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Dilations of Irreversible Evolutions in Algebraic Quantum Theory

By

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PREFACE

The evolution of a Hamiltonian system is reversible. The evolution of a real system is not: it always returns to a state of thermal equilibrium at a temperature determined by its surroundings. The explanation of this phenomenon is the fundamental problem of statistical mechanics. Beginning around 1870 with the work of Boltzmann and Gibbs, herculean efforts have been made to solve this in the context of the classical mechanics of systems with a finite number of degrees of freedom. The main problem remains open, but some beautiful theorems have been discovered: a new branch of mathematics, Ergodic Theory, has arisen.* More recently, there has been intense activity in the context of the quantum mechanics of systems with an infinite number of degrees of freedom. Again the harvest, so far, has largely been mathematical. One line of development can be traced to the seminal paper of Ford, Kac and Mazur (1965). In particular, this paper was studied in 1970-71 by an Oxford seminar run by one of us (JTL) in collaboration with E. B. Davies. Both of us owe Brian Davies a debt of gratitude for what we have learned from him. These notes arose from a Dublin seminar which in 1975-76 studied one of his papers (Davies 1973a), and we thank [illegible], [illegible] and [illegible] for many stimulating discussions during this period. We have attempted to present a self-contained account of the mathematical results which are necessary for work in this field. We do not claim to give a complete catalogue of results; for reviews of the literature see Gorini et al. (1978a) and Davies (1976b). The first draft was written in Dublin in 1975-76. The second draft was completed in 1976-77 by one of us (JVP) while in Oslo. He is grateful to Erling Størmer and his colleagues for their warm hospitality and the stimulating atmosphere of their group.

It is a pleasure to thank Mrs. Eithne Maguire whose patience and skill in typing have produced the camera-ready copy; and Miss Evelyn White, the technical editor of this series, whose professional expertise we have relied on.

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* For a description of the present situation in an historical context, see Lebowitz and Penrose (1973).

in preparing the manuscript. Needless to say, those imperfections which remain are attributable solely to the authors.

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J. T. Lewis

Dublin 5. 11. 77.

TABLE OF CONTENTS

INTRODUCTION

The purpose of these notes is to consider the problem of whether irreversible evolutions of a quantum system can be obtained as restrictions of reversible dynamics in some larger system. In the classical theory of Markov processes, the Fokker-Planck semigroup $\{T_t : t \geq 0\}$ can be factored as $T_t = E \circ U_t \circ J$, $t \geq 0$, where J is an embedding, U_t is a group of automorphisms, and E is a conditional expectation. Is such a factorization of an irreversible evolution possible in algebraic quantum theory? In particular, we consider the mathematical formulation of this question in Hilbert space and C*-algebra settings.

Positivity is a central theme in any probability theory; the theory of noncommutative stochastic processes is no exception. In the first section we give a brief account of the theory of reproducing kernel Hilbert spaces. This allows us to give a short, unified treatment of various well-known dilation theorems, such as the Naimark - Sz.-Nagy unitary dilation of positive definite functions on groups, the GNS-Stinespring construction for C*-algebras, and related Schwarz-type inequalities, the construction of Fock space, and the algebras of the canonical commutation and anticommutation relations.

In our first attempt to construct reversible dynamics from irreversible systems we consider, in Chapter 3, the category whose objects are Hilbert spaces and whose morphisms are contractions. Here we show how one can dilate certain families of morphisms to automorphisms (unitary operators). As shown by Lewis and Thomas (1974, 1975), this is the mechanism behind the construction of the FKM-model (Ford, Kac & Mazur, 1965). This Hilbert space theory is then lifted to a C*-algebra setting using the algebras of the canonical commutation and anticommutation relations. We are thus led naturally to the C*-algebraic setting of quantum theory, where the bounded observables of the system are represented by the self-adjoint elements of the algebra, and the states by positive linear functionals.

From this point on, we concern ourselves with the category whose objects are C*-algebras, and whose morphisms are completely positive contractions.

Complete positivity is a property whose study may be motivated both by mathematical and by physical arguments. It is a much stronger property than positivity. However, for commutative C*-algebras the concepts of complete positivity and positivity coincide; for this reason the distinction does not arise in classical probability theory. It follows from the Schwarz inequality for completely positive maps that a contraction which has an inverse which is also a contraction is in fact an algebraic *-isomorphism (and hence merits the name 'isomorphism'). This is not so if one has mere positivity. Completely positive maps have an interesting physical interpretation (Kraus 1971, Lindblad 1976a). They arise physically with the study of operations on systems in interaction. We adopt the view that reversible behaviour is described by a one-parameter group of *-automorphisms of a C*-algebra, and irreversible Markovian behaviour is described by a semigroup of completely positive maps (Lindblad, 1976a).

In Chapter 8 we are concerned with the mathematical formulation of the embedding of a quantum mechanical system in a larger one, and the dual operation of restriction to a subsystem. We thus require a non-commutative analogue of the conditional expectation of classical probability theory. This will be an injection of the states of the system into those of a larger system (the Schrödinger picture), or the dual operation of averaging or projecting of the observables of a large system onto those of a subsystem (the Heisenberg picture). Thus we seek a projection N from a unital C*-algebra A onto a unital C*-algebra B. Since in the dual picture normalised states of B must go into normalised states of A, we require that N be positive and that it take 1_A to 1_B. It is shown that such a map is automatically completely positive. This observation provides us with a second argument for taking irreversible evolutions to be described by completely positive maps: the restriction to a subsystem of a reversible evolution is necessarily completely positive.

An abstract dilation theorem for completely positive maps is obtained for C*-algebras in Chapter 13. For the remainder of this work, we concentrate on W*-algebras and norm continuous semigroups of completely positive normal maps. A study of generators of such semigroups in Chapters 14 and 15 leads to a unitary dilation in Chapter 17, via the isometric representation of Chapter 16.

We have not given any account of approximate dilations involving a limiting process such as the weak coupling and the singular coupling limits. We recommend the excellent reviews by Gorini et al. (1978a) and Davies (1977a).

0. PRELIMINARIES

We give here a brief summary of the prerequisites for the main text, and establish some notation. We assume that the reader is familiar with the fundamental elements of functional analysis on Banach spaces, in particular with the theory of Hilbert spaces and algebras of operators on Hilbert spaces, such as can be found in Dunford & Schwartz (1963), Reed & Simon (1972, 1975), Yosida (1965), Dixmier (1969a,b), and Kato (1976). We work throughout with vector spaces over the complex field, although much of the work with the CAR and CCR algebras is valid on real spaces.

0.1 BANACH SPACES AND ONE-PARAMETER SEMIGROUPS

If X and Y are Banach spaces, $B(X,Y)$ denotes the Banach space of all bounded linear operators from X into Y. We write X^* for $B(X,\mathbb{C})$ and $B(X)$ for $B(X,X)$. A contraction T from X into Y is an element of $B(X,Y)$ such that $\|T\| \leq 1$. If $\|Tx\| = \|x\|$ for all x in X, then T is called an isometry.

If X is a Banach space, a one-parameter semigroup $\{T_t : t \geq 0\}$ is a map $T : \mathbb{R}^+ \to B(X)$ such that $T_0 = I$ and $T_s T_t = T_{s+t}$ for all s, t in $\mathbb{R}^+$. The semigroup is said to be strongly continuous if the maps $t \mapsto T_t(x)$ are norm continuous for each x in X, or equivalently, if $t \mapsto f(T_t(x))$ is continuous at zero for all x in X, and all f in X^* (Dunford & Schwartz 1963, p.616, Yosida 1965, p. 233). In this case, there exists a closed densely defined linear operator L such that $Lx = \lim_{t \downarrow 0} (T_t x - x)/t$ on the domain $D(L)$, and $D(L)$ is precisely the set of x in X for which this limit exists in the norm topology (Dunford & Schwartz 1963, p. 620, Yosida 1965, pp. 237, 241). The operator L is called the generator of the semigroup. The domain of L is globally invariant under the semigroup. Moreover, $\frac{d}{dt} T_t x = L T_t x = T_t L x$ for all x in $D(L)$ (Dunford & Schwartz 1963, p. 619, Yosida 1965, p. 239). Thus we write the formal symbol e^{tL} for T_t. There exist positive numbers M and ω such that $\|e^{tL}\| \leq M e^{\omega t}$ for all $t \geq 0$; for all complex λ with $\mathrm{Re}\,\lambda > \omega$ we then have that λ lies in $\rho(L)$, the resolvent set of L, and $(\lambda - L)^{-1} = \int_0^\infty e^{tL} e^{-\lambda t}\, dt$ (Dunford & Schwartz 1963, pp. 619, 622, Yosida 1965, pp. 232, 240). Conversely,

$e^{tL} = \lim_{n\to\infty} (1 - tL/n)^{-n}$ gives the semigroup in terms of the resolvent of the generator (Hille & Phillips 1957, p. 352). Moreover, e^{tL} is a contraction semigroup if and only if the following equivalent conditions hold:

For all x in $D(L)$, there exists f in X^* with $\|f\| = 1$, $f(x) = \|x\|$, and $\mathrm{Re}\, f(Lx) \le 0$. (1.1)

For all $\lambda \ge 0$ and x in $D(L)$, we have

$$\lambda \|x\| \le \|(\lambda - L)x\| . \qquad (1.2)$$

(Dunford & Schwartz 1958, p. 626; Yosida 1965, p. 249; Lumer & Phillips 1961.) The semigroup e^{tL} is norm continuous if and only if L is in $B(X)$ (Dunford & Schwartz 1958, p. 621). In this case e^{tL} can be given by the usual power series expansion $e^{tL} = \sum_{n=0}^{\infty} (tL)^n/n!$. If L is bounded, e^{tL} is a contraction semigroup if and only if $\inf\{(\|1 + tL\| - 1)/t : t > 0\} = \lim_{t \to 0} (\|1 + tL\| - 1)/t \le 0$ (Lumer & Phillips 1961).

If L generates a strongly continuous one-parameter semigroup e^{tL}, and Z is a bounded operator on X, then $L + Z$ generates a strongly continuous one-parameter semigroup $e^{t(L+Z)}$ which satisfies

$$e^{t(L+Z)}(x) = e^{tL}(x) + \int_0^t e^{(t-s)L}\, Z\, e^{s(L+Z)}(x)\, ds$$

for $t \ge 0$ and x in X (Dunford & Schwartz 1958, p. 631; Kato 1966, p. 495). The perturbed semigroup is also given by the Lie-Trotter product formula

$$e^{t(L+Z)}x = \lim_{n\to\infty} [e^{tL/n}\, e^{tZ/n}]^n(x), \quad t \ge 0,$$

for all x in X (Trotter 1959; Chernoff 1974).

1.2 Banach *-algebras and C*-algebras

A Banach algebra A is a complete normed algebra with $\|xy\| \le \|x\|\,\|y\|$ for all x, y in A. If A possesses an identity, written 1_A or 1, we require $\|1\| = 1$. In this case A is said to be unital. An approximate identity for a Banach algebra A is a net $\{u_\lambda : \lambda \in \Lambda\}$ in A such that $\|u_\lambda\| \le 1$ for all λ, and such that for each x in A we have $xu_\lambda \to x$ and $u_\lambda x \to x$ in the norm topology as $\lambda \to \infty$. A *-algebra A (also called an algebra with involution) is an algebra equipped with a conjugate-linear involutive antiautomorphism $x \mapsto x^*$. An element x in a *-algebra A is said to be self-adjoint (or hermitian) if $x = x^*$; the set of self-adjoint elements of A is denoted by A_h. Each element x in A has a unique decomposition $x = x_1 + ix_2$ with x_1 and x_2 in A_h. A linear map T between *-algebras A and B is said to be self-adjoint if $T(A_h) \subseteq B_h$, or equivalently if $T(x^*) = T(x)^*$ for all x in A. An element x in a unital *-algebra is said to be isometric if $x^*x = 1$, and unitary if both x and x^* are isometric. A Banach *-algebra is a Banach algebra with an isometric involution $x \to x^*$; e.g., if G is a locally compact group, then $L^1(G)$ with the usual operations is a Banach *-algebra with approximate identity (Loomis 1953).

A C*-algebra A is a Banach *-algebra such that $\|x^*x\| = \|x\|^2$ for all x in A. If A is a Banach *-algebra, then the algebra $\bar{A}$ obtained from A by adjoining an identity is a Banach algebra containing A as a Banach subalgebra; moreover, if A is a C*-algebra, then so is $\bar{A}$ (Sakai 1971, §1.1.7). Every C*-algebra has an approximate identity (Dixmier 1964a, §1.7.2). If T is a bounded linear map from a C*-algebra A into a Banach space, then $\|T\| = \sup\{\|Tu\| : u$ unitary in $A\}$, because A is the norm-closed convex hull of its unitaries (Russo & Dye 1966). If π is a *-homomorphism from a C*-algebra A into another C*-algebra B, then π is a contraction and $\pi(A)$ is norm closed in B; if π is faithful it is an isometry (Dixmier 1964a, §1.3.7, Sakai 1971, §§1.2.8, 1.17.4). A norm-closed *-subalgebra of a C*-algebra A is a C*-algebra, and is said to be a C*-subalgebra of A. For any Hilbert space H, the algebra $B(H)$ is a C*-algebra, and its C*-subalgebras are known as C*-algebras on H, or concrete C*-algebras. A *-representation of a *-algebra A on a Hilbert space H is a *-homomorphism from A into $B(H)$. The Gelfand-Naimark-Segal representation theorem says that every C*-algebra has a faithful representation as a concrete C*-algebra on a Hilbert space (Dixmier, 1964a, §2.6.1, Sakai, 1971, §1.16.6). If X is a locally compact Hausdorff space, then $C_0(X)$ (the space of continuous functions which vanish at infinity, equipped with the supremum norm) is a commutative C*-algebra. Conversely, every commutative C*-algebra is isomorphic to some $C_0(X)$ (Dixmier, 1964a, §1.4.1, Sakai, 1971, §§1.2.1, 1.2.2).

D.3 W*-ALGEBRAS

A W*-algebra A is a C*-algebra which is a dual Banach space (that is, there exists a Banach space F such that $A = F^*$). In this case F is uniquely determined up to isometric isomorphism, and is called the pre-dual of A, written A_* (Sakai, 1971, §1.13.3). The weak *-topology $\sigma(A,A_*)$ is also known as the ultraweak, or σ-weak (operator), topology. Every W*-algebra has an identity (Sakai, 1971, §1.7). If A is a W*-algebra and B is a $\sigma(A,A_*)$-closed *-subalgebra of A, then B is a W*-algebra with predual $A_*/B_\perp$. Here $B_\perp$ is the annihilator of B in A_* (Sakai, 1971, §1.1.4). Then B is said to be a W*-subalgebra of A. The prefix "W*-" applied, for example, to a homomorphism means a weak *-continuous homomorphism. Thus a W*-homomorphism φ from a W*-algebra A into a W*-algebra B is a weak *-continuous homomorphism, and in this case φ(A) is a W*-subalgebra of B (Sakai, 1971, §1.16.2).

When H is a Hilbert space, B(H) is a W*-algebra; the pre-dual of B(H) can be identified with the Banach space T(H) of all trace-class operators on H, under the pairing $\langle b,a \rangle = \mathrm{trace}(ab)$ of a in T(H) and b in B(H) (Sakai, 1971, §1.15.3). The W*-subalgebras of B(H) are also called W*-algebras on H. Consider a W*-algebra A on a Hilbert space H. If A contains the identity of B(H), we say that A is a *von Neumann algebra* on H. In general, the identity 1_A of A is merely a projection on H; but A can be viewed also as a von Neumann algebra on $1_A H$. If H is a Hilbert space and X a subset of B(H), then the commutant X' of X is defined as $X' = \{y \in B(H) : xy = yx, \forall x \in X\}$. If A is a *-subalgebra of B(H) containing the identity of B(H), then A is a von Neumann algebra if and only if $A = A''$ (Dixmier, 1969b, p. 42; Sakai, 1971, §1.20.3). Sakai's representation theorem says that every W*-algebra has a faithful W*-representation as a von Neumann algebra on a Hilbert space (Sakai, 1971, §1.16.7).

If A is a C*-algebra, then A^{**} is a W*-algebra, and can be identified with the von Neumann algebra generated by A in its universal representation (Sakai, 1971, §1.17.2). If T is a bounded linear map from a C*-algebra A into a C*-algebra B, then T can be uniquely extended to an ultraweakly continuous map from A^{**} into B^{**}. If B is in fact a W*-algebra then T can be uniquely extended is an ultraweakly continuous map (see W*-role 8 (Sakai, 1971, §1.21.13)).

D.4 Order

A (partial) ordering of a set is a reflexive, transitive relation, denoted by $\leq$. If V is a vector space (over the complex field, as usual), a wedge P in V is a subset satisfying $P + P \subseteq P$ and $\mathbb{R}^+ . P \subseteq P$. An *ordered vector space* is a vector space V equipped with a wedge V^+. The elements of V which are in V^+ are said to be positive. The wedge V^+ of positive elements induces an ordering $\geq$ in V: for x and y in V, $x \geq y$ if $x - y$ is in V^+. A linear map T between ordered vector spaces V and W is said to be positive if $T(V^+) \subseteq W^+$. If A is a *-algebra, we introduce the wedge A^+ of all finite sums $\sum a^*a$ with a in A; we note that $A^+ \subseteq A_h$. If A is a C*-algebra, then $A^+ = \{a^*a : a \in A\}$ and A^+ is a cone (that is, $A^+ \cap -A^+ = \{0\}$); each element a in A_h has a unique decomposition $a = a_+ - a_-$ with a_+ and a_- in A^+ and $a_+a_- = 0$ (Sakai, 1971, §1.4). A linear map T between *-algebras A and B is positive if and only if $T(a^*a) \geq 0$ for all a in A. Any positive linear map from a Banach *-algebra with approximate identity into a C*-algebra is automatically continuous (Sinclair, 1976, §13.11). Moreover, if A and B are unital C*-algebras, then a bounded linear map T from A into B, satisfying $T(1_A) = 1_B$, is positive if and only if T is of norm one (Russo & Dye, 1966).

If A is a C*-algebra, we use the notation $x_\lambda \uparrow x$ to mean that $\{x_\lambda : \lambda \in \Lambda\}$ is a net of self-adjoint elements of A, filtering upwards, with least upper bound x. Then a positive map T between C*-algebras A and B is said to be normal if $x_\lambda \uparrow x$ in A implies $Tx_\lambda \uparrow Tx$ in B. A positive map between W*-algebras is normal if and only if it is weak *-continuous (Sakai, 1971, §§1.7.4, 1.13.2).

D.5 Tensor products

If A and B are Banach spaces, we denote their algebraic tensor product by $A \otimes B$. Completions are denoted as follows: $A \hat{\otimes} B$ denotes the projective tensor product (Grothendieck, 1955). If A and B are Hilbert spaces, $A \otimes B$ denotes the Hilbert space tensor product (Reed & Simon, 1972). If (X, μ) is a measure space, and H is a Hilbert space, we let $L^2(X;H)$ denote the space of (equivalence classes of) functions $f : X \to H$ satisfying:

$\langle f(\cdot), \xi \rangle$ is measurable for all ξ in H; (0.5.1)

there is a separable subspace H_0 in H such that f(x) lies in H_0 for almost every x; (0.5.2)

$\| f(\cdot) \|$ is in $L^2(X)$. (0.5.3)

Then $L^2(X;H)$ is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle = \int_X \langle f(x), g(x) \rangle \, d\mu(x).$$

The map $f \otimes \xi \mapsto f(\cdot)\xi$ extends uniquely to a unitary map of $L^2(X) \otimes H$ onto $L^2(X;H)$ [illegible].

We define the C*- and W*-tensor products for concrete algebras as follows. Let A, B be C*-algebras on Hilbert spaces H, K; the C*-tensor product $A \otimes B$ is the C*-algebra on $H \otimes K$ generated by $A \odot B$. If A and B are W*-algebras, the W*-tensor product $A \bar{\otimes} B$ is the W*-algebra on $H \otimes K$ generated by $A \odot B$. For abstract algebras we take representations, since the definitions of C*- and W*-algebras which we have given are representation-independent (Sakai, 1971, §§1.22.5, 1.22.11).

Let (X,μ) be a decomposable measure space (that is, a direct sum of finite measure spaces). Then $L^\infty(X)$, the space of all essentially bounded, locally measurable functions, is a commutative W*-algebra, whose predual can be naturally identified with $L^1(X)$. Conversely, every commutative W*-algebra is *-isomorphic to $L^\infty(X)$, for some (X,μ) (Sakai, 1971, §1.18). The map $\pi : L^\infty(X) \to B(L^2(X))$ given by

$$(\pi(f)g)(x) = f(x)g(x)$$

is a faithful *-representation of $L^\infty(X)$ as a maximal abelian von Neumann algebra on $L^2(X)$ (Sakai, 1971, §2.9.1). Let M be a W*-algebra with separable pre-dual. Then $L^\infty(X;M)$, the space of all M-valued, essentially bounded, weak *-locally measurable functions on X, is a W*-algebra with pre-dual $L^1(X;M_*)$, the Banach space of all M_*-valued Bochner μ-integrable functions on X. Moreover, the mapping $f \otimes x \mapsto f(\cdot)x$ extends uniquely to a W*-isomorphism of the W*-algebras

$L^\infty(\Omega) \bar{\otimes} M$ onto $L^\infty(\Omega;M)$. Under this identification, the pre-dual $L^1(\Omega;M_*)$ of $L^\infty(\Omega;M)$ is naturally identified with the product $L^1(\Omega) \hat{\otimes} M_*$ of $L^\infty(\Omega) \bar{\otimes} M$ (Sakai, 1971, §1.22.13).

1. Positive-definite kernels

Throughout this chapter X denotes a set and H a Hilbert space. A map $K : X \times X \to B(H)$ is called a *kernel* and the set of such kernels is a vector space denoted by $K(X;H)$.

1.1 Definition A kernel K in $K(X;H)$ is said to be *positive-definite* if, for each positive n and each choice of vectors $h_1,\ldots,h_n$ in H and elements $x_1,\ldots,x_n$ in X, the inequality

$$\sum_{i,j} \langle K(x_i, x_j) h_j, h_i \rangle \geq 0 \tag{1.1}$$

holds.

1.2 Example Let H' be a Hilbert space, let V be a map from X into $B(H,H')$, and set

$$K(x, y) = V(x)^* V(y). \tag{1.2}$$

Then

$$\sum_{i,j} \langle K(x_i, x_j) h_j, h_i \rangle = \| \sum_j V(x_j) h_j \|^2 \geq 0 ,$$

so that K is positive-definite.

The principal result of this chapter is that a kernel K is positive-definite if and only if it can be expressed in the form (1.2).

1.3 Definition Let K be a kernel in $K(X;H)$. Let H_V be a Hilbert space and $V : X \to B(H,H_V)$ a map such that $K(x, y) = V(x)^* V(y)$ for all x, y in X. Then V is said to be a *Kolmogorov decomposition* of K; if $H_V = \vee\{V(x)h : x \in X, h \in H\}$ then V is said to be *minimal*. Two Kolmogorov decompositions V and V' are said to be *equivalent* if there is a unitary mapping $U : H_V \to H_{V'}$ such that $V'(x) = UV(x)$ for all x in X. A minimal Kolmogorov decomposition is universal in the following sense:

1.4 Lemma *Let K be in $K(X;H)$ and let V be a minimal Kolmogorov decomposition of K. Then to each Kolmogorov decomposition V' of K there corresponds a unique isometry $W : H_V \to H_{V'}$ such that $V'(x) = WV(x)$ for all x in X. Moreover, if V' is minimal then W is unitary.*

Proof. Since V is minimal the set of elements of the form $\sum_j V(x_j)h_j$ is dense in H_V. The map $W(\sum V(x_j)h_j) = \sum V'(x_j)h_j$ is well-defined and isometric since

$$\langle V(y)k, V(x)h\rangle = \langle K(x,y)h, k\rangle = \langle V'(y)k, V'(x)h\rangle ,$$

and hence it extends by continuity to an isometry $W : H_V \to H_{V'}$. The rest is routine.

We have yet to show the existence of a Kolmogorov decomposition for an arbitrary positive-definite kernel; we remedy this by constructing a decomposition canonically associated with the kernel. We employ a Hilbert space of H-valued functions spanned by those of the form $x \mapsto K(x, y)h$, using the positivity of K to get an inner product. For this purpose it is convenient to reformulate Definition 1.1. But first we need another definition.

