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U(1) Anomaly and Index Theorem for Compact and Euclidean Manifolds

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ABSTRACT

It is shown that the usual U(1)-anomaly for $\not{D}$ exists for any supersymmetric (QM) operator Δ, and that it is the value at the origin of the Laplace transform of the supersymmetric partition function $Z(t)$ (in contrast to the η-invariant which is the value at the origin of the Mellin transform of $Z(t)$). The known equality of the anomaly A, the flux Φ and the AS-index I for compact manifolds without boundaries is generalized to the case of Euclidean manifolds, where the fractional discrepancy between $A=\Phi$ and I is shown to be a sum over zero-energy phase-shifts (of the Bohm-Aharonov type). The relationship between the results for Euclidean manifolds and compact manifolds with boundaries is illustrated by using as an example the 2-dimensional Dirac operator.

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1. Introduction

Let γ_μ, $\mu=1\ldots2n$ be the (hermitian) Dirac matrices in 2n-dimensions, $\gamma=i^n\gamma_1\gamma_2\ldots\gamma_{2n}$ the generalization of Dirac's γ_5 ($\gamma^2=1$) $\not{D}=\not{\partial}+\not{A}$ the 2n-dimensional Dirac operator, and $M(\theta)=m\exp(2i\theta\gamma)$ θ=constant, the chirally covariant mass. Then it is well-known[4][5] that on compact manifolds without boundaries there is an equality

$$\Phi = A = I, \tag{1.1}$$

between the flux Φ, the anomaly A, and the Atiyah-Singer (AS) index I, defined diversely as

$$\Phi = \int dV\ \epsilon_{\alpha\beta\gamma\delta\ldots\mu\nu}F_{\alpha\beta}F_{\gamma\delta}\ldots F_{\mu\nu},\ \text{where } F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \tag{1.2}$$

and V is the 2n-dimensional volume,

$$A = \lim_{m=0} A(m^2) \quad \text{where} \quad A(m^2) = \frac{1}{2i}\frac{\partial}{\partial\theta}\ln\det(\not{D}+iM), \tag{1.3}$$

and $I = (n_+-n_-)$ where $n_\pm$ is the number of eigenstates of $\not{D}$ with zero eigenvalue and $\gamma=\pm1$. It is also known[6][7] (indeed has been previously[8] reported at these conferences) that for compact manifolds with boundaries, the equality (1.1) generalizes to

$$\Phi = A = I+\eta(0)\ , \tag{1.4}$$

where $\eta(s)$, called the η-invariant, is a contribution from the boundary.

The purpose of the present talk is to present a different generalization of (1.1), namely to (non-compact) Euclidean manifolds, in which case the formula (1.1) generalizes to

$$\Phi = A = I + \frac{\delta}{\pi} = (n_+-n_-) + \frac{1}{\pi}(\delta_+-\delta_-), \tag{1.5}$$

where $\delta_\pm$ are sums over the zero-energy phase-shifts for scattering by the Hamiltonians $H_\pm = \frac{1}{2}(1\pm\gamma)\not{D}^2$, and to consider the relationship between (1.3) and (1.4). In passing, we shall show that all these considerations apply not merely to $\not{D}^2$ but to any supersymmetric Hamiltonian[9], and that $A(m^2)$ is actually (m^2 times) the Laplace transform of the supersymmetric partition function $Z(t)$.

2. The Supersymmetric Anomaly and its Relationship to the Partition Function

It is well-known that the properties of the Dirac operator $(\not{D}+im)$ discussed in the Introduction are consequences of the fact that $\not{D}$ and M are a scalar and vector respectively with respect to rigid chiral transformations i.e.

$$e^{i\gamma\phi}\not{D}e^{i\gamma\phi} = \not{D}, \text{ and } e^{i\gamma\phi}M(\theta)e^{i\gamma\phi} = M(\theta+\phi), \text{ where } \phi=\text{constant}. \tag{2.1}$$

