Sgríbhinní Institiúid Árd-Léinn Bhaile Átha Cliath
Sraith. A, Uimh. 26

Communications of the Dublin Institute for Advanced Studies
Series A (Theoretical Physics), No. 26

DIFFERENTIAL FORMS IN GENERAL RELATIVITY

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Second Edition

DUBLIN

Institiúid Árd-Léinn Bhaile Átha Cliath
Dublin Institute for Advanced Studies
1979

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Preface to 2nd Edition

In the decade since the first edition of this Communication was written, differential forms have become a staple ingredient of elementary textbooks. Nevertheless, this informal account, which has been unavailable for some years, is still in demand. In this new edition the original text is unchanged apart from minor corrections and improvements, but I have added a chapter on tensor-valued forms and Lie derivatives, and the bibliography has been brought up to date.

I should like to express my appreciation to Mrs. E. Maguire for an excellent typing job. I owe a special debt of gratitude to Miss Eva Wills of the Dublin Institute for Advanced Studies, but for whose patient encouragement this Communication would never have reappeared.

W. Israel
University of Alberta,
10 October 1978.

Introduction

The following pages are intended as an introduction to some modern techniques in gravitational theory which make use of differential forms. In particular, I have tried to give a leisurely account of the elegant complex vectorial formalism of Cahen, Debever, and DeFrise, which is a compact equivalent of the better known Newman-Penrose spin coefficient formalism. I hoped to make the subject easily accessible to anyone whose acquaintance with relativity and its mathematical background does not extend beyond one of the traditional texts. Accordingly, the exposition is interlaced with many simple examples, intermediate steps of calculations have usually been given in full and contact with classical tensor notation is maintained at every point.

The work arose from an informal seminar given in 1968 while I had the privilege of being a Visiting Professor at the School of Theoretical Physics, Dublin Institute for Advanced Studies. To Professor J. L. Synge and Professor C. Lanczos I am much indebted for encouraging its expansion into the present form and for their hospitality. Discussions with Professor Synge, Dr. M. Misra and Dr. G. Ludwig helped to clarify the exposition at many points.

CHAPTER I: BASIC IDEAS

1.1. DIFFERENTIAL FORMS

The representation of a tensor by a multilinear function of vector arguments is familiar. For instance, the second-order tensor $T_{\alpha\beta}$ can be associated with the bilinear scalar function $T(\underset{\sim}{u},\underset{\sim}{v}) \equiv T_{\alpha\beta}u^{\alpha}v^{\beta}$. Specifying the value of $T(\underset{\sim}{u},\underset{\sim}{v})$ for all $\underset{\sim}{u}$ and $\underset{\sim}{v}$ (actually, a finite set of pairs of basis vectors obviously suffices) constitutes a *definition* of the tensor, a definition which has the merit of being co-ordinate independent.

A *differential form* is such a multilinear scalar function associated with a tensor which is (effectively) completely skew. For example, if $F_{\alpha\beta}$ is any second-order tensor, then

$$\phi(d\underset{\sim}{x},d\underset{\sim}{y}) \equiv 2!\ F_{\alpha\beta}dx^{[\alpha}dy^{\beta]} \tag{1}$$

is a differential form of degree two, or a 2-form. It is convenient to write the vector arguments as differentials, for reasons which will soon appear. Square brackets enclosing a set of indices indicate anti-symmetrization, e.g.

$$dx^{[\alpha}dy^{\beta]} \equiv (1/2!)\ (dx^{\alpha}dy^{\beta} - dx^{\beta}dy^{\alpha}).$$

(Division by the factorial is a universal but rather unfortunate convention. It is in order to undo the effects of this that factorial coefficients have to be inserted in most of the definitions.) The square bracket operation in (1) effectively sifts out the skew part $F_{[\alpha\beta]}$ of the original tensor.

A 1-form

$$\theta(d\underset{\sim}{x}) = A_{\alpha}dx^{\alpha} \tag{2}$$

corresponds to a vector. A 0-form has no vector argument and is therefore simply a scalar.

Equality and addition of two p-forms, and multiplication of a form by a number are defined in an obvious way in terms of corresponding operations on the associated skew-symmetrized tensors.

1.2. WEDGE PRODUCT

The exterior product, or wedge product, of a p-form and a q-form is the (p+q)-form obtained by taking the tensor product of the associated tensors and anti-symmetrizing. As an example, for the forms (1) and (2),

$$\psi \equiv \phi \wedge \theta$$

iff $$\psi(\underset{\sim}{dx},\underset{\sim}{dy},\underset{\sim}{dz}) = 3!\ F_{\alpha\beta}A_{\gamma}dx^{[\alpha}dy^{\beta}dz^{\gamma]} .$$

Multiple wedge products like $\alpha \wedge \beta \wedge \gamma$ can, of course, be defined similarly and are associative.

