\underline{\epsilon}_{A+1})$$

with $\underline{e}_i$ a basis for E_{A0} so that $i = 0, 1, \ldots , 2^{A-1}-1$ and $\underline{f}_j$ a basis of E_{A1} so that $j = 1, 2, \ldots ,2^{A-1}$.

Correspondingly the elements of $\hat{\eta}$ are in one-to-one correpondence to the vector

$$(\ell,\ k,\ \phi) = (\ell_{\mu\lambda},\ k_{\sigma},\ \phi_{\alpha}) \quad ,$$
$$= (\ell^{j,(A+1)}_{\mu\lambda},\ k^{j,(A+1)}_{\sigma},\ \phi^{i,(A+1)}_{\alpha}) \quad , \tag{42}$$

with $\mu,\lambda,\sigma,\alpha,i,j$ taking the values defined for them above. The characters of η are then given by

$$\chi_{(\ell,k,\phi)} = expi(\ell,\ k,\ \phi).(y,\ x,\ \tau) \quad . \tag{43}$$

The action of a group element $g\epsilon\mathcal{G}_A$ on a character is then specified by equation (28) to be

$$(\chi_{(\ell,k,\phi)})g = \chi_{((\ell,k,\phi)Ad(g))}. \tag{44}$$

We note that the 'translation' element of $\hat{\eta}$ is often denoted by p corresponding to the Hermitian generator of the Poincaré group $P_{\sigma} = \frac{i}{\hbar} K_{\sigma}$ this is related to our parameter k by $\frac{1}{\hbar} p^0_{\sigma} = k^0_{\sigma}$.

The action of $g = [\Lambda|t|\tau]\epsilon\mathcal{G}_A$ on $\hat{\eta}$ is given by

$$(\ell,\ k,\ \phi)\begin{bmatrix} Ad(\Lambda) & 0 & 0 \\ (V+TU)Ad(\Lambda) & I & 2T\Gamma(\Lambda) \\ UAd(\Lambda) & 0 & \Gamma(\Lambda) \end{bmatrix}$$

$$= (\ell Ad(\Lambda) + k(V+TU)Ad(\Lambda) + \phi UAd(\Lambda),\ k\ ,\ \phi\Gamma(\Lambda) + 2kT\Gamma(\Lambda)), \tag{46}$$

with the matrices V, T, U as specified in section II. The representation $\Gamma(\Lambda)$ is given by

$$\Gamma(\Lambda) = S^{*}\begin{bmatrix} \Gamma^{0,\frac{1}{2}}(\Lambda) & 0 \\ 0 & \Gamma^{\frac{1}{2},0}(\Lambda) \end{bmatrix}(S^{-1}) = \exp -\tfrac{1}{2}y^{\mu\lambda}(\gamma^{M}_{\mu}\gamma^{M}_{\lambda}) \tag{47}$$

with the similarity transformation S as defined in the appendix.

The matrix $Ad(\Lambda)$ is equivalent to the representation $\begin{bmatrix} \Gamma^{0,1}(\Lambda) & 0 \\ 0 & \Gamma^{1,0}(\Lambda) \end{bmatrix}$.

It is convenient to have an explicit representation in this decomposed form and to have the corresponding characters for this decomposition. We find that a suitable matrix representation of $ad(L_{\mu\lambda})$ is given by

$$ad(L_{12}) = \begin{bmatrix} 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & 0 & \\ 0 & 0 & 0 & & & \\ & & & 0 & 1 & 0 \\ & 0 & & -1 & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix}, \quad ad(L_{13}) = \begin{bmatrix} 0 & 0 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 0 & -1 & 0 & & & \\ & & & 0 & 0 & 0 \\ & 0 & & 0 & 0 & 1 \\ & & & 0 & -1 & 0 \end{bmatrix},$$

$$ad(L_{23}) = \begin{bmatrix} 0 & 0 & -1 & & & \\ 0 & 0 & 0 & & 0 & \\ 1 & 0 & 0 & & & \\ & & & 0 & 0 & -1 \\ & 0 & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \end{bmatrix}, \quad ad(L_{14}) = \begin{bmatrix} 0 & 0 & i & & & \\ 0 & 0 & 0 & & 0 & \\ -i & 0 & 0 & & & \\ & & & 0 & 0 & -i \\ & 0 & & 0 & 0 & 0 \\ & & & i & 0 & 0 \end{bmatrix},$$