But since for $\not{D}$, this is equivalent to the statement that $\not{D}$ and γ anti-commute, the same will hold for any self-adjoint operator Δ that anti-commutes with γ i.e. any Δ of the form

$$\Delta = \begin{pmatrix} 0 & \Delta_- \\ \Delta_+ & 0 \end{pmatrix} \qquad \Delta_\pm = (\Delta\,)^+, \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2}$$

in the basis in which γ is diagonal, and since (2.2) is just the definition of a supersymmetric operator(9) one sees that the properties will hold for any supersymmetric operator. In particular if $\Delta_\pm$ are first-order differential operators Δ will be a quantum-mechanical (QM) supersymmetric operator(9), so for any QM soperator Δ one may define an anomaly, an index, a scattering phase-shift for $H_\pm=\Delta_\pm\Delta_\mp$ and an η-invariant, and obtain the results (1.1)(1.4)(1.5)

Let us consider, in particular, the anomaly for Δ, defined as

$$A(m^2) = \frac{1}{2i}\frac{\partial}{\partial\theta}\mathrm{tr}\ln(\Delta+iM) = \frac{1}{2}\,\mathrm{tr}(\Delta+M)^{-1}\frac{\partial M}{\partial\theta}. \tag{2.3}$$

Since $\partial M/\partial\theta = 2iM\gamma$ and $M\Delta+\Delta M^+$ is zero, this expression reduces to

$$A(m^2) = \mathrm{tr}\Big(\frac{i\Delta+M^+}{\Delta^2+M^+M}\Big)(M\gamma) = \mathrm{tr}\gamma\Big(\frac{M^+M}{\Delta^2+M^+M}\Big) = \mathrm{tr}\Big(\gamma\,\frac{m^2}{\Delta^2+m^2}\Big)$$

$$= \mathrm{tr}\Big(\frac{m^2}{m^2+\Delta_-\Delta_+} - \frac{m^2}{m^2+\Delta_+\Delta_-}\Big). \tag{2.4}$$

Indeed (2.4) may be regarded as a working definition of $A(m^2)$.

By using the working definition it is easy to relate $A(m^2)$ to the supersymmetric partition function. For if one uses the identity

$$(m^2+\Delta)^{-1} = \int_0^\infty dt\; e^{-t(m^2+\Delta)}, \tag{2.5}$$

one sees that $A(m^2)$ may also be written as

$$A(m^2) = m^2\int_0^\infty dt\; e^{-m^2 t}\,Z(t) \qquad \Big(=\int_0^\infty dy\; e^{-y}Z\big(\tfrac{y}{m^2}\big)\Big), \tag{2.6}$$

where $Z(t)$ is the supersymmetric partition function, defined as

$$Z(t) = \mathrm{tr}\,\gamma e^{-\Delta^2 t} = \mathrm{tr}\big(e^{H_- t} - e^{-H_+ t}\big), \qquad H_\mp = \Delta_\mp\Delta_\pm \tag{2.7}$$

Thus $A(m^2)/m^2$ is the Laplace transform of $Z(t)$, and, in particular, $A=A(0)=Z(\infty)$ i.e. the anomaly is just the large t (low-energy) limit of the supersymmetric partition function.

3. The Formula A=I=Φ for Compact Manifolds without Boundaries

From the working definition of $A(m^2)$ it is also easy to obtain the equality A=I of the Introduction for compact manifolds without boundary. The point is that for such manifolds the spectra of $\Delta_\mp$ and $\Delta_\pm$ are discrete, and since

$$(\Delta_\mp\Delta_\pm)f = \lambda f \Rightarrow (\Delta_\pm\Delta_\mp)g = \lambda g, \text{ where } g = \Delta_\pm f, \tag{3.1}$$

the non-zero parts of the spectra are equal. Thus only the zero eigenvalues of $\Delta_\mp\Delta_\pm$ contribute to $A(m^2)$ in (2.4) and one sees by inspection that $A(m^2)=n_+-n_-=I$ where $n_\pm$ are the multiplicities. Note that for these manifolds the result is actually true for all m^2 i.e. $A(m^2)$ is independent of m^2 (and $Z(t)$ is independent of t) so the limit $m\to 0$ is not necessary.