If α and β are forms of degrees a and b, it is easy to verify that

$$\alpha \wedge \beta = (-1)^{ab}\ \beta \wedge \alpha . \tag{3}$$

(The factor $(-1)^{ab}$ arises from pulling each of the a indices of the first factor on the left through the b indices of the second factor, entailing ab successive interchanges.) Thus, the wedge product anti-commutes if *both* factors are odd, but commutes otherwise. In particular,

$$\theta \wedge \theta = 0 \quad \text{if} \quad \theta \text{ is } \textit{odd}. \tag{4}$$

But $f \wedge f = f^2$ for a 0-form (scalar) f, and for the 2-form (1), $\phi \wedge \phi$ corresponds to the tensor $F_{[\alpha\beta}F_{\gamma\delta]}$, which does not vanish in general.

1.3. Exterior Differential

We proceed to consider tensor *fields* in a Riemannian space. The associated p-forms are then scalar fields which depend on p vector-field arguments.

The exterior differential of a p-form is the (p+1)-form obtained by taking the partial or covariant derivative (it is immaterial which!) of the associated p^{th} order tensor. Thus, the exterior differential of the 2-form $\phi = 2!\ F_{\alpha\beta}dx^{[\alpha}dy^{\beta]}$ is the 3-form

$$d\phi\ (\underset{\sim}{dx},\underset{\sim}{dy},\underset{\sim}{dz}) = 3!\ \partial_{\alpha}F_{\beta\gamma}dx^{[\alpha}dy^{\beta}dz^{\gamma]}$$

$$= 3!\ F_{\beta\gamma|\alpha}\ dx^{[\alpha}dy^{\beta}dz^{\gamma]} , \tag{5}$$

where the stroke denotes covariant differentiation. (The additional terms involving the affine connection (assumed symmetric) cancel out upon anti-symmetrization.) If f is a 0-form,

$$df(\underset{\sim}{dx}) = (\partial_{\alpha}f)dx^{\alpha} ,$$

in agreement with the elementary definition.

We shall frequently use the formula

$$d(f\Omega) = f\,d\Omega + df \wedge \Omega\,, \tag{6}$$

which holds for any form Ω and scalar f. This is actually a special case of the general identity

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^a\, \alpha \wedge d\beta\,, \qquad a \equiv \deg.\alpha\,. \tag{7}$$

The factor $(-1)^a$ arises from pulling the differentiation index ∂_α through the a indices of the first factor in order to operate on the second.

Applying the operator d twice in succession involves the skew part of $\partial_\alpha \partial_\beta$, which vanishes. Hence

$$d^2\Omega \equiv 0 \tag{8}$$

for any form Ω.

1.4. Co-ordinate Differentials

If a definite co-ordinate system $(x^1, \ldots, x^n)$ is singled out, the functions x^μ are scalar fields. Closely linked with them are n vector fields $\underset{\sim}{e}^{(\mu)} \equiv \text{grad}\, x^\mu$ (often called co-ordinate basis vectors). Their components (in the special co-ordinate system) are $e^{(\mu)}{}_\alpha = \delta^\mu_\alpha$.

The exterior differential of the 0-form x^1 is

$$dx^1(d\underset{\sim}{y}) = (\partial_\alpha x^1)dy^\alpha = \delta^1_\alpha\, dy^\alpha \tag{9}$$

which is the 1-form associated with $\underset{\sim}{e}^{(1)}$.

It will be noticed that in (9) we have used the symbol dx^1 in a new and quite distinct sense. Whereas previously it was simply a *number*, i.e. a component of the vector argument $d\underset{\sim}{x}$, it now denotes a 1-form, i.e. a *function* of a vector argument. However, if the vector argument in (9) is $d\underset{\sim}{x}$ (as we shall normally assume), the distinction is unimportant for practical purposes. This dual use of the symbol dx^α is very convenient and, once recognized, seldom leads to difficulty.

As a simple example, consider cartesian co-ordinates x^1, x^2, x^3 in Euclidean 3-space. We construct the 3-form

$$\begin{aligned} dx^1 \wedge dx^2 \wedge dx^3\,(d\underset{\sim}{u}, d\underset{\sim}{v}, d\underset{\sim}{w}) &= 3!\; \delta^1_\alpha\, \delta^2_\beta\, \delta^3_\gamma\, du^{[\alpha} dv^\beta dw^{\gamma]} \\ &= \epsilon_{\alpha\beta\gamma} du^\alpha dv^\beta dw^\gamma\,, \end{aligned}$$

which associates with each triplet of vectors the volume of the 3-cell formed by them. By choosing these vectors along the co-ordinate axes with lengths dx^1, dx^2, dx^3, we obtain

$$dx^1 \wedge dx^2 \wedge dx^3 = dx^1 dx^2 dx^3 \qquad (10)$$

i.e. the standard form of the volume element. We have merely to bear in mind the implication that (10) is a *numerical* relation which holds for a special (but natural) choice of the vector arguments.

