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Effect of the Self-Induced Torsion of the Dirac Sources on Gravitational Singularities

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Abstract

The effect of the torsion induced by the Dirac field on gravitational singularities is investigated. Examples of the Dirac sources which violate the energy condition for singularity theorems in the Einstein-Cartan theory are presented. The self-induced Dirac torsion appears to prevent rather than enhance singularity formation, while the intrinsic mass of the Dirac field plays no essential role in forming or preventing singularities.

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INTRODUCTION

Recently interest in the Einstein-Cartan theory has been revived with such an expectation that the introduction of intrinsic spin effects into general relativity via the torsion term may possibly avert the singularity formation in gravitational collapse and cosmology.[1,2] As is well-known, the singularity theorems[3] show under very general assumptions that singularities cannot be prevented in general relativity insofar as a certain energy condition is satisfied. In a recent paper,[4] Hehl, von der Heyde and Kerlick obtained the dominant energy condition for singularity theorems in the Einstein-Cartan theory and showed that all known cosmological models which are free from singularities due to the effect of torsion violate the energy condition. In a subsequent paper,[5] Kerlick wrote down the energy condition for the Dirac field in the form

$$W = \sum_{\mu\nu} u^\mu u^\nu - \frac{1}{2} mc^2 \bar{\psi}\psi \geq 0 \quad , \tag{1}$$

where

$$\sum_{\mu\nu} = -\frac{1}{2}\hbar c \left[\nabla_\mu \bar{\psi} \gamma_\nu \psi - \bar{\psi} \gamma_\nu \nabla_\mu \psi \right] \quad , \tag{2}$$

∇_μ being the covariant differential operator with respect to the Christoffel connection, and made an observation that the formation of singularities will be enhanced rather than averted when the Dirac field is taken as the source for the metric and the torsion.

The purpose of this paper is to present some examples of the Dirac sources which do not satisfy the energy condition (1). The vanishing mass limit of the energy condition will also be briefly discussed. Because of the classical nature of the theory, we confine ourselves to c-number fields. As for the notations we follow Ref. 5 unless otherwise stated.

DIRAC SOURCES WITH W ≤ 0

The Dirac field ψ, serving as a source to the metric and the torsion, obeys the nonlinear equation

$$\gamma^{\mu}\nabla_{\mu}\psi+\frac{3}{8}\ell^{2}(\bar{\psi}\gamma_{\mu}\gamma_{5}\psi)\gamma^{\mu}\gamma_{5}\psi+\frac{mc}{\hbar}\psi=0, \tag{3}$$

where $\ell^{2}=8\pi G\hbar/c^{3}$. This generalized Dirac equation for the field in the self-induced torsion has not been solved exactly, but we can talk about the energy condition for certain classes of the field without specifying their explicit forms.

Let us first consider a field ψ under the dynamical constraint

$$\nabla_{\mu}\psi=\frac{3}{8}\ell^{2}(\bar{\psi}\gamma_{\mu}\psi)\psi-\frac{mc}{4\hbar}\gamma_{\mu}\psi. \tag{4}$$

This is a solution of the Dirac equation (3), as is easily checked by using the following identities for a c-number four-component field:[6]

$$(\bar{\psi}\gamma_{\mu}\psi)\gamma^{\mu}\psi=(\bar{\psi}\psi)\psi-(\bar{\psi}\gamma_{5}\psi)\gamma_{5}\psi, \tag{5a}$$

$$(\bar{\psi}\gamma_{\mu}\gamma_{5}\psi)\gamma^{\mu}\gamma_{5}\psi=(\bar{\psi}\gamma_{5}\psi)\gamma_{5}\psi-(\bar{\psi}\psi)\psi. \tag{5b}$$

The adjoint relation of the constraint (4) is

$$\nabla_{\mu}\bar{\psi}=-\frac{3}{8}\ell^{2}(\bar{\psi}\gamma_{\mu}\psi)\bar{\psi}+\frac{mc}{4\hbar}\gamma_{\mu}\bar{\psi} \tag{6}$$

Substituting (4) and (6) into (2), we obtain

$$\Sigma_{\mu\nu}=\frac{3}{8}\hbar c\ell^{2}(\bar{\psi}\gamma_{\mu}\psi)(\bar{\psi}\gamma_{\nu}\psi)-\frac{1}{4}mc^{2}\bar{\psi}\psi g_{\mu\nu}, \tag{7}$$

and hence

$$W=\frac{3}{8}\hbar c\ell^{2}(\bar{\psi}\gamma_{\mu}\psi)(\bar{\psi}\gamma_{\nu}\psi)u^{\mu}u^{\nu}-\frac{1}{4}mc^{2}\bar{\psi}\psi. \tag{8}$$

In a frame with $u^{\mu}=(1,0,0,0)$,

$$W=-\frac{3}{8}\hbar c\ell^{2}(\psi^{\dagger}\psi)^{2}-\frac{1}{4}mc^{2}\bar{\psi}\psi. \tag{9}$$

Here $(\psi^{\dagger}\psi)^{2}$, being positive-definite, varies depending on the spacetime position, whereas the bilinear scalar $\bar{\psi}\psi$, as is seen from (4) and (6), remains constant everywhere in spacetime. Therefore the energy condition (1) is violated unless the initial scalar density is taken to be negative.