1.5 Definition Let $F_0 = F_0(X;H)$ denote the vector-space of H-valued functions on X having finite support. Let $F = F(X;H)$ denote the vector-space of all H-valued functions on X. We identify F with a sub-space of the algebraic anti-dual $F_0^\dagger$ of F_0 by defining the pairing $g, f \mapsto (g, f)$ of F and F_0 by

$$(g, f) = \sum_{x \in X} \langle g(x), f(x)\rangle .$$

(Since f has finite support only a finite number of terms in the sum are non-zero.) Given K in $K(X;H)$ we define the associated convolution operator $K : F_0(X;H) \to F(X;H)$ by

$$(Kf)(x) = \sum_{y \in X} K(x, y) f(y).$$

Then Definition 1.1 may be reformulated as:

1.6 Definition The kernel K in $K(X;H)$ is *positive-definite if and only if the associated convolution operator* $K : F_0(X;H) \to F(X;H)$ *is positive*:

$$(Kf, f) \geq 0 \quad \text{for all } f \text{ in } F_0(X;H).$$

Next we need a vector-space result:

1.7 Lemma *Let V be a complex vector-space, and let V' be its algebraic anti-dual, with the pairing $V' \times V \to \mathbb{C}$ written $v', v \mapsto (v', v)$. Let $K : V \to V'$ be a linear mapping such that $(Kv, v) \geq 0$ for all v in V. Then there is a well-*

defined inner-product on the range-space is given by

$$\langle Av_1, Av_2 \rangle = [Av_1, v_2].$$

Proof: The sesquilinear form $v_1, v_2 \mapsto a(v_1, v_2) = [Av_1, v_2]$ is positive, so that the Schwarz inequality holds:

$$|a(v_1, v_2)|^2 \le a(v_1, v_1)\, a(v_2, v_2).$$

It follows that the set $V_0 = \{v \in V : a(v, v) = 0\}$ coincides with the subspace ker A, and so the natural projection $\pi : V \to V/\ker A$ carries the form $a(\cdot, \cdot)$ into an inner-product $\langle \cdot, \cdot \rangle_A$ on $V/\ker A$ given by $\langle \pi(v_1), \pi(v_2) \rangle_A = a(v_1, v_2)$. The vector-space isomorphism $A' : V/_{\ker A} \to AV$ given by $A' \circ \pi = A$ carries the inner-product $\langle \cdot, \cdot \rangle_A$ into an inner-product $\langle \cdot, \cdot \rangle$ on AV, given by

$$\langle Av_1, Av_2 \rangle = \langle A'\pi(v_1), A'\pi(v_2) \rangle = \langle \pi(v_1), \pi(v_2) \rangle_A$$
$$= a(v_1, v_2) = [Av_1, v_2].$$

1.8 THEOREM *For each positive-definite K in $K(X;H)$ there exists a unique Hilbert space $H(K)$ of H-valued functions on X such that*

(a) $H(K) = \bigvee\{K(\cdot, x)h : x \in X, h \in H\}$;

(b) $\langle F(x), h \rangle = \langle F, K(\cdot, x)h \rangle$ *for all F in $H(K)$, x in X and h in H.*

Proof: Since the kernel K is positive-definite the associated convolution operator K of $F_0 = F_0(X;H)$ into F^* defined in 1.4 satisfies the hypotheses of Lemma 1.7. Let $\overline{KF}_0$ be the abstract completion of KF_0 with respect to the norm got from the inner-product $\langle Kf_1, Kf_2 \rangle = [Kf_1, f_2]$, and identify KF_0 with a dense subset of $\overline{KF}_0$. For each x in X and h in H define the function h_x in F_0 by putting $h_x(y) = h$ if $y = x$ and $h_x(y) = 0$ otherwise. Then $(Kh_x)(y) = K(y,x)h$. Define K_x on H by $K_x h = Kh_x$ for all x in X and h in H. Then $\|K_x h\| \le \|K(x,x)\|^{\frac{1}{2}} \|h\|$, so that K_x is a bounded linear map from H into $\overline{KF}_0$. A straightforward calculation shows that on KF_0 we have $K_x^* F = F(x)$. The mapping of $\overline{KF}_0$ into the space of all H-valued functions on X which sends F into the function $x \mapsto K_x^* F$ is linear, injective and compatible with the identification of KF_0 with a dense subset of $\overline{KF}_0$. Thus we can regard $\overline{KF}_0$ as a Hilbert space $H(K)$ of H-valued functions on X. We have proved that $H(K)$ satisfies (a) and (b);

the reproducing property clearly holds. $H(K)$ is called the *reproducing-kernel Hilbert space determined by* K.

1.9 THEOREM *A kernel has a Kolmogorov decomposition if and only if it is positive-definite.*

Proof: It follows from Example 1.2 that a kernel having a Kolmogorov decomposition is positive-definite. If K is a positive-definite kernel, take $V(x) = K_x : H \to H(K)$ as in the proof of Theorem 1.8; then $K(x,y) = V(x)^* V(y)$; thus $(V, H(K))$ is a Kolmogorov decomposition of K. From Theorem 1.8 it is minimal.

1.10 REMARK It follows from Theorem 1.8 that a positive-definite kernel is hermitian symmetric: $K(x,y)^* = V(y)^* V(x) = K(y,x)$.

1.11 DEFINITION The set $K^+(X;H)$ of positive-definite kernels in $K(X;H)$ forms a cone; we define the induced partial ordering: put $K \geq K'$ if and only if $K - K'$ is in $K^+(X;H)$. The next result says that H is functorial:

1.12 THEOREM *Let K and K' be positive-definite kernels; then $K \geq K'$ if and only if there is a (necessarily unique) contraction $C : H(K) \to H(K')$ such that $K'_x = CK_x$ for all x in X.*

Proof: Let K, K' be in $K^+(X;H)$. Then $K \geq K'$ if and only if $(Kv, v) \geq (K'v, v)$ for all v in $F_0(X;H)$. This holds if and only if $(Kv, Kv) \geq (K'v, K'v)$ for all v in $F_0(X;H)$. This is the case if and only if there is a contraction $C : H(K) \to H(K')$ such that $K'v = CKv$ for all v in $F_0(X;H)$. The result follows by considering the generating set $\{h_x : h \in H,\ x \in X\}$ in $F_0(X;H)$, since $K'_x h = K'h_x = CKh_x = CK_x h$ for all x in X and h in H. Putting this result together with Lemma 1.4 we have:

1.13 COROLLARY *Let K and K' be positive-definite kernels with Kolmogorov decompositions V and V' respectively. Then $K \geq K'$ if and only if there is a positive contraction T in $B(H_V)$ such that*

$$K'(x,y) = V(x)^* TV(y)$$

for all x, y in X.

1.14 Theorem. *Let K be in $S^+(\Omega,\mathcal{E})$; then for each $z \in \Omega$ and each x in Ω we have*

$$K(\cdot\,,\cdot) \geq K(\cdot\,,z)(I_{\mathcal{E}} + K(z,z))^{-1} K(z\,,\cdot).$$

In particular, the Schwarz inequality holds:

$$K(\cdot\,,\cdot)\,(1 + \|K(z,z)\|) \geq K(\cdot\,,z)\,K(z\,,\cdot).$$

Proof: Let V be a minimal Kolmogorov decomposition for K. Then we have

$$K(x,z)\,(I_{\mathcal{E}} + K(z,z))^{-1} K(z,y)$$
$$= V(x)^* V(z)\,(I_{\mathcal{E}} + V(z)^* V(z))^{-1} V(z)^* V(y)$$

for all x, y, z in Ω. Thus by Theorem 1.9 it is enough to show that the operator

$$W = (I_{\mathcal{E}} + V(z)^* V(z))^{-\frac{1}{2}} V(z)^*$$

is a contraction. But

$$WW^* = (I_{\mathcal{E}} + V(z)^* V(z))^{-\frac{1}{2}} V(z)^* V(z)\,(I_{\mathcal{E}} + V(z)^* V(z))^{-\frac{1}{2}}$$
$$= (I_{\mathcal{E}} + V(z)^* V(z))^{-1} V(z)^* V(z) \leq I$$

by the spectral theorem.

2. Positive-definite functions

The principal results of this chapter are two well-known representation theorems: the Naimark–Sz.-Nagy characterisation of positive-definite functions on groups (Corollary 2.8) and the Stinespring decomposition for completely-positive maps on Banach *-algebras (Theorem 2.13). We exploit the existence and uniqueness of minimal Kolmogorov decompositions for certain functions on semigroups with involution.

2.1 Definition Let S be a semigroup, and let $J : S \to S$ be a map of S into itself such that (i) $J^2 = I_S$, (ii) $J(st) = J(t)J(s)$ for all s, t in S; then J is said to be an involution. An element s of a semigroup with involution (S,J) is said to be an isometry if

$$J(t)\,J(s)\,su = J(t)\,u \qquad (2.1)$$

for all u, t in S. The set S_J of isometries in (S,J) is a sub-semigroup.

2.2 Examples 1. Let G be a group and let $J(g) = g^{-1}$ for all g in G; then $G_J = G$.

2. Let B be a *-algebra with unit, and let $J(a) = a^*$; then $B_J = \{a \in B : a^*a = 1\}$ so that the elements of B_J are isometries in the usual sense, and the elements of $B_J \cap J(B_J)$ are the unitaries.

2.3 Definition Let H be a Hilbert space and let (S,J) be a semigroup with involution; then a function $T : S \to B(H)$ is said to be *positive-definite* if the kernel $s,t \mapsto T(J(s)t)$ is positive-definite. A *Kolmogorov decomposition* for a positive-definite function is a Kolmogorov decomposition for its associated kernel.

2.4 Example Let (G,J) be a group, as in Example 2.2(1) above. Let $\pi : G \to B(K)$ be a unitary representation of G. Let $W : H \to K$ be an isometry; then the function

$$T(g) = W^*\pi(g)W \qquad (2.2)$$

is positive-definite and has a Kolmogorov decomposition V where $V(g) = \pi(g)W$. We shall see that every positive-definite function on a group can be put in this form.

2.5 THEOREM *Let* (S,J) *be a semigroup with involution, let* $T : S \to B(H)$ *be a positive-definite function on* S*, and let* V *be a minimal Kolmogorov decomposition for* T*. Then there exists a unique homomorphism* ϕ *of* S_1 *into the semigroup of isometries on* H_V*, such that*

$$\phi(t)\, V(s) = V(ts)$$

for all t *in* S_1 *and all* s *in* S*. It follows that*

$$T(J(s)ut) = V(s)^*\, \phi(u)\, V(t)$$

for all u *in* S_1 *and all* s, t *in* S*, and that the restriction of* ϕ *to* $S_1 \cap J(S_1)$ *is a *-map:*

$$\phi(u)^* = \phi(J(u)).$$

Moreover, if S *is a topological semigroup then continuity in the weak operator topology of the map* $s \mapsto T(s)$ *entails the same for* $u \mapsto \phi(u)$*.*

Proof. For all s, t in S we have $V(us)^*V(ut) = T(J(us)ut) = T(J(s)t) = V(s)^*V(t)$ whenever u is in S_1. Hence, by Lemma 1.3, the minimality of V entails the existence of a unique isometry $\phi(u) : H_V \to H_V$, such that

$$\phi(u)\, V(s) = V(us) \qquad (2.3)$$

for all s in S. It follows from (2.3) that $\phi(u)\,\phi(u') = \phi(uu')$ for all u, u' in S_1. Now suppose that u is in $S_1 \cap J(S_1)$. Then for all s, t in S we have

$$\begin{aligned} V(s)^*\phi(u)^*V(t) &= [\phi(u)V(s)]^*V(t) = V(us)^*V(t) \\ &= T(J(us)t) = T(J(s)\,J(u)t) \\ &= V(s)^*\phi(J(u))\, V(t)\,, \end{aligned}$$

so that $\phi(u)^* = \phi(J(u))$ by uniqueness. The continuity assertion is clear.

2.6 COROLLARY *Let* G *be a group, and let* $T : G \to B(H)$ *be a positive-definite function on* G*. Then there exists a Hilbert space* H_V*, a unitary representation* $\pi : G \to B(H_V)$ *and a map* V *in* $B(H, H_V)$ *such that*

$$T(g) = V^*\pi(g)\, V \qquad (2.4)$$

for all g *in* G*. If the decomposition (2.4) is minimal then it is unique up to unitary equivalence.*

2.7 DEFINITION Let A be a *-algebra with involution $J(a) = a^*$. A map $T : A \to B(H)$ is said to be *completely positive* if it is linear and positive-definite. It follows that if V is a minimal Kolmogorov decomposition for a completely positive map then $V : A \to B(H, H_V)$ is linear.

2.8 Examples 1. Let $W : H \to H_W$ be an isometry, and let A be a *-subalgebra of $B(H_W)$. Then $T : A \to B(H)$ given by $T(a) = W^*aW$ is completely positive.
2. Let $\pi : A \to B(H)$ be a *-representation of a *-algebra A. Then π is completely positive.

2.9 Definition An algebra B with involution J is said to be a U^*-algebra if it is the linear span of $B_J \cap U(B_J)$. If B has a unit, then u is in $B_J \cap U(B_J)$ if and only if $J(u)u = 1 = uJ(u)$.

2.10 Example An element of Banach *-algebra with identity, A, can be expressed as a linear combination of four unitaries in A. Hence every such algebra is a U^*-algebra.

2.11 Theorem *Let (B,J) be a U^*-algebra, let $T : B \to B(H)$ be completely positive, and let V be a minimal Kolmogorov decomposition for T. Then there exists a unique *-representation $\pi : B \to B(H_V)$ such that*

$$\pi(a)\,V(b) = V(ab)$$

for all a, b in B. It follows that

$$T(b^*ac) = V(b)^*\pi(a)\,V(c)$$

for all a, b, c in B.

Proof: Let $\phi : B_J \cap U(B_J) \to B(H_V)$ be the *-homomorphism of Theorem 2.5. Then for a in B we have $a = \sum_{i=1}^{m} z_i u_i$, where $z_1, \ldots, z_m$ are complex numbers and $u_1, \ldots, u_m$ are in $B_J \cap U(B_J)$, and $\pi(a) = \sum_{i=1}^{m} z_i \phi(u_i)$. Then $\pi(a)V(b) = V(ab)$ for all b in B, so that π is a well-defined *-homomorphism from B into $B(H_V)$.

From this follows the Stinespring decomposition for a completely positive map on a unital C^*-algebra.

2.12 Corollary *Let A be a unital C^*-algebra and let $T : A \to B(H)$ be completely positive. Then there exists, uniquely up to unitary equivalence, a *-representation π of A on a Hilbert space H_V and a bounded linear map $V : H \to H_V$ such that*

$$T(a) = V^*\pi(a)V$$

for all a in A and $H_\pi = \bigvee\{\pi(a)Vh : a \in A,\ h \in H\}$.

Stinespring decompositions can also be obtained for more general algebras (for example, for some non-unital algebras) in such a way that the Stinespring representation is actually defined on a larger algebra. Rather than give the details in very abstract situations, we give an example of an extension of Stinespring's theorem. The result is quite adequate for our needs; the proof illustrates the essential technique.

2.13 Theorem. *Let A be a Banach *-algebra with approximate identity, and let T be a completely positive map from A into B(H). Then there exists, uniquely up to unitary equivalence, a Hilbert space* H_π, *a *-representation π of A on* H_π, *and a map V in* $B(H,H_\pi)$ *such that*

$$T(a) = V^*\pi(a)V$$

for all a in A, and

$$H_\pi = \bigvee\{\pi(a)Vh : a \in A,\ h \in H\}.$$

Proof: Let ψ be a minimal Kolmogorov decomposition for T, and let A' denote the unital Banach *-algebra obtained from A by adjoining an identity. Then A is an ideal in A' and

$$\psi(ua)^*\psi(uc) = T((ua)^*(uc)) = T(a^*c) = \psi(a)^*\psi(c)$$

for all a, c in A and all unitaries u in A'. Hence, since A' is a U*-algebra, there exists a unique representation π' of A' on H_ψ such that $\pi'(b)\psi(a) = \psi(ba)$ for all b in A' and a in A. Let π denote the restriction of π' to A. It follows from B0.4 that T is bounded and hence so is ψ(·), since $\|\psi(a)\|^2 = \|T(a^*a)\|$ for all a in A. We identify $B(H,H_\psi)$ with the dual of the space of trace-class operators from H_ψ into H. Let $\{u_\lambda\}$ be an approximate identity for A; then the net $\{\psi(u_\lambda)\}$ is bounded in $B(H,H_\psi)$ and so has a weak *-limit V say. We see that $\pi(a)V = \lim \pi(a)\psi(u_\lambda) = \lim \psi(au_\lambda) = \psi(a)$ for all a in A. The result follows.

Note that the above theorem applies to a non-unital C*-algebra and to the group algebra $L^1(G)$ of a locally compact group G. It is apt at this point to discuss the intimate relationship between positive-definite functions on groups and completely positive maps on algebras, and in particular the relationship between the Naimark-Sz.-Nagy construction and the Stinespring decomposition. In the first place, consider a unital C*-algebra A, and let U denote a subgroup of its group of unitaries such that Lin(U) = A. Clearly a completely positive map on A restricts to a positive-definite function on U. Conversely, if T is a linear map on A such that its restriction to U is positive-definite, then T is completely positive. For if a_i, $i = 1, \ldots, n$, are elements of A, then there exist complex numbers λ_{mi} and elements u_m of U, $m = 1, \ldots, N$, such that $a_i = \Sigma_m \lambda_{mi} u_m$ (since $A = \mathrm{Lin}(U)$). From the linearity of T we have

$$T(a_i^* a_j) = \Sigma_{p,q} \bar{\lambda}_{pi} T(u_p^{-1} u_q) \lambda_{qj} ;$$

regarding the right-hand side as a matrix-element of the product of three matrices, we see that $[T(a_i^* a_j)]$ is positive since $[T(u_p^{-1} u_q)]$ is. Moreover, T is a homomorphism if and only if its restriction to U is a unitary representation. Thus the restriction map takes the Stinespring decomposition into the Naimark-Sz.-Nagy representation.

This connection can be taken further. Suppose G is a locally compact group, and T is a strongly continuous positive-definite function on G taking values in B(H). Then it is easy to verify that

$$T'(f) = \int_G f(g)\, T(g)\, dg ,$$

where dg is a left-invariant Haar measure on G, defines a completely positive map T' of the Banach *-algebra $L^1(G)$ into B(H). Moreover it can be shown, using the existence of an approximate identity for $L^1(G)$, that each completely positive map on $L^1(G)$ arises in this way. T' is a homomorphism of $L^1(G)$ if and only if T is a unitary representation of G. Thus the (minimal) Naimark-Sz.-Nagy representation of T on G (Corollary 2.7),

$$T(g) = V^* W(g)\, V,$$

gives the (minimal) Stinespring decomposition of $L^1(G)$ (Theorem 2.13),

$$T'(f) = V^* W'(f)\, V,$$

and vice-versa.

5. Dilations of semigroups of contractions

In this chapter we discuss some dilation theorems for semigroups of operators on Hilbert space. They are of two kinds, one typified by Cooper's Theorem (5.1), the other by Sz.-Nagy's Theorem (5.2). In §10 we will present yet a third kind.

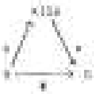
5.1 Theorem *Let $\{T_s : s \in \mathbb{R}^+\}$ be a strongly continuous semigroup of isometries on a Hilbert space H. Then there exists a Hilbert space H_U, a unitary group $\{U_s : s \in \mathbb{R}\}$ on H_U, and an isometry $V : H \to H_U$, such that $VT_s = U_sV$ for all s in $\mathbb{R}^+$.*

If we assume less about $\{T_s\}$ we get the weaker result:

5.2 Theorem *Let $\{T_s : s \in \mathbb{R}^+\}$ be a strongly continuous semigroup of contractions on a Hilbert space H. Then there exists a Hilbert space H_U, a unitary group $\{U_s : s \in \mathbb{R}\}$ on H_U, and an isometry $V : H \to H_U$ such that $T_s = V^*U_sV$ for all s in $\mathbb{R}^+$.*

We now discuss the extent to which the results of Theorems 5.1 and 5.2 generalize when $\mathbb{R}^+$ is replaced by an arbitrary abelian semigroup S. We will obtain Theorem 5.1 as a special case of Theorem 5.4 and Theorem 5.2 as a special case of Theorem 5.11. Finally, we show (Theorem 5.12) that when the semigroup $\{T_s\}$ in the statement of Theorem 5.2 is strongly contracting to zero, the unitary group $\{U_s\}$ satisfies an abstract Langevin equation. Only Theorems 5.1, 5.2 and 5.12 will be required in the applications to irreversible evolutions.

In this chapter each abelian semigroup S is assumed to have a zero. We are given a homomorphism $T : S \to L(H)$ of S into the semigroup of isometries on a Hilbert space H. We want to use T to construct a homomorphism $U : S \to U(H_U)$ of S into the group of unitaries on some Hilbert space H_U and to examine its relation to T. Now to each abelian semigroup S there corresponds a group $K(S)$ and a homomorphism $k : S \to K(S)$ which is universal in the sense that every homomorphism of S into a group G factors through $K(S)$: there exists a unique homomorphism α such that the diagram commutes.



The first step, then, is to use T to construct a homomorphism from $K(S)$ into the group of unitaries on some Hilbert space. It turns out that this is always possible. First we recall the construction of $K(S)$.

3.3 DEFINITION Let S be an abelian semi-group. Let $\Delta : S \to S \times S$ be the diagonal map, and let $\pi : S \times S \to S \times S/\Delta(S)$ be the natural projection. Then $S \times S/\Delta(S)$ is a group (since $\pi(s,t) + \pi(t,s) = \pi(0,0)$, every element has an inverse), which is called the Grothendieck group of S, and denoted $K(S)$. The map $s \mapsto \pi(0,s)$ is a homomorphism, which we denote by $\gamma_S : S \to K(S)$. If S is itself a group then γ_S is an isomorphism. The construction is functorial: if $\alpha : S \to S'$ is a homomorphism of semi-groups, then there is a unique homomorphism $K(\alpha) : K(S) \to K(S')$ such that the diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\gamma_S} & K(S) \\ \alpha \downarrow & & \downarrow K(\alpha) \\ S' & \xrightarrow{\gamma_{S'}} & K(S') \end{array}$$

The universal property of $(\gamma_S, K(S))$ follows from this. The homomorphism γ_S is injective if and only if the cancellation law holds in S: $s + u = t + u$ implies that $s = t$. When S is a topological semi-group we give $K(S)$ the quotient topology: this makes γ_S continuous.

3.4 THEOREM *Let S be an abelian semi-group. Let $\gamma : S \to K(S)$ be the canonical homomorphism of S into the Grothendieck group of S. Let $T : S \to B(H)$ be a homomorphism of S into the semi-group of isometries on a Hilbert space H. Then there is a positive-definite function F' on $K(S)$ such that*

$$F'_{\gamma(t)-\gamma(s)} = T_s^* T_t \qquad (3.1)$$

for all (s, t) in $S \times S$.

Proof. Consider the function $s,t \mapsto T_s^* T_t$ on $S \times S$. Since T_u is an isometry we have $T_u^* T_u = I$ and the function is constant on $\Delta(S)$-cosets and determines a unique function F' on $K(S)$ such that (3.1) holds. To prove that F' is positive-definite, consider a fixed n-tuple $x_1, \ldots, x_n$ in $K(S)$ and choose fixed representations (s_i, t_i) of x_i, $i = 1, \ldots, n$.

Put $s_1' = t_1 + t_2 + \dots + t_n$;

$s_2' = s_1 + t_2 + \dots + t_n$;

$\dots \qquad \dots$

$s_n' = s_1 + s_2 + \dots + t_n$;

then $s_j + t_i = s(s_i', s_j')$

so that $T'_{s_j + t_i} = T^*_{s_i'} T_{s_j'}$;

and it is clear that T' is positive-definite.

3.5 DEFINITION A semi-group homomorphism $T : S \to B(H)$ of an abelian semi-group into the bounded operators on a Hilbert space H such that $T_0 = I$ is said to have a *unitary dilation in the strong sense* if there exists a Hilbert space H_U, an isometry $V : H \to H_U$, and a unitary representation $U : A(S) \to B(H_U)$ of the Grothendieck group of S such that

$$VT_s = U_{q(s)} V . \qquad (3.2)$$

The relation (3.2) implies the weaker

$$T_s = V^* U_{q(s)} V \qquad (3.3)$$

since V is an isometry. If (3.3) holds we say that T has a unitary dilation.

3.6 THEOREM *Let S be an abelian semi-group and let $T : S \to B(H)$ be a homomorphism such that $T_0 = I$. Then T has a unitary dilation (U, V) in the strong sense if and only if T_s is an isometry for all s in S. If (U, V) is minimal then it is unique up to a unitary equivalence. If S is a topological semi-group then the continuity of $s \mapsto T_s$ in the weak-operator topology implies the same for $s \mapsto U_s$.*

Proof: The "only if" part is obvious since the $U_{q(s)}$ and V are isometries. If the T_s are isometries then it follows from Theorem 3.4 that the associated function T' on $A(S)$ is positive-definite. The remainder of the proof follows the lines of that of Theorem 2.3 and its Corollary, but the particular form (3.1) of T' yields more: (3.2) holds. It is enough to use the minimal Kolmogorov decomposition T' associated with the reproducing kernel Hilbert space $H(T')$ of the

Lemma 3.? [illegible]. We take H_0 to be [illegible] and the isometry $V : H \to H_0$ to be

$$[illegible] = T^*_{-s} h$$

for all s in [illegible]. The representation $\pi : S(S) \to B(H_0)$ is given by

$$(\pi_s\, [illegible])(s') = [illegible]_{s+s'} h;$$

using (3.?) we get (3.?) when s = [illegible].

Turning to semigroups of contractions which are not necessarily commuting, we ask if they have a unitary dilation (in the sense of (1.3)). To adapt the proof of Theorem 3.3 to this case we have to assume more about S.