1.5. Integral Theorems

Let V_2 be any reasonable 2-space whose boundary ∂V_2 consists of one or more closed curves. We loosely think of V_2 partitioned, in an arbitrary continuous fashion into infinitesimal 2-cells $(d\underset{\sim}{x}, d\underset{\sim}{y})$, and ∂V_2 into infinitesimal straight segments $d\underset{\sim}{z}$. Then it is easy to give a heuristic derivation of Stokes' theorem in the form

$$2! \iint_{V_2} \partial_\lambda A_\mu dx^{[\lambda} dy^{\mu]} = \int_{\partial V_2} A_\mu dz^\mu$$

where the sense of $d\underset{\sim}{z}$ relative to $d\underset{\sim}{x}, d\underset{\sim}{y}$ has to be suitably prescribed. (The condition is that the pair $(\underset{\sim}{E}, d\underset{\sim}{z})$ have the same orientation as $(d\underset{\sim}{x}, d\underset{\sim}{y})$, i.e.

$$\varepsilon(d\underset{\sim}{x})\ \varepsilon(d\underset{\sim}{y})\ E_\lambda dz_\mu dx^{[\lambda} dy^{\mu]} > 0$$

on ∂V_2, where $\underset{\sim}{E}$ is any vector pointing outwards from V_2 and $\varepsilon(\underset{\sim}{A}) \equiv \mathrm{sign}(\underset{\sim}{A}.\underset{\sim}{A}) = -1, +1$ or 0 accordingly as $\underset{\sim}{A}$ is space-like, time-like or null.)

This result, and its higher-dimensional analogues, for instance

$$4! \iiiint_{V_4} \partial_\kappa F_{\lambda\mu\nu} dx^{[\kappa} dy^\lambda dz^\mu dt^{\nu]} = 3! \iiint_{\partial V_4} F_{\lambda\mu\nu} du^{[\lambda} dv^\mu dw^{\nu]} \qquad (11)$$

(orientation of $\underset{\sim}{E}$, $d\underset{\sim}{u}$, $d\underset{\sim}{v}$, $d\underset{\sim}{w}$ the same as $d\underset{\sim}{x}$, $d\underset{\sim}{y}$, $d\underset{\sim}{z}$, $d\underset{\sim}{t}$ on ∂V_4) can all be subsumed under the elegant statement

$$\int_{V_n} d\omega = \int_{\partial V_n} \omega \qquad (12)$$

where ω is a differential form of degree $(n-1)$. Equation (12) is, of course, to be understood as a numerical relation, in which it is implied that the vector arguments of $d\omega$ and ω are the cell vectors of arbitrary (compatible) partitions of V_n and ∂V_n respectively.

It is well-known that this generalized Stokes theorem contains the divergence theorem as a special case. In a four-dimensional space, for instance, let us choose

$$F_{\lambda\mu\nu} = |g|^{\frac{1}{2}} \varepsilon_{\kappa\lambda\mu\nu} A^{\kappa}$$

in (11), and note that (for positively oriented 4-cells) the invariant elements of 4-volume and 3-area are given by

$$4!\ dx^{[\kappa} dy^{\lambda} dz^{\mu} dt^{\nu]} = |g|^{-\frac{1}{2}} \varepsilon^{\kappa\lambda\mu\nu}\, dV_4 ,$$

$$d\Sigma_{\kappa} = |g|^{\frac{1}{2}} \varepsilon_{\kappa\lambda\mu\nu} du^{\lambda} dv^{\mu} dw^{\nu} .$$

Then

$$\int_{V_4} A^{\kappa}\big|_{\kappa}\, dV_4 = \int_{\partial V_4} A^{\kappa}\, d\Sigma_{\kappa} .$$

Chapter II: Riemannian Geometry

2.1. Introduction

The traditional approach to this subject makes heavy use of Christoffel symbols. These are notoriously clumsy to work with, and have no invariant significance. Differential forms clear the way for a much more elegant and flexible approach. Basically, this is nothing but a streamlined version of the classical theory of Ricci rotation coefficients.

In a Riemannian n-space, let $\underset{\sim}{e}^{(1)}, \ldots, \underset{\sim}{e}^{(n)}$ be any set of n vector fields which are linearly independent and therefore form a complete vector basis ("frame") at each point. It will be recalled that the Ricci rotation coefficients $\gamma^a{}_{bc}$ associated with the $\underset{\sim}{e}^{(a)}$ are a set of n^3 numbers (scalars under co-ordinate transformations) which give the components (with respect to the frame) of the covariant derivatives of the $\underset{\sim}{e}^{(a)}$ (see equation (26) below). It is known that the Riemann tensor is expressible in terms of the $\gamma^a{}_{bc}$ and their first partial derivatives.