$$ad(L_{24}) = \begin{bmatrix} 0 & 0 & 0 & & & \\ 0 & 0 & i & & 0 & \\ 0 & -i & 0 & & & \\ & & & 0 & 0 & 0 \\ & 0 & & 0 & 0 & -i \\ & & & 0 & i & 0 \end{bmatrix}, \quad ad(L_{34}) = \begin{bmatrix} 0 & -i & 0 & & & \\ i & 0 & 0 & & 0 & \\ 0 & 0 & 0 & & & \\ & & & 0 & i & 0 \\ & 0 & & -i & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix}.$$

corresponding to this representation the vector ℓ takes the form

$$\begin{aligned} &\left(\tfrac{1}{\sqrt{2}}(\ell_{13} + i\ell_{24}), \tfrac{1}{\sqrt{2}}(\ell_{23} - i\ell_{14}), \tfrac{1}{\sqrt{2}}(\ell_{12} - i\ell_{34}), \tfrac{1}{\sqrt{2}}(\ell_{13} - i\ell_{24}),\right. \\ &\qquad \left.\tfrac{1}{\sqrt{2}}(\ell_{23} + i\ell_{14}), \tfrac{1}{\sqrt{2}}(\ell_{12} + i\ell_{34})\right) \\ &= (r_1, r_2, r_3, r_1^{*}, r_2^{*}, r_3^{*}) \ . \end{aligned} \tag{48}$$

VI. THE UNITARY IRREDUCIBLE REPRESENTATIONS OF THE SUPER POINCARE GROUP $\mathcal{SO}_0(3,1;E_0)$.

The representations of $\mathcal{G}_0$ are the well known representations of the Poincaré group. In our terminology we consider elements of the form $[\Lambda|t|\tau]$ which admit the semidirect decomposition

$$[\Lambda|t|\tau] = [I|t|\tau][\Lambda|0|0] \qquad . \tag{49}$$

The elements of $\hat{\eta}$ are the four vector $k^0 = (k_1^0,\ k_2^0,\ k_3^0,\ k_4^0)$ and the action of Λ on k^0 leaves invariant the quadratic form

$$k_\mu^0 k_\lambda^0 g^{\mu\lambda} = -(k_1^0)^2 - (k_2^0)^2 - (k_3^0)^2 + (k_4^0)^2 = \frac{Mc}{\hbar}^2 \tag{50}$$

where we have written the constant on the right hand side in its standard form with M being the 'mass', c the velocity of light and $\hbar$ Plank's constant.

There are six distinct types of orbit, for which the set of representative characters forms a Borel set, so that we obtain every representation. We list these (for more details cf. Cornwell[13]).

(i) The orbit consists of four vectors k for which $\frac{Mc}{\hbar}^2 > 0$ and $k_4^0 > 0$. The vectors of the carrier space of these representations are interpreted as particles of mass M and spin j.

(ii) The orbit consists of four vectors k for which $\frac{Mc^2}{\hbar} > 0$ and $k_4^0 < 0$. The particles corresponding to these representations are considered to be 'non-physical'

(iii) The orbit consists of four vectors k for which $\frac{Mc^2}{\hbar} < 0$. These representations are again considered to be 'non-physical'.

(iv) The orbit consists of four vectors k for which $\frac{Mc^2}{\hbar} = 0$ but $k_4^0 > 0$. After inducing to G certain of these representations are interpreted as massless particles of helicity λ.

(v) The orbit consists of four vectors k for which $\frac{Mc^2}{\hbar} = 0$ but $k_4^0 < 0$. These are again discarded on physical grounds.

(vi) The orbit consists of the four vector $k^0 = (0, 0, 0, 0)$. The resulting representations are interpreted as space time symmetries.

Our interest in the later papers in this series[11,14] is in massive particles. To show how we obtain a representation of $\mathcal{G}_0$ in this case we quote the result, first obtained by Wigner[7] (our terminology is that of Cornwell[13]) in terms of the four vector k^0 that we are using.

$$\Phi^{\hat{k}^0, j}([\Lambda | t | 0])\phi_{k^0, m}$$
$$= \exp i\{(k^0\Lambda^{-1})_\alpha t_0^\alpha\} D^j([B(k^0\Lambda^{-1}, \hat{k}^0)^{-1}\Lambda B(k^0, \hat{k}^0) | 0 | 0])_{mm'}\phi_{k^0\Lambda^{-1}, m'} \tag{51}$$

Here the representative character is $\hat{k}^0 = (0, 0, 0, \frac{Mc}{\hbar})$ and D^j is the $(2j + 1)\times(2j + 1)$ dimensional representation of $SU(2)$, the covering group of $SO(3,\mathbb{R})$. The coset representative $B(k^0, \hat{k}^0)$ is the inverse of the coset representative $J(k^0, \hat{k}^0)$ and is known as the Lorentz boost from $\hat{k}^0$ to k^0. (It is as given by Cornwell[13] with p replaced by k^0.)