3.7 Remark The following two properties of S are equivalent:

(i) $S(S) \cap [-S(S)] = \{0\}$, (3.4)

(ii) If s, t, u, v are in S and

$s + u = v$ and $u + t = s$ [illegible],

then $s = u + w$ for some w in S. (3.5)

3.8 Theorem *Let S be an abelian semigroup for which (3.4) holds, and let $T : S(S) \to B(H)$ satisfy*

(i) $T_0 = I$,

(ii) $T_s^* = T_{-s}$ for all s in $S(S)$,

(iii) $T_s T_{s'} = T_{s+s'}$ whenever s, s' and $s+s'$ are not in $[-S(S)]$.

Then T is positive-definite if and only if T_s is a contraction for each s in $S(S)$; in which case T has a unitary dilation.

Proof. Choose a fixed n-tuple of elements $s_1, \ldots, s_n$ of $S(S)$, ordered so that $s_j - s_i$ is not in $[-S(S)]$ if $i < j$. Consider the $n \times n$ matrix with entries

$$t_{ij} = T_{s_j - s_i} ,$$

and define

$$r_{ij} = \begin{cases} t_{ij} & \text{if } i < j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_{ii} = [illegible](I - t^*_{i-1,i}\, t_{i-1,i})^{[illegible]}, \quad i > 1,$$

$$d_{11} = I.$$

We claim that

$$t = u^* s u. \qquad (3.6)$$

Since t is positive if and only if [illegible], and s is positive if and only if the [illegible] are nonnegative. It remains to prove (3.6). Notice that $t_{ij} t_{jk} = t_{ik}$ whenever $i \le j \le k$, and that $t^*_{ij} = t_{ji}$. If $i \le j$ then

$$(u^* s u)_{ij} = \sum_{k=1}^{i} (u^*)_{ik} (su)_{kj}.$$

Thus for $i = 1$ we have

$$(u^* s u)_{1j} = u^*_{11} s_{11} u_{1j} = t_{11} t_{1j} = t_{1j}.$$

For $j \ge i > 1$ we have

$$(u^* s u)_{ij} = \sum_{k=1}^{i} (u^*)_{ik} s_{kk} u_{kj}$$

$$= \sum_{k=1}^{i} t^*_{ki} s_{kk} t_{kj}$$

$$= t^*_{1i} t_{1j} + (t^*_{2i} t_{2j} - t^*_{1i} t_{1j})$$

$$+ \dots$$

$$+ (t^*_{ii} t_{ij} - t^*_{i-1,i} t_{i-1,j})$$

$$= t_{ij}.$$

This establishes that [illegible]. The existence of the unitary dilation follows from Corollary 2.6.

3.9 Remark The following conditions on S are equivalent:

(i) [illegible] (3.7)

(ii) whenever s, t are in S there exist u, v, w in S such that

either [illegible]
or [illegible] (3.8)

3.10 Definition We say that an abelian semigroup S is totally ordered if (3.6) and (3.7) hold.

3.11 Theorem Let S be a totally ordered abelian semigroup, and let [illegible] be a homomorphism satisfying (i) $T_0 = I$, (ii) $\|T_s\| \le 1$, and the condition (iii) [illegible] if [illegible] then $T_s = T_t$.

Then there is a unique positive-definite function T' *on* $S(G)$ *such that*

$$T'_{\gamma(s)} = T_s \quad \text{and} \quad T'_{-\gamma(s)} = T_s^* \tag{3.9}$$

for all s *in* S*; hence* T *has a unitary dilation.*

Proof: Since (3.4) and (3.7) hold, there is a well-defined function T' on $S(G)$ which is uniquely determined by (3.9). It is easy to check that T' satisfies conditions (i), (ii) and (iii) of Theorem 3.8; the result follows.

We began this chapter by looking at one extreme case of a semi-group of contractions, where the contractions preserve the norm of each vector. We end the chapter with a look at the opposite extreme, in which the norm of each vector goes to zero eventually under repeated action of each contraction. In this case, a minimal unitary dilation of the semi-group satisfies an abstract Langevin equation.

3.12 Definition Let S be a locally compact semi-group. A semi-group $\{T_s : s \in S\}$ of contractions on a Hilbert space H is said to contract strongly to zero (at infinity) if for all h in H we have

$$\lim_{s \to \infty} \| T_s h \| = 0.$$

First we require an alternative construction of a unitary dilation of a semi-group of contractions over $\mathbb{R}^+$, which contracts strongly to zero.

3.13 Theorem *Let* $\{T_t : t \in \mathbb{R}^+\}$ *be a strongly continuous semi-group of contractions on a Hilbert space* H *which contracts strongly to zero. Then there is a Hilbert space* K *and an isometry* $W : H \to L^2(\mathbb{R}, K)$ *such that*

$$T_t = W^* U_t W, \quad t \geq 0, \tag{3.10}$$

where $\{U_t : t \in \mathbb{R}\}$ *is the strongly continuous unitary group of right-translations on* $L^2(\mathbb{R}, K)$*:*

$$(U_t f)(x) = f(x - t).$$

Proof: Let A denote the infinitesimal generator of T_t. Since $t \mapsto \| T_t h \|^2$ is monotonic decreasing we have, for all h in $D(A)$,

$$-\langle Ah, h \rangle - \langle h, Ah \rangle = -\frac{d}{dt}\Big|_{t=0} \langle T_t h, T_t h \rangle \geq 0. \tag{3.11}$$

Let N_0 denote the null space of this quadratic form:

$$N_0 = \{h \in D(S) \mid \langle Sh, h\rangle + \langle h, Sh\rangle = 0\};$$

let Λ be the quotient map of $D(S)$ onto $D(S)/N_0$. Then by (3.11) and the Schwarz inequality there exists an inner product $\langle \cdot, \cdot \rangle_0$ on $D(S)/N_0$ such that

$$\langle \Lambda h, \Lambda k\rangle_0 = -\langle Sh, k\rangle - \langle h, Sk\rangle \qquad (3.12)$$

for all h, k in $D(S)$. Let K denote the separable Hilbert space got by completing $D(S)/N_0$. Then, for all h in $D(S)$ and $t \geq 0$, we have by (3.11) and (3.12)

$$\int_t^\infty \|\Lambda T_s h\|_0^2 \, ds = \|h\|^2 - \|T_t h\|^2 . \qquad (3.13)$$

Letting $t \to \infty$, remembering that T_t converges strongly to zero, we see that there is an isometric embedding W of H in $L^2(\mathbb{R}^+; K)$ given on $D(S)$ by

$$(Wh)(s) = \Lambda T_s h \qquad (3.14)$$

for all $s \geq 0$.

We regard $L^2(\mathbb{R}^+; K)$ as a subspace of $L^2(\mathbb{R}; K)$ in the obvious way. Then we have, for each h in $D(S)$ and $t \geq 0$,

$$(U_t Wh)(s) = \begin{cases} \Lambda T_{s-t} h, & s \geq t, \\ 0, & s < t \end{cases}$$

$$= (WT_t h)(s) + u_t(s),$$

where u_t is in $L^2(\mathbb{R}^+; K) \ominus WH$. Thus for each $t \geq 0$ we have

$$T_t = W^* U_t W,$$

so that U_t is a unitary dilation of T_t on $H_0 = L^2(\mathbb{R}; K)$. It will be shown later that this dilation is minimal. It is, in fact, a consequence of the Langevin equation (3.17) which we now propose to study.

Let $\xi : \mathbb{R} \to B(K, H_0)$ be the map given by

$$(\xi_t k)(s) = \begin{cases} \chi_{[0,t]}(s) k, & t \geq 0, \\ -\chi_{[t,0]}(s) k, & t < 0, \end{cases}$$

for each k in K, where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$ in $\mathbb{R}$. Then ξ is a minimal Kolmogorov decomposition of the positive definite kernel $s, t \mapsto s \wedge t \, 1_K$ on $\mathbb{R} \times \mathbb{R}$:

$$\xi_s^* \xi_t = s \wedge t \, 1_K . \qquad (3.15)$$

for all s, t in $\mathbb{R}$. The following lemma is useful in proving Theorem 3.15.

3.14 Lemma *Let $\{T_t : t \in \mathbb{R}^+\}$ be a strongly continuous semi-group of contractions on a Hilbert space H, and let B be its generator. Then $D(B)$ may be regarded as a Hilbert space with respect to the norm given, for h in $D(B)$, by*

$$|h|^2 = \|h\|^2 + \|Bh\|^2 , \qquad (3.16)$$

in which $D(B^2)$ is dense.

Proof: Since the generator B is a closed operator, its domain $D(B)$ is a Hilbert space with respect to the norm (3.16). On it we define the semigroup $S_t = h \mapsto T_t h$. The strong continuity of $t \mapsto T_t$ implies the same for S_t; hence the domain of the generator of S_t is dense, and the proof is completed.

3.15 Theorem *Let $\{T_t = e^{tB} : t \in \mathbb{R}^+\}$ be a strongly continuous semi-group of contractions, contracting strongly to zero, on a Hilbert space H. Let $\{U_t\}$ be a minimal unitary dilation of T_t. Then there exists:*

(i) *a Hilbert space K, and a bounded linear operator*
$A : (D(B), |\cdot|) \to K$,

(ii) *a map $\xi : \mathbb{R} \to B(K, \mathcal{K})$ satisfying*

$$\xi_s^* \xi_t = [illegible]$$

for s, t in $\mathbb{R}$ and

$$\mathcal{K} = \vee\{\xi_s k : s \in \mathbb{R},\ k \in K\} ,$$

such that

$$U_t Wh - U_s Wh = \int_s^t du\, U_u WBh + (\xi_t - \xi_s)Ah \qquad (3.17)$$

for all h in $D(B)$.

Proof: Take for $\{U_t\}$ the dilation of Theorem 3.13; take for the map ξ the minimal Kolmogorov decomposition ([illegible]); then (3.17) is easily verified by integration-by-parts for h in $D(B^2)$ and hence, by Lemma 3.14 for all h in $D(B)$. That the dilation $\{U_t\}$ is minimal now follows from (3.17) and the minimality of ξ.

3.16 Remark It is also possible to treat the semi-group $\mathbb{N}$ using this procedure. In this case, let T be a contraction on the Hilbert space H such that the semi-group $\{T^n : n \in \mathbb{N}\}$ contracts strongly to zero at infinity. We can show that

$$\sum_{j=-\infty}^{\infty} \| D_T T^{j} h \|^2 = \| h \|^2 , \qquad (3.40)$$

for all h in H, where $D_T = (1 - T^*T)^{\frac{1}{2}}$. We take $N = \overline{D_T H}$ and $\kappa : H \to N$ the map given by $\kappa h = D_T h$. We embed H isometrically in $H_U = l^2(\mathbb{Z}; N)$ by

$$(uh)(j) = \begin{cases} \kappa T^{-j} h , & j \le 0 \\ 0 , & j > 0 \end{cases}$$

The unitary group $\{U^n : n \in \mathbb{Z}\}$ is defined on $l^2(\mathbb{Z})$ by translations

$$(U^n f)(j) = f(j - n)$$

for j, n in $\mathbb{Z}$ and f in H_U. Then $\{U^n\}$ is a minimal unitary dilation for T. We now define $\zeta_n : \mathbb{Z} \to B(N, H_U)$, as in the continuous case, so that

$$\zeta_m^* \zeta_n = \delta_{mn} 1_N$$

for all m, n in $\mathbb{Z}$, and

$$H_U = \bigvee \{ \zeta_n x : n \in \mathbb{Z} , x \in N \} .$$

In this case we have the discrete Langevin equation

$$U^n uh = U^m uh + \sum_{j=m}^{n-1} U^j u(T - 1)h + (\zeta_n - \zeta_m) \kappa h , \qquad (3.41)$$

valid for all h in H.

4. C^*-ALGEBRAS AND POSITIVITY

The main results in this chapter concern a positive linear map T from one C^*-algebra A into another C^*-algebra B. If either A or B is commutative, then T is completely positive (Theorems 4.3 and 4.2). This allows us to deduce certain Schwarz-type inequalities in Corollary 4.4, and the identities of [illegible] in Corollary 4.5. In the proofs we make use of a characterisation of the positivity of an element of the matrix C^*-algebra $M_n(A)$ over a C^*-algebra A (Lemma 4.1).

We end by deriving the canonical decomposition of a normal completely positive map on a von Neumann algebra (Theorem 4.8).

If A is a $*$-algebra, and n is a positive integer, we let $M_n(A)$ denote the $*$-algebra of all $n \times n$ matrices over A under the natural operations. If $\{e_{ij} : 1 \leq i, j \leq n\}$ is a system of matrix units for $M_n \equiv M_n(\mathbb{C})$, then the $*$-algebraic isomorphism $[a_{ij}] \mapsto \sum a_{ij} \otimes e_{ij}$ allows us to identify $M_n(A)$ with the algebraic tensor product $A \otimes M_n$. If A is a C^*-algebra, represented say on a Hilbert space H, then $M_n(A)$ is also a C^*-algebra and can be faithfully represented on $H^n = H \oplus \ldots \oplus H \cong H \otimes \mathbb{C}^n$ as follows:

$$[a_{ij}]_{i,j=1}^n (\xi_j)_{j=1}^n = \left(\sum_{j=1}^n a_{ij}\xi_j\right)_{i=1}^n , \quad [a_{ij}] \in M_n(A),\ (\xi_j) \in H^n .$$

Let A and B be $*$-algebras, and let T be a linear map from A into B. Let T_n denote the product mapping $T \otimes \iota_n$ from $M_n(A)$ into $M_n(B)$ where ι_n denotes the identity mapping on $M_n(\mathbb{C})$. Then T_n acts elementwise on each matrix over A,

$$T_n : [a_{ij}] \mapsto [T(a_{ij})] .$$

Suppose now B is a C^*-algebra. Then T_n is positive (§1.4) if and only if $T_n(a^*a) \geq 0$ for each a in $M_n(A)$. Now if $a = [a_{ij}] \in M_n(A)$, a^*a is the sum $\sum_{k=1}^n [a_{ki}^* a_{kj}]$. Thus T_n is positive if and only if $[T(a_i^* a_j)]$ is a positive matrix for all $a_1, \ldots, a_n$ in A. In particular, T completely positive is equivalent to T_n positive for all $n \geq 1$. It would thus seem useful to study the order structure of matrix algebras more closely.

4.1 LEMMA. *Let A be a C^*-algebra, and $a = [a_{ij}]$ be an element of $M_n(A)$.*

(a) *The following conditions are equivalent:*

(i) $a \geq 0$;

(ii) a *is a finite sum of matrices, each of the form* $(b_i^* b_j)$ *where* $b_1, \dots, b_n \in A$;

(iii) $\sum a_i^* a_{ij} a_j \geq 0$, *for all sequences* $a_1, \dots, a_n$ *in* A.

(b) *If A is commutative, then the above three conditions are also equivalent to:*

(iv) $\sum a_{ij} \bar{z}_i z_j \geq 0$, *for all sequences* $z_1, \dots, z_n$ *in* $\mathbb{C}$.

(c) *If for the C*-algebra A condition (iv) is equivalent to conditions (i) - (iii), then A must be commutative.*

Proof:

(a) (i) ⇔ (ii) has already been observed;

(ii) ⇒ (iii) is trivial;

(iii) ⇒ (i): If we represent A on a Hilbert space H, we can decompose H into cyclic orthogonal subspaces. Thus we can assume H has a cyclic vector $f \in H$. Then

$$\sum \langle a_{ij} a_j f, a_i f \rangle = \langle (\sum a_i^* a_{ij} a_j) f, f \rangle \geq 0 ,$$

for all $a_1, \dots, a_n$ in A. Thus, since f is cyclic, $\sum \langle a_{ij} f_j, f_i \rangle \geq 0$ for all $f_1, \dots, f_n$ in H. That is, $[a_{ij}]$ is positive.

(b) (iv) ⇔ (iii): Represent A as $C_0(X)$, the continuous functions vanishing at infinity on a locally compact Hausdorff space X.

Then $\sum a_{ij} \bar{z}_i z_j \geq 0$, for all $z_1, \dots, z_n \in \mathbb{C}$,

⇔ $\sum a_{ij}(x) \bar{z}_i z_j \geq 0$, for all $z_1, \dots, z_n \in \mathbb{C}$, $x \in X$,

⇔ $[a_{ij}(x)] \geq 0$ (in $M_n(\mathbb{C})$), for all $x \in X$,

⇔ $\sum a_{ij}(x) \overline{a_i(x)} a_j(x) \geq 0$, for all $a_1, \dots, a_n \in A$, $x \in X$,

⇔ $\sum a_{ij} a_i^* a_j \geq 0$, for all $a_1, \dots, a_n \in A$.

(i), (ii) ⇒ (iv) is trivial.

(c) Suppose A has the property that if $a \in M_2(A)$ satisfies

$$\sum a_{ij} \bar{z}_i z_j \geq 0, \quad \text{for all } z_1, z_2 \in \mathbb{C}, \tag{4.1}$$

then a is positive. The C*-algebra obtained from A by adjoining an identity has the same property. Thus we can assume A is unital. Take

$b \in A$, and consider the matrix

$$x = \begin{bmatrix} 1 & b \\ b^* & bb^* \end{bmatrix}$$

which clearly satisfies (4.1), so that x is positive. But

$$bb^* - b^*b = [b^*, -1] \begin{bmatrix} 1 & b \\ b^* & bb^* \end{bmatrix} \begin{bmatrix} b \\ -1 \end{bmatrix} \geq 0,$$

and so $bb^* \geq b^*b$, for all b in A. By symmetry each element of A is normal and so A is commutative.

4.2 THEOREM *Let A, B be C^*-algebras, with B commutative. Then automatically any positive linear map from A into B is completely positive.*

Proof: Suppose $[a_{ij}] \in M_n(A)$ is positive. Then

$$\sum a_{ij} \bar{\xi}_i \xi_j \geq 0 \quad \text{for all } \xi_1, \ldots, \xi_n \in \mathbb{C}.$$

Then if T is any positive map from A into B, and

$$T(\sum a_{ij} \bar{\xi}_i \xi_j) \geq 0, \quad \text{for all } \xi_1, \ldots, \xi_n \in \mathbb{C}.$$

Hence

$$\sum T(a_{ij}) \bar{\xi}_i \xi_j \geq 0, \quad \text{for all } \xi_1, \ldots, \xi_n \in \mathbb{C}.$$

The conclusion follows from Lemma 4.1(b).

Positive linear maps whose domains are commutative C^*-algebras automatically are completely positive, as the following theorem shows.

4.3 THEOREM *Let A, B be C^*-algebras with A commutative. Then any positive linear map from A into B is completely positive.*

Proof: By going to the second dual, we can assume that A is a W^*-algebra and that the given positive linear map T from A into B is ultraweakly continuous. We represent A as $L^\infty(\Omega, \mu)$ for some localizable measure space (Ω, μ), with predual $L^1(\Omega, \mu)$, and we take B to act on a Hilbert space H. Then for all $f, g \in H$, the map

$$a \mapsto \langle T(a)f, g \rangle$$

is ultraweakly continuous on $L^\infty(\Omega, \mu)$. Hence there exists $h(f, g)$ in $L^1(\Omega, \mu)$ such that

$$\langle T(a)f, g \rangle = \langle a, h(f, g) \rangle .$$

Moreover $f, g \mapsto h(f, g)$ is sesquilinear, and $h(f, f) \geq 0$ since T is positive.

Let $f_1, \ldots, f_n$ be elements of B. Then for all $z_1, \ldots, z_n$ in $\mathbb{C}$:

$$\sum z_i \bar{z}_j T(f_i f_j^*) = T\left(\sum z_i f_i \left(\sum z_j f_j\right)^*\right) \geq 0.$$

$$\Rightarrow \quad \sum z_i \bar{z}_j T(f_i f_j^*)(\omega) \geq 0 \quad \text{for almost all } \omega \text{ in } \Omega, \tag{4.2}$$

$$\Rightarrow \quad [T(f_i f_j^*)(\omega)] \geq 0 \qquad \text{a.e.}$$

Thus, for all $a_1, \ldots, a_n$ in $L^\infty(\Omega)$, $\langle [T(f_i f_j^*)](a_j), (a_i) \rangle = \sum \langle T(f_i f_j^*) a_j, a_i \rangle =$
$= \int_\Omega \overline{a_i(\omega)}\, a_j(\omega)\, T(f_i f_j^*)(\omega)\, d\mu(\omega) \geq 0$, by (4.2).

4.4 Corollary *Let T be a positive linear map from a C^*-algebra A into another C^*-algebra B. If a is a normal element of A, then*

$$\|T\|\, T(a^*a) \geq T(a)^* T(a). \tag{4.3}$$

More generally:

$$\|T\|\, T(a^*a + aa^*) \geq T(a)^* T(a) + T(a) T(a)^* \tag{4.4}$$

for all a in A.

Proof: If C is the commutative C^*-algebra generated by a normal element a, then the restriction of T to C is completely positive, by Theorem 3.5. Hence we can apply the Schwarz inequality of Theorem 1.10. If a is an arbitrary element of A, we can apply (4.3) to the self-adjoint elements $a + a^*$ and $i(a - a^*)$. The inequality in (4.4) then follows by addition.

4.5 Corollary *Let T be a positive contraction from a C^*-algebra A into another C^*-algebra B, and p a self-adjoint element of A, such that $T(p^2) = T(p)^2$. Then*

$$T(ap + pa) = T(a)T(p) + T(p)T(a) \tag{4.5}$$

and

$$T(pap) = T(p)T(a)T(p) \tag{4.6}$$

for all a in A.

Proof: Fix ϕ, a state on B, and consider the sesquilinear form D on A,

$$D : (x,y) \mapsto \phi[T(xy^* + y^*x) - T(x)T(y)^* - T(y)^*T(x)].$$

By Corollary 4.4, we have $D(x,x) \geq 0$ for all x in A. However $D(p,p) = 0$, by assumption, and so $D(x,p) = 0$ by the Cauchy-Schwarz inequality applied to D; hence (4.5) holds. Then (4.6) follows easily from Jordan identities.

The Stinespring representation theorem can also be used to obtain a description of completely positive normal maps:

4.5 THEOREM *Let M be a von Neumann algebra on a Hilbert space H, and let K be another Hilbert space. If φ is a completely positive ultraweakly continuous map from M into $B(K)$, then there exist $\{A_i : i \in N\}$ in $B(K,H)$ such that, for all x in M,*

$$\varphi(x) = \sum A_i^* x A_i .$$

If H is infinite-dimensional, we can choose N such that its cardinality is at most that of a complete orthonormal set for H.

Proof: By the Stinespring decomposition, we can assume that φ is a normal representation with cyclic vector f. Then since $\langle \varphi(\cdot)f, f \rangle$ is a normal state on M, there exist vectors $\{f_i : i \in \propto\}$ in H such that $\sum \| f_i \|^2 < \infty$ and $\langle \varphi(x)f, f \rangle = \sum \langle xf_i, f_i \rangle$ for all x in M. Since $\| xf_i \| \le \| \varphi(x)f \|$ for all x in M, there exist contractions A_i from K into H such that $A_i \varphi(x)f = xf_i$. Then, for all x, z in M, we have

$$\begin{aligned} \langle \varphi(x)\varphi(z)f, \varphi(z)f \rangle &= \langle \varphi(z^*xz)f, f \rangle = \sum \langle z^*xzf_i, f_i \rangle \\ &= \sum \langle xzf_i, zf_i \rangle = \sum \langle x A_i\varphi(z)f, A_i\varphi(z)f \rangle \\ &= \sum \langle A_i^* x A_i \varphi(z)f, \varphi(z)f \rangle . \end{aligned}$$

Since f is a cyclic vector for φ, we have $\varphi(x) = \sum A_i^* x A_i$ for all x in M. The series converges in the ultraweak topology. The usual counting arguments in a Hilbert space give the cardinality result.

5. Conditional expectations

As we mentioned in the Introduction, we wish to define a class of C*-algebraic maps which generalize the class of conditional expectations of classical probability theory. In this chapter, A will denote a unital C*-algebra, and B a unital C*-subalgebra of A. To merit the description "conditional expectation", we will require the following properties of a linear map of A onto B:

CE1: N is a projection of norm one such that $N(1_A) = 1_B$.

CE2: $N(a_1 N(a_2)) = N(a_1)N(a_2)$, for all a_1, a_2 in A, or equivalently, $N(ab) = N(a)b$ for all a in A, and b in B.

CE3: N is completely positive.

It is easily verified that these properties hold in the following examples:

5.1 Examples 1. Let $\{p_i : i \in I\}$ be a mutually orthogonal family of projections in a W*-algebra A, let $p = \sum p_i$ and let $N(a) = \sum p_i a p_i$ for all a in A. Then N is a projection of A onto the intersection of pAp with the relative commutant $\{p_i : i \in I\}^c = \{a \in A : ap_i = p_i a$ for all i in I$\}$.

2. Let A and B be W*-algebras, and identify B with $1 \otimes B$ as a W*-subalgebra of the W* tensor product $A \otimes B$. Let ϕ be a normal state of A, then $\phi \otimes 1$ is a projection of $A \otimes B$ onto B; it is the dual of the injection of states:

$$\psi \to \phi \otimes \psi \quad \text{for all } \psi \text{ in } B_* .$$

(Similarly for C*-algebras with spatial or minimal tensor product.)