The combination of A=I and the general result A=Φ implies, of course, that Φ=I, and it may be of interest to verify this directly for a simple case, namely the (axisymmetric) Dirac operator on the 2-sphere S_2. Using stereographic coordinates (ρ,ϕ) where $\rho=\tan\theta/2$ and (θ,ϕ) are the polar angles, and the gauge $A_\rho=0$, the Dirac operator on S_2 may be written in the form (2.2) with

$$\Delta_\epsilon = D_1+i\epsilon D_2 = e^{i\epsilon\phi}\big[\epsilon\partial_\rho + \tfrac{\nu}{\rho}\big], \text{ where } \epsilon=\mp \text{ and } \nu=\frac{1}{i}\frac{\partial}{\partial\phi} - A_\phi = m-\Phi(\rho), \tag{3.2}$$

$\Phi(\rho)$ being the flux through the 'cap' of rim ρ.

For this operator the index equation $\Delta\psi=0$ reduces to

$$\big(\epsilon\partial_\rho + \tfrac{\nu}{\rho}\big)\psi_\epsilon = 0, \quad\text{or}\quad \frac{\partial\ln\psi_\epsilon}{\partial\ln\rho} = -\epsilon\nu, \tag{3.3}$$

or the components ϕ_ε of ϕ, where the inner-product for 2-spinors of rank s = 0, 1/2, 1,... is constructed with the measure $(1+\rho^2)^{2(s-1)}\rho d\rho$ n the usual manner. Without even solving the equation one sees rom the measure that square-integrability at $\theta=0$ (where $\rho=0$, and $\nu=m$ because $\Phi(0)=0$) and at $\theta=\pi$ (where $\rho=\infty$ and $\nu=m-\Phi$ where Φ is the otal flux) require that $\varepsilon m>-\frac{1}{2}$ and $\varepsilon(m-\Phi)<-2s+\frac{1}{2}$ respectively, nd hence, since m and Φ are integers, the necessary and sufficient ondition for square-integrability on S_2 is $0<\varepsilon m<\varepsilon\Phi-2s$. Thus $n_\varepsilon=\varepsilon\Phi-2s+1$, $n_{-\varepsilon}=0$ and the AS-index $I_s=\varepsilon(n_\varepsilon-n_{-\varepsilon})$ is $\Phi-\varepsilon(2s-1)$. In partic-lar $I_{\frac{1}{2}}=\Phi$, as required. As a bonus one sees that the contributions o I come from the angular momenta m such that $|m|<|\Phi|$ and sgn m=sgn Φ.

t may also be of interest to mention that the case s=1 occurs naturally n a completely different physical context, namely in the theory of onopole instability(3)(10), where the index $|\Phi|-1=2|q|-1$ is just the umber of negative modes of a monopole of charge q.

4. Generalized Levinson Theorem

A generalization of the formula A=1 for the case of continuous pectra of $H_\pm=\Delta_\pm\Delta_\mp$ may be obtained from the working definition (2.4) of $A(m^2)$ by writing

$$A(m^2) = (n_+-n_-) + \int\left(\frac{m^2}{m^2+\varepsilon}\right)(d\mu_+^c(\varepsilon)-d\mu_-^c(\varepsilon)), \text{ where } H_\pm^c=\int\varepsilon d\mu_\pm^c(\varepsilon), \tag{4.1}$$

i.e. where $\mu_\pm^c(\varepsilon)$ are the spectral measures for the continuum part of $H_\pm$. At first sight the measures in (4.1) do not seem to have a direct physical interpretation (like the interpretation of $n_\pm$ as bound states of $H_\pm$) but it turns out that they do indeed have such an interpretation, namely as scattering phase-shifts, at least in the case that $H_\pm^c$ can be interpreted as scattering Hamiltonians. In fact, for scattering Hamiltonians one has