Now we want to construct a representation of $\mathcal{G}_L$ from a representation of $\mathcal{G}_0$. First consider inducing to a representation of $\mathcal{G}_1 = \mathcal{G}_1' \circledS \mathcal{G}_0$. We choose the character of $\mathcal{G}_1'$ defined by $\phi^1 = (0, 0, 0, 0)$ and the representation of $\mathcal{G}_0$ given by $\Phi_{\mathcal{G}_0}([\Lambda | t | 0])$. The orbit consists of a single element and the coset representative is $J(\phi^1, \hat{\phi}^1) = I$, so that the operators of the induced representation are given by

$$\Phi_{\mathcal{G}_1}([\Lambda | t | \tau]) = \begin{cases} \Phi_{\mathcal{G}_0}([\Lambda | t | \tau]) & \text{,if } ([\Lambda | t | \tau])\epsilon\mathcal{G}_0 \text{ ,} \\ 0 & \text{,otherwise .} \end{cases}$$

This argument repeats at each step in the sequence $\mathcal{G}_0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \dots \mathcal{G}_L$ so that the operators $\Phi_{\mathcal{G}_L}([\Lambda | t | \tau])$ of a representation of $\mathcal{G}_L$ obtained by inducing from a representation $\Phi_{\mathcal{G}_0}([\Lambda | t | \tau])$ of $\mathcal{G}_0$ are obtained simply by setting

$$\Phi_{\mathcal{G}_L}([\Lambda | t | \tau]) = \begin{cases} \Phi_{\mathcal{G}_0}([\Lambda | t | \tau]) & \text{,if } ([\Lambda | t | \tau])\epsilon\mathcal{G}_0 \text{ ,} \\ 0 & \text{,otherwise .} \end{cases} \tag{52}$$

A consequence of this is that any operator corresponding to a group transformation of $\mathcal{G}_L/\mathcal{G}_0$ leaves the state vectors unchanged, thus the eigenvalues of the operators $\underline{e}_i K_\sigma$, $\underline{e}_i L_{\mu\lambda}$ and $\underline{f}_j Q_\alpha$ are zero for $i = 1, 2, \ldots, \mathcal{N}-1$; $j = 1, 2, \ldots \mathcal{N}$, and each σ, μ, λ, $\alpha = 1, 2, 3, 4$.

VII. THE UNITARY IRREDUCIBLE REPRESENTATIONS OF THE SUPER POINCARE GROUP $\overline{SO}_0(3,1;E_1)$.

A basis of the Lie algebra of $\mathcal{G}_0$ is $\{\underline{\varepsilon}_0 L_{\mu\lambda}, \underline{\varepsilon}_0 K_\sigma\}$ and a basis of the Lie algebra of $\mathcal{G}_1'$ is $\{\underline{\varepsilon}_1 Q_\alpha\}$. The action of the Lie algebra of $\mathcal{G}_0$ on the Lie algebra of $\mathcal{G}_1'$ is given by

$$[\underline{\varepsilon}_0 L_{\mu\lambda}, \underline{\varepsilon}_1 Q_\alpha] = -\tfrac{1}{2}\underline{\varepsilon}_1 (\gamma_\mu \gamma_\lambda)_{\alpha\beta} Q_\beta \tag{53a}$$

and $$[\underline{\varepsilon}_0 K_s, \underline{\varepsilon}_1 Q_\alpha] = 0 \quad . \tag{53b}$$

Thus we can deduce immediately that the translation subgroup generated by $\{\underline{\varepsilon}_0 K_\sigma\}$ is part of the stability group for any character of $\mathcal{G}_1'$.

Now consider equation (53a) with $\alpha = 1$. Using the Majorana representation of the Dirac matrices, given in the Appendix, we find that the group generated by $\{(\underline{\varepsilon}_0 L_{12} + \underline{\varepsilon}_0 L_{14}), (\underline{\varepsilon}_0 L_{23} - \underline{\varepsilon}_0 L_{34})\}$ leaves the character $\chi_{(0;\phi_1^1,0,0,0)}$ of $\mathcal{G}_1'$ invariant. It is also clear from equation (53a) that the orbit of the character corresponding to the choice of any non-zero $\phi_1^1 \in \mathbb{R}$ is the complete set of non-zero characters of $\mathcal{G}_1'$. Thus if we set $\phi_1^1 = 1$ the representations of $\mathcal{G}_1$ are given by