The main result (Theorem 5.3) is that CE1 entails both CE2 and CE3. We are thus led to:

5.2 Definition Let B be a unital C*-subalgebra of a unital C*-algebra A. A conditional expectation N is a projection of norm one from A onto B such that $N(1_A) = 1_B$.

Taking C = B'', we see in the following theorem that a conditional expectation is automatically completely positive (CE3), and has the module mapping property [illegible].

5.3 THEOREM *Let B be a unital C*-subalgebra of a unital C*-algebra A. Let Φ be a linear map of norm one from A into a C*-algebra C such that the restriction of Φ to B is a homomorphism onto a weakly dense subalgebra of C, with $\Phi(1_A) = 1_C$. Then Φ is completely positive, and* [illegible] *for all a in A, b in B.*

Proof: That Φ is positive follows from [illegible]. By going to the second duals we can assume that A, B, C are all von Neumann algebras and Φ is normal. It is enough to consider C in an irreducible representation, and so we may assume that C = B(H) for some Hilbert space H. Let e be the central projection in [illegible] such that [illegible] is the two-sided ideal [illegible]. Then $\Phi(e) = 1_C$. For the moment we will only consider the restriction Φ_0 of Φ to eAe, so that the [illegible] of [illegible] is [illegible]. Via a spatial isomorphism, we may assume [illegible] and [illegible] for all [illegible]. Then, by Corollary [illegible], we have [illegible] for all a in [illegible] and all [illegible] in B(H). After some computation, we find that [illegible]. Thus [illegible] lies in [illegible] and [illegible], where ω is a normal state on [illegible]. Hence [illegible], which is completely positive, and [illegible] for all a in [illegible] and b in [illegible]. Then for all a in A, b in B, we have

$$\Phi(ab) = \Phi(eab)\ [\text{illegible}]$$

[illegible], by Corollary [illegible],

[illegible], since e is central in B,

[illegible];

the theorem follows.

6. FOCK SPACE

In this chapter we recall some elementary results about Fock space, and show how the Boson and Fermion Fock spaces arise naturally with the Kolmogorov decompositions of certain positive-definite functions.

Let H be a Hilbert space. For each positive integer n, let H_n denote the n-fold tensor product $\otimes^n H$, and let H_0 denote the one-dimensional Hilbert space spanned by a single unit vector Ω, called the Fock vacuum vector. Fock space $F(H)$ is then defined as

$$F(H) = \oplus_{n=0}^{\infty} H_n .$$

Let T be a contraction from H to another Hilbert space K, let T_n denote the contraction $\otimes^n T$ from H_n into K_n, and put $T_0 = 1$; we define $F(T)$ to be the contraction from $F(H)$ into $F(K)$ given by

$$F(T) = \oplus_{n=0}^{\infty} T_n .$$

The assertions in the following lemma are then easily verified.

6.1 LEMMA. 1. *F is a functor on the category whose objects are Hilbert spaces and whose morphisms are contractions:*

$$F(ST) = F(S)\,F(T), \quad F(1) = 1. \tag{6.1}$$

2. *$F(0)$ is the projection on the Fock vacuum vector Ω:*

$$F(0) = \Omega \otimes \bar{\Omega} , \tag{6.2}$$

3. *F is a *-map:*

$$F(T^*) = F(T)^*. \tag{6.3}$$

We will not be interested in the whole of Fock space, but only in two of its subspaces, namely the Boson and the Fermion Fock spaces.

For each positive integer n, let S_n denote the group of all permutations on n symbols. There is a natural unitary action of S_n on the Hilbert space H_n given by

$$\pi(f_1 \otimes \dots \otimes f_n) = f_{\pi^{-1}(1)} \otimes \dots \otimes f_{\pi^{-1}(n)}$$

for all π in S_n and $f_1, \dots, f_n$ in H.

6.2 REMARK. Let T be a contraction between Hilbert spaces H and K. Then T_n

intertwines the actions of S_n on H_n and K_n: $T_n\pi = \pi T_n$ for all π in S_n.

Let $P_n = (n!)^{-1}\sum_{\pi \in S_n} \pi$. Then P_n is the projection from H_n onto the space H_n^s of symmetric tensors of degree n. Symmetric (or Boson) Fock space $F^s(H)$ is then defined by

$$F^s(H) = \oplus_{n=0}^{\infty} H_n^s .$$

Now let $T : H \to K$ be a contraction. It follows from Remark 6.2 that T_n maps H_n^s into K_n^s, and so $F(T)$ induces a contraction $F^s(T) : F^s(H) \to F^s(K)$. Note that F^s inherits the properties (6.1) to (6.3) of the functor F in Lemma 6.1.

Let $\varepsilon(\pi)$ denote the signature of the permutation π, and let $Q_n = (n!)^{-1}\sum_{\pi \in S_n} \varepsilon(\pi)\pi$. Then Q_n is the projection from H_n onto the space H_n^a of antisymmetric tensors of degree n over H. Antisymmetric (or Fermion) Fock space $F^a(H)$ is defined by

$$F^a(H) = \oplus_{n=0}^{\infty} H_n^a .$$

Again, if $T : H \to K$ is a contraction, it follows from Remark 6.2 that T_n maps H_n^a into K_n^a, and so $F(T)$ induces a contraction $F^a(T) : F^a(H) \to F^a(K)$, and F^a inherits the properties (6.1) to (6.3) from the functor F.

For use later in the study of some algebras naturally associated with the Fock spaces, we relate the Fock spaces to Kolmogorov decompositions of some positive-definite kernels.

First we look at Boson Fock space. Let h be a vector in the Hilbert space H, and let h_n denote the n-fold tensor product $h \otimes \ldots \otimes h$ which lies in H_n^s, with $h_0 = 1$. Then $\langle h_n, k_n \rangle = \langle h, k \rangle^n$ for all h, k in H, thus $h \mapsto h_n$ is a minimal Kolmogorov decomposition of the positive-definite kernel $h, k \mapsto \langle h, k \rangle^n$ on $H \times H$. Now define $\mathrm{Exp} : H \to F^s(H)$ by

$$\mathrm{Exp}(h) = \oplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} h_n .$$

6.3 THEOREM *The map $\mathrm{Exp} : H \to F^s(H)$ is a minimal Kolmogorov decomposition for the positive-definite kernel $h, k \mapsto \exp\langle h, k \rangle$ on $H \times H$. Moreover, $\{\mathrm{Exp}(h) : h \in H\}$ is a linearly independent total set of vectors for $F^s(H)$.*

Proof. That $\mathrm{Exp}(\cdot)$ is a Kolmogorov decomposition for the kernel $\exp\langle \cdot , \cdot \rangle$ is follows by computation:

$$\langle \mathrm{Exp}(h), \mathrm{Exp}(k) \rangle = \exp \langle h, k \rangle . \qquad (6.4)$$

Totality is a consequence of the relation

$$\frac{d^n}{dt^n} \mathrm{Exp}(th)\Big|_{t=0} = (n!)^{\frac{1}{2}} h_n ;$$

It remains to prove the asserted linear independence. Suppose $h_1, \ldots, h_n$ in H and $z_1, \ldots, z_n$ in $\mathbb{C}$ satisfy $\sum_{j=1}^{n} z_j \, \mathrm{Exp}(h_j) = 0$. Then, by the reproducing property (6.4), $\sum_{j=1}^{n} z_j \exp t\langle h_j, k \rangle = 0$ for all t in $\mathbb{R}$ and k in H. But $e^{\lambda t}$ is an eigenvector of the linear operator $\frac{d}{dt}$ corresponding to the eigenvalue λ, and eigenvectors corresponding to distinct eigenvalues are linearly independent. Thus, for each k in H, we have $\langle h_i, k \rangle = \langle h_j, k \rangle$ for some $i \neq j$. Hence the set $\{h_i\}$ cannot be distinct.

6.4 COROLLARY *There is a natural identification of $\Gamma^s(H \oplus K)$ with $\Gamma^s(H) \otimes \Gamma^s(K)$ under which*

$$\mathrm{Exp}(h \oplus k) = \mathrm{Exp}(h) \otimes \mathrm{Exp}(k)$$

and

$$\Gamma^s(S \oplus T) = \Gamma^s(S) \otimes \Gamma^s(T) .$$

Proof: This is a consequence of the uniqueness of a minimal Kolmogorov decomposition (Lemma 1.4), Theorem 6.3, and the relation:

$$\langle \mathrm{Exp}(h_1) \otimes \mathrm{Exp}(k_1), \mathrm{Exp}(h_2) \otimes \mathrm{Exp}(k_2) \rangle = \exp \langle h_1 \oplus k_1, h_2 \oplus k_2 \rangle .$$

Next we consider Fermion Fock space. Let $f_1, \ldots, f_n$ lie in the Hilbert space H, and define $f_1 \wedge \ldots \wedge f_n$ by

$$f_1 \wedge \ldots \wedge f_n = (n!)^{\frac{1}{2}} Q_n (f_1 \otimes \ldots \otimes f_n) .$$

Thus we have

$$\langle f_1 \wedge \ldots \wedge f_n, g_1 \wedge \ldots \wedge g_n \rangle = (n!) \langle Q_n(f_1 \otimes \ldots \otimes f_n), g_1 \otimes \ldots \otimes g_n \rangle$$

$$= \sum \varepsilon(\sigma) \langle f_{\sigma^{-1}(1)} \otimes \ldots \otimes f_{\sigma^{-1}(n)}, g_1 \otimes \ldots \otimes g_n \rangle$$

$$= \sum \varepsilon(\sigma) \langle f_{\sigma^{-1}(1)}, g_1 \rangle \ldots \langle f_{\sigma^{-1}(n)}, g_n \rangle$$

$$= \det [\langle f_i, g_j \rangle] .$$

then the map $(f_j)_{j=1}^n \mapsto f_1 \wedge \ldots \wedge f_n$ of H^n into Λ_a^n is a minimal Kolmogorov decomposition for the positive-definite kernel $((f_i),(g_j)) \mapsto \det(\langle f_i, g_j \rangle)$ on $H^n \times H^n$.

In what follows we will drop the subscript a on Λ when there is no risk of confusion arising.

7. REPRESENTATIONS OF THE CANONICAL COMMUTATION RELATIONS

In this section we recall some definitions and formulae associated with the canonical commutation relations. The main result (Theorem 7.1) is a characterization of generating functions.

Let H be a Hilbert space. In Theorem 1.3 we noted that the map $e(\cdot)$ is a minimal Kolmogorov decomposition for the positive-definite kernel $\exp\langle\cdot,\cdot\rangle$ on $H \times H$. Consider now the linearly independent total set of normalized vectors

$$\{\psi(x) = (\exp(-\tfrac{1}{2}\|x\|^2))\, e(x) : x \in H\}.$$

Then

$$\langle \psi(x), \psi(y)\rangle = \exp(-\|x-y\|^2/2)\exp(i \operatorname{Im}\langle x,y\rangle)$$

for all x, y in H, so that $\psi(\cdot)$ is a minimal Kolmogorov decomposition for the positive-definite kernel

$$(x,y) \mapsto \exp(-\|x-y\|^2/2)\exp(i \operatorname{Im}\langle x,y\rangle). \tag{7.1}$$

In other words, $\Gamma^s(H)$ can be identified with the reproducing kernel Hilbert space for the kernel (7.1). Note that the map

$$(x,y) \mapsto \exp(i \operatorname{Im}\langle x,y\rangle) \tag{7.2}$$

defines a multiplier in the sense of group representation theory.

7.1 DEFINITION Let $(G,\cdot)$ be a group. A multiplier σ on G is a map from $G \times G$ into the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, such that

$$\sigma(g, e) = \sigma(e, g) = 1, \tag{7.3}$$

$$\sigma(g, g')\sigma(g\cdot g', g'') = \sigma(g, g'\cdot g'')\sigma(g', g''), \tag{7.4}$$

for all g, g', g'' in G. A σ-representation of a group G with multiplier σ is a map U from G into the unitary operators on some Hilbert space such that

$$U(e) = 1, \tag{7.5}$$

$$U(g)U(g') = \sigma(g, g')U(g\cdot g'), \tag{7.6}$$

for all g, g' in G. A projective representation is a σ-representation for some multiplier σ.

7.2 REMARK The equations (7.3) and (7.4) of a multiplier are merely consistency conditions for the existence of σ-representations. For example, (7.4) reflects the associative law.

Since $\{E(h) : h \in H\}$ is a linearly independent total set of normalised vectors, there is a well-defined unitary $W(h)$, for each h in H, such that

$$W(h)\, E(k) = E(h + k)\, \omega(h, k) \qquad (7.7)$$

for all k in H. Moreover, $W(h)$ obeys the canonical commutation relations:

$$W(h)\, W(k) = W(h + k)\, \omega(h, k). \qquad (7.8)$$

7.3 DEFINITIONS *A representation of the CCR (canonical commutation relations) is a projective representation of a Hilbert space H with multiplier ω given by (7.2).* The C^*-algebra generated by a representation W of the CCR is denoted by $W(H)$. Thus $W(H)$ is the norm-closed linear span of the unitaries $\{W(h) : h \in H\}$. The representation of the CCR defined by (7.7) is called the *Fock representation*. A representation W of the CCR is said to be *non-singular* if the map $\lambda \mapsto W(\lambda h)$ is weakly continuous on $\mathbb{R}$ for each h in H, or equivalently, if W is strongly continuous on all finite-dimensional subspaces of H. In this case, by Stone's theorem, there is for each h in H a self-adjoint operator $R(h)$, called a *field operator*, such that $W(h) = \exp iR(h)$.

7.4 REMARKS 1. The Fock representation is non-singular.

2. It is sometimes instructive to regard the field operators $R(h)$ as the random variables of a non-commutative probability theory. They satisfy, at least formally, the commutation relation

$$R(h)\, R(k) - R(k)\, R(h) = -2i\, \mathrm{Im} < h,k > 1,$$

as a consequence of the $W(h)$ satisfying (7.8).

3. Defining the annihilation operator $a(h)$ by $a(h) = 2^{-\frac{1}{2}}(R(h) + iR(ih))$, and the creation operator $a^*(h)$ by $a^*(h) = 2^{-\frac{1}{2}}(R(h) - iR(ih))$, we have

$$a(h)\, a^*(k) - a^*(k)\, a(h) = < h,k > 1.$$

4. The Weyl operator $W(h) = e^{iR(h)}$ can be written in terms of annihilation and creation operators as follows:

$$W(h) = \exp(i\, 2^{-\frac{1}{2}} a^*(h)) \exp(i\, 2^{-\frac{1}{2}} a(h)) \exp(-\|h\|^2/4).$$

7.5 DEFINITIONS A representation W of the CCR over H is said to be *cyclic* if there exists a (unit) vector Ω in H_W such that

$$H_W = \bigvee\{W(h)\Omega : h \in H\}.$$

We then call Ω the vacuum vector of the representation. The *generating functional* μ of a cyclic representation W with vacuum vector Ω is the function defined on H by

$$x \mapsto \mu(x) = \langle W(x)\Omega, \Omega \rangle .$$

7.6 Remarks 1. We shall see (Theorem 7.8) that the Fock representation is irreducible; hence every non-zero vector is cyclic. In particular, the Fock vacuum vector is cyclic.

2. The generating functional is useful for the calculation of the expectation values of various operators (such as polynomials in the field operators, in the case of non-singular representations) in the vacuum state of a cyclic representation. For a non-singular representation the generating functional is given by

$$\mu(x) = \langle e^{i\Phi(x)} \Omega, \Omega \rangle ,$$

analogous to the characteristic function of a probability distribution. The analogy will be strengthened in Theorem 7.8.

A generalisation of the notion of cyclic representation has proved useful:

7.7 Definitions Let H, K be Hilbert spaces; a representation W of the CCR over H is said to be *K-cyclic* if there exists a V in $B(K, H_W)$ such that

$$H_W = \bigvee \{ W(x)Vk : x \in H, k \in K \} .$$

Let (W,V) be a K-cyclic representation of the CCR over H, and define a map $M : H \to B(K)$ by

$$x \mapsto M(x) = V^* W(x) V .$$

Then M is called the *generating function* of (W,V).

The following theorem, which is simply a 'projective version' of the Naimark-Sz.-Nagy representation theorem for groups, provides a characterisation of generating functions:

7.8 Theorem *Let H, K be Hilbert spaces, and M a map from H into B(K). Then there exists a K-cyclic representation (W,V) having M as its generating function if and only if the kernel*

$$h, k \mapsto F(k - h)\, \omega(h, k) \tag{7.8}$$

is positive-definite on $H \times H$. In this case (π, Ω) is uniquely determined up to unitary equivalence; the representation π is non-singular if and only if the map $t \mapsto F(h + tk)$ is weakly continuous on $\mathbb{R}$ for all h, k in H.

Proof. Let F be the generating function of a W-cyclic representation (π, Ω), then

$$F(k - h)\, \omega(h, k) = \langle \pi(h)\Omega, \pi(k)\Omega \rangle ,$$

and so (7.8) is a positive-definite kernel. Conversely, suppose the kernel (7.8) is positive-definite with a minimal Kolmogorov decomposition $V(\cdot)$, so that

$$V(h)^* V(k) = F(k - h)\, \omega(h, k)$$

for all h, k in H. Then, for all h, h', h'' in H, we have

$$\begin{aligned} V(h + h'')^* V(h' + h'')\, \omega(h', h'')\, \overline{\omega(h, h'')} &= F(h' - h)\, \omega(h + h'', h' + h'')\, \omega(h', h'')\, \overline{\omega(h, h'')} \\ &= F(h' - h)\, \omega(h, h') , \\ &= V(h)^* V(h') . \end{aligned}$$

Thus, by the uniqueness of the minimal Kolmogorov decomposition, there exists a well-defined unitary $W(h'')$ such that

$$W(h'')\, V(h) = V(h + h'')\, \omega(h, h'') ,$$

It is readily seen that W is a representation of the CCR over H, with cyclic vector $\Omega = V(0)$, such that F is the generating function of (π, Ω). The remainder of the proof is clear.

Thus we see that the Fock representation of the CCR is determined by the generating functional

$$h \mapsto \langle W(h)\Omega, \Omega \rangle = \exp(-\|h\|^2/4) . \tag{7.10}$$

More generally, we have:

7.9 THEOREM *For each $\lambda \geq 1$ there exists a cyclic representation π_λ of the CCR over H, acting on a Hilbert space $F_\lambda(H)$, with cyclic vector Ω_λ, and generating functional μ_λ given by*

$$\mu_\lambda(h) = \exp(-\lambda \|h\|^2/4) . \tag{7.11}$$

The representation π_λ is irreducible.

Proof: We can check directly that μ_λ is positive-definite, and then apply Theorem 7.5. Alternatively, we can write down a cyclic representation W_λ having (7.11) as generating functional. We choose the second approach. Let J be a conjugation on H (that is, an antilinear map satisfying $J^2 = 1$ and $\langle Jh, Jh' \rangle = \langle h', h \rangle$ for all h, h' in H). Given $\lambda \geq 1$, choose $\alpha, \beta \geq 0$ such that $\alpha^2 + \beta^2 = \lambda$, $\alpha^2 - \beta^2 = 1$, and put

$$W_\lambda(h) = W(\alpha h) \otimes W(\beta Jh). \qquad (7.12)$$

Then W_λ, defined on

$$F_\lambda(H) = F(H) \otimes F(H),$$

is a cyclic representation of the CCR with cyclic vector $\Omega_\lambda = \Omega \otimes \Omega$. An easy calculation shows that

$$\langle W_\lambda(h)\, \Omega_\lambda, \Omega_\lambda \rangle = \exp(-\tfrac{1}{4}\|h\|^2 \lambda).$$

To show that W_λ is an irreducible representation for each $\lambda \geq 1$, it is enough (by a tensor product argument) to show this for the case where H is a one-dimensional Hilbert space, which we identify with $\mathbb{C}$ or $\mathbb{R}^2$. In this case, consider the Schrödinger representation of the CCR over $\mathbb{C}$, defined on $L^2(\mathbb{R})$ as follows:

$$(W(s,t)g)(x) = e^{i(2x+s)t/2}\, g(x+s) \qquad (7.13)$$

for g in $L^2(\mathbb{R})$. One verifies that this defines a representation of the CCR over $\mathbb{C}$; moreover, by considering the cyclic vector $\Omega(x) = \pi^{-1/4} e^{-x^2/2}$, one can see that the Schrödinger representation has the same generating functional (7.11) as the Fock representation, so that the representations are unitarily equivalent. We show that the Schrödinger representation (7.13) on $L^2(\mathbb{R})$ is irreducible; a similar argument will show that W_λ, given by (7.12), is an irreducible representation of the CCR over $\mathbb{C}$ on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}^2)$.

Let T be an element of $W(\mathbb{C})'$, where W is the Schrödinger representation (7.13). Then, in particular, T commutes with $W(0,t)$ for all t in $\mathbb{R}$. But $W(0,t)$ is multiplication by the function $x \mapsto e^{itx}$; a density argument shows that T commutes with multiplication by an arbitrary bounded measurable function. In other words, T is in the commutant of $L^\infty(\mathbb{R})$. But $L^\infty(\mathbb{R})$ is a maximal abelian von Neumann algebra (4.3); hence T is itself a multiplication operator. Moreover, T commutes with $W(o,y)$ for all y in $\mathbb{R}$. But $W(o,y)$ is a translation operator, and so T must be multiplication by a constant function; hence the Schrödinger representation is irreducible.

8. Representations of the Canonical Anti-Commutation Relations

In chapter 7 we studied the CCR field operators $B(h)$ through their exponentials $W(h) = e^{iB(h)}$. This was done for technical convenience, since the $B(h)$ are necessarily unbounded. Nevertheless, this procedure carries a bonus: the generating functions are very useful in computations. In this chapter we turn to canonical anti-commutation relations, where the situation is very different: the field operators are necessarily bounded, and there is no useful analogue of a generating function. However, there is an associated projective representation of a discrete group (Theorem 8.6) which will prove useful in chapter 9.

8.1 Definitions. Let H be a Hilbert space. A representation of the canonical anti-commutation relations over H is a conjugate linear map a from H into the bounded linear operators on some Hilbert space, which satisfies the canonical anti-commutation relations (CAR):

$$a(f)^* a(g) + a(g) a(f)^* = \langle f, g \rangle 1, \tag{8.1}$$

$$a(f) a(g) + a(g) a(f) = 0, \tag{8.2}$$

for all f, g in H. The norm closure of the linear span of the monomials in $\{a(h) : h \in H\}$ and $\{a(h)^* : h \in H\}$ is a C^*-algebra denoted by $A(H)$. As a Banach space, $A(H)$ is linearly generated by the Wick monomials

$$a(h_1)^* \ldots a(h_n)^* a(h_{n+1}) \ldots a(h_{n+m}),$$

with $h_1, \ldots, h_{n+m}$ in H, or alternatively, by the anti-Wick monomials

$$a(h_1) \ldots a(h_n) a(h_{n+1})^* \ldots a(h_{n+m})^* .$$

8.2 Remarks. It follows from (8.1) that $\|a(h)\| = \|h\|$, since $a(h)a(h)^* \geq 0$ so that $a(h)^*a(h) \leq \|h\|^2$. Consequently, $h \mapsto a(h)$ is automatically continuous. Moreover, if $\{f_n\}$ is an orthonormal basis for H, we have $a(h) = \sum_n \langle f_n, h \rangle a(f_n)$ in the sense of norm convergence, so that $a(h)$ can be recovered from the a_n, where $a_n = a(f_n)$. (For notational convenience, we assume that H is separable, but this is not necessary.) Trivial computations yield:

8.3 Lemma. Let $\{a_n\}_{n=1}^N$ satisfy the discrete version of the CAR:

$$a_n a_m^* + a_m^* a_n = \delta_{nm} 1 \tag{8.3}$$

$$a_n a_m + a_m a_n = 0 \tag{8.4}$$

For each $n \geq 1$, put

$$U_{2n-1} = i(a_n^* - a_n), \quad U_{2n} = a_n^* + a_n . \tag{8.5}$$

Then $\{U_n\}_{n=1}^{\infty}$ is a sequence of unitaries satisfying

$$U_n U_m + U_m U_n = 2\delta_{nm} 1. \tag{8.2}$$

Conversely, if $\{U_n\}_{n=1}^{\infty}$ is a sequence of unitaries satisfying (8.2), then the sequence $\{a_n = \frac{1}{2}(U_{2n} + iU_{2n-1}) : n = 1, 2, \ldots\}$ satisfies the relations (8.3) and (8.4).

Before going further, we look at an example: the Fock representation of the CAR.