$$\mu_+-\mu_- = (n_+-n_-)+\frac{1}{\pi}(\sigma_+-\sigma_-) = n_+-n_- + \frac{1}{\pi}\sum_\ell\mu(\ell)(\sigma_+^\ell-\sigma_-^\ell), \tag{4.2}$$

where $\sigma_\pm^\ell(\varepsilon)$ are the phase-shifts for scattering by $H_\pm$ at angular momentum ℓ, and $\mu(\ell)$ is just a weight-factor such as $(2\ell+1)$. The formula (4.2) is actually a special case of a general formula which relates spectral measures to phase-shifts, namely

$$\mu(\varepsilon) - \mu_0(\varepsilon) = \Sigma n_i\ \delta(\varepsilon-\varepsilon_i) + \frac{1}{\pi}\sum_\ell \mu(\ell)\delta_\ell(\varepsilon), \tag{4.3}$$

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where $\mu_0(\varepsilon)$ is the density for the free Hamiltonian and (n_i,ε_i) are the multiplicities and energies of the bound states. This formula is evidently a generalization of Levinson's theorem(11) and it shows that the phase-shifts are just the spectral densities, which evidently play the role of multiplicities for the continuum states. A direct proof of (4.3) is given in ref. 1 and an indirect but ultra-rigorous proof is given in ref. 12. A simple intuitive feeling for the result may be obtained by considering the Schrodinger equation and its derivative with respect to energy,

$$H\psi_\varepsilon=\varepsilon\psi_\varepsilon,\quad H\dot\psi_\varepsilon=\varepsilon\dot\psi_\varepsilon+\psi_\varepsilon \text{ where } \dot\psi=\frac{\partial\psi}{\partial\varepsilon} \text{ and } \psi\to\cos(kr+\delta),\ r\to\infty \tag{4.4}$$

(for a given angular momentum) and constructing the space-integral of the Wronskian. One obtains(13) in this way the identity

$$\delta_\ell(\varepsilon) + R(\varepsilon) = \int d\Omega(\psi\partial_r\dot\psi-\dot\psi\partial_r\psi) = \int dr(\psi\Delta\dot\psi-\dot\psi\Delta\psi) = \int dV\ \psi_\varepsilon^2(x)=\rho(\varepsilon), \tag{4.5}$$

where R is a remainder that contributes only to the free case and at $\varepsilon=0$, and since with the normalization of ψ given in (4.4) the quantity $\rho(\varepsilon)$ is the spectral measure $\mu(\varepsilon)$ (modulo $R(\varepsilon)$) one sees how $\delta(\varepsilon)$ is related to $\mu(\varepsilon)$.

5. Generalization of $A=(n_+-n_-)$ to Euclidean Manifolds

Once the relationship between the spectral measures and the phase-shifts is established, the generalization of the formula A=I to Euclidean manifolds (or to any manifolds, such as asymptotically Euclidean manifolds, that admit phase-shifts) is trivial. Indeed from (4.2) one has

$$A(m^2)=(n_+-n_-)+\frac{1}{\pi}\int\left(\frac{m^2}{m^2+\varepsilon}\right)d\sigma(\varepsilon), \text{ where } \sigma(\varepsilon)=\sigma_+^\ell(\varepsilon)-\sigma_-^\ell(\varepsilon)=\mu_+(\varepsilon)-\mu_-(\varepsilon) \tag{5.1}$$

and in particular

$$A = A(0)=(n_+-n_-)+\frac{1}{\pi}\Delta\sigma(0) = (n_+-n_-)+\frac{1}{\pi}\Sigma\mu(\ell)(\sigma_+^\ell(0)-\sigma_-^\ell(0)), \tag{5.2}$$

where $\Delta\sigma(0)$ denotes the jump in $\sigma(\varepsilon)$ at zero-energy i.e. the sum of the zero energy phase-shifts. Note that, in contrast to the compact case, it was necessary to take the limit $m\to0$ to obtain (5.2). (Note also that, in contrast to the conventional single-state distributions $(f,\mu(\varepsilon)f)$ the distributions $\mu(\varepsilon)$ can have non-integer (and right-handed)discontinuities because the trace is an infinite sum).