$$(\Delta_{\mathcal{H}} \exp i\phi_1^1.\theta)\uparrow\mathcal{G}_1 \tag{54}$$

with $\Delta_{\mathcal{H}}$ some unitary irreducible representation of the group $\mathcal{H}$ generated by $\{(\underline{\varepsilon}_0 L_{12} + \underline{\varepsilon}_0 L_{14}), (\underline{\varepsilon}_0 L_{23} - \underline{\varepsilon}_0 L_{34}), \underline{\varepsilon}_0 K_\sigma\}$. The Lie algebra of $\mathcal{H}$ is given by

$$[(\underline{\varepsilon}_0 L_{12} + \underline{\varepsilon}_0 L_{14}), (\underline{\varepsilon}_0 L_{23} - \underline{\varepsilon}_0 L_{34})] = 0 \tag{55a}$$

$$[(\varepsilon_0 L_{12} + \varepsilon_0 L_{14}),\ \varepsilon_0 K_\sigma] = g_{1\sigma}\varepsilon_0 K_1 - g_{2\sigma}\varepsilon_0 K_1 + g_{1\sigma}\varepsilon_0 K_4 - g_{4\sigma}\varepsilon_0 K_1 \tag{55b}$$

$$[(\varepsilon_0 L_{23} - \varepsilon_0 L_{34}),\ \varepsilon_0 K_\sigma] = g_{2\sigma}\varepsilon_0 K_3 - g_{3\sigma}\varepsilon_0 K_2 - g_{3\sigma}\varepsilon_0 K_4 + g_{4\sigma}\varepsilon_0 K_3 \tag{55c}$$

and $$[\varepsilon_0 K_\sigma,\ \varepsilon_0 K_\rho] = 0. \tag{55d}$$

This algebra possesses the obvious semidirect structure, which we can write symbolically as $K \circledS L$ with both subalgebras generating abelian groups, which we will call respectively $\mathcal{H}'$ and η. These will thus both have one dimensional representations given by their characters. To determine the nature of the group $\mathcal{H}'$ we exponentiate, using the expressions for $L_{\mu\lambda}$ given in equation (32). Thus $h\epsilon\mathcal{H}'$ is given by

$$h = \exp a(\varepsilon_0 L_{12} + \varepsilon_0 L_{14})\exp b(\varepsilon_0 L_{23} - \varepsilon_0 L_{34})$$

$$= \begin{bmatrix} 1 & a & 0 & -a \\ -a & 1 - \frac{a^2}{2} & 0 & \frac{a^2}{2} \\ 0 & 0 & 1 & 0 \\ -a & -\frac{a^2}{2} & 0 & 1 + \frac{a^2}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{b^2}{2} & b & \frac{b^2}{2} \\ 0 & -b & 1 & b \\ 0 & -\frac{b^2}{2} & b & 1 + \frac{b^2}{2} \end{bmatrix} .$$

Hence it is isomorphic to $\mathbb{R}^2$ and has characters labeled by continuous parameters. The two generators of $\mathcal{H}'$ allow us to construct four possible stability groups. It is convenient to deal with these in descending size order.

Case (i) Stability group = $\mathcal{H}'$.

Since both generators of $\mathcal{H}'$ *commute with* $(\varepsilon_0 K_2 + \varepsilon_0 K_4)$ we can identify the characters of η corresponding to this choice of stability group as $\exp ik^0.t = \exp i(0,\ k^0,\ 0,\ k^0).t$ for some $k^0\epsilon\mathbb{R}$. The representations of $\mathcal{H}'$ are also given by the characters

$\exp i\ell^0.y = \exp i(\ell^0_{12},\ 0,\ \ell^0_{12},\ \ell^0_{23},\ 0,\ -\ell^0_{23}).y$ for $\ell^0_{12}, \ell^0_{23}\epsilon\mathrm{R};\ \ell^0_{12}, \ell^0_{23} \neq 0.$

Then since the stability group is equal to $\mathcal{H}'$ we obtain the one dimensional representations of $\mathcal{H}$ given by

$$\Delta_{\mathcal{H}} = \exp i\ell^0.y\ \exp ik^0.x$$

which are labeled by three real numbers. The corresponding representations of $\mathcal{G}_1$ are then given by

$$(expi\ell^0.y\ expik^0.x\ expi\phi^1.\tau)\uparrow\mathcal{G}_1$$

and we can use equation (29) to obtain these explicitly. Since they do not seem to have any physical significance we do not carry out the induction.

Case (ii) Stability group generated by $(\varepsilon_0L_{12} + \varepsilon_0L_{14})$.