In the newer formulation we shift attention from these objects to sets of forms associated with them. These are: (i) the set of 1-forms $\theta^a(d\underset{\sim}{x}) \equiv \underset{\sim}{e}^{(a)}{}_\alpha dx^\alpha$ associated with the basis vectors $\underset{\sim}{e}^{(a)}$; (ii) the n^2 "connection 1-forms" $\omega^a{}_b \equiv \gamma^a{}_{bc}\theta^c$ and (iii) the "curvature 2-forms" $\Omega^a{}_b \equiv \frac{1}{2}R^a{}_{bcd}\theta^c \wedge \theta^d$, where $R^a{}_{bcd}$ are the frame components of the Riemann tensor. (Notice how suppression of the indices c and d saves writing without loss of information, since the vector arguments of the forms are arbitrary.)

The essence of Riemannian geometry is then summarized in the "equations of structure", which relate the exterior differentials $d\theta^a$ to $\omega^a{}_b$, and $d\omega^a{}_b$ to $\Omega^a{}_b$.

The traditional approach is included as a special case: we have only to tie the frame to the co-ordinate net ($\underset{\sim}{e}^{(a)}$ = grad x^a) to reduce $\gamma^a{}_{bc}$ to the Christoffel symbols. However, the formalism will also accommodate choices of $\underset{\sim}{e}^{(a)}$ having a direct physical or geometrical significance, e.g. orthonormal tetrads propagated along material world-lines, or Sachs null tetrads defined by a principal null vector of the Riemann tensor.

In addition to the conceptual gain, there are also enormous computational advantages resulting from the fact that, in both of the examples just mentioned, the matrix $\underset{\sim}{e}^{(a)} \cdot \underset{\sim}{e}^{(b)}$ is constant. In this case, the connection forms have the property $\omega_{ab} = -\omega_{ba}$, so that there are only six nontrivial $\omega^a{}_b$ in a 4-space (as opposed to forty Christoffel symbols) which can be obtained, usually very easily, from $d\theta^a$. We shall see from examples how much this can simplify computation of the Riemann tensor.

2.2. Basic 1-forms

We consider n linearly independent vector fields $\underset{\sim}{e}_{(a)}(x^\mu)$ in a Riemannian n-space. Greek indices will be used throughout to refer to the co-ordinates x^μ and tensor components with respect to them. Indices a,b,c, ... in the first half of the Latin alphabet refer to the frame $\underset{\sim}{e}_{(a)}$, and behave as scalar (labelling) indices under co-ordinate transformations. (Indices m,n,p, ... will not appear until the next chapter.) Thus, $e_{(a)}{}^\alpha$ are the contravariant components of the vector $\underset{\sim}{e}_{(a)}$. We define the *"frame components"* (tetrad components in four dimensions) of any tensor $T_{\alpha\beta\ldots}$ by

$$T_{ab\ldots} \equiv T_{\alpha\beta\ldots}\, e_{(a)}{}^\alpha\, e_{(b)}{}^\beta \ldots \quad . \tag{13}$$

We introduce the matrix of scalar products of the basis vectors:

$$g_{ab} \equiv \underset{\sim}{e}_{(a)}\cdot\underset{\sim}{e}_{(b)} \quad . \tag{14}$$

Written out at length, (14) is

$$g_{ab} = g_{\alpha\beta}\, e_{(a)}{}^\alpha\, e_{(b)}{}^\beta \tag{15}$$

showing that g_{ab} are the frame components of the metric tensor.

Since the $\underset{\sim}{e}_{(a)}$ are linearly independent, the matrix g_{ab} has a symmetric inverse g^{ab}, satisfying $g^{ab}g_{ac} = \delta^b_c$. Defining the *dual basis*

$$\underset{\sim}{e}^{(a)} \equiv g^{ab}\underset{\sim}{e}_{(b)} \quad , \tag{16}$$

we easily show

$$\underset{\sim}{e}^{(a)}\cdot\underset{\sim}{e}_{(b)} = \delta^a_b \quad , \tag{17}$$

so that $e^{(a)}{}_\alpha$, $e_{(b)}{}^\beta$ are inverse matrices.

We now introduce the 1-forms

$$\theta^a = e^{(a)}{}_\alpha dx^\alpha \tag{18}$$

associated with the dual basis vectors. Solving for dx^α yields

$$dx^{\alpha} = e_{(a)}{}^{\alpha}\theta^{a} \quad . \tag{19}$$

Equations (18) and (19) may be interpreted in either of the ways indicated in Section 1.4, i.e. as numerical relations with the vector argument $d\underset{\sim}{x}$ of θ^a understood, or as linear relations between forms.

Because the $\underset{\sim}{e}^{(a)}$ form a complete vector basis, the θ^a form a basis for all 1-forms. Explicitly, the expansion of any 1-form $\alpha = A_{\alpha}dx^{\alpha}$ as a linear combination of the θ^a is

$$A_{\alpha}dx^{\alpha} = A_{\alpha}e_{(a)}{}^{\alpha}\theta^{a} = A_{a}\theta^{a} \quad .$$

Similarly,

$$2!\ F_{\alpha\beta}dx^{[\alpha}dy^{\beta]} = F_{ab}\ \theta^{a} \wedge \theta^{b} \quad , \tag{20}$$

showing that $\theta^a \wedge \theta^b$ form a basis for all 2-forms, and so on.