As before $A=\Phi$ and (5.2) together imply that

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$$\Phi = (n_+ - n_-) + \frac{1}{\pi}(\sigma_+(0) - \sigma_-(0)), \tag{5.3}$$

and it may be of interest to verify this directly for the 2-dimensional Dirac operator. Since the E_2-operator is the same as on S_2 except that the measure is $\rho d\rho$ instead of $(1+\rho^2)^{2(s-1)}\rho d\rho$ the number of bound states (n_+-n_-) can be inferred from the S_2 results, and is just the S_2 number minus $2(s-1)$. In particular for $S=1/2$ the index drops by one unit, and this is easily seen to correspond to the fact that the $\partial \ln\psi/\partial\ln\rho = \nu$ wave-function is no longer square-integrable for $-1<\nu<0$. However, in compensation, the condition $-1<\nu<0$ is just the one given in the Appendix for the wave-function to have the maverick phase-shift $-f\pi/2$ (instead of the generic $f\pi/2$) where f is the fractional part of Φ. Hence for this wave-function Φ loses one unit in (n_+-n_-) but gains a fractional part $(\delta_+-\delta_-)=f\pi$ from this state and thus satisfies (5.3).

The formula (5.3) is remarkable in that it incorporates (for this special case) three well-known theorems which, a priori, would appear to be unrelated, namely, (i) the Atiyah-Singer theorem (recovered when $f=1$) (ii) the Levinson theorem (recovered when $\Phi=0$), and (iii) the (supersymmetric) Bohm-Aharonov theorem (recovered when $(n_+-n_- = 0)$. It should be mentioned that formula (5.3) was found independently by some other authors also.[14]

6. The η-Invariant

The relationship between Euclidean manifolds (E) and compact manifolds with boundaries (B) is somewhat obscured by the fact that the latter are usually discussed in terms of the so-called η-invariant $\eta(s)$[6], rather than $A(m^2)$ or $Z(t)$, and hence before going on to the E-B relationship, one should first consider η. This invariant differs from $A(m^2)$ in two ways, namely it is the Mellin, rather than the Laplace, transform of a partition function $\tilde{Z}(t)$, and $\tilde{Z}(t)$ is $Z(t)-Z(\infty)$ rather than $Z(t)$. (Note that $\tilde{Z}(t)$ can be obtained from $Z(t)$ by changing the measure $\mu(\epsilon)$, which was normalized so that $\mu(\infty)=\mu(-\infty)=0$, to the measure $\mu(\epsilon)$, which is normalized so that $\mu(0)=\mu(-\infty)$ i.e. by letting $\mu(\epsilon)=\theta(\epsilon)[\mu(\epsilon)-\mu(\infty)]$). In other words the η-invariant is defined as

$$\eta(s) = \frac{-1}{\Gamma s}\int_0^\infty \frac{dt}{t}\, t^s\, \tilde{Z}(t), \quad s>0, \text{ where } \tilde{Z}(t)=Z(t)-Z(\infty). \tag{6.1}$$

Note that the integral converges, for small s at least, and that $\eta(s)$ can be defined for any partition function $Z(t)$ and thus, in itself, has nothing to do with boundaries. The main interest in $\eta(s)$ is its limit as $s\to 0$, and by separating the range of integration into $0<t<\lambda$ and $\lambda<t<\infty$ where $0<\lambda<1$, and making the transformation $t\to y=t^s$ in $0<t<\lambda$

one easily sees that the limit is just

$$\eta(0) = -\tilde{Z}(\infty) = Z(\infty) - Z(0). \tag{6.2}$$

On the other hand, from the definition of $Z(0)$ it is easy to see that $Z(0)=n_+-n_-$ and hence

$$\eta(0) = Z(\infty) - (n_+-n_-). \tag{6.3}$$

Thus $\eta(0)$ is just the fractional part of anomaly. Thus it corresponds to the phase-shift contribution for E-manifolds, and, as can be seen in ref. 6, to the boundary contribution for B-manifolds.