This generator commutes with $\{\varepsilon_0K_3, \varepsilon_0K_2 + \varepsilon K_4\}$, and from equation (55) we can see that

$$[(\varepsilon_0L_{23} - \varepsilon_0L_{34}), \varepsilon_0K_3] = \varepsilon_0K_2 + \varepsilon_0K_4.$$

Thus the orbit of any fixed element of $\hat{\eta}$, $k^0 = (0, 0, k_3^0, 0)$ consists of all elements of the form $(0, a, k_3^0, a)$ for $a\epsilon\mathbb{R}$. To obtain all possible characters we need to allow k_3^0 to range over all non zero values ($k_3^0 = 0$ is covered by case (i) above).

The representations of the stability subgroup of $\mathcal{H}'$ are also given by characters of the form $expi\ell^0.y$ with $\ell^0 = (\ell_{12}^0, 0, \ell_{12}^0, 0, 0, 0)$ for $\ell_{12}^0\epsilon\ \mathbb{R}$.

The representations of $\mathcal{H}$ can now be obtained by induction ie. $\Delta = (expi\ell^0.y\ expik^0.t)\uparrow\mathcal{H}$. This will then be an infinite dimensional representation labeled by two real numbers, one of which must be non zero.

Case (iii) Stability group generated by $(\varepsilon_0L_{23} - \varepsilon_0L_{34})$.

The analysis is similar to case (ii). We again have infinite dimensional representations labeled by two real numbers.

Case (iv) Stability group is the identity of $\mathcal{H}$.

This case includes all the characters we have not covered in the other cases. The representations in this case are labeled by a set of (at most) three real numbers.

What is clear from this analysis is that any representation obtained in this step is labeled by a set of continuous parameters and that none of them has an integer label that can be associated with particle spin.

VIII. THE UNITARY IRREDUCIBLE REPRESENTATIONS OF THE SUPER POINCARE GROUP $\overline{SO}_0(3,1;E_2)$.

A basis of the Lie algebra of $\mathcal{G}_1$ is $\{\epsilon_0 L_{\mu\lambda}, \epsilon_0 K_\sigma, \epsilon_1 Q_\alpha\}$ and a basis of the Lie algebra of $\mathcal{G}'_2$ is $\{\epsilon_{1,2} L_{\mu\lambda}, \epsilon_{1,2} K_\sigma, \epsilon_2 Q_\alpha\}$. The action of the Lie algebra of $\mathcal{G}_1$ on the Lie algebra of $\mathcal{G}'_2$ is given by

$$
\begin{aligned}
[\epsilon_0 L_{\lambda\mu}, \epsilon_{1,2} L_{\sigma\rho}] &= g_{\lambda\sigma}\epsilon_{1,2} L_{\mu\rho} - g_{\lambda\rho}\epsilon_{1,2} L_{\mu\sigma} - g_{\mu\sigma}\epsilon_{1,2} L_{\lambda\rho} + g_{\mu\rho}\epsilon_{1,2} L_{\lambda\sigma} ,\\
[\epsilon_0 K_\sigma, \epsilon_{1,2} L_{\lambda\mu}] &= g_{\mu\sigma}\epsilon_{1,2} K_\lambda - g_{\lambda\sigma}\epsilon_{1,2} K_\mu ,\\
[\epsilon_1 Q , \epsilon_{1,2} L_{\lambda\mu}] &= 0 ,\\
[\epsilon_0 L_{\lambda\mu}, \epsilon_{1,2} K_\sigma] &= g_{\lambda\sigma}\epsilon_{1,2} K_\mu - g_{\mu\sigma}\epsilon_{1,2} K_\lambda ,\\
[\epsilon_0 K_\sigma, \epsilon_{1,2} K_\rho] &= 0 ,\\
[\epsilon_1 Q_\alpha, \epsilon_{1,2} K_\sigma] &= 0 ,\\
[\epsilon_0 L_{\lambda\mu}, \epsilon_2 Q_\alpha] &= -\tfrac{1}{2}(\gamma_\lambda \gamma_\mu)_{\alpha\beta}\epsilon_2 Q_\beta ,\\
[\epsilon_0 K_\sigma , \epsilon_2 Q_\alpha] &= 0
\end{aligned}
\tag{56}
$$

and

$$
[\epsilon_1 Q_\alpha , \epsilon_2 Q_\beta] = -(\gamma^\sigma C)_{\alpha\beta}\epsilon_{1,2} K_\sigma .
$$

The characters of $\mathcal{G}'_2$ are in one to one correspondence to the vector $(\ell^{1,2}_{\mu\lambda}, k^{1,2}_\sigma, \phi^2_\alpha)$. It is convenient to deal first with the three cases such that the representative character in a given orbit is chosen such that: (i) $\ell^{1,2}_{\mu\lambda} = 0$, $\phi^2_\alpha = 0$ for each μ, λ, α; (ii) $\ell^{1,2}_{\mu\lambda} = 0$, $k^{1,2}_\sigma = 0$ for each μ, λ, σ and (iii) $k^{1,2}_{\sigma} = 0$, $\phi^{2}_{\alpha} = 0$ for each σ, α.