From (15) and (18) we obtain

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} = g_{ab}\theta^{a}\theta^{b} \quad . \tag{21}$$

As a simple example, let us consider Gaussian polar co-ordinates ρ, ϕ on a 2-space. The metric is

$$ds^2 = d\rho^2 + [f(\rho,\phi)]^2\ d\phi^2 \quad .$$

(If $f = \rho$, these are plane polar co-ordinates; if $f = \sin\rho$, spherical polars on the unit sphere.) We choose $\underset{\sim}{e}_{(1)}$, $\underset{\sim}{e}_{(2)}$ to be unit vectors along the ρ, ϕ directions so that (14) and (16) yield $g_{ab} = \delta_{ab}$, $\underset{\sim}{e}^{(a)} = \underset{\sim}{e}_{(a)}$. For an arbitrary displacement $dx^{\alpha} = (d\rho, d\phi)$ we then have

$$\theta^1 = \underset{\sim}{e}^{(1)}.d\underset{\sim}{x} = d\rho \ , \qquad \theta^2 = \underset{\sim}{e}^{(2)}.d\underset{\sim}{x} = f\ d\phi \tag{22}$$

and the metric is

$$ds^2 = (\theta^1)^2 + (\theta^2)^2 \quad .$$

2.3. Connection 1-forms

Let the covariant differential

$$D\underset{\sim}{e}_{(a)} \equiv \underset{\sim}{e}_{(a)}(x + dx) - \underset{\sim}{e}_{(a)}^{\parallel}(x)$$

denote the absolute change of the vector field $\underset{\sim}{e}_{(a)}$ in the small displacement from x^α to $x^\alpha + dx^\alpha$. Here, $\underset{\sim}{e}_{(a)}^{\parallel}(x)$ denotes the vector $\underset{\sim}{e}_{(a)}(x)$ *parallel-transferred* to the point $x + dx$. Written in terms of components,

$$[D\underset{\sim}{e}_{(a)}]^\alpha = e_{(a)}{}^{\alpha}{}_{|\gamma}\, dx^\gamma \quad . \tag{23}$$

More generally, we shall write $D\Phi^{\cdot\cdot}{}_{\cdot\cdot} \equiv \Phi^{\cdot\cdot}{}_{\cdot\cdot|\gamma}dx^\gamma$ for any tensor $\Phi^{\cdot\cdot}{}_{\cdot\cdot}$.

Since $D\underset{\sim}{e}_{(b)}$ is a vector at x^α, it can be expressed as a linear combination of the local base vectors:

$$D\underset{\sim}{e}_{(b)} = \omega^c{}_b\, \underset{\sim}{e}_{(c)} \quad , \tag{24}$$

where the coefficients $\omega^c{}_b(d\underset{\sim}{x})$ are 1-forms. If we take the scalar product of both sides with $\underset{\sim}{e}^{(a)}$, and note that $e^{(a)}\cdot\underset{\sim}{e}_{(c)} = \delta^a_c$, and that $\underset{\sim}{e}^{(a)}\cdot D\underset{\sim}{e}_{(b)} + \underset{\sim}{e}_{(b)}\cdot D\underset{\sim}{e}^{(a)} = D(\underset{\sim}{e}^{(a)}\cdot\underset{\sim}{e}_{(b)}) = D(\delta^a_b) = 0$, we find

$$\begin{aligned}\omega^a{}_b = \underset{\sim}{e}^{(a)}\cdot D\underset{\sim}{e}_{(b)} &= -\underset{\sim}{e}_{(b)}\cdot D\underset{\sim}{e}^{(a)} \\ &= -e^{(a)}{}_{\beta|\gamma}\, e_{(b)}{}^{\beta}\, dx^\gamma \quad .\end{aligned} \tag{25}$$

Now, the usual definition of Ricci's rotation coefficients is

$$\gamma^a{}_{bc} \equiv -e^{(a)}{}_{\beta|\gamma}\, e_{(b)}{}^{\beta}\, e_{(c)}{}^{\gamma} \quad . \tag{26}$$

So we have

$$\omega^a{}_b = \gamma^a{}_{bc}\theta^c \quad , \tag{27}$$

which is the decomposition of the connection 1-forms $\omega^a{}_b$ with respect to the basis θ^c .

From (25) we infer

$$D\underset{\sim}{e}^{(a)} = -\,\omega^a{}_b\, \underset{\sim}{e}^{(b)} \quad . \tag{28}$$

This is equivalent to the expanded version

$$e^{(a)}{}_{\beta|\gamma} = -\gamma^a{}_{bc}\, e^{(b)}{}_\beta\, e^{(c)}{}_\gamma \tag{29}$$

obtained from (26).