The second example is concerned with the source which causes an effect similar to that of the cosmological term.[7] Consider a field constrained by

$$\nabla_{\mu}\psi=\frac{3}{32}\ell^{2}(\bar{\psi}\gamma^{\sigma}\psi)\gamma_{\mu}\gamma_{\sigma}\psi-\frac{mc}{4\hbar}\gamma_{\mu}\psi. \tag{10}$$

Again using the identities (5a) and (5b), one can show that this field also satisfies the Dirac equation (3). The adjoint of (10) is

$$\nabla_{\mu}\bar{\psi}=-\frac{3}{32}\ell^{2}(\bar{\psi}\gamma^{\sigma}\psi)\bar{\psi}\gamma_{\sigma}\gamma_{\mu}+\frac{mc}{4\hbar}\bar{\psi}\gamma_{\mu} \tag{11}$$

Substitution of (10) and (11) into (2) yields

$$\Sigma_{\mu\nu}=-\lambda g_{\mu\nu} \tag{12}$$

with

$$\lambda=\frac{1}{4}mc^{2}\bar{\psi}\psi-\frac{3}{32}\hbar c\ell^{2}(\bar{\psi}\gamma_{\sigma}\psi)(\bar{\psi}\gamma^{\sigma}\psi). \tag{13}$$

As a result, we have

$$W = -\frac{3}{32}\hbar c l^2(\bar{\Psi}\gamma_\sigma\Psi)(\bar{\Psi}\gamma^\sigma\Psi) - \frac{1}{4}mc^2\bar{\Psi}\Psi \quad . \tag{14}$$

From (10) and (11) it follows that λ in (13) is a constant in spacetime. If λ is chosen to be positive, then the stress-energy tensor (12), when it serves as a source to geometry, will play a role of the cosmological term. By making use of the identity (5a) it is easy to show that $(\bar{\Psi}\gamma_\sigma\Psi)(\bar{\Psi}\gamma^\sigma\Psi)$ is positive-definite. Thus, for $\lambda > 0$, $\bar{\Psi}\Psi$ has a positive lower limit. Computing the nucleon distribution by $\rho \sim m\bar{\Psi}\Psi$ in a case where $|\bar{\Psi}\Psi| >> |\bar{\Psi}\gamma_5\Psi|$ we observe that $\rho \sim \bar{\rho} = (4m^2c/3\hbar l^2) \sim 10^{54}g/cm^3$ for $0 < \lambda < \bar{\rho}c^2/8$. As is evident from (14), however, the energy condition is violated by this class of the Dirac field regardless of the sign of λ unless $\bar{\Psi}\Psi$ can assume a negative value.

Since the Dirac equation with the torsion effect is nonlinear in character, it is not immediately clear whether a given solution is physically significant. A solution which may seem more realistic than those presented above is the one considered by Kerlick.[5] For the Dirac field which depends only on time, he obtained under appropriate assumptions

$$W = \frac{1}{2}mc^2\bar{\Psi}\Psi + \frac{3}{8}\hbar c l^2(\bar{\Psi}\gamma_\sigma\gamma_5\Psi)(\bar{\Psi}\gamma^\sigma\gamma_5\Psi) \quad . \tag{15}$$

With the help of the identities (5a) and (5b), we can rewrite (15) as

$$W = \frac{1}{2}mc^2\bar{\Psi}\Psi - \frac{3}{8}\hbar c l^2(\bar{\Psi}\gamma_\sigma\Psi)(\bar{\Psi}\gamma^\sigma\Psi) \quad , \tag{16}$$

which is not positive-definite. For the case where $|\bar{\Psi}\Psi| >> |\bar{\Psi}\gamma_5\Psi|$, we find that $W \gtrsim 0$ if $\rho \sim m\bar{\Psi}\Psi \lesssim 10^{54}g/cm^3$. Thus, contrary to Kerlick's observation, the singularity formation may be averted at densities higher than the critical density $\bar{\rho} \approx 10^{54}g/cm^3$.

THE ZERO MASS LIMIT

Now we study the zero mass limit of the energy condition for the Dirac sources. In the limit $m \to 0$, the Dirac equation (3) becomes Heisenberg's nonlinear equation[8] defined in a Riemannian background. The vanishing mass in the Dirac equation carrying the torsion term does not mean that all solutions of the equation reduce to the neutrino field. In fact, Heisenberg looked for all possible states of matter in the nonlinear character of the equation without any presumed mass. This is a feature that differentiates the Dirac equation in the Einstein-Cartan theory from that in a Riemannian spacetime. We notice that the limiting process does shift the effective range of the Dirac source but does not alter the general nature of the energy condition for the Dirac sources. In the zero mass limit, the energy condition (1) becomes

$$W = \Sigma_{\mu\nu} u^\mu u^\nu \geqslant 0 \quad . \tag{17}$$

With the choice of $u^\mu = (1,0,0,0)$, $W = \Sigma_{00}$ which is negative for all sources we have considered.

The neutrino limit[9] may be characterized by the two-component condition; say, $\Psi = \gamma_5\Psi$. In c-number theory, the two-component condition demands not only the mass term to vanish but also the torsion term to disappear.[6] As is obvious from (5b), $(\bar{\Psi}\gamma_\mu\gamma_5\Psi)\gamma^\mu\gamma_5\Psi = 0$ if $\Psi = \gamma_5\Psi$. This implies that the c-number two-component neutrino source will not induce the torsion term. The vanishing torsion effect must have some serious effect on the energy condition.

At a glance, W in (17) appears to approach a definite value in the limit. As one can modify the original Dirac equation by adding the vanishing torsion term with an arbitrary factor, the sign of W becomes indefinite. Therefore, for the c-number neutrino field, the energy condition (17) is insignificant.

In conclusion, the self-induced torsion effect is indeed the essence of the violation of the energy condition for the Dirac sources, while the intrinsic mass of the Dirac field seems to have no essential effect on the formation or prevention of singularities. Relevance of the c-number solutions in the singularity considerations remains yet to be answered.

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