Case (i) Representations obtained when inducing from a character of the form $\chi_{(0,k,0)}$.

We can see from equation (56) that the stability subgroup must contain $\{\varepsilon_1 Q_\alpha, \varepsilon_0 K_\sigma\}$ and that, depending on our choice of character, it will also contain some subgroup of the Lorentz group generated by $\{\varepsilon_0 L_{\lambda\mu}\}$. In fact In fact we can recognise that the possibilities are the same as for the representations given in section VI. That is we have five possible types of orbit, for which we can choose the representative characters

(i) $k^{1,2} = (0,\ 0,\ 0,\ k^{1,2}_4)$ with $k^{1,2}_4 > 0$,

(ii) $k^{1,2} = (0,\ 0,\ 0,\ k^{1,2}_4)$ with $k^{1,2}_4 < 0$,

(iii) $k^{1,2} = (k^{1,2}_4,\ 0,\ 0,\ 0)$,

(iv) $k^{1,2} = (0,\ 0,\ k^{1,2}_4,\ k^{1,2}_4)$ with $k^{1,2}_4 > 0$

and (v) $k^{1,2} = (0,\ 0,\ k^{1,2}_4,\ k^{1,2}_4)$ with $k^{1,2}_4 < 0$.

We do not consider the possibility $k^{1,2} \equiv 0$ since this gives us the representations of $\mathcal{G}_1$ that we already have.

For each of the five possibilities we have to consider representations of the appropriate stability group. This again is a semidirect product of the form (*Some subgroup of the Lorentz group*) Ⓢ (*The group generated by* $\{\varepsilon_1 Q_\alpha\ ,\ \varepsilon_0 K_\sigma\}$).

If we require that any representation we obtain is a candidate for the representation of a physical particle, we need the representation to be labeled by a number that takes integer or half integer values, which we can then associate with the spin of the particle. This in turn implies that the stability group must contain a representation of a covering group of a rotation group in two or three dimensions. The obvious candidates warranting consideration are types (i) and (iv) above.

Type (i). The stability group is generated by $\{\varepsilon_0 L_{ij}, \varepsilon_0 K_\sigma, \varepsilon_1 Q_\alpha\}$ with $i, j = 1, 2, 3$ and $\sigma, \alpha = 1, 2, 3, 4$. This admits the semidirect structure $\mathcal{H}' \circledS \eta'$ with $\mathcal{H}'$ generated by $\{\varepsilon_0 L_{ij}\}$ and η' generated by $\{\varepsilon_0 K_\sigma, \varepsilon_1 Q_\alpha\}$.

Now suppose we choose some character of η' such that $\hat{\phi}^1 \neq 0$ then the stability subgroup of $\mathcal{H}'$ will be generated by the intersection of $\{\varepsilon_0 L_{ij}\}$ and the set $\{(\varepsilon_0 L_{12} + \varepsilon_0 L_{14}), (\varepsilon_0 L_{23} - \varepsilon_0 L_{34})\}$. Thus for this character choice the stability subgroup of $\mathcal{H}'$ *is just the identity I*. The resulting induced representations will not be labeled by an integer or half integer valued parameter. Thus we conclude that we require $\hat{\phi}^1 \equiv 0$.

Now consider the choice of a character such that $\hat{k}^0 \neq 0$. The obvious choice is that $\hat{k}$ does not alter the stability group, so that we put $\hat{k}^0 = (0, 0, 0, \hat{k}_4^0)$ with $\hat{k}_4^0 \neq 0$. The representations of $\mathcal{H}$ are then given by equation (52), which we can then induce to a representation of $\mathcal{G}_2$. Alternatively we can recognise that the result can be obtained in one step by considering the character $\chi_{(0, \hat{k}, 0)}$ with $\hat{k} = (0, 0, 0, \hat{k}_4^0; 0, 0, 0, \hat{k}_4^{1\cdot 2})$. The result is then the same as equation (51) with k^0 and $\hat{k}^0$ replaced by $k = k^0\varepsilon_0 + k^{1\cdot 2}\varepsilon_{1\cdot 2}$ and $\hat{k} = (0, 0, 0, \hat{k}^0\varepsilon_0 + \hat{k}^{1\cdot 2}\varepsilon_{1\cdot 2})$ respectively, Λ an element of the super Lorentz group with $L = 2$ and $\phi_{k^0, m}$ replaced by $\phi_{k, m}$. Specifically we have

$$\Phi^{\hat{k}, j}([\Lambda | t | 0])\phi_{k, m}$$
$$= \exp i\{(k\Lambda^{-1})_\alpha t^\alpha\} D^j([B(k\Lambda^{-1}, \hat{k})^{-1}\Lambda\, B(k, \hat{k}) | 0 | 0])_{mm'}\phi_{k\Lambda^{-1}, m'}. \tag{57}$$