8.6 Example Let $f, h_1, \ldots, h_n$ be elements of a Hilbert space H. Let $L = \mathrm{Lin}\{h_1, \ldots, h_n\}$ and put $f = f_1 + f_2$, where f_1 is in L and f_2 in $L^{\perp}$. Then

$$f \wedge h_1 \wedge \ldots \wedge h_n = f_2 \wedge h_1 \wedge \ldots \wedge h_n ,$$

and so, by considering determinants,

$$\| f \wedge h_1 \wedge \ldots \wedge h_n \|^2 = \| f_2 \|^2 \| h_1 \wedge \ldots \wedge h_n \|^2 \leq \| f \|^2 \| h_1 \wedge \ldots \wedge h_n \|^2 .$$

Thus there is a well-defined linear map, denoted by $a(f)_n^*$, from H_n^a to H_{n+1}^a such that

$$a(f)_n^* (h_1 \wedge \ldots \wedge h_n) = f \wedge h_1 \wedge \ldots \wedge h_n , \tag{8.7}$$

and

$$\| a(f)_n^* \| \leq \| f \| .$$

Hence we can define a bounded linear operator $a(f)^* : F(H) \to F(H)$ which extends the family $\{a(f)_n^*\}$. Now let f be a unit vector in H and put $K = \{f\}^{\perp}$. Then $a(f)_n^*$ maps K_n^a (regarded as a subspace of H_n^a) isometrically onto $f \wedge K_n^a$ and annihilates $f \wedge H_{n-1}^a$, the orthogonal complement of K_n^a in H_n^a. Thus, $a(f)^*$ maps $F^a(K)$ isometrically onto $F^a(H) \ominus F^a(K)$ and annihilates $F^a(H) \ominus F^a(K)$. That is, $a(f)^* a(f) + a(f) a(f)^* = 1$, or more generally, $a(f)^* a(f) + a(f) a(f)^* = \langle f, f \rangle$, for all f in H. So by polarization

$$a(f)^* a(g) + a(g) a(f)^* = \langle f, g \rangle 1$$

for all f, g in H. We also have

$$a(f)a(g) + a(g)a(f) = 0$$

for all f, g in H, since $f \wedge g + g \wedge f = (f + g) \wedge (f + g) = 0$. The representation of the CAR constructed by (8.7) is called the Fock representation.

8.5 THEOREM: *The Fock representation of the CAR is irreducible.*

Proof: Consider the state ω (called the Fock state) on the algebra $A(H)$ given by the cyclic Fock vacuum vector Ω; $\omega(A) = \langle A\Omega, \Omega \rangle$. The Fock vacuum vector Ω is annihilated by every Wick monomial except the identity. Thus, if ρ is any state on $A(H)$ with $\rho \leq \lambda\omega$, we have $\rho(x^*x) \leq \lambda\|x\Omega\|^2 = 0$ for every Wick monomial except the identity. Thus, by the Schwarz inequality, ρ annihilates every Wick monomial except the identity, and so clearly $\rho = \omega$, and so ω is a pure state.

Finally, we show how to transform a representation of the CAR so that it looks like a representation of the CCR.

8.6 THEOREM *Let H be a Hilbert space, let a be a representation of the CAR over H, and let $A(H)$ be the C*-algebra generated by a. Then there exists a projective representation of the group $\bigoplus_1^N \mathbb{Z}_2$, where $N = \dim H$, which also generates $A(H)$.*

Proof: For notational convenience we will assume that H is separable, but this is not necessary. Let $\{h_n : n = 1, \ldots, N\}$ be an orthonormal basis for H, and put $a_n = a(h_n)$. Then, by Lemma 8.3, there is a sequence $\{U_n\}$ of unitaries which generates the a_n. If $g = \{g_n : n = 1, \ldots, N\}$ is an element of $G = \bigoplus_1^N \mathbb{Z}_2$, $g_n = 0$ unless n is in a finite set on which $g_n = 1$. Define U_g for g in G by

$$U_g = \prod_{n=1}^{N} U_n^{g_n}, \tag{8.8}$$

Then we have

$$U_g U_{g'} = \sigma(g, g') U_{g+g'}, \tag{8.9}$$

where σ is a multiplier taking values ± 1. Thus $A(H)$ is generated by the projective representation (8.8) of the discrete group G.

9. SLAWNY'S THEOREM

In this chapter we study projective representations of groups, in order to prove that two representations of the CCR (or of the CAR) over a fixed Hilbert space generate isomorphic C*-algebras.

9.1 DEFINITIONS Consider a locally compact abelian group G with continuous multiplier σ. Throughout this chapter, we will restrict attention to strongly continuous σ-representations. This will involve no loss of generality, since in applications the group G is given the discrete topology. Let R be the map from G into the unitary operators on $L^2(G)$ given by

$$(R(g)f)(g') = \sigma(g', g)\, f(g' + g).$$

Then R is a strongly continuous σ-representation called the *σ-regular representation*. It is unitary, because the inner product on $L^2(G)$ is taken with respect to Haar measure on G, which is translationally invariant. The regular representation ρ of G is the σ-regular representation in the particular case in which $\sigma(\cdot\,,\cdot) = 1$.

9.2 LEMMA *Let G be a locally compact abelian group, and σ a continuous multiplier for G. Let U be a strongly continuous σ-representation for G on a Hilbert space H. Then the σ-representations $\rho \otimes U$ and $R \otimes 1_H$ are unitarily equivalent, where ρ is the regular representation, and R the σ-regular representation.*

Proof: Identify $L^2(G) \otimes H$ with $L^2(G,H)$, as in (10.1). Define the unitary operator A on $L^2(G,H)$ by $(Af)(g) = U_g f(g)$. Then a straightforward calculation yields

$$A\,(\rho \otimes U) = (R \otimes 1_H)\,A.$$

9.3 DEFINITION Let G be a locally compact abelian group. Then the space $\hat{G}$ of continuous characters on G can be endowed with the structure of a locally compact abelian group. The *Fourier transform* is the unitary map $f \mapsto \hat{f}$ of $L^2(G)$ onto $L^2(\hat{G})$, which on $L^1 \cap L^2$ is given by

$$\hat{f}(\chi) = \int_G f(g)\, \overline{\chi(g)}\, dg .$$

where dg is Haar measure on G. The Fourier transform implements a unitary equivalence between the regular representation R of G on $L^2(G)$ and the representation $\hat{R}$ of G on $L^2(\hat{G})$ given by

$$(\hat{R}_g h)(\chi) = \chi(g)\, h(\chi)$$

for all χ in $\hat{G}$.

9.4 LEMMA *Let G be a locally compact abelian group, and σ a continuous multiplier for G. Then the C*-algebra generated by the σ-representation $\hat{R} \otimes \sigma$ and the C*-algebra generated by the σ-regular representation R are isomorphic.*

Proof: The representations R and $R \otimes I_\sigma$ generate isomorphic C*-algebras. Thus the result follows from the remarks following Definition 9.2 and from Lemma 9.2, since unitarily equivalent representations generate isomorphic C*-algebras.

9.5 DEFINITION Let G be a locally compact abelian group, and σ a continuous multiplier for G. Then there is a canonical homomorphism χ from G into $\hat{G}$, called the *natural map*, given by

$$\chi_g(h) = \sigma(g, h)\, \sigma(h, g)^{-1} .$$

9.6 LEMMA *Suppose that the natural map $\chi : G \to \hat{G}$ is injective. Then $\chi(G)$ is dense in $\hat{G}$.*

Proof: Put $H = \chi(G)$; then $(\hat{G}/\bar{H})^\wedge \cong H^\perp$, where $H^\perp$ is the annihilator in G of H (or of its closure $\bar{H}$). But $H^\perp = \{0\}$, since χ is injective; and so $\bar{H} = \hat{G}$.

9.7 LEMMA *Let G be a locally compact abelian group, and let σ be a continuous multiplier for G, such that the associated natural map of G into $\hat{G}$ is injective. Let U be a strongly continuous σ-representation of G. Then the C*-algebra generated by U is isomorphic to the C*-algebra generated by $\hat{R} \otimes \sigma$.*

Proof: We will show that there is an isomorphism of the C*-algebra generated by $\hat{R} \otimes \sigma$ onto the C*-algebra generated by U such that $\sum f(g)(\hat{R} \otimes \sigma)_g \mapsto \sum f(g)\, U_g$ for each function f on G with finite support. The problem is to show that this map is well-defined. We have

$$\| \sum f(g)(\tilde{\pi} \circ \omega)_g \| = \operatorname*{ess\,sup}_{\hat{a} \in \hat{G}} \| \sum f(g)\, \hat{a}(g)\, U_g \| ,$$

$$= \operatorname*{ess\,sup}_{h \in G} \| \sum f(g) \rho_h(g)\, U_g \| , \text{ by Lemma 9.6,}$$

$$= \operatorname*{ess\,sup}_{h \in G} \| \sum f(g)\, U_h U_g U_h^* \| , \text{ since } U \text{ is a } \sigma\text{-representation,}$$

$$= \operatorname*{ess\,sup}_{h \in G} \| U_h \left(\sum f(g)\, U_g \right) U_h^* \|$$

$$= \| \sum f(g)\, U_g \| .$$

Putting together the conclusions of Lemmas 9.4 and 9.7, we have:

9.8 THEOREM *Let G be a locally compact abelian group, and σ a continuous multiplier for G such that the associated natural map $\rho : G \to \hat{G}$ is injective. Let U^1 and U^2 be strongly continuous σ-representations of G, and let A^1 and A^2 be the C*-algebras which they generate. Then there exists a unique isomorphism β from A^1 onto A^2 such that $\beta(U^1_g) = U^2_g$.*

We now apply Theorem 9.8 to the case in which G is a Hilbert space H; we give it the discrete topology in order to make it locally compact.

9.9 EXAMPLE Take G to be a Hilbert space H endowed with the discrete topology, and define $\sigma : H \times H \to \mathbb{T}$ by

$$\sigma(g, h) = \exp(i \operatorname{Im}\langle g, h \rangle / 2) .$$

Then σ is a multiplier. The associated natural map $h \mapsto \rho_h$ of G into $\hat{G}$ is given by

$$\rho_h(g) = \exp(i \operatorname{Im}\langle h, g \rangle),$$

and is clearly injective. Thus from Theorem 9.8 we have:

9.10 THEOREM *Let H be a Hilbert space, and let W^1 and W^2 be representations of the CCR over H. Let $W^1(H)$ and $W^2(H)$ be the C*-algebras which they generate. Then there exists a (necessarily unique) isomorphism $\beta : W^1(H) \to W^2(H)$ such that*

$$\beta(W^1(h)) = W^2(h)$$

for each h in H.

9.11 EXAMPLE Take G to be $\bigoplus_1^\infty \mathbb{Z}_2$ and σ to be the multiplier defined in (8.3).

Then the natural map $h \mapsto \chi_h$ is given by $\chi_h(g) = (-1)^{\sum h_i g_i}$, and this is injective. Thus from Theorem [illegible] we have

5.12 **THEOREM** Let H be a Hilbert space and let a^1 and a^2 be representations of the CAR over H. Let $A^1(H)$ and $A^2(H)$ be the C*-algebras which they generate. Then there exists a (necessarily unique) isomorphism $\alpha : A^1(H) \to A^2(H)$ such that

$$\alpha(a^1(h)) = a^2(h)$$

for each h in H.

10. COMPLETELY POSITIVE MAPS ON THE CCR ALGEBRA

Now that we have completed the construction of the C*-algebras of the CCR and CAR over a Hilbert space H, we turn to the study of their morphisms, the completely positive maps. In particular, we investigate those morphisms, known as quasi-free maps, which are induced by morphisms of the Hilbert space H. In this chapter we treat the CCR algebra $W(H)$.

The following simple fact will prove to be useful:

10.1 THEOREM *Let H be a Hilbert space, B a C^*-algebra, and F a map from H into B. Then there exists a completely positive map $T : W(H) \to B$ such that $T(W(h)) = F(h)$ for all h in H, if and only if the following kernel is positive-definite on $H \times H$:*

$$h, k \mapsto F(k - h)\, \omega(h, k).$$

Proof: The result follows from Theorems 7.8 and 9.10. Alternatively, noting that $W(H)$ is the closed linear span of the unitaries $\{W(h) : h \in H\}$, one can argue as in §2.

The following is the most general result on quasi-free completely positive maps which we will need:

10.2 THEOREM *Let H, K be Hilbert spaces, A a linear map from H into K, and f a map from H into $\mathbb{C}$. Then there exists a completely positive map $T : W(H) \to W(K)$ such that*

$$T[W(h)] = W(Ah)\, f(h)$$

for all h in H, if and only if the following kernel is positive-definite on $H \times H$:

$$h, k \mapsto f(k - h)\, \frac{\omega(h, k)}{\omega(Ah, Ak)} \tag{10.1}$$

Proof. Define $F : H \to W(K)$ by $F(h) = W(Ah) f(h)$. Then for all h, k in H, we have

$$F(k - h)\,\omega(h, k) = W(Ah)^* W(Ak)\, f(k - h)\, \omega(h, k)/\omega(Ah, Ak).$$

Thus if the kernel (10.1) is positive-definite then so is the kernel $h, k \mapsto F(k - h)\,\omega(h, k)$, and the existence of the required completely positive map T is a consequence of Theorem 10.1. Conversely, if the kernel $h, k \mapsto F(k - h)\,\omega_\lambda(k, h)$ is positive definite, it has a Kolmogorov decomposition (9.1), so that

$$F(k - h)\,\frac{\omega(h_1, k)}{\omega(h_1, h)} = \langle W(h)\,\xi(h)^*\,\xi(k)\,W(k)\rangle^*,$$

and the result follows.

In Theorem 7.8 we noted that for each Hilbert space H, and each $\lambda \geq 1$, there exists a cyclic representation $(W_\lambda, \Omega_\lambda)$ of the CCR over H, with generating functional ω_λ given by

$$\omega_\lambda(h) = \langle W_\lambda(h)\Omega_\lambda, \Omega_\lambda\rangle = \exp(-\tfrac{\lambda}{4}\|h\|^2).$$

(The Fock generating functional is got by putting $\lambda = 1$.) The representation W_λ acts on the space $F_\lambda(H)$ and is irreducible. We will denote by $W_\lambda(H)$ the concrete C^*-algebra generated by the representation W_λ. Since $\omega_\lambda(h \oplus k) = \omega_\lambda(h)\omega_\lambda(k)$, it follows that we can identify $F_\lambda(H \oplus K)$ with $F_\lambda(H) \otimes F_\lambda(K)$, and $W_\lambda(h \oplus k)$ with $W_\lambda(h) \otimes W_\lambda(k)$, and hence $W_\lambda(H \oplus K)$ with the spatial C^*-tensor product (10.1), written $W_\lambda(H) \otimes W_\lambda(K)$, which is the C^*-algebra generated by the algebraic tensor product $W_\lambda(H) \odot W_\lambda(K)$.

10.3 THEOREM *Let $\lambda \geq 1$ be fixed. Let H, K be Hilbert spaces; for each contraction $T : K \to H$ there is a completely positive map $W_\lambda(T) : W_\lambda(H) \to W_\lambda(K)$ of C^*-algebras such that*

$$W_\lambda(T)[W_\lambda(h)] = W_\lambda(T^*h)\, e^{-\frac{\lambda}{4}(\|h\|^2 - \|T^*h\|^2)} \qquad (10.2)$$

for all h in H. Moreover, W_λ is functorial:

$$W_\lambda(ST) = W_\lambda(T)W_\lambda(S), \quad W_\lambda(1) = 1.$$

It has the additional properties:

$$W_\lambda(S \oplus T) = W_\lambda(S) \otimes W_\lambda(T),$$

$W_\lambda(0)$ is the state determined by ω_λ.

Proof: We apply Theorem 10.1, checking that the kernel which appears is positive-definite, to prove that $W_\lambda(T)$ is completely positive. The rest of the proof is straightforward.

10.4 COROLLARY *The generating functional ω_λ is invariant under $W_\lambda(T)$ for each contraction T.*

Proof. For each contraction $T : H \to K$ we have

$$\omega_\lambda \circ W_\lambda(T) = W_\lambda(\mathrm{CON}_\lambda(T)) = W_\lambda(0,T) = W_\lambda(0) = \omega_\lambda .$$

10.5 REMARK In the case in which $\lambda = 1$ (the Fock representation), there is a connection between the functor W and the Fock functor F. To see this, recall that to each contraction $T : H \to K$ there corresponds a contraction $F(T) : F(H) \to F(K)$ such that

$$F(T)\, W(h)\, \Omega = F(T)\, \mathrm{Exp}(ih)$$
$$= F(T)\, \mathrm{Exp}(2^{-\frac{1}{2}} ih)\, e^{-\|h\|^2/4}$$
$$= \mathrm{Exp}(2^{-\frac{1}{2}} iTh)\, e^{-\|h\|^2/4}$$
$$= \Omega(Th)\, e^{-(\|h\|^2 - \|Th\|^2)/4}$$
$$= W(Th)\, \Omega\, e^{-(\|h\|^2 - \|Th\|^2)/4} .$$

But we have seen that there is a completely positive map $W(T)$ such that

$$W(T)\,[W(h)] = W(Th)\, e^{-(\|h\|^2 - \|Th\|^2)/4} .$$

Thus, for all h in H, we have

$$F(T)\, W(h)\, \Omega = W(T)\,[W(h)]\, \Omega .$$

There is an analogous contraction $F_\lambda(T)$ in the general case in which $\lambda \geq 1$.

10.6 THEOREM *Let $\lambda \geq 1$ be fixed. Let H, K be Hilbert spaces; for each contraction $T : H \to K$ there is a contraction $F_\lambda(T) : F_\lambda(H) \to F_\lambda(K)$ such that*

$$F_\lambda(T)\, W_\lambda(h)\, \Omega_\lambda = W_\lambda(Th)\, \Omega_\lambda\, e^{-\frac{1}{4}(\|h\|^2 - \|Th\|^2)} \qquad (10.1)$$

for all h in H. Moreover, F_λ is functorial:

$$F_\lambda(ST) = F_\lambda(S)\, F_\lambda(T), \quad F_\lambda(1) = 1.$$

It has the additional properties:

$$F_\lambda(T)^* = F_\lambda(T^*),$$
$$F_\lambda(S \oplus T) = F_\lambda(S) \otimes F_\lambda(T),$$

$F_\lambda(0)$ *is the projection on the vacuum.*

Proof: For each x in $W_t(H)$ we have

$$\|W_t(T)[x]\,\Omega_t\|^2 = \langle W_t(T)[x^*]\,W_t(T)[x]\,\Omega_t, \Omega_t\rangle$$

$$\le \langle W_t(T)[x^*x]\,\Omega_t, \Omega_t\rangle \quad \text{by the Schwarz inequality (Theorem 1.?)}$$

$$= \langle x^*x\,\Omega_t, \Omega_t\rangle \quad \text{by the invariance of } \omega_t \text{ (Corollary 10.4)}$$

$$= \|x\,\Omega_t\|^2 .$$

Hence there is a well-defined contraction $F_t(T) : F_t(H) \to F_t(K)$ such that $F_t(T)[x\,\Omega_t] = W_t(T)[x]\,\Omega_t$ for all x in $W_t(H)$. The only remaining assertion which is not immediately apparent is that $F_t(T^*) = F_t(T)^*$. This can be verified by calculating $\langle F_t(T)\,W_t(h)\,\Omega_t, W_t(k)\,\Omega_t\rangle$ and $\langle W_t(h)\,\Omega_t, F_t(T^*)\,W_t(k)\,\Omega_t\rangle$, using the definitions.

Thus we have a functor W_t from the category of Hilbert spaces and contractions to the category of unital C*-algebras and completely positive identity preserving maps; and a functor F_t on the category of Hilbert spaces and contractions; the functors W_t and F_t are related by the following result:

10.7 THEOREM *Let $t > 0$ be fixed. Let $T : H \to K$ be a contraction. Then the map*

$$x \mapsto W_t(T)[x] - F_t(T)\,x\,F_t(T)^*$$

from $W_t(H)$ into $B(F_t(K))$ is completely positive. We have $W_t(T) = F_t(T)(\cdot)F_t(T)^*$ if and only if T is a coisometry. Moreover, we have

$$\omega_t(W_t(T)[x]\,y) = \omega_t(x\,W_t(T^*)[y])$$

for all x in $W_t(H)$ and y in $W_t(K)$.

Proof: Suppose $W_t(T) = F_t(T)(\cdot)F_t(T)^*$. Then, by evaluating at the identity, we see that $F_t(TT^*) = F_t(T)\,F_t(T)^* = 1$, and so $TT^* = 1$. Conversely, if $TT^* = 1$, we can show that $W_t(T) = F_t(T)(\cdot)F_t(T^*)$ by using (10.2) and (10.3) to evaluate $W_t(T)[W_t(h)]\,W_t(k)\,\Omega_t$ and $F_t(T)\,W_t(h)\,F_t(T)^*\,W_t(k)\,\Omega_t$ for all h, k in K. Now let $T : H \to K$ be a contraction. Then there exists a Hilbert space L and isometries $V_1 : H \to L$ and $V_2 : K \to L$ such that $T = V_2^*\,V_1$. Then we have the following Stinespring decomposition for $W_t(T)$:

$$W_L(T) = W_L(V_2^* V_1) = W_L(V_2)^* W_L(V_1)$$
$$= F_L(V_2)^* (W_L(V_1)(\cdot)) F_L(V_1), \qquad (10.4)$$

since V_2^* is a co-isometry. Moreover, we have $F_L(T) = F_L(V_2)^* F_L(V_1)$. Thus it is enough to prove that $W_L(T) - F_L(T)(\cdot) F_L(T)^*$ is completely positive when T is an isometry. An isometry can be factored into a unitary and an injection, and so it is enough to consider the case in which T is the canonical injection $T : H \to H \oplus H'$, for some Hilbert space H'. In this case we have $W_L(T)[x] = x \otimes 1$, for each element x of $W_L(H)$, where 1 is the identity on $F_L(H')$. On the other hand, we have $F_L(T)\,\xi = \xi \otimes \Omega$, for each ξ in $F_L(H)$, where Ω is the vacuum vector in $F_L(H')$. Thus we have

$$x \mapsto W_L(T)[x] - F_L(T)\, x\, F_L(T)^* = x \otimes (1 - \omega),$$

where ω is the projection on Ω, and the map $x \mapsto x \otimes (1 - \omega)$ is completely positive.

Finally, for all x in $W_L(H)$ and y in $W_L(K)$, we have

$$\omega_L(W_L(T)[x]\, y) = \langle W_L(T)[x]\, y\,\Omega_L, \Omega_L \rangle$$
$$= \langle y\Omega_L, W_L(T)[x^*]\,\Omega_L \rangle$$
$$= \langle y\Omega_L, F_L(T)\, x^*\, \Omega_L \rangle$$
$$= \langle x\, F_L(T^*)\, y\,\Omega_L, \Omega_L \rangle$$
$$= \omega_L(x\, W_L(T^*)[y]) .$$

10.8 Remark In the course of the proof we obtained a Stinespring decomposition (10.4) for $W_L(T)$. If we identify H with a subspace of L, we have

$$W_L(T)[x] = F_L(V_2)^*(x \otimes 1)\, F_L(V_2)$$

for all x in $W_L(H)$, and so $W_L(T)$ has an ultraweak extension to a completely positive map on $B(F_L(H))$ (which is, in fact, $W_L(H)''$ since the representation W_L is irreducible, by Theorem 7.20). Thus the ultraweak extension $W_L(T) : B(F_L(H)) \to B(F_L(K))$ is unique.

10.9 Remark We have constructed a C^*-algebra $W_L(H) \otimes W_L(K)$ by taking the spatial tensor product. It is interesting to note that the CCR-algebra is nuclear: given any C*-algebra B there is a unique way of completing the *-algebra W(H) ⊗ B to get a C*-algebra.

10.10 THEOREM For any Hilbert space H, the CCR algebra W(H) is nuclear.

Proof: Showing that W(H) is nuclear is equivalent (see Effros [1977]) to showing that the weak closure of the CCR algebra in any representation is injective (that is, given any representation π of the CCR, there is a projection of norm one of $B(H_\pi)$ onto π(H)''). But a von Neumann algebra is injective if and only if its commutant is injective (see Effros [1977]). Thus, given any representation π of the CCR, we seek a projection of norm one from $B(H_\pi)$ onto π(H)'. If h is an element of H, let α(h) denote the automorphism of $B(H_\pi)$ given by

$$\alpha(h)\, x \;\to\; \pi(h)^* x\, \pi(h)$$

for all x in $B(H_\pi)$. Then α is a representation of the abelian group H on $B(H_\pi)$. But any abelian group is amenable (see Greenleaf [1969]), so there exists an invariant mean m on H. Then Φ = m(α(·)) is a projection from $B(H_\pi)$ onto the fixed-point algebra of α, namely π(H)'. Thus π(H)' is injective, and the result follows.

10.11 REMARK The CAR algebra A(H) is nuclear. If H is a finite-dimensional Hilbert space, say of dimension n, then A(H) can be identified with the full matrix algebra $M_{2^n}(\mathbb{C})$. It follows that, for any infinite-dimensional Hilbert space H, the CAR algebra A(H) is uniformly hyperfinite (that is, it is an inductive limit of full matrix algebras), and hence is nuclear (see Effros [1977]).

11. COMPLETELY POSITIVE MAPS ON THE CAR ALGEBRA

The results on quasi-free completely positive maps on the CAR are not as extensive as those on the CCR algebra, because of the lack of any useful analogue of the generating functional. However, we have the following analogue of Theorem 10.3:

11.1 THEOREM *Let $T : H \to K$ be a contraction between Hilbert spaces. Then there exists a completely positive map $A(T) : A(H) \to A(K)$, whose action on Wick monomials is given by*

$$a(h_1)^* \dots a(h_n)^* a(h_{n+1}) \dots a(h_{n+m})$$
$$\mapsto a(Th_1)^* \dots a(Th_n)^* a(Th_{n+1}) \dots a(Th_{n+m}). \qquad (11.1)$$

Moreover, A is functorial:

$$A(ST) = A(S)\,A(T), \quad A(1) = 1.$$

We have the additional property:

$$A(0) \text{ is the Fock state.}$$

Proof: First, let $V : H \to K$ be an isometry. Then the map $h \mapsto a(Vh)$ is a representation of the CAR. Hence there is a faithful homomorphism $A(V) : A(H) \to A(K)$ such that $A(V)[a(h)] = a(Vh)$.