One can make contact with the original example[5] of APS, in which the operator Δ in (2.2) is given by $\Delta_\pm=\pm\partial_\rho+B$ on the range $\rho>0$, with boundary conditions $\psi_+(0)=0$ and $\psi'_-(0)=-\omega\psi_-(0)$ respectively, where ω is any positive eigenvalue of the x-independent operator B (and $\psi_\pm$ are interchanged for $\omega<0$), by noting that the scattering states (for $\omega>0$) are $\psi_+=\sin kx$ and $\psi_-=\sin(kx+\delta)$, where $\tan\delta=k/\omega$ and $k^2=\epsilon^2-\omega^2$. Hence by the generalized Levinson theorem one has

$$\mu(\epsilon) = \tan^{-1}\frac{k}{\omega}, \quad Z(t) = \int_0^\infty e^{-(\omega^2+k^2)t}\,\frac{\omega dt}{(k^2+\omega^2)} = \mathrm{Er}(\omega\sqrt{t}), \tag{6.4}$$

where Er denotes the error function (which vanishes at infinity).

7. Boundary Conditions for B-Manifolds

As already mentioned, the essential difference between Euclidean (E) manifolds, and compact manifolds with boundaries (B), lies not in the use of the η-invariant for the latter, but in the boundary conditions. Hence to compare the two cases we shall restrict our considerations to those B-manifolds that are embedded in E-manifolds, in particular the interiors of spheres $\rho=a$ in E_n, where ρ is the radial variable. Following APS we shall also restrict our attention to supersymmetric operators Δ such that

$$\Delta_\mp = \mp\partial_\rho + B(\rho), \tag{7.1}$$

where $B(\rho)$ is a hermitian operator, non-singular on $\rho=a$, and where, for simplicity, we have used the measure $d\rho$, with respect to which $\partial\rho$ is anti-hermitian, so that $\Delta_\mp$ are hermitian conjugates. The simplest examples for $B(\rho)$ are $B(\rho)=B$ and $B(\rho)=B/\rho$, where B is constant, the first being the APS example already considered in section 6, and the second being obtained from the 2-dimensional Dirac operator of section 3 by renormalizing the wave-functions $\psi_\pm$ with factors $\rho^{1/2}$ and $\rho^{1/2}e^{-i\psi}$ respectively (for $\rho>0$) and setting $\nu=B$.

Although the operator Δ in (7.1) is hermitian (since $\Delta_\pm$ are
rmitian conjugate) it is self-adjoint on $\rho<a$ only if the $\Delta_\pm$ are tually adjoint, and it is easy to see, using partial integration, at this will be true if, and only if,

$$\psi_+(a)\psi_-(a) = 0, \tag{7.2}$$

ere $\psi_\pm$ are the $\gamma=\pm 1$ components of the wave-function. There are ny ways to satisfy (7.2) but the way chosen by APS is to let ω note the (discrete-non-zero) eigenvalues of B(a) and to let

$$\psi_+(a)=0 \text{ for } \omega>0, \text{ and } \psi_-(a)=0 \text{ for } \omega<0. \tag{7.3}$$

te that since $\Delta_+\psi_+$ is a ψ_- and $\Delta_-\psi_-$ is a ψ_+ these conditions imply at

$$(\Delta_-\psi_-)_a=0 \text{ for } \omega>0, \text{ and } (\Delta_+\psi_+)_a=0 \text{ for } \omega<0, \tag{7.4}$$

d that these conditions, together with (7.3), imply the self-jointness of $\Delta_\mp\Delta_\pm$.