We note that this formula is valid for all k such that $\hat{k}_4^0 > 0$ and $\hat{k}_4^{1\cdot 2} \in \mathbb{R}$, with at least one of them non-zero, it then subsumes the representations given by equation (52). The representation is complex valued even though it is written in terms of Grassman parameters. We have 'encoded' $\mathcal{G}_2$ in Grassman form. The eigenvalues of the 'translation' generators are

$$\epsilon_0 K_\sigma \phi_{k,m} = ik^0_\sigma \phi_{k,m} \tag{58}$$

and
$$\epsilon_{1\hat{\ }2} K_\sigma \phi_{k,m} = ik^{1\hat{\ }2}_\sigma \phi_{k,m} \quad . \tag{59}$$

It can now be induced to a representation of $\mathcal{G}_L$, in which case in addition to equations (58) and (59) we have

$$\epsilon_i K_\sigma \phi_{k,m} = 0 \quad \text{for} \quad i = 1,\ 2,\ \ldots\ ,\ -1;\ \epsilon_i \neq \epsilon_{1\hat{\ }2} \quad . \tag{60}$$

Type (iv). The arguments given above for type (i) can be repeated, we obtain 'massless' type representations with a spin index. We do not consider these.

Any other choice of k will result in a reduction of the (Lorentz) stability group of $k^{1\hat{\ }2}$. We do not consider these.

Case (ii) Representations obtained when inducing from a character of the form $\chi_{(0,\ 0,\ q)}$.

We can see from equation (56) that the generators of the stability group contain $\{\epsilon_0 K_\sigma\}$ and that $\{\epsilon_1 Q_\alpha\}$ are excluded as generators of the stability group, which must include some subset of $\{\epsilon_0 L_{\mu\lambda}\}$. In fact we can see that we have a repeat of the arguments of section VII. The representations we obtain will be a copy of those cited there. Since these are labeled by a set of continuous parameters none of them are suitable for representing particles with spin.

Case (iii) Representations obtained when inducing from a character of the form $\chi_{(\ell,\ 0,\ 0)}$.

We can see from equation (56) that the stability group generators must include $\{\epsilon_1 Q_\alpha\}$, and that $\{\epsilon_0 K_\alpha\}$ are excluded. The stability group is thus generated by some subset of $\{\epsilon_0 L_{\mu\lambda}\}$ and by $\{\epsilon_1 Q_\alpha\}$. *If we choose a* representation of the stability group to be such that $\phi^1 \neq 0$ then any representation we obtain will be labeled only by continuous parameters with no discrete labels. We thus choose $\phi^1 \equiv 0$ and examine the possible subsets of $\{\epsilon_0 L_{\mu\lambda}\}$ that can generate stability groups.

Recall that the action of the Lorentz subgroup on the characters $\{\ell^{1,2}_{\mu\lambda}\}$ is given by the adjoint representation of the Lorentz group, a matrix representation of which can be obtained from equation (38) and the corresponding six-vector for the reduced representation is given by equation (46). We clearly need only consider one of the 3×3 subrepresentations of this, and choose to use the top left hand one acting on the vector $(r_1^{1,2},\ r_2^{1,2},\ r_3^{1,2}) = r^{1,2}$ with $r \in \mathbb{C}$.

This representation is isomorphic to $SO(3,\mathbb{C})$ considered as a real (six parameter) Lie group. Since it is an orthogonal group it leaves invariant the quadratic form $(z)^2 = (r_1^{1,2})^2 + (r_2^{1,2})^2 + (r_3^{1,2})^2$ with $z \in \mathbb{C}$. The parameter z serves as a label for the representations (except for the case when $r_1^{1,2} = r_2^{1,2} = r_3^{1,2} = 0$ which we ignore, since it just gives us the representations of $\mathcal{G}_1$). We can identify two types of orbit.

Type (i) $z = 0$.