Next, let $T : H \to K$ be a co-isometry. Consider the completely positive map of $A(H)$ into $A(K)$ given, in the Fock representation, by

$$x \mapsto F(T)\, x\, F(T)^*.$$

Direct calculation on a total set of vectors in Fock space shows that, on Wick monomials, we have

$$F(T)\, a(h_1)^* \dots a(h_{n+m})\, F(T)^* = a(Th_1)^* \dots a(Th_{n+m}).$$

Finally, let $T : H \to K$ be a contraction. Then there exists a Hilbert space L and isometries $V_1 : H \to L$, $V_2 : K \to L$ such that $T = V_2^* V_1$. Put

$$A(T)[x] = F(V_2)^* A(V_1)[x]\, F(V_2) \qquad (11.2)$$

for all x in $A(H)$. Then $A(T)$ is a completely positive map whose action on Wick monomials is given by (11.1). The remaining assertions follow from this.

11.2 REMARK The relation between the functors A and F will be seen formally as follows: we have

$$W(x)\exp(\|x\|^2/4) = \exp(i\, 2^{-\frac{1}{2}}\, a(x)^*)\exp(i\, 2^{-\frac{1}{2}}\, a(x)).$$

The right-hand side is a sum of Wick monomials and, applying the rule of the A-functor to them, we have

$$W(x) \mapsto W(Tx)\, e^{-\frac{1}{4}(\|x\|^2 - \|Tx\|^2)} ,$$

as for the W-functor.

In the Fock representation the functors A and F are related as follows:

11.3 Theorem *For each contraction $T : H \to K$ between Hilbert spaces we have*

$$F(T)\, x\, \Omega = A(T)[x]\,\Omega$$

for all x in $A(H)$. We have $A(T) = F(T)\cdot F(T)^*$ if and only if T is a co-isometry, and $A(T)$ is a homomorphism if and only if T is an isometry. Moreover, for the Fock state ω we have

$$\omega(A(T)[x]\, y) = \omega(x\, A(T^*)[y])$$

for all x in $A(H)$ and y in $A(K)$.

Proof. As for Theorem 11.2.

12. DILATIONS OF QUASI-FREE DYNAMICAL SEMI-GROUPS

We now use the Hilbert space dilation theory which we described in Chapter 3, together with the quasi-free completely positive maps constructed in Chapters 11 and 10, to obtain examples of dilations of dynamical semi-groups on the C^*-algebraic level.

12.1 EXAMPLE Let $\{T_t : t \geq 0\}$ be a strongly continuous semi-group of contractions on a Hilbert space H. Then, by Theorem 3.2, there is an isometric embedding V of H into another Hilbert space K, on which there is a semi-group $\{U_t : t \in \mathbb{R}\}$ of unitaries such that

$$T_t = V^* U_t V, \quad t \geq 0.$$

Hence, for each $\lambda \geq 0$, there is a strongly continuous semigroup $\{W_\lambda(T_t) : t \geq 0\}$ of completely positive maps on $W_\lambda(H)$ such that

$$W_\lambda(T_t) = W_\lambda(V^*)\, W_\lambda(U_t)\, W_\lambda(V), \quad t \geq 0.$$

Now $W_\lambda(V)$ is an embedding of $W(H)$ as a C^*-subalgebra of $W(K)$, and $W_\lambda(V^*)$ is a conditional expectation of $W(K)$ onto $W(H)$. Furthermore,

$$W_\lambda(U_t) = F_\lambda(U_t)(\cdot)\, F_\lambda(U_t)^*$$

is a unitarily implemented group of automorphisms of $W_\lambda(K)$. If we identify H as a subspace of K, we have

$$W_\lambda(T_t)[x] = E \otimes \omega_\lambda(F_\lambda(U_t)(x \otimes 1)\, F_\lambda(U_t)^*), \quad t \geq 0,$$

for all x in $W_\lambda(H)$. In particular, we have

$$W_\lambda(T_t)[W_\lambda(h)] = W_\lambda(T_t h)\, e^{-\frac{1}{4}(\|h\|^2 - \|T_t h\|^2)},$$

$t \geq 0$, for all h in H.

12.2 EXAMPLE Let $\{T_t : t \geq 0\}$ be a semi-group of isometries on a Hilbert space H. Then, by Theorem 3.1, we have the stronger dilation

$$VT_t = U_t V, \quad t \geq 0.$$

In this case, at the C^*-algebraic level we have

$$W_\lambda(V)\, W_\lambda(T_t) = W_\lambda(U_t)\, W_\lambda(V), \quad t \geq 0, \qquad (12.1)$$

Identifying H as a subspace of K, we have

$$W_\lambda(T_t)[x] \otimes 1 = F_\lambda(U_t)(x \otimes 1)\, F_\lambda(U_t)^*, \quad t \geq 0,$$

for all x in $W_1(H)$. This is a very strong form of dilation. It transforms the semi-group of homomorphisms $\{W_1(T_t) : t \geq 0\}$ into the unitarily implemented group of automorphisms $\{W_1(U_t) : t \in \mathbb{R}\}$.

12.3 EXAMPLE Let $\{T_t : t \geq 0\}$ be a semi-group of contractions on a Hilbert space H, such that there is an isometric embedding V of H into a Hilbert space K on which there is a strongly continuous semi-group of isometries $\{S_t : t \geq 0\}$ and

$$VT_t = S_t^* V , \quad t \geq 0.$$

(In Chapter 15 we will show that such a co-isometric dilation exists for certain semi-groups.) For the CCR algebra we have the following interesting isometric representation:

$$W_1(V)\, W_1(T_t) = F_1(S_t)^*\, W_1(V)\, [\cdot]\, F_1(S_t) , \quad t \geq 0.$$

Identifying H with a subspace of K, this gives

$$W_1(T_t)\, [x] \oplus 0 = F_1(S_t)^*\, W_1(V)\, [x]\, F_1(S_t), \quad t \geq 0,$$

for all x in $W_1(H)$.

Analogous results hold for the CAR algebra. In the remaining chapters we will be concerned with finding dilations of more general dynamical semi-groups on operator algebras. We notice, by using a tensor-product construction, that a dilation of the type (12.1) exists trivially for any semi-group of homomorphisms. In the C*-algebra case, this method gives a dilation of a family of completely positive maps – the subject of the next chapter.

13. DILATIONS OF COMPLETELY POSITIVE MAPS ON C^*-ALGEBRAS

In Chapter 12 we gave some examples of dilations in a C^*-algebraic setting. We now take a more abstract approach. We show that a family of completely positive maps on a C^*-algebra can be dilated to a group of C^*-automorphisms on a larger C^*-algebra.

13.1 THEOREM *Let A be a unital C^*-algebra of operators on a Hilbert space H; let $\{T_g : g \in G\}$ be a family of completely positive maps $T_g : A \to A$ indexed by the elements of a locally compact group G, and strongly continuous in the sense that $g \mapsto T_g(a)\,\xi$ is norm continuous for all a in A and ξ in H. Suppose that $T_e = I$ and $T_g(1) = 1$ for all g in G. Then there exists a C^*-algebra B on a Hilbert space K, a strongly continuous unitary representation U of G on K such that $U_g B U_g^* = B$ for all g in G, an isometric $*$-homomorphism $\pi : A \to B$, and a conditional expectation E of B onto A such that*

$$T_g(a) = E(U_g\, \pi(a)\, U_g^*)$$

for all g in G and a in A.

Proof. Let $H' = L^2(G,H)$, and define a completely positive map $T : A \to B(H')$ by

$$(T(a) f)(g) = T_g(a)\, f(g) .$$

Let U' be the strongly continuous unitary representation of G on H', defined by $(U'_g f)(h) = f(hg)$, and let A' be the C^*-algebra generated by $T(A)$ and $U'(G)$. Let $\{f_\lambda\}$ be an L^2-approximate identity on G. For each λ, define an isometric embedding $V_\lambda : H \to H'$ by

$$(V_\lambda\, \xi)(g) = f_\lambda(g)\, \xi .$$

Then $\lim_{\lambda \to \infty} V_\lambda^*\, a\, V_\lambda$ exists in the weak operator topology for all a in A', and

$$\lim_{\lambda \to \infty} V_\lambda^*\, U_g'^*\, T(a)\, U_g'\, V_\lambda = T_g(a).$$

Since T is completely positive, there exists a representation π of A on a Hilbert space K, and an isometry $V : H' \to K$, such that $T(a) = V^* \pi(a) V$ for all a in A, and π is faithful since

$$\lim_{\lambda \to \infty} V_\lambda^*\, V^*\, \pi(a)\, V V_\lambda = a$$

for all a in A. Let U_g be the strongly continuous unitary representation of G

on K defined by

$$R_g = V U'_g V^* + 1 - VV^*$$

for all g in G. Let B be the C*-subalgebra of L(H) generated by the set $\{U_g \otimes [x] U_g^* : g \in G, x \in A\}$. Then we have $V^*BV \subseteq A''$; thus, for each x in B, the limit $\psi(x) = \lim_{i \to \infty} R_i^* V x V^*R_i$ exists in the weak operator topology, and

$$\Gamma_g[x] = \psi(U_g \otimes [x] U_g^*)$$

for all x in A and g in G.

14. GENERATORS OF DYNAMICAL SEMIGROUPS

In this chapter we examine the generators of norm-continuous one-parameter semigroups of positive maps and, in particular, of completely positive maps on C^*-algebras. We sharpen the well-known result for reversible processes: derivations generate automorphism groups.

Recall that a *derivation* on an algebra A is a map L whose domain $D(L)$ is a subalgebra of A, such that

$$L(ab) = L(a)b + aL(b)$$

for all a, b in A.

14.1 THEOREM *Let $\{e^{tL} ; t \geq 0\}$ be a strongly continuous semigroup on a Banach algebra A. Then e^{tL} for each $t \geq 0$ is a homomorphism if and only if L is a derivation.*

Proof: Let L be a derivation, let x, y be elements of $D(L)$, and put

$$f(t) = e^{tL}(xy) - e^{tL}(x)\, e^{tL}(y), \quad t \geq 0;$$

then $t \mapsto f(t)$ is continuously differentiable,

$$f'(t) = L\, e^{tL}(xy) - L\, e^{tL}(x)\, e^{tL}(y) - e^{tL}(x)\, L\, e^{tL}(y), \quad t \geq 0,$$

and for h in $D(L)$ we have

$$\frac{d}{ds}\, e^{(t-s)L}(h) = -e^{(t-s)L} Lh, \quad 0 \leq s \leq t.$$

Thus we have

$$\begin{aligned} f(t) - e^{tL} f(0) &= \int_0^t \frac{d}{ds} \{e^{(t-s)L} f(s)\}\, ds \\ &= -\int_0^t e^{(t-s)L} L f(s)\, ds + \int_0^t e^{(t-s)L} f'(s)\, ds \\ &= \int_0^t e^{(t-s)L} \{L(e^{sL}(x)\, e^{sL}(y)) - (L e^{sL}(x))\, e^{sL}(y) \\ &\qquad - e^{sL}(x)\, (L\, e^{sL}(y))\}\, ds \\ &= 0, \text{ since } L \text{ is a derivation.} \end{aligned}$$

Thus if $f(t)$ is identically zero, we have

$$e^{tL}(xy) = e^{tL}(x)\, e^{tL}(y), \quad t \geq 0,$$

for all x, y in $D(L)$. The result follows, since $D(L)$ is dense in A. The proof of the converse is trivial.

Next we have analogous results for the generators of positive semigroups on C*-algebras. First recall that if S is a set of states on a C*-algebra A, then S is said to be *full* if $f(a) \geq 0$ for all f in S implies that $a \geq 0$ whenever a is a self-adjoint element of A. Moreover, if f belongs to S implies that $y \mapsto f(x^*yx)/f(x^*x)$ belongs to S for all x in A such that $f(x^*x) \neq 0$, then S is said to be *invariant*.

14.2 THEOREM *Let L be a bounded self-adjoint linear map on a unital C*-algebra A. Then the following conditions are equivalent:*

1. e^{tL} *is positive for all positive t.*
2. $(\lambda - L)^{-1}$ *is positive for all sufficiently large positive λ.*
3. *If y is in A_+, then $ya = 0$ implies $a^*L(y)a \geq 0$.*
4. *For some full, invariant set of states S: If f is in S and y is in A_+, then $f(y) = 0$ implies $f(L(y)) \geq 0$.*
5. $L(a^2) - aL(1)a \geq L(a)a + aL(a)$ *for all self-adjoint a in A.*
6. $L(1) + u^*L(1)u \geq L(u^*)u + u^*L(u)$ *for all unitary u in A.*

Proof: 4. ⇒ 3. Let S be a full, invariant set of states satisfying 4.. Let y in A_+ and a in A be such that $ya = 0$. Then $f(a^*ya) = 0$ for all f in S. Hence, by 4. and the invariance of S, we have $f(a^*L(y)a) \geq 0$ for all f in S, and so $a^*L(y)a \geq 0$ since S is full.

3. ⇒ 2. Let λ be greater than $\|L\|$. In order to show that $(\lambda - L)^{-1} \geq 0$, it is enough to show that $x \geq 0$ whenever x is self-adjoint and $(\lambda - L)x \geq 0$. Let $x = x^+ - x^-$ with x^+ and x^- positive and $x^+x^- = 0$. Then, by 3., we have $x^-L(x^+)x^- \geq 0$, so that

$$
\begin{aligned}
0 &\leq x^-\left[(1 - \lambda^{-1}L)(x)\right]x^- \\
&= x^- x\, x^- - \lambda^{-1}x^-L(x)\,x^- \\
&= -(x^-)^3 - \lambda^{-1}x^-L(x^+)x^- + \lambda^{-1}x^-L(x^-)x^- .
\end{aligned}
$$

Thus $0 \leq (x^-)^3 \leq \lambda^{-1}x^-L(x^-)x^-$, and so $\|x^-\|^3 \leq \lambda^{-1}\|L\|\,\|x^-\|^3$, since $\|a\| \leq \|b\|$ whenever $0 \leq a \leq b$. Hence $x^- = 0$, since $\lambda^{-1}\|L\| < 1$.

2. ⇒ 1. we have $e^{tL} = \lim_{n \to \infty} \left(1 - \frac{t}{n}L\right)^{-n}$.

1. ⇒ 5. Let $K = -L(1)/2$, and put $L'(x) = Kx + xK$. Then $e^{tL'}(x) = e^{tK}x\,e^{tK}$,

so that $\{e^{tL^*} : t \in \mathbb{R}\}$ is a group of positive maps. Applying the Lie-Trotter formula to $L^* + L = L^\#$, we have $e^{tL^\#} \geq 0$ for all $t \geq 0$. Using Kadison's Schwarz inequality (Corollary 4.4) and the fact that $e^{tL^\#}(1) = 1$, we have $e^{tL^\#}(x^2) \geq [e^{tL^\#}(x)]^2$ for $t \geq 0$. Differentiating at $t = 0$, we have $L^\#(x^2) \geq L^\#(x)x + xL^\#(x)$ for all self-adjoint x in A, and so the result follows on substituting $L^\# = L + L^*$.

3. ⇒ 4. Let y be in A_+, f in A_+^* with $f(y) = 0$. Then $f(y^{\frac{1}{2}}x) = f(xy^{\frac{1}{2}}) = 0$ for all x in A, by the Schwarz inequality. Hence

$$L(y) - y^{\frac{1}{2}} L(1) y^{\frac{1}{2}} \geq L(y^{\frac{1}{2}})y^{\frac{1}{2}} + y^{\frac{1}{2}} L(y^{\frac{1}{2}})$$

implies that $f(L(y)) \geq 0$.

4. ⇒ 5. By the reduction employed above, it is enough to prove this when $L(1) = 0$.

5. ⇒ 6. Since $e^{tL} \geq 0$ and $e^{tL}(1) = 1$ for all $t \geq 0$, we have $\| e^{tL} \| = 1$ for all $t \geq 0$. Thus $\| e^{tL}(u) \| \leq 1$ for all unitaries u in A and all $t \geq 0$. Hence $e^{tL}(u^*) e^{tL}(u) \leq 1$ for all $t \geq 0$; differentiating this inequality at $t = 0$, we have $L(u^*)u + u^*L(u) \leq 0$ for all unitaries u in A.

6. ⇒ 1. Since we have assumed that $e^{tL}(1) = 1$ for all $t \geq 0$, it is enough (by [illegible]) to prove that e^{tL} is a contraction for all $t \geq 0$. By [illegible], this is the case if $\lim_{t \to 0} (\| 1 + tL \| - 1)/t \leq 0$. Moreover

$$\| 1 + tL \| = \sup\{\| u + tL(u) \| : u \text{ unitary}\}$$

(see [illegible]). But if u is unitary and $t \geq 0$, we have

$$\begin{aligned} \| u + tL(u) \|^2 &= \| 1 + t(L(u^*)u + u^*L(u)) + t^2 L(u)^* L(u) \| \\ &\leq \| 1 + t^2 L(u)^* L(u) \| \\ &\leq 1 + t^2 \| L \|^2 . \end{aligned}$$

Thus $\| 1 + tL \| \leq (1 + t^2 \| L \|^2)^{\frac{1}{2}}$, and so

$$\lim_{t \to 0} (\| 1 + tL \| - 1)/t \leq \lim_{t \to 0} ((1 + t^2 \| L \|^2)^{\frac{1}{2}} - 1)/t = 0.$$

Hence e^{tL} is a contraction for each $t \geq 0$.

A self-adjoint linear map on a C^*-algebra is automatically continuous if it satisfies condition 5. of Theorem 14.2. We prove the following:

14.3 THEOREM *Let L be a self-adjoint linear map on a unital C^*-algebra A, with the following property:*

If x is in A_+, f is in A_+^*, and $f(x) = 0$, then $f(L(x)) \geq 0$. (14.1)

Then L is bounded, and so e^{tL} is positive for all $t \geq 0$.

Proof: The map $x \mapsto L(x) - \frac{1}{2}[L(1)x + xL(1)]$ satisfies condition (14.1) whenever L does, so we may assume that $L(1) = 0$. We will show that, in this case, L is dissipative on A_h (in the sense of §3.1):

$$\lambda\|a\| \leq \|\lambda a - La\| \text{ for all } a \text{ in } A_h \text{ and } \lambda > 0. \qquad (14.2)$$

In order to prove this for some self-adjoint a, we may assume that there exists a positive f in A^* such that $f(a) = \|a\|$ and $\|f\| = 1$. Then $f(\|a\|1 - a) = 0$, and so $f(L(\|a\|1 - a)) \geq 0$; that is, we have $f(L(a)) \leq 0$. Let λ be strictly positive. Then $\lambda f(a) \leq f(\lambda a - La) \leq \|f\|\,\|\lambda a - La\|$. Hence $\lambda\|a\| \leq \|f\|\,\|\lambda a - La\|$ for all self-adjoint a in A. It follows that L is closed on A_h, and so L is bounded: let $\{x_n \in A_h\}$ be a sequence satisfying $x_n \to 0$, $Lx_n \to y$. Then for all h in A_h, and $\lambda > 0$, we have

$$\lambda\|\lambda x_n + h\| \leq \|(\lambda - L)(\lambda x_n + h)\|.$$

Letting $n \to \infty$, we have $\lambda\|h\| \leq \|\lambda(h - y) - L(h)\|$. Let $\lambda \to \infty$, we have $\|h\| \leq \|h - y\|$ for all h in A_h. Hence $y = 0$. It then follows that e^{tL} is positive for all $t \geq 0$. Alternatively, this follows from (14.2) which shows that $(1 - \lambda^{-1}L)^{-1}$ is a contraction for all $\lambda > \|L\|$, and hence is positive since it preserves the identity (see §3.4).

The results listed in Theorem 14.2 relate mostly to the Jordan structure of a C^*-algebra, but they will be used to prove a result about its C^*-structure (Theorem 14.4). First we consider an example: let $A = M_n(\mathbb{C})$ and let $L(a) = a^t - a$ (where $a \mapsto a^t$ is the transpose mapping). Then L satisfies the hypotheses of Theorem 14.3, but not those of Theorem 14.4.

14.4 THEOREM *Let L be a bounded self-adjoint linear map on a C^*-algebra A. Then the following conditions are equivalent:*

i) $e^{tL}(a^*a) \geq e^{tL}(a^*)\, e^{tL}(a)$, $t \geq 0$, *for all a in A.*

2. $L(a^*a) \geq L(a^*)a + a^*L(a)$ *for all* a *in* A.

Proof: 1. ⇒ 2. This follows by differentiating the inequality in 1. at $t = 0$.

2. ⇒ 1. Suppose 2. holds. Adjoin an identity 1 to A, and extend L to the enlarged algebra by putting $L(1) = 0$. Then, by Theorem 14.2, e^{tL} is positive on the enlarged algebra for all $t \geq 0$. Fix a in A and define

$$F(t) = e^{tL}(a^*a) - e^{tL}(a^*)\, e^{tL}(a), \quad t \geq 0.$$

Then $F'(t) = L\, e^{tL}(a^*a) - [L\, e^{tL}(a^*)]\, e^{tL}(a) - e^{tL}(a^*)[L\, e^{tL}(a)]$, so that

$$F(t) - e^{tL}F(0) = \int_0^t \frac{d}{ds}\left[e^{(t-s)L} F(s)\right] ds$$

$$= -\int_0^t e^{(t-s)L} LF(s)\,ds + \int_0^t e^{(t-s)L} \frac{d}{ds} F(s)\,ds$$

$$= \int_0^t e^{(t-s)L}\Big\{L(e^{sL}(a^*)\, e^{sL}(a)) - [L\, e^{sL}(a^*)]\, e^{sL}(a) - e^{sL}(a^*)\,[L\, e^{sL}(a)]\Big\}\, ds.$$

But, by hypothesis, $L(e^{sL}(a^*)\, e^{sL}(a)) \geq [L\, e^{sL}(a^*)]\, e^{sL}(a) + e^{sL}(a^*)[L\, e^{sL}(a)]$ for all a in A and $s \geq 0$. Moreover, $e^{(t-s)L}$ is positive for $0 \leq s \leq t$. Hence $F(t) \geq e^{tL}F(0) = 0$ for all $t \geq 0$. This means that

$$e^{tL}(a^*a) \geq e^{tL}(a^*)\, e^{tL}(a), \quad t \geq 0,$$

for all a in A.

Before we go on to prove some characterizations of the generators of norm continuous one-parameter semigroups of completely positive maps on C*-algebras, we will give a result which has a slightly more general setting, and which we will need in the proof of Theorem 15.3.

14.5 Lemma *Let A be a C*-subalgebra of a C*-algebra B, and let $L : A \to B$ be a self-adjoint bounded linear map. Then the following conditions are equivalent:*

1. *For all a in A, the kernels*

$$x, y \mapsto L(x^*a^*ay) + x^*L(a^*a)y - L(x^*a^*)ay - x^*a^*L(ay)$$

are positive-definite on A × A.

2. *The kernels*

$$(x_1, y_1), (x_2, y_2) \mapsto L(x_1^*y_1^*y_2x_2) + x_1^*L(y_1^*y_2)x_2 - L(x_1^*y_1^*y_2)x_2 - x_1^*L(y_1^*y_2x_2)$$

are positive-definite on $(A \times A) \cup (B \times A)$.

3. *The following holds for all n in* $\mathbb{N}$:

$\sum_{i,j=1}^{n} b_i^* L(a_i^* a_j) b_j \geq 0$ *for all* $a_1, \ldots, a_n$ *in A and*

$b_1, \ldots, b_n$ *in B for which* $\sum_{i=1}^{n} a_i b_i = 0$.

Proof: 1. ⇒ 2. This is trivial.

1. ⇒ 3. Let $a_1, \ldots, a_n$ in A and $b_1, \ldots, b_n$ in B satisfy $\sum a_i b_i = 0$. Then for all e in A we have

$$\sum_{i,j} b_i^* (L(a_i^* e^* e a_j) + a_i^* L(e^* e) a_j$$

$$- L(a_i^* e^* e) a_j - a_i^* L(e^* e a_j)) b_j \geq 0,$$

thus
$$\sum_{i,j} b_i^* L(a_i^* e^* e a_j) b_j \geq 0,$$

since
$$\sum_i a_i b_i = \sum_i b_i^* a_i^* = 0.$$

Taking e to be an approximate identity for A, we have

$$\sum_{i,j} b_i^* L(a_i^* a_j) b_j \geq 0.$$

3. ⇒ 2. Suppose $c_1, \ldots, c_n$, $a_1, \ldots, a_n$ in A and $f_1, \ldots, f_n$ in B are arbitrary. Define

$$e_i = \begin{cases} c_i & , 1 \leq i \leq n, \\ c_{i-n} a_{i-n} & , n < i \leq 2n, \end{cases}$$

and

$$h_i = \begin{cases} -a_i f_i & , 1 \leq i \leq n, \\ f_{i-n} & , n < i \leq 2n. \end{cases}$$

Then $\sum_{i=1}^{2n} e_i h_i = 0$, so that

$$\sum_{i,j=1}^{2n} h_i^* L(e_i^* e_j) h_j \geq 0,$$

substituting for e_i and h_i, we have

$$\sum_{i,j=1}^{n} f_i^* L(a_i^* c_i^* c_j a_j) f_j + \sum_{i,j=1}^{n} f_i^* a_i^* L(c_i^* c_j) a_j f_j$$

$$\geq \sum_{i,j=1}^{n} f_i^* L(a_i^* c_i^* c_j) a_j f_j + \sum_{i,j=1}^{n} f_i^* a_i^* L(c_i^* c_j a_j) f_j .$$

Thus 2. holds.