In the equation $g_{ab} = \underset{\sim}{e}_{(a)} \cdot \underset{\sim}{e}_{(b)}$, let us take the differential of both sides. (Notice that it is here immaterial whether we interpret "differential" as covariant, ordinary or exterior differential, since each g_{ab} is a scalar.) Thus

$$dg_{ab} = Dg_{ab} = \underset{\sim}{e}_{(a)} \cdot D\underset{\sim}{e}_{(b)} + \underset{\sim}{e}_{(b)} \cdot D\underset{\sim}{e}_{(a)}$$

$$= \underset{\sim}{e}_{(a)} \cdot \underset{\sim}{e}_{(c)} \omega^c{}_b + \underset{\sim}{e}_{(b)} \cdot \underset{\sim}{e}_{(c)} \omega^c{}_a \quad .$$

If we agree to raise and lower Latin indices with the aid of the matrices g^{ab} and g_{ab} (note: this is consistent with (16)!), then this can be rewritten

$$dg_{ab} = \omega_{ab} + \omega_{ba} \quad . \tag{30}$$

The symmetrized 1-forms $2\omega_{(ab)}$ are therefore the exterior differentials of the 0-forms g_{ab} .

2.4. Co-ordinate Frame

In order to see clearly how the traditional approach is contained in the present formalism we make a special choice of frame. Given a specific co-ordinate system x^α, we select the "co-ordinate frame" $\underset{\sim}{e}^{(a)} \equiv \text{grad}\; x^a$. Then $e^{(a)}{}_\alpha = \delta^a_\alpha$, $e_{(a)}{}^\alpha = \delta^\alpha_a$ (i.e. the vectors $\underset{\sim}{e}_{(a)}$ are tangent to the co-ordinate curves) and $dx^\alpha = \theta^\alpha$. The frame components of any tensor are thus numerically equal to its ordinary components. From (26), we easily find

$$\gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} \quad ,$$

so the rotation coefficients are now the Christoffel symbols.

Since the numerical equality between $g_{\alpha\beta}$ and g_{ab} implies equality of their ordinary (not covariant!) differentials, (30) now reduces to

$$dg_{\alpha\beta} = (\gamma_{\alpha\beta\gamma} + \gamma_{\beta\alpha\gamma})\, \theta^\gamma \quad ,$$

or:

$$\partial_\gamma\, g_{\alpha\beta} = g_{\alpha\mu}\, \Gamma^\mu{}_{\beta\gamma} + g_{\mu\beta}\, \Gamma^\mu{}_{\alpha\gamma} \quad ,$$

which is the familiar result $g_{\alpha\beta|\gamma} = 0$.

2.5. Equations of Structure

Returning to the general formalism, let us evaluate the exterior differential of $\theta^a = e^{(a)}{}_\beta dx^\beta$. By definition,

$$d\theta^a = 2!\, e^{(a)}{}_{\beta|\gamma}\, dx^{[\gamma} dy^{\beta]}$$

$$= -\, 2!\, \gamma^a{}_{bc} e^{(c)}{}_\gamma\, e^{(b)}{}_\beta dx^{[\gamma} dy^{\beta]} \qquad \text{by (29)}$$

$$= -\, \gamma^a{}_{bc}\, \theta^c \wedge \theta^b \quad . \qquad (31)$$

Since $\gamma^a{}_{bc}\, \theta^c = \omega^a{}_b$, this can be written

$$d\theta^a = -\, \omega^a{}_b \wedge \theta^b \quad . \qquad (32)$$

Equations (32) have been called by Cartan the "first equations of structure". The information they convey is that the skew parts $\gamma^a{}_{[bc]}$ of the rotation coefficients can be easily evaluated by taking the curls of the dual basis vectors $\underset{\sim}{e}^{(a)}$, i.e. by exterior (partial) differentiation.

In the special case of a co-ordinate frame, the $\underset{\sim}{e}^{(a)}$ are gradients and (32) reduces to the trivial statement $d\theta^\alpha = d^2 x^\alpha = 0$. (Alternatively, this follows from (31), upon noting that the Christoffel symbols are symmetric in their two lower indices.)

Suppose we are given the 1-forms θ^a and the matrix g_{ab} as functions of the co-ordinates. We can immediately find $d\theta^a$ and dg_{ab} by simple differentiation. *The 1-forms* $\omega^a{}_b$ *are then completely determined by* (30) *and* (32). To see this, we note (i) that $\gamma_{(ab)c}$ are determined as the numerical coefficients in the expansion of the known 1-forms dg_{ab} in terms of the basis θ^c:

$$dg_{ab} = 2\gamma_{(ab)c}\, \theta^c \quad ;$$

and (ii) that $\gamma_{a[bc]}$ can be read off as coefficients in the expansion of the known 2-forms

$$g_{ab} d\theta^b = -\, 2\gamma_{a[bc]}\, \theta^c \wedge \theta^b \quad .$$

These results fix $\omega^a{}_b$ uniquely:

$$\omega_{ab} = \gamma_{abc}\,\theta^c = (\gamma_{(ab)c} + \gamma_{(ac)b} - \gamma_{(bc)a} +$$

$$+ \gamma_{a[bc]} + \gamma_{b[ca]} - \gamma_{c[ab]})\,\theta^c\,. \qquad (33)$$

This formula is cumbersome. Fortunately, it is hardly ever necessary to use it. Things are particularly easy if g_{ab} is a constant matrix, as happens for an orthonormal frame or a Sachs null tetrad. For (30) then asserts merely that $\omega_{ab} = -\omega_{ba}$, and only (32) remains to be solved. The answer can often be written down by simple guesswork (this is permissible, since the solution is known to be unique).