We choose as representative character $r = (0,\ i,\ 1)$. The stability group is then the abelian group generated by $\{(\epsilon_0 L_{12} - \epsilon\, L_{14}),\ (\epsilon_0 L_{23} + \epsilon_0 L_{34})\}$. We note that in the representation defined by equation (45) $(\epsilon_0 L_{12} - \epsilon_0 L_{14}) = i(\epsilon_0 L_{23} + \epsilon\, L_{34})$. To examine the structure of the stability group we exponentiate, using the this representation to obtain

$$\exp a((\varepsilon_0 L_{12} - \varepsilon_0 L_{14}) = \begin{bmatrix} 1 & a & ia \\ -a & 1-\frac{a^2}{2} & \frac{ia^2}{2} \\ ia & \frac{ia^2}{2} & 1+\frac{a^2}{2} \end{bmatrix},$$

with $a \epsilon \mathbb{C}$. Thus the stability group is isomorphic to $\mathbb{C}$ and has representations labeled by continuous parameters.

Type (ii) $z \neq 0$.

There are two possibilities for a representative character χ_r either $r = (0, 0, z)$ or $r = (0, 0, z)$. *In either case the stability group is isomorphic to* $SO(2,\mathbb{C})$. Consequently in this case we do obtain a character label that takes integer values, ie. the parameter corresponding to the subgroup $SO(2,\mathbb{R})$. Representations of this type are invariant on superspace, so that they are of no physical interest.

These three cases cover a large number of the possible orbits in the step $\mathcal{G}_1 \to \mathcal{G}_2$ but not all of them. The others can be obtained by choosing non-zero character representatives of the form $\chi_{(\ell, k, \phi)}$ with at least two of $\{\ell^{1,2}, k^{1,2}, \phi^{1,2}\}$ non-zero. The stability group in each of these cases will be the intersection of the stability groups of the appropriate characters of cases (i), (ii) and (iii). What is clear is that very few of these representations will have a discrete label. For this reason we do not investigate them here.

IX. CONCLUDING REMARKS.

We have considered many of the representations of $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$. If we continued to examine $\mathcal{G}_3$, $\mathcal{G}_4$, . . . we would obtain a larger number of representations at each step, many of which would be similar to the ones we have already obtained and the majority would be parametrized by continuous parameters.

It is clear that any unitary irreducible representation we obtain that can be associated with an elementary particle - ie. it has a spin index, will have only a single spin value. The representations constructed here act on a complex Hilbert space, and have no explicit Grassman structure so that they are not directly related to the Salam-Strathdee[10] representations. How this is achieved is the subject of the last paper in this series[11].

It is also clear that an examination of any supergroup in the ways described here will reach similar conclusions, ie. there will be no 'mixing' of spins within a unitary irreducible representation.

APPENDIX

Our conventions follow Cornwell[13]. We use the metric $g_{\mu\lambda} = diag(-1, -1, -1, 1)$ with $\mu, \lambda = 1, 2, 3, 4$; which is used to raise and lower indices.

For $\mu = 1, 2, 3$ $(\gamma^{\mu})^{\dagger} = -\gamma^{\mu}$ and $(\gamma^4)^{\dagger} = \gamma^4$. $\gamma^{\mu}\gamma^{\lambda} + \gamma^{\lambda}\gamma^{\mu} = 2Ig^{\mu\lambda}$. The matrix γ^5 is defined by $\gamma^5 = i\gamma^1\gamma^2\gamma^3\gamma^4$ so that $(\gamma^5)^{\dagger} = \gamma^5$ also we put $\gamma^5 = -\gamma_5$. The charge congugation matrix C is defined by $C^{-1}\gamma^{\mu}C = -(\gamma^{\mu})^t$. It is necessarily antisymmetric ie. $C^t = -C$. It is chosen such that $C^{\dagger}C = I$ and such that in our Majorana representation $C = -\gamma_4^M$.

Our Majorana representation γ^M *is given by*

$$\gamma_1^M = \begin{bmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{bmatrix}, \quad \gamma_2^M = \begin{bmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}, \quad \gamma_3^M = \begin{bmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{bmatrix},$$

$$\gamma_4^M = \begin{bmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{bmatrix}, \quad \gamma_5^M = \begin{bmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \quad C^M = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix},$$

Our chiral representation γ^C is given by

$$\gamma_j^C = \begin{bmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{bmatrix} \text{ for } j = 1, 2, 3; \quad \gamma_4^C = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix},$$

$$\gamma_5^C = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad C^C = \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix}.$$

These two representations are related by the similarity transformation S given by

$$S = S^{-1} = S^{\dagger} = \tfrac{1}{2}\begin{bmatrix} I - \sigma_2 & I + \sigma_2 \\ I + \sigma_2 & -I + \sigma_2 \end{bmatrix}.$$

So that $\gamma_{\mu}^M = S\gamma_{\mu}^C S^{-1}$, but note that $C^M = SC^C(S)^t$.

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