10.6 DEFINITION Let A be a C*-subalgebra of a C*-algebra B. A linear map $L : A \to B$ is said to be *conditionally completely positive* if it satisfies the conditions of Lemma 14.5.

We conclude this chapter with a characterization of the generators of quantum dynamical semigroups.

14.7 Theorem *Let L be a self-adjoint bounded linear map on a C^*-algebra A. Then L is conditionally completely positive if and only if e^{tL} is completely positive for all $t \geq 0$.*

Proof: Suppose L is conditionally completely positive. Then L satisfies condition 3. of Lemma 14.5. By going to the second dual (if necessary) we can assume that A is unital. Then, taking $a = 1$, the result follows from the implication 3. ⇒ 1. of Theorem 14.3, and the converse follows from the implication 1. ⇒ 3.

15. CANONICAL DECOMPOSITION OF CONDITIONALLY COMPLETELY POSITIVE MAPS

In Chapter 14 we gave a characterization of the generators of norm-continuous one-parameter semigroups of completely positive maps. They are the conditionally completely positive maps, characterized by certain inequalities. For a large class of von Neumann algebras, a more detailed description of conditionally completely positive maps can be given, in terms of a canonical decomposition (Theorem 15.1). This result can be stated using a cohomology theory for operator algebras, and one is tempted at this point to introduce all the machinery of cohomology; resisting the temptation, we make use instead of a little shorthand. Let A be a von Neumann subalgebra of a von Neumann algebra B; we write $H^1(A,B) = 0$ if the following is true: if $\delta : A \to B$ is a derivation (that is, a linear map such that $\delta(xy) = \delta(x)y + x\delta(y)$ for all x, y in A), then there exists $\hat{\omega}$ in B such that $\delta(x) = \hat{\omega}x - x\hat{\omega}$ for all x in A.

Let A be a von Neumann subalgebra of a von Neumann algebra B, and let $L : A \to B$ be a *-linear map such that both L and $-L$ are conditionally completely positive:

$$L(a^*b^*ca) + a^*L(b^*c)a = L(a^*b^*c)a + a^*L(b^*ca)$$

for all a, b, c in A. Putting

$$L_0(x) = L(x) - \tfrac{1}{2}(L(1)x + xL(1))$$

for all x in A, we see that L_0 is a derivation of A into B. If $H^1(A,B) = 0$, there exists a self-adjoint h in B such that $L_0 = i$ ad h. Hence we have $L(x) = k^*x + xk$ for all x in A, where $k = \frac{1}{2}L(1) + ih$. Conversely, if k is an element of B, then the map $L : A \to B$ given by $L(x) = k^*x + xk$ is such that both L and $-L$ are conditionally completely positive. It is trivial that a completely positive map is conditionally completely positive. We are now ready to describe the canonical decomposition for conditionally completely positive maps.

15.1 THEOREM *Let A be a W^*-algebra. Then the following conditions on A are equivalent:*

1. *whenever A is faithfully represented as a W^*-algebra on a Hilbert space H we have $H^1(A,B(H)) = 0$.*

2. *Whenever R is faithfully represented as a W*-algebra on a Hilbert space H, and $L : A \to B(H)$ is a conditionally completely positive ultraweakly continuous *-linear map, there exists k in B(H) and a completely positive map $\psi : A \to B(H)$ such that*

$$L(a) = \psi(a) + k^*a + ak$$

for all a in A.

Proof: 1. ⇒ 2. Let R be faithfully represented on a Hilbert space H, and let $L : A \to B(H)$ be a *-linear ultraweakly continuous map such that if D is the tri-linear map defined by

$$D(x, y, z) = L(xyz) + xL(y)z - L(xy)z - xL(yz)$$

for all x, y, z in A, then the map $(a_1, a_2), (b_1, b_2) \mapsto D(a_1^*, a_2^* b_2, b_1)$ is positive-definite on $(A \times A) \times (A \times A)$. Then, by the results of Chapters 1 and 2, there exists a Hilbert space K, a normal representation π of A on K, and a linear map $V : A \to B(H, K)$, such that $D(x, y, z) = V(x^*)^* \pi(y) V(z)$ for all x, y, z in A, and $K = \vee\{\pi(a) V(b)\xi : a, b \in A, \xi \in H\}$. Then, for all x, y, a, b in A, we have

$$V(x^*)^*\pi(y)[V(ab) - \pi(a) V(b) - V(a)b]$$

$$= D(x, y, ab) - D(x, ya, b) - D(x, y, a)b = 0 .$$

Hence, by minimality of K, we have

$$V(ab) = \pi(a) V(b) + V(a)b$$

for all a, b in A. Let θ denote the following normal faithful representation of A on $H \oplus K$:

$$\theta(a) = \begin{pmatrix} a & 0 \\ 0 & \pi(a) \end{pmatrix} ,$$

where we identify elements of $B(H \oplus K)$ with 2 × 2 matrices in the obvious way. Let W be the following linear map of θ(A) into $B(H \oplus K)$:

$$W(\theta(a)) = \begin{pmatrix} 0 & 0 \\ V(a) & 0 \end{pmatrix} .$$

Then $W(\theta(a)\theta(b)) = \theta(a) W(\theta(b)) + W(\theta(a)) \theta(b)$ for all a, b in A. Hence, since $H^1(\theta(A), B(H \oplus K)) = 0$, there exists

$$\tilde{W} = \begin{pmatrix} \cdot & \cdot \\ \ell & \cdot \end{pmatrix}$$

in $B(H \oplus K)$ such that $W(\theta(a)) = \tilde{W}\theta(a) - \theta(a)\tilde{W}$.

In particular, $\theta(x) = \pi(x)r - rx$ for all x in A. Thus, for all x, y, z in A, we have

$$L(xyz) + xL(y)z - L(xy)z - xL(yz)$$
$$= D(x, y, z) = \theta(x^*)^* \pi(y)\, \theta(z)$$
$$= [\pi(x^*)r - rx^*]^* \pi(y)[\pi(z)r - rz]$$
$$= \phi(xyz) + x\phi(y)z - \phi(xy)z - x\phi(yz),$$

where ϕ is the completely positive map $x \mapsto r^*\pi(x)r$. From the discussion preceding the statement of the theorem, and since $H^1(A,B(H)) = 0$, we see that there exists k in $B(H)$ such that $L(x) + \phi(x) = k^*x + xk$ for all x in A.

2. ⇒ 1. Let A be faithfully represented on a Hilbert space H, and let $L : A \to B(H)$ be a derivation. Put $k_0 = \frac{1}{2}L(1)$, where 1 is the identity of A, and define $L_0 : A \to B(H)$ by $L_0(x) = L(x) - k_0x - xk_0$. Then $L_0(1) = 0$. Thus, without loss of generality, we may assume that $L(1) = 0$, and that L is a *-map. Hence, by condition 2., there is an element k of $B(H)$, and a completely positive map $\phi : A \to B(H)$, such that $L(x) = \phi(x) + k^*x + xk$ for all x in A. Take a minimal Stinespring decomposition $\phi(x) = r^*\pi(x)r$, where π is a representation of A in a Hilbert space K and r is an element of $B(H,K)$. Then, as above, we have

$$0 = L(xyz) + xL(y)z - L(xy)z - xL(yz)$$
$$= [\pi(x^*)r - rx^*]^* \pi(y)[\pi(z)r - rz]$$

for all x, y, z in A. Hence we have $\pi(z)r = rz$ for all z in A. In particular, putting $s = \frac{1}{2}\phi(1)$, we have $\phi(x) = sx = xs$ for all x in A. But we can assume that $\phi(1) = 0$, so that $k = k^* + \phi(1) = L(1) = 0$ and $k + s = -k^* - s = h$, say. Then we have

$$L(x) = \phi(x) + kx + xk^*$$
$$= (\tfrac{1}{2}\phi(x) + k^*)x + x(k + \tfrac{1}{2}\phi(x))$$
$$= hx - xh$$

for all x in A, so that $H^1(A,B(H)) = 0$.

15.2 Remark Let R be a von Neumann algebra on a Hilbert space H, and let $\{T_t : t \geq 0\}$ be a norm-continuous semigroup of completely positive normal maps on K. Thus it follows from Theorems 14.7 and 15.1 that, under suitable conditions on the algebras, there exists k in $B(H)$ and $\psi : A \to B(H)$ a completely positive normal map such that the generator L of T_t is given by

$$L(x) = \psi(x) + k^*x + xk$$

for all x in A. If T_t preserves the identity of A, then $L(I) = 0$ and so $k + k^* = -\psi(I) \le 0$. Hence k is the generator of a contraction semigroup $\{W_t : t \ge 0\}$ say, on H. Let $\{S_t : t \ge 0\}$ be the contraction semigroup on $B(H)$ given by $S_t(x) = W_t^* x W_t$ for all x in $B(H)$. The generator of S_t is the map $x \mapsto k^*x + xk$. By Banach space perturbation theory we have

$$T_t(x) = S_t(x) + \int_0^t S_{t-s} \circ \psi \circ T_s(x)\,ds, \quad t \ge 0,$$

for all x in A. More generally, we make the following definition.

15.3 DEFINITION Let A be a von Neumann algebra on a Hilbert space H. A *dynamical semigroup of Lindblad type* on A is a weakly continuous semigroup $\{T_t : t \ge 0\}$ of normal completely positive unital maps such that there exists a strongly continuous contraction semigroup $\{W_t : t \ge 0\}$ on H, and a completely positive normal map $\psi : A \to B(H)$, such that

$$T_t(x) = S_t(x) + \int_0^t S_{t-s} \circ \psi \circ T_s(x)\,ds, \quad t \ge 0,$$

for all x in A, where $S_t(x) = W_t^* x W_t$.

15.4 REMARK A dynamical semigroup of Lindblad type on A has an extension to a dynamical semigroup of Lindblad type on $B(H)$.

16. Isometric representations of quantum dynamical semigroups

In Chapter 3 the problem of dilating was considered at the Hilbert space level. The results were used in Chapter 13, together with the CAR and CCR functors, to obtain examples of dilations of dynamical semigroups at the C^*-algebra level. We now begin consideration of the general problem of dilating dynamical semigroups. As in the Hilbert space situation (Chapter 3), and thus in the CAR and CCR algebras (Chapter 13), there are various ways of formulating the concept of a dilation. The first general form which we treat for arbitrary operator algebras is the isometric representation version (compare §17.2 and §17.3).

16.1 Theorem *Let A be a von Neumann algebra on a Hilbert space H, and let $\{T_t : t \geq 0\}$ be a weakly continuous dynamical semigroup of bounded type on A. Then there exists a Hilbert space K and a strongly continuous semigroup $\{V_t : t \geq 0\}$ of isometries on $H \otimes K$, such that*

$$T_t(x) \otimes 1 = V_t^*(x \otimes 1)V_t , \quad t \geq 0,$$

for all x in A.

Proof. We can assume (see Chapter 14) that $A = B(H)$, and that there exists a contraction semigroup $\{S_t : t \geq 0\}$ on H, and a normal completely positive map Ψ on $B(H)$, such that

$$T_t(x) = R_t(x) + \int_0^t R_{t-s} \circ \Psi \circ T_s(x)\, ds, \quad t \geq 0, \tag{16.1}$$

for all x in $B(H)$, where $R_t(x) = S_t^* x S_t$. The pre-adjoint semigroups ${}_*T_t$ and ${}_*R_t$ on the pre-dual $T(H)$ satisfy

$$ {}_*T_t(\rho) = {}_*R_t(\rho) + \int_0^t {}_*T_s \circ {}_*\Psi \circ {}_*R_{t-s}(\rho)\, ds, \quad t \geq 0 \tag{16.2}$$

for all ρ in $T(H)$. By Theorem 4.8, there exists a family $\{A_\alpha : \alpha \in X\}$ of bounded operators on H such that

$$\Psi(x) = \sum_\alpha \Psi_\alpha(x) , \quad \Psi_\alpha(x) = A_\alpha^* x A_\alpha , \tag{16.3}$$

for all x in $B(H)$. Because of the particular form (16.3) of the perturbation Ψ, we can write the Dyson series for (16.1) and (16.2) in an attractive, and useful, way.

Let Y_n be the set of all sequences $\{(s_i, \alpha_i)\} \in X \times [0,\infty) : 0 \leq s_1 \leq \ldots \}$, regarded as a Borel subset of $\bigcup_{n=0}^{\infty} \prod_{k=1}^{n} X \times [0,\infty)$ in an obvious way. Let Y_∞ be the Borel subset of X_∞ consisting of all sequences of finite length, and for each $t > 0$ let X_t be the Borel subset of X_∞ given by all finite sequences $\{(x_j,t_j) : 0 < t_1 < \ldots < t_n < t\}$. For each $t > 0$, there is a Borel isomorphism $j_t : X_t \times Y_\infty \to Y_\infty$ defined by

$$(\{(x_j,t_j)\}_{j=1}^{n}, \{(y_j,s_j)\}_{j=1}^{m}) \mapsto$$

$$(x_1,t_1), \ldots, (x_n,t_n), (y_1,s_1+t), \ldots, (y_m,s_m+t).$$

The inverse map is given by

$$\{(y_j,s_j)\}_{j=1}^{n} \mapsto (y_1,s_1), \ldots, (y_p,s_p), (y_{p+1},s_{p+1}-t), \ldots, (y_n,s_n-t),$$

where p is the unique integer such that $s_p < t \le s_{p+1}$. We denote by X_0 the subset consisting of the single sequence $\emptyset$ of zero length. We define the measure μ_t on X_t to be the product measure constructed from counting measure on each component X, and Lebesgue measure on each component $[0,\infty)$; we assign three measure to the point $\emptyset$ in X_0. We define a measure μ_∞ on X_∞ in an analogous fashion. For each ω in X_t, define $({}_tD,V,S)(\omega)$ by

$$({}_tD,V,S)(\omega) = {}_0D_{t_1} \circ {}_0V_{x_1} \circ {}_0D_{t_2-t_1} \circ {}_0V_{x_2} \circ \ldots \circ {}_0V_{x_n} \circ {}_0D_{t-t_n}$$

where $\omega = \{(x_j,t_j) : 0 < t_1 < \ldots < t_n < t\}$. Then the Neumann series

$$ {}_0T_t(\rho) = {}_0S_t(\rho) + \int_0^t ({}_0S_{t_1} \circ {}_0V \circ S_{t-t_1})(\rho)\, dt_1$$

$$+ \int_0^t \int_0^{t_2} ({}_0S_{t_1} \circ {}_0V \circ {}_0S_{t_2-t_1} \circ {}_0V \circ {}_0S_{t-t_2})(\rho)\, dt_1 dt_2$$

$$+ \ldots \qquad (10.4)$$

can be written as

$$ {}_0T_t(\rho) = \int_{X_t} ({}_tS,V,S)(\omega)(\rho)\, d\mu_t(\omega)\ , \qquad (10.5)$$

and the adjoint series as

$$T_t(a) = \int_{X_t} (({}_tS,V,S)(\omega))^{*}(a)\, d\mu_t(\omega)\ . \qquad (10.6)$$

We take K to be $L^2(Y_\infty)$, and define the operator U_t on $L^2(Y_\infty,\mu)$ for $t \ge 0$:

$$(U_t f)(\omega) = (UVS)(\omega_1) f(\omega_2)\ ,$$

where $(\omega_t, s_t) = \lambda_t^{-1}(\omega)$. The element $(BAB)(\omega')$ of $B(H)$ is given by

$$(BAB)(\omega') = B_{s_1} A_{x_1} B_{s_2-s_1} A_{x_2} \cdots A_{x_n} B_{t-s_n} ,$$

for each $\omega' = \{(x_j, s_j) : 0 \le s_1 < \dots < s_n \le t\}$ in X_t.

We prove that G_t is a strongly continuous semigroup of isometries on $L^2(Y^\infty; H)$. First we check that G_t is an isometry, by using (16.6), and by observing that the measure μ_∞ is the product of the measures μ_t and μ_∞ under the Borel isomorphism $\lambda_t : X_t \times Y_\infty \to Y_\infty$. That is,

$$\langle G_t f, G_t f \rangle = \int_{Y_\infty} \langle (BAB)(\omega_t) f(\omega_t), (BAB)(\omega_t) f(\omega_t) \rangle \, d\mu_\infty(\omega)$$

$$= \int_{Y_\infty} \langle E(\lambda, B, A, B)(\omega_t)^*(1) f(\omega_t), f(\omega_t) \rangle \, d\mu_\infty(\omega)$$

$$= \int_{Y_\infty} \int_{X_t} \langle E(\lambda, B, A, B)(\omega_t)^*(1) f(\omega_t), f(\omega_t) \rangle \, d\mu_t(\omega_t) \, d\mu_\infty(\omega_t)$$

$$= \int_{Y_\infty} \langle T_t(1) f(\omega_t), f(\omega_t) \rangle \, d\mu_\infty(\omega_t)$$

$$= \int_{Y_\infty} \langle f(\omega_t), f(\omega_t) \rangle \, d\mu_\infty(\omega_t) = \langle f, f \rangle .$$

Here we have used the normalization condition $T_t(1) = 1$. Next we show that $\{G_t : t \ge 0\}$ is a semigroup. Indeed, we have

$$(G_{t_1}(G_{t_2} f))(\omega) = (BAB)(\omega_{t_1})(G_{t_2} f)(\omega_{t_1})$$

$$= (BAB)(\omega_{t_1})(BAB)(\omega_{t_1, t_2}) f(\omega_{t_1+t_2})$$

$$= (BAB)(\omega_{t_1+t_2}) f(\omega_{t_1+t_2}) = (G_{t_1+t_2} f)(\omega) ,$$

where we have used the following immediate consequences of the definitions:

$$(BAB)(\omega_{t_1})(BAB)(\omega_{t_1, t_2}) = (BAB)(\omega_{t_1+t_2}) ,$$

$$\omega_{t_1, t_2} = \omega_{t_1+t_2} .$$

Now that we have shown that $(Q_t : t \geq 0)$ is a semigroup of isometries, it is enough to verify that it is weakly continuous at $t = 0$ on elements of the algebraic tensor product $L^2(\Gamma_\infty) \otimes h$; this we do by noting that [illegible]. Finally, we derive the isometric representation property of Q_t: taking a in $B(h)$ and f in $L^2(\Gamma_\infty; h)$, we have

$$< Q_t^*(a \otimes 1) Q_t f, f > = < (a \otimes 1) Q_t f, Q_t f >$$

$$= \int_{\Gamma_\infty} < [illegible] > d\mu_\infty(\omega)$$

$$= \int_{\Gamma_\infty} < [illegible] > d\mu_\infty(\omega)$$

$$= \int_{\Gamma_\infty} \int_{\Gamma_t} [illegible] \, d\mu_t(\omega_1) \, d\mu_\infty(\omega_2)$$

$$= \int_{\Gamma_\infty} < T_t(a) f(\omega_2), f(\omega_2) > d\mu_\infty(\omega_2), \quad \text{by [illegible]},$$

$$= < (T_t(a) \otimes 1) f, f >.$$

The theorem follows.

Theorems 26.1 and 15.1 together show that all norm-continuous dynamical semigroups on a large class of W*-algebras possess isometric representations. We have, as a by-product, the following Hilbert space dilation theorem (mentioned in Chapter 2, and used in Chapter 27 to dilate some quasifree dynamical semigroups):

26.2 THEOREM *Let h be a Hilbert space, and H a self-adjoint (possibly unbounded) operator on h; let k be a positive bounded operator on h. Then there exists an isometric embedding W of h into a Hilbert space K and a strongly continuous semigroup $(Q_t : t \geq 0)$ of isometries on K, such that*

$$W e^{(iH-k)t} = Q_t^* W, \quad t \geq 0. \qquad (26.2)$$

Proof. Let T_t be the dynamical semigroup of Lindblad type on $B(h)$ constructed from the contraction semigroup $P_t = e^{(iH-k)t}$ on h, together with the completely positive map Ψ given by $\Psi(a) = k^{\frac{1}{2}} a k^{\frac{1}{2}}$ for a in $B(h)$. By Theorem 26.1, there exists a strongly continuous semigroup Q_t of isometries on $K = L^2(\Gamma_\infty; h)$ such

that $T_t(x) \otimes f = D_t^*(x \otimes f) D_t$. Consider the isometric embedding W of H in K given by $f \mapsto \delta_0 \otimes f$. Then, by the definition of Q_t , we have $(Q_t f)(z) = \theta_t f(z)$.
Thus $W^* D_t = Q_t W^*$, and so $Q_t^* W = W D_t^*$.

17. UNITARY DILATIONS OF DYNAMICAL SEMIGROUPS

In Chapter 16 we obtained isometric representations for semigroups of Lindblad type on W^*-algebras. We now investigate unitary dilations of such semigroups, using Cooper's unitary dilation of isometric semigroups (Theorem 3.1). In order to carry through the construction we need to place further restrictions on either the algebra or the semigroup. In the first place, we use σ-finite injective von Neumann algebras. For simplicity, we give a detailed discussion for $B(H)$ only.

17.1 THEOREM. *Let H be a Hilbert space, and let $\{T_t : t \geq 0\}$ be a weakly continuous dynamical semigroup of Lindblad type on $B(H)$. Then there exists a von Neumann algebra M on a Hilbert space L, an embedding π of $B(H)$ as a von Neumann subalgebra of M, a conditional expectation E of M onto $B(H)$, and a strongly continuous unitary group $\{U_t : t \in \mathbb{R}\}$ on L, such that*

$$U_t^* M U_t = M \quad \text{for all } t \text{ in } \mathbb{R},$$

and

$$T_t(x) = E[U_t^*\, \pi(x)\, U_t], \quad t \geq 0,$$

for all x in $B(H)$. Moreover, we have

$$T_t = T_t^* = E[U_t\, \pi(x)\, U_t^*], \quad t \geq 0,$$

for all x in $B(H)$.

Proof. We use the notation of Theorem 16.1. Let $\{V_t : t \geq 0\}$ be the semigroup of isometries such that $1 \otimes T_t(x) = V_t^*(1 \otimes x)V_t$ for all x in $B(H)$. By Cooper's Theorem (Theorem 3.1), there exists a Hilbert space, an isometric embedding $W_2 : L^2(\Gamma_\infty, H) \to L$, and a strongly continuous unitary group $\{U_t : t \in \mathbb{R}\}$ on L such that

$$W_2\, V_t = U_t\, W_2, \quad t \geq 0. \tag{17.1}$$

Let $\pi_1 : B(H) \to 1 \otimes B(H) \subseteq B(L^2(\Gamma_\infty) \otimes H)$ be the canonical embedding $x \to 1 \otimes x$, and let $\pi_2 : B(L^2(\Gamma_\infty, H)) \to B(L)$ be the embedding given by $\pi_2(a) = W_2\, a\, W_2^*$. Define a conditional expectation E_2 of $B(L)$ onto $B(L^2(\Gamma_\infty, H))$ by $E_2(a) = W_2^*\, a\, W_2$. Let W_1 be the isometry from H into $L^2(\Gamma_\infty, H)$ given by $W_1 f = \delta_\emptyset \otimes f$. Then the map $a \to E_1(a) = W_1^*\, a\, W_1$ is a conditional expectation of $B(L^2(\Gamma_\infty, H))$ onto $B(H)$.

[illegible] Finally, we take H to be [illegible], embedding [illegible] in H with [illegible], and projecting back onto [illegible] with [illegible]. For x in [illegible] and t ≥ 0, we have

$$[illegible]$$

On the other hand, for y in [illegible], we have

$$[illegible]$$

This is more than enough to prove the theorem.

17.2 Remarks In the course of the proof of Theorem 17.1 we noted that the embedding [illegible] given by [illegible] is distinct from the embedding [illegible] given by [illegible], where P_0 is the projection in $L^2(T_\infty)$ given by the characteristic function of the singleton {0} in T_∞. However, it turns out that the embedding [illegible] has the same, and that [illegible] is a conditional expectation with respect to [illegible] (as well as with respect to [illegible]). Moreover, for x in B(H) and t ≥ 0, we have

$$[illegible]$$

so that

$$[illegible]$$

while

$$[illegible]$$

More generally, for each Borel subset E of T_∞ and its associated projection P_E in $L^2(T_\infty)$, we have

$$[illegible]$$

Thus we have a simultaneous dilation of the Markov kernels in Davies' non-commutative probability theory.

In Theorem 17.1 we constructed an automorphism group, namely $\{U_t^{*}(\cdot)\, U_t : t \in \mathbb{R}\}$, on the algebra $B(L)$, which projects onto the given dynamical semigroup $\{T_t : t \geq 0\}$ on $B(H)$. In order to treat a von Neumann subalgebra M of $B(H)$ which is globally invariant under T_t, we must either project from $B(H)$ onto M, or work with some subalgebra of $B(L)$. To follow the first alternative, we need the concept of an injective von Neumann algebra.