Let us return to our simple example of Gaussian polar co-ordinates (see (22)). We had $g_{ab} = \delta_{ab}$, $\theta^1 = d\rho$, $\theta^2 = f(\rho,\phi)d\phi$. Recalling that $d^2\Omega = 0$ and $d(f\Omega) = fd\Omega + df \wedge \Omega$ for any form Ω and scalar f, we find

$$d\theta^1 = d^2\rho = 0 \qquad (34)$$

$$d\theta^2 = fd^2\phi + df \wedge d\phi = f_\rho d\rho \wedge d\phi + f_\phi d\phi \wedge d\phi$$

$$= (f_\rho/f)\,\theta^1 \wedge \theta^2 \qquad (35)$$

where the subscripts indicate partial differentiation. Since $\omega_{ab} = -\omega_{ba}$, we have $\omega^1{}_1 = \omega^2{}_2 = 0$, and (32) reduces to

$$d\theta^1 = -\omega^1{}_2 \wedge \theta^2\,, \qquad d\theta^2 = -\omega^2{}_1 \wedge \theta^1\,.$$

We compare this with (34) and (35). It is clear that if we guess

$$\omega^2{}_1 = (f_\rho/f)\,\theta^2 = -\omega^1{}_2 \qquad (36)$$

then equations (32) are all satisfied, and this is therefore the unique solution. The Ricci rotation coefficients, if wanted, can be read off immediately from (36):

$$\gamma^2{}_{12} = f_\rho/f = -\gamma^1{}_{22}\,; \qquad \text{other } \gamma^a{}_{bc} = 0.$$

2.6. CURVATURE 2-FORMS

We shall next derive the simple relations ("second equations of structure") which connect the curvature tensor to the exterior differentials of the connection forms $\omega^a{}_b$.

By definition, the exterior differential of

$$\omega^a{}_b = - e^{(a)}{}_{\beta|\gamma}\, e_{(b)}^{\beta}\, dx^{\gamma} \tag{37}$$

is

$$d\omega^a{}_b = - 2(e^{(a)}{}_{\beta|\gamma}\, e_{(b)}^{\beta})_{|\delta}\, dx^{[\delta} dy^{\gamma]} \tag{38}$$

Making use of the Ricci commutation relation

$$2e^{(a)}{}_{\beta|[\gamma\delta]} = e^{(a)}{}_{\alpha}\, R^{\alpha}{}_{\beta\gamma\delta} \tag{39}$$

and of

$$e_{(b)|\delta}^{\beta}\, dx^{\delta} = e_{(c)}^{\beta}\, \omega^c{}_b \tag{40}$$

yields

$$d\omega^a{}_b = e^{(a)}{}_{\alpha}\, e_{(b)}^{\beta}\, R^{\alpha}{}_{\beta\gamma\delta}\, dx^{[\gamma} dy^{\delta]} - \omega^a{}_c \wedge \omega^c{}_b \ . \tag{41}$$

Thus, if we define the *curvature 2-forms*

$$\Omega^a{}_b \equiv \tfrac{1}{2} R^a{}_{bcd}\, \theta^c \wedge \theta^d \ , \tag{42}$$

where $R^a{}_{bcd}$ are the frame components of the Riemann tensor, then (41) may be written

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \ . \tag{43}$$

These are the "second equations of structure" of Cartan. Notice that *all* components of the curvature tensor can be recovered from the 2-forms $\Omega^a{}_b$ by expanding them in terms of the basis $\theta^c \wedge \theta^d$.

Returning to the example of Gaussian polar co-ordinates (see (36)), we have obtained so far:

$$g_{ab} = \delta_{ab} \ , \qquad \theta^1 = d\rho \ , \qquad \theta^2 = f(\rho,\phi)d\phi \ ,$$

$$\omega^1{}_1 = \omega^2{}_2 = 0 \ , \qquad \omega^2{}_1 = - \omega^1{}_2 = (f_\rho/f)\, \theta^2 \ .$$

The non-vanishing curvature 2-forms are

$$\begin{aligned}\Omega^2{}_1 &= - \Omega^1{}_2 = d\omega^2{}_1 + \omega^2{}_p \wedge \omega^p{}_1 \\ &= d[(f_\rho/f)\theta^2] + \omega^2{}_1 \wedge \omega^1{}_1 + \omega^2{}_2 \wedge \omega^2{}_1 \\ &= (f_\rho/f)d\theta^2 + d(f_\rho/f) \wedge \theta^2 \ .\end{aligned}$$