Comparison of E and B Manifolds

In order to compare E and B manifolds in the simplest way pos-ble let us restrict ourselves to the case of the 2-dimensional Dirac erator with $B\neq 0$ for $\rho<a$ and $B=0$ for $\rho>a>0$ so that there is a tural (Bohm-Aharonov type) boundary at $\rho=a$. Then we may compare e E and B cases by comparing the APS boundary conditions of the evious section with the boundary conditions at $\rho=a$ which are induced the usual L_2-conditions on the whole 2-space $0<\rho<\infty$. For con-enience we shall write the common Dirac operator in the form

$$\not{D} = \begin{pmatrix} 0 & -\partial_\rho + \frac{\nu}{\rho} \\ \partial_\rho + \frac{\nu+1}{\rho} & 0 \end{pmatrix}, \quad \not{D}^2 = \begin{pmatrix} -\Delta_\rho + (\frac{\nu+1}{\rho})^2 & 0 \\ 0 & -\Delta_\rho + \frac{\nu^2}{\rho^2} \end{pmatrix},$$

$$\Delta_\rho = \partial_\rho^2 + \frac{1}{\rho}\partial_\rho, \quad \nu=m-\Phi \tag{8.1}$$

here, in order to make contact with conventional Bessel functions, we ave reverted to the measure $\rho d\rho$. Indeed from (8.1) it is clear that he $\gamma=\pm 1$ components (f,g say) of the wave-function satisfying $\not{D}^2\psi=k^2\psi$ will be Bessel functions of argument $k\rho$ and order $\nu+1$ and ν espectively. Furthermore, in the Euclidean case they will xtrapolate, for small k, to the solutions that go like $\rho^{|\nu+1|}$ and

$\rho^{|\nu|}$ at the origin (where $\nu\to m$). Using these conditions and the APS boundary conditions (7.3)(7.4) one may construct the following comparison table:

Mfld	$m\geq 0$: $(\omega>0)$		$m\leq -1$: $n\geq 0$: $(\omega<0)$	
B	$f(a)=0$	$[(\partial-\frac{\nu}{\rho})g]_a=0$	$[(\partial-\frac{\mu}{\rho})f]_a=0$	$g(a)=0$
	$\gamma_f(a)=\infty$	$\gamma_g(a)=\nu$	$\gamma_f(a)=\mu$	$\gamma_g(a)=\infty$
E	$f(0)\sim\rho^{m+1}$	$g(0)\sim\rho^m$	$f(0)\sim\rho^n$	$g(0)\sim\rho^{n+1}$
	$f(a)\sim(ka)^{\nu+1}$	$g(a)\sim(ka)^\nu$	$f(a)\sim(ka)^\mu$	$g(a)\sim(ka)^{\mu+1}$
	$\gamma_f(a)\sim\nu+1$	$\gamma_g(a)\sim\nu$	$\gamma_f(a)\sim\mu$	$\gamma_g(a)\sim\mu+1$

In the table the symbol ~ denotes equality up to order k^2, $n=-(m+1)$ and $\mu=-(\nu+1)$ have been introduced to exhibit the symmetry between the APS-sectors $\omega \gtrless 0$, where $\omega=\nu+1/2$, and it is easily verified that $m\geq 0$ and $m\leq -1$ are equivalent to $\omega \gtrless 0$. It is assumed that the flux is a proper fraction $(0<|\Phi|<1)$ since the integer part can always be ab-sorbed in m.

From the table it is evident that the E and B conditions at $\rho=a$ are not the same but that they become the same for $k\to 0$ (even for $\gamma(a)=\infty$, in the sense that $\psi(0)=0$ and $\psi(ka)=0$ become the same). In particular the maverick case $-1<\gamma<0$ which produces the fractional part of the anomaly (as discussed in the Appendix) is the same for the E and B manifolds.

Appendix: Relationship between Phase-Shift and Boundary Condition

The relationship between the phase-shift and the boundary conditions for $H = \Delta_+\Delta_-$ was quoted in the text, and is a special case of the relationship for any radial Schrodinger equation

$$H\psi = [-\Delta_\rho^2+V(\rho)]\psi(\rho) = k^2\psi(\rho), \quad \lim_{\rho=\infty} V(\rho) = \omega^2<\infty, \tag{A.1}$$

i.e. for any radial Hamiltonian. Since any scattered solution $\psi(\rho)$ of (A.1) may be written as