17.3 DEFINITION A von Neumann algebra M is injective if, whenever M is embedded as a von Neumann subalgebra of another von Neumann algebra M', there exists a conditional expectation (not necessarily normal) of M' onto M. Thus we see that weakly continuous dynamical semigroups of Lindblad type on injective von Neumann algebras possess unitary dilations in the sense of Theorem 17.1. However, it is known (see Effros [1977]) that a von Neumann algebra is injective if and only if it is hyperfinite. Thus in general the first alternative is not feasible. Turning to the second alternative, we seek a von Neumann subalgebra $\tilde{M}$ of $B(L)$ which is at least invariant under $\{U_t^{*}(\cdot)\, U_t : t \in \mathbb{R}\}$, and contains $\alpha(M)$. We also employ the following device: we do not attempt to project $\tilde{M}$ directly onto $1 \otimes M$ via the map E_0, but rather onto some algebra $Z \otimes M$, where Z is a judiciously chosen von Neumann subalgebra of $B(L^2(Y_\infty))$. The following diagram may clarify matters:

$$\begin{array}{ccccccccc}
M & \longleftrightarrow & 1 \otimes M & \longrightarrow & Z \bar{\otimes} M & \longrightarrow & B(L^2(Y_\infty, H)) & \xrightarrow{\;E_1\;} & B(L) \\
M & \longleftrightarrow & 1 \otimes M & \longleftarrow & Z \bar{\otimes} M & & \xleftarrow{\qquad E_2 \qquad} & & \tilde{M}
\end{array}$$

This programme is performed in the following theorem:

17.4 THEOREM Let H be a separable Hilbert space; let $\{T_t : t \geq 0\}$ be a weakly continuous dynamical semigroup of Lindblad type on $B(H)$, so that there exists a strongly continuous contraction semigroup S_t on H, and an ultraweakly continuous completely positive linear map Ψ on $B(H)$, such that

$$T_t(x) = \tilde{S}_t(x) + \int_0^t (T_{t-s} \circ \Psi \circ \tilde{S}_s)(x)\, ds,$$

with $T_t(x) = W_t^* x W_t$. *Suppose that W has a decomposition*

$$W(x) = \int_X A_x^* x A_x \, d\lambda(x),$$

where (X,λ) is a σ-finite measure space, and $x \mapsto A_x$ is weakly measurable. If M is a von Neumann algebra on H such that

A_x *lies in M for almost all x in X,* (17.2)

$W_t^* M W_t \subseteq M$ *for all $t \geq 0$,* (17.3)

then the dynamical semigroup $\{T_t : t \geq 0\}$ on M has a unitary dilation. That is, there exists a von Neumann algebra $\tilde{M}$ on a Hilbert space L, a strongly continuous unitary group $\{U_t : t \in \mathbb{R}\}$ on L, an embedding π of M as a von Neumann subalgebra of $\tilde{M}$, and a normal conditional expectation E of $\tilde{M}$ onto M such that:

$U_t^* \tilde{M} U_t \subseteq \tilde{M}$ *for all $t \in \mathbb{R}$,* (17.4)

$T_t(x) = E(U_t^* \pi(x) U_t)$ *for all x in M and $t \geq 0$.* (17.5)

Proof. For clarity, we give the details of the proof for the case where $M = B(H)$ and λ is a counting measure. We employ the notation and construction used in the proof of Theorem 16.1. Thus we have a strongly continuous isometric semigroup $\{V_t : t \geq 0\}$ on $L^2(\Gamma_\infty, H)$, and an isometric embedding W_0 of $L^2(\Gamma_\infty, H)$ into a Hilbert space L on which there is a strongly continuous unitary group $\{U_t : t \in \mathbb{R}\}$, such that $W_0 V_t = U_t W_0$ for $t \geq 0$. Take $\mathcal{A}$ to be the commutative von Neumann algebra $L^\infty(\Gamma_\infty)$, and take $M^\#$ to be $L^\infty(\Gamma_\infty, M)$, which is a W^*-algebra with predual $M^\#_* = L^1(\Gamma_\infty, M_*)$. The mapping $f \otimes x \mapsto f(\cdot)x$ has a unique extension to a W^*-isomorphism of $L^\infty(\Gamma_\infty) \bar{\otimes} B(H)$ onto $L^\infty(\Gamma_\infty, M)$ (see [illegible]).

Put $\tilde{M} = \{U_t^* \pi_0(M^\#) U_t : t \in \mathbb{R}\}''$, where $\pi_0 : M^\# \to B(L)$ is again defined as $\pi_0(x) = W_0 x W_0^*$. We will show that $N_t(\tilde{M}) \subseteq M^\#$ where $N_t : \tilde{M} \to B(L^2(\Gamma_\infty, H))$ is defined as $N_t(x) = W_0^* x W_0$. For this, it is convenient to have the explicit form of the action of V_t^* on a vector f. We get this by computing $\langle V_t g, f \rangle$ for arbitrary g:

$$\langle V_t g, f \rangle = \int_{\Gamma_\infty} \int_{\Gamma_t} \langle [illegible](\omega_1)\, g(\omega_1),\, f(\sigma_t(\omega_t, \omega_1)) \rangle \, d\mu_t(\omega_t)\, d\mu_\infty(\omega_1)$$

$$= \int_{\Gamma_\infty} \int_{\Gamma_t} \langle g(\omega_1),\, ([illegible](\omega_1))^* f(\sigma_t(\omega_t, \omega_1)) \rangle \, d\mu_t(\omega_t)\, d\mu_\infty(\omega_1).$$

where Q_t^* is given by

$$(Q_t^* f)(w) = \int_{X_t} [(D\theta B)(w')]^* f(k_t(w',w)) \, d\mu_t(w') .$$

In what follows we use w^t to denote $k_t(w',w)$, where w' is a variable of integration running through X_t; we remark that $w^t_0 = w'$, and $w^t_t = w$. We claim that $M_t(M) \subseteq M^1$. For $t \geq 0$ and x in M^1, we have

$$M_t(Q_t^* \, m_t(x) \, U_t) = M_t^* \, Q_t^* \, m_t \, x \, M_t^* \, U_t \, M_t$$
$$= Q_t^* \, x \, Q_t .$$

We take x in $L^\infty(Y_\infty, M)$ and compute $Q_t^* \, x \, Q_t$ as an element of $B(L^2(Y_\infty, H))$, and show that it lies in $L^\infty(Y_\infty, M)$; we have

$$(Q_t^* \, x \, Q_t f)(w) = \int_{X_t} [(D\theta B)(w')]^* (x Q_t f)(w^t) \, d\mu_t(w')$$

$$= \int_{X_t} [(D\theta B)(w')]^* x(w^t) \, (D\theta B)(w^t_t) \, f(w^t_t) \, d\mu_t(w')$$

$$= \int_{X_t} [(D\theta B)(w')]^* x(w^t) \, (D\theta B)(w')f(w) \, d\mu_t(w')$$

$$= \int_{X_t} [(I, \theta, V_t, B)(w')]^* \, x(w^t) \, d\mu_t(w') \, f(w) .$$

Thus $(Q_t^* \, x \, Q_t)(w) = \int_{X_t} [(I, \theta, V_t, B)(w')]^* \, x(w^t) \, d\mu_t(w')$ lies in $L^\infty(Y_\infty, M)$, and so $Q_t^* \, M^1 Q_t \subseteq M^1$. For $n \geq 1$ and $t_i \geq 0$, $i = 1, \ldots, n$, we define x_n by

$$x_n = M_t(Q_{t_1}^* \, m_t(x_1) \, U_{t_1} \, Q_{t_2}^* \, m_t(x_2) \, U_{t_2} \cdots Q_{t_n}^* \, m_t(x_n) \, U_{t_n}) .$$

It follows that

$$x_n = Q_{t_1}^* \, x_1 \, Q_{t_2}^* \, Q_{t_1} \, x_2 \, Q_{t_3}^* \, Q_{t_2} \cdots Q_{t_{n-1}} \, x_n \, Q_{t_n} ,$$

observing that for all $s, t \geq 0$ we have $M_t^* \, U_s \, U_t^* \, M_t = Q_s^* \, Q_t$, as a consequence of Theorem 3.7. We have to show that x_n lies in M^1. In order to state an induction hypothesis we introduce z_n defined by

$$z_n = Q_{t_1}^* \, x_1 \, Q_{t_2}^* \, Q_{t_1} \, x_2 \, Q_{t_3}^* \, Q_{t_2} \cdots x_n \, Q_{t_{n+1}}^* \, Q_{t_n} .$$

and notice that $C_n\big|_{t_{n+1}=0} = a_n$. By direct calculation of the first case above, we have

$$(b_1 f)(w) = \int_{S_{t_1}}\int_{S_{t_2}} \tilde{k}_1(w',w'';w)\, f(w^{t_1 t_2}_{s_1})\, d\mu_{t_1}(w')\, d\mu_{t_2}(w'')$$

where

$$\tilde{k}_1(w',w'';w) = [[\text{illegible}](w'')]^{*}\, a_1(w^{t_1})\, [[\text{illegible}](w'')]^{*}\, [\text{illegible}](w^{t_1 t_2}_{s_1}).$$

Suppose that, for $n \geq 1$, we have

$$(b_n f)(w) = \int_{S_{t_1}}\cdots\int_{S_{t_{n+1}}} \tilde{k}_n(w',\ldots,w^{(n+1)};w)\, f(w^{t_1 t_2}_{s_1}\cdots{}^{t_{n+1}}_{s_n})\, d\mu_{t_1}(w') \cdots d\mu_{t_{n+1}}(w^{(n+1)}); \qquad (17.6)$$

then

$$(b_{n+1} f)(w) = \int_{S_{t_1}}\cdots\int_{S_{t_{n+1}}} \tilde{k}_n(w',\ldots,w^{(n+1)};w)(a_{n+1} \circ [\text{illegible}])\, f(w^{t_1 t_2}_{s_1}\cdots{}^{t_{n+1}}_{s_n})\, d\mu_{t_1}(w')\cdots d\mu_{t_{n+1}}(w^{(n+1)})$$

$$+ \int_{S_{t_1}}\cdots\int_{S_{t_{n+2}}} \tilde{k}_{n+1}(w',\ldots,w^{(n+2)};w)\, f(w^{t_1 t_2}_{s_1}\cdots{}^{t_{n+1}}_{s_n}{}^{t_{n+2}}_{s_{n+1}})\, d\mu_{t_1}(w')\cdots d\mu_{t_{n+2}}(w^{(n+2)}) \qquad (17.7)$$

where

$$\tilde{k}_{n+1}(w',\ldots,w^{(n+2)};w) = \tilde{k}_n(w',\ldots,w^{(n+1)};w)\, a_{n+1}(w^{t_1 t_2}_{s_1}\cdots{}^{t_{n+1}}_{s_n})\, [[\text{illegible}](w^{(n+2)})]^{*} [\text{illegible}](w^{t_1 t_2}_{s_1}\cdots{}^{t_{n+2}}_{s_{n+1}}).$$

But (17.6) holds for $n = 1$, and hence, by (17.7), for all $n \geq 1$. Evaluating $(b_n f)(w)$ at $t_{n+1} = 0$, we have

$$(a_n f)(w) = \int_{S_{t_1}}\cdots\int_{S_{t_n}} \tilde{k}_n(w',\ldots,w^{(n)};w)\, f(w^{t_1 t_2}_{s_1}\cdots{}^{t_n}_{s_{n-1} t_n})\, d\mu_{t_1}(w')\cdots d\mu_{t_n}(w^{(n)}).$$

It follows directly from the definitions that $w^{t_1 t_2}_{s_1}\cdots{}^{t_n}_{s_{n-1} t_n} = w$, so that

$$(a_n f)(w) = a_n(w)\, f(w).$$

where

$$a_n(m) = \int_{K_{t_2}} \cdots \int_{K_{t_n}} \tilde{a}_n(w^1, \dots, w^{(n)}) \dots d\mu_{t_1}(w^1) \dots d\mu_{t_n}(w^{(n)}),$$

Then a_n lies in M^t, so then $E_t(M) \subseteq M^t$ by continuity. We complete the proof by taking any conditional expectation $\tilde{E}_t$ of M^t onto $Z(M)$. For example, let ϕ be a normal state on $L^\infty(Y_\infty)$ (that is, ϕ is an element of $L^1(Y_\infty)$) and put $\tilde{E}_t = \phi \otimes 1 : L^\infty(Y_\infty) \bar{\otimes} M \to M$. (If we take $\phi = \delta_0$, then $\tilde{E}_t(a) = a(0)$ for a in M^t; in fact, in the notation of Theorem 17.1, the restriction of $E_t(\cdot) = \omega_t^0(\cdot)[\mu_t]$ to M^t coincides with $\tilde{E}_t$ in this case.) We then put $\alpha = \omega_2 \circ \alpha_1$ and $N = \tilde{E}_1 \circ E_2$, and we have

$$T_t(m) = N(\Omega_t^\alpha \text{ word } (\mu_t)), \quad t \geq 0,$$

for all m in M.

17.5 REMARK. The map $t \mapsto \Omega_t^\alpha(\cdot)[\mu_t]$ is weakly continuous. It cannot be norm-continuous, even though $t \mapsto T_t$ may be, unless T_t is a homomorphism of M. Indeed, suppose $t \mapsto T_t$ is strongly continuous with generator L, $t \mapsto \Omega_t^\alpha(\cdot)[\mu_t]$ is strongly continuous with generator A, and $Z = D(A) \cap M$ is a core for L ([illegible]). Then for x in Z we have [illegible], and so x is in $D(L)$ and $L(x) = N(A(x))$. Then for x, y in the subalgebra Z, we have

$$L(xy) = N(A(xy)) = N(A(x)y + xA(y))$$
$$= N(A(x))y + xN(A(y))$$
$$= L(x)y + xL(y),$$

and so L is a derivation if Z is a core for L; in this case it follows from Theorem 14.4 that T_t is a semigroup of homomorphisms.

References

1. The main result of this chapter is Theorem 1.8. For scalar-valued kernels on $X \times X$, it was proved by Kolmogorov (1941); he showed that a kernel is the correlation kernel of a stochastic process if and only if it is positive-definite (Parthasarathy & Schmidt, 1972). For operator-valued kernels, versions of Theorem 1.8 with various restrictive assumptions on X, can be found in the literature (Pedrick, 1957, Kunze, 1967, Ponomarenko, 1968, Allen, Narcowich & Williams, 1975).

The idea of using the image space rather than the quotient space (Naimark, 1943a) goes back to Aronszajn (1950). It has been exploited by Halmos (1950) and Schreiber & Wiskemann (1970) for Hilbert space dilations, and by Pierce (1957) and Casey (1970) in group representation theory.

Remarks on the origins of Theorem 1.10 will be found in the notes on Chapter 11.

2. The dilation theorem for positive-definite functions on groups (Corollary 2.8) is due to Naimark (1943b). It was extended to *-semigroups by Sz.-Nagy (1955). The canonical decomposition of a completely positive scalar-valued map (that is, of a state) on a C*-algebra is known as the GNS construction (Gelfand & Naimark, 1943, Segal, 1947). It was extended by Stinespring (1955) to operator-valued completely positive maps on unital C*-algebras; the original proof was simplified by Arveson (1969), and the result was extended to a larger class of unital *-algebras by Powers (1974). Lance (1973) obtained the Stinespring decomposition for non-unital C*-algebras by going to the second dual. The result for Banach *-algebras with approximate identities (Theorem 2.10) is due to Evans (1975). For some related results, see Paschke (1973). As can be seen from the proof of Theorem 2.13, the Stinespring decomposition for a completely positive map whose domain consists of a subspace N*N, where N is a left ideal in an algebra A, can be obtained in such a way that it is constructed on the whole of A. This is the decomposition used by Evans (1977a) to study unbounded completely positive maps on C*-algebras whose domains consist of hereditary *-subalgebras.

The relationship between the Stinespring decomposition for algebras and the Naimark dilation for groups has been described several times in the literature (see Evans, 1976). If G is a locally compact group, there is a canonical bijection between completely positive maps on $L^1(G)$ and those on $C^*(G)$, the enveloping C^*-algebra of $L^1(G)$. If G is abelian, $C^*(G)$ can be identified via the Fourier transform with $C_0(\hat{G})$, the continuous functions vanishing at infinity on $\hat{G}$, the dual of G.

3. The theory of dilations of continuous semigroups began with Cooper (1947) who discovered Theorem 3.1. It is interesting to note that his motivation came from quantum mechanics (Cooper 1950a,b). Theorem 3.3, on the dilation of semigroups of contractions, is due to Sz.-Nagy (1953); it is a powerful tool in Hilbert space theory (Sz.-Nagy & Foias, 1970).

The idea of the proof of Theorem 3.4 comes from Sz.-Nagy (1955), who discovered the connection between positive-definite functions on $\mathbb{Z}$ and *-semigroups of contractions indexed by $\mathbb{N}$. This method was generalized by Mlak (1966) and Evans (1976), and their work is the basis of our exposition.

The construction of a unitary dilation of a contraction semigroup contracting strongly to zero (Theorem 3.10) is due to Lax & Phillips (1967); this method can be modified to give an alternative proof of Theorem 3.3 (Sz.-Nagy & Foias, 1970, §1.10.2). The abstract Langevin equation in Theorem 3.13 was obtained by Lewis & Thomas (1974) in connection with an analysis of the Ford-Kac-Mazur model (Lewis & Thomas, 1975); see also Lewis & Pulè (1975).

4. There is an extensive recent literature on completely positive maps on C^*-algebras and the tensor-product construction; see the review by Effros (1977). The equivalence of (i) and (ii) in Lemma 4.1(a) occurs in the work of Andrews (1974) and Paschke (1973). The proof given here of Lemma 4.1(c) is due to Skau (private communication). Stinespring (1955) showed that a positive map from an arbitrary C^*-algebra into a commutative C^*-algebra is completely positive; he used a slightly different method from the one given here (Theorem 4.2). That any positive map from a commutative C^*-algebra into an arbitrary C^*-algebra is completely positive was shown by Naimark (1943), and by Stinespring (1955). The

Schwarz inequality (4.3) in Corollary 4.4 was first obtained for self-adjoint elements by Kadison (1952), who used an entirely different method. Corollary 4.4 and its proof were first recorded by Størmer (1963, 1967) along with the Schwarz inequality of Theorem 4.14 for completely positive maps (with essentially the same proof as in Chapter 4). For other Schwarz-type inequalities, with various positivity assumptions, see Araki (1959), Choi (1974), Evans (1976a, 1976b), Lieb & Ruskai (1974). Corollary 4.5 is due to Arveson (unpublished), and is recorded by Størmer (1967). The proof given here is due to Evans & Høegh-Krohn (1977) and uses an observation of Evans (1976b).

Kraus (1971) obtained the canonical decomposition of a normal completely positive map on the von Neumann algebra of all bounded operators on a Hilbert space. Choi (1975) showed that if, in Theorem 4.8, H and K are finite-dimensional the decomposition can be chosen so that the cardinality of the set X is at most dim (H). dim (K).

5. Conditional expectations on classical probability spaces were characterised by Moy (1954) in terms of positive maps with the module property (5.2). The study of analogues of conditional expectations in the non-commutative setting was begun by Umegaki (1954). A detailed discussion of Examples 5.1 and 5.2 can be found in Davies (1976b): the first arises in measurement theory, the second in the composition of quantum systems. Theorem 5.4 is due to Tomiyama (1957) and Arveson (unpublished); the proof given here is taken verbatim from Størmer (1967). The definition of a conditional expectation adopted in this chapter is quite adequate for many purposes in non-commutative probability theory, but not for all; see Davies & Lewis (1970) and Accardi (1974, 1976) for more general concepts.

6. - 8. These chapters provide an exposition of some of the folk-lore of mathematical physics. The fundamental paper on Fock space is by Fock (1932). The characterisation of a generating functional of the CCR is due to Araki (1960) and to Segal (1961) independently. The extension to the operator-valued case was given by Evans (1977). The extremal universally invariant states whose generating functionals are of the form (7.11) were introduced by Segal (1962).

Our treatment of the CAR algebra and its representations is in the spirit of Hugenholtz & Kadison (1975).

9. The main results of this section are due to Evans (1977), we follow the exposition of Evans (1979). The key lemma (Lemma 9.2) is due to Hall (1962). The construction employed in its proof has been used by Howland (1974) and by Evans (1976b) in the study of scattering by time-dependent perturbations.

10. Quasi-free dynamical semigroups associated with representations of the CCR were investigated in the thesis of Thomas (1973); see also Lewis & Thomas (1975a). In the algebraic context they were studied by Davies (1972a, 1977a, 1976b) and also by Demoen, Vanheuverzwijn & Verbeure (1976, 1977), Evans (1976), Emch, Albeverio & Eckmann (1977), Evans & Lewis (1976b) and Lindblad (1976c). Necessity in Theorem 10.2 was proved by Evans & Lewis (1976b), whilst sufficiency was shown by Demoen et al. (1977). In fact, Demoen et al. (1977) introduce the multiplier [illegible], and use it to construct a CCR algebra $W_\sigma(H)$ over H. They exploit the fact that the function F of Theorem 10.2 gives rise to a completely positive map if and only if it is a generating functional of a state of the algebra $W_\sigma(H)$.

Theorems 10.3, 10.4, 10.5 are an elaboration of the work of Evans & Lewis (1976b). Incidentally, the proof of Theorem 10.10 is due to Kastler (private communication), who uses it to give an elementary proof of the fact that any type I von Neumann algebra is injective.

11. Theorem 11.1 appears in Hugenholtz & Kadison (1975). For related work see Nelson (1973), Streater & Wulfsohn (1975).

12. Dilations of quasi-free dynamical semigroups induced by contraction semigroups can be found in the [illegible] model (Ford et al., 1965, Thomas, 1971, Lewis & Thomas, 1975a,b). They have been studied in detail by Davies (1972a), Evans (1976), Emch et al. (1977), Evans & Lewis (1976).

Evans (1976) has shown that a one-parameter unitary group on a Hilbert space gives rise to a norm-continuous group of automorphisms on the CAR algebra if and only if the generator on the Hilbert space is bounded. Lindblad (1976c) has studied quasi-free dynamical semigroups on the CAR algebra in detail.

It follows from this work that a contraction semigroup with a trace class generator on a Hilbert space induces a norm-continuous dynamical semigroup on the CAR algebra.

13. Theorem 13.1 and its proof are due to Davies (1978). It is unclear how to extend the construction to the category of W*-algebras. Evans (1975, 1976a) had previously obtained this result for discrete groups. There is no problem in modifying his construction to deal with von Neumann algebras.

The C*-algebra generated by π(A) and S*(G) is a generalisation of the C*-crossed product of a C*-algebra by a group of automorphisms (Turumaru, 1958; Doplicher, Kastler & Robinson, 1966).

14. Theorem 14.1 was obtained by Evans (1976), who generalised the well-known result for strongly continuous one-parameter groups. That (i) implies (vi) in Theorem 14.2 is due to Kraus (1970), who observed also that (vi) implies (i) is implicit in the work of Lindblad (1976a). The equivalences (i) - (iv) of Theorem 14.2, and also Theorems 14.5, 14.6, are due to Evans & Hanche-Olsen (1977). Theorem 14.3 is an improvement on the work of Kishimoto (1976). See also Sullivan's (1975) proof of Lumer & Phillips's (1961) result: a densely defined dissipative linear map is closable. Theorem 14.4 was first proved for identity-preserving semigroups on unital C*-algebras by Lindblad (1976a). He used a different method.

The concept of conditionally completely positive maps was introduced by Evans (1977c). Lemma 14.5 is built on the work of Evans (1976), Lindblad (1976a,c) and Davies (1977). Theorem 14.7 is a strengthening of the result of Evans (1977) for unital C*-algebras. For the analogous result for semigroups of positive-definite functions on groups, see Parthasarathy & Schmidt (1972).

For earlier work on the generators of dynamical semigroups, and dissipativity, see Kossakowski (1972a,b, 1973) and Ingarden & Kossakowski (1975). For recent work on the generators of strongly continuous dynamical semigroups, see Davies (1977a - d). Note also the characterisations of the generators of positive semigroups in a function space context by Simon (1976) and by Evans, Schrader & Uhlenbrock (1977).

15. The canonical decomposition of norm-continuous semigroups of completely positive normal maps on a von Neumann algebra was first obtained independently by Gorini, Kossakowski & Sudarshan [1976a] for finite-dimensional matrix algebras, and by Lindblad [1976a] for hyperfinite von Neumann algebras. The implication (i) => (iii) in Theorem 15.1 is an improved version of Lindblad [1976a]; the converse is due to Evans [1977a].

If A is a von Neumann algebra on a Hilbert space H, it is known that $H^1(A,B(H)) = 0$ if: (i) A is type I or hyperfinite (Johnson, 1972; Ringrose, 1972), (ii) A is properly infinite (Christensen, 1976). It is widely conjectured that $H^1(A,B(H)) = 0$ for all von Neumann algebras.

16, 17. These chapters are an improved version of the work of Evans & Lewis [1976a] which was inspired by Davies [1972a].

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