$$\psi(\rho)=(\phi(\rho)e^{i\delta}+\bar{\phi}(\rho)e^{-i\delta}), \text{ where } \phi(\rho)\to\frac{e^{ik}}{\rho^c}, \ 2c=d-1, \text{ as } \rho\to\infty \tag{A.2}$$

where $\phi(\rho)$ is the solution with 'canonical' asymptote and δ is the phase-shift, one sees at once that the relationship between the phase-shift and boundary conditions at $\rho=a$ is

$$\gamma(a) = \frac{e^{i\delta}\phi'(a)+e^{-i\delta}\bar{\phi}'(a)}{e^{i\delta}\phi(a)+e^{-i\delta}\bar{\phi}(a)} \Leftrightarrow e^{2i\delta} = \frac{\gamma(a)\phi(a)-\phi'(a)}{\gamma(a)\bar{\phi}(a)-\bar{\phi}'(a)}$$

$$\text{where } \gamma(a) = \frac{\phi'(a)}{\phi(a)}, \qquad \text{(A.3)}$$

and a minus sign is dropped since δ is defined only modulo π. In particular, if $\gamma(a)$ is real (as is usual for regularity at the origin) then δ is just the phase of the quantity $\gamma(a)\phi(a)-\phi'(a)$. For example, for the APS model of sections 6, 7, the 'canonical' solution $\phi(\rho)$ is just $\exp(ik\rho)$ itself and thus, in general,

$$e^{2i\delta} = e^{2ika}\left(\frac{\gamma(a)-ik}{\gamma(a)+ik}\right). \qquad \text{(A.4)}$$

In particular, for the APS boundary condition $\gamma(0)=-\omega$ one has $\delta=\tan^{-1}k/\omega$ as before.

Our main interest here, however, is the 2-dimensional Dirac model of section 8 for which, according to (8.1), (A.1) reduces to the Bessel equation of order $\lambda=\nu+1$ or ν. From the standard asymptotic properties of Bessel functions it is easy to see that the combination

$$\phi(\rho) = e^{\frac{-i\pi\lambda}{2}} J_\lambda(k\rho) - e^{\frac{i\pi\lambda}{2}} J_{-\lambda}(k\rho) \to (2e^{\frac{-3\pi i}{4}}\sin\pi\lambda)(k\rho)^{-1/2}e^{ik\rho}, \qquad \text{(A.5)}$$

is the solution with canonical asymptote, and, by using the identity $(\partial-\lambda/\rho)J_\lambda=-J_{\lambda+1}$ it is then easy to see that the quantity $\gamma\phi-\phi'$ at $\rho=a$ is

$$\gamma(a)\phi(a)-\phi'(a) = e^{\frac{-i\pi\lambda}{2}}[(\gamma-\lambda)J_\lambda(ka)+kJ_{\lambda+1}(ka)]-e^{\frac{+i\pi\lambda}{2}}[(\gamma+\lambda)J_{-\lambda}(ka)+kJ_{-\lambda+1}(k \qquad \text{(A.6)}$$

For general k the phase of (A.6) is quite complicated, but for the limit of interest $k\to0$ it simplifies because $J_\lambda(ka)\simeq(ka)^\lambda$ and one term in (A.6) always dominates. In fact, it is easy to see that unless $\gamma^2=\lambda^2<1$ and $\gamma<0$ i.e. unless $\gamma^2=\lambda^2$ and $-1<\gamma<0$, the phase-shift is $\pm\frac{\pi\lambda}{2}$ according as $\lambda \gtrless 0$. In other words the generic phase-shift is $\frac{\pi|\lambda|}{2}$. In the exceptional case $\gamma^2=\lambda^2$, $-1<\gamma<0$ the coefficient of the leading term vanishes and (if it vanishes to order k^2) the phase-shift just reverses to $-\frac{\pi|\lambda|}{2}$. Thus the maverick (non-generic) phase-shift

$\frac{-\pi|\lambda|}{2}$ occurs only for the narrow range $-1<\gamma<0$ and $\gamma^2=\lambda^2+0(k^2)$.

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