We have $d\theta^2 = -\omega^2{}_1 \wedge \theta^1 = (f_\rho/f)\,\theta^1 \wedge \theta^2$, and

$$d(f_\rho/f) \wedge \theta^2 = (f_\rho/f)_\rho d\rho \wedge \theta^2 = (f_{\rho\rho}/f - f_\rho{}^2/f^2)\,\theta^1 \wedge \theta^2 \ .$$

Hence

$$\Omega^2{}_1 = -\Omega^1{}_2 = (f_{\rho\rho}/f)\,\theta^1 \wedge \theta^2 \ . \tag{44}$$

Equating coefficients of $\theta^c \wedge \theta^d$ in (42) and (44), we find

$$R^2{}_{112} = f_{\rho\rho}/f$$

is the only non-trivial component of the Riemann tensor.

2.7. Identities for the Curvature

In Cartan's formalism, the *Bianchi identities* arise very simply as the integrability conditions of the second equations of structure (43). We take the exterior differential of (43), and note that $d^2\omega^a{}_b = 0$, and that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$$

if α, β are 1-forms. The result is

$$\begin{aligned} d\Omega^a{}_b &= d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b \\ &= (\Omega^a{}_c - \omega^a{}_d \wedge \omega^d{}_c) \wedge \omega^c{}_b - \omega^a{}_c \wedge (\Omega^c{}_b - \omega^c{}_d \wedge \omega^d{}_b) \end{aligned}$$

by a second use of (43). The triple wedge products cancel, and we are left with

$$d\Omega^a{}_b = \Omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \Omega^c{}_b \ . \tag{45}$$

To verify that these really are the Bianchi identities, we specialize to a co-ordinate frame and to Riemannian co-ordinates, so that $\omega^\alpha{}_\beta = \Gamma^\alpha{}_{\beta\gamma}dx^\gamma$ at some selected point. Then (45) reduces to

$$\partial_\varepsilon R^\alpha{}_{\beta\gamma\delta}dx^{[\varepsilon}dy^\gamma dz^{\delta]} = 0$$

which is equivalent to $R^\alpha{}_{\beta[\gamma\delta|\varepsilon]} = 0$.

Another identity follows from the first equations of structure, $d\theta^a = -\omega^a{}_b \wedge \theta^b$. Taking exterior differentials, we find

$$\begin{aligned} 0 = d^2\theta^a &= -d\omega^a{}_b \wedge \theta^b + \omega^a{}_b \wedge d\theta^b \\ &= -(\Omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b) \wedge \theta^b - \omega^a{}_b \wedge \omega^b{}_c \wedge \theta^c \end{aligned}$$

i.e.

$$\Omega^a{}_b \wedge \theta^b = 0 . \tag{46}$$

This can be written $R_{abcd}\theta^b \wedge \theta^c \wedge \theta^d = 0$, which is nothing but the cyclic identity $R_{a[bcd]} = 0$. It is not difficult to show that the cyclic identity together with the skew-symmetry of the Riemann tensor in both index pairs implies the remaining algebraic identities $R_{abcd} = R_{cdab}$. Thus, all symmetries of the Riemann tensor are summed up in the two identities for the curvature 2-forms:

$$\Omega_{ab} = -\Omega_{ba} \quad , \qquad \Omega^a{}_b \wedge \theta^b = 0.$$

2.8. An Example (Vaidya Metric)

To gain some idea of the computational ease afforded by these techniques, let us work through a typical example. We consider the non-diagonal line-element

$$ds^2 = 2drdu + [1 - 2m(u)/r]du^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{47}$$

where m(u) is an arbitrary function. This line-element was shown by Vaidya to represent the spherically symmetric exterior field of a radiating star, whose mass at retarded time u is m(u). If m(u) is constant, it reduces to a form of the Schwarzschild metric.

To simplify the computations as much as possible, we look for a basis $\theta^a = e^{(a)}{}_\alpha dx^\alpha$ such that the coefficients g_{ab} in the expression $ds^2 = g_{ab}\theta^a\theta^b$ for the metric are constants. A simple choice is

$$\left.\begin{aligned} \theta^1 &= dr + \tfrac{1}{2}(1 - 2m/r)du , & \theta^4 &= du \\ \theta^2 &= r\, d\theta \ , & \theta^3 &= r \sin\theta\, d\phi \ . \end{aligned}\right\} \tag{48}$$

This choice is also physically significant, since it implies that $e^{(4)}{}_\alpha = \delta^4_\alpha = \partial_\alpha u$, so that $\underset{\sim}{e}^{(4)}$ is a null vector tied to the retarded time u. Similarly, $\underset{\sim}{e}^{(1)}$ is a null vector associated with the direction of "advanced time".

The line-element (47) now appears as

$$ds^2 = 2\,\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2 \quad ,$$

so that the matrix g_{ab} and its inverse are given by:

$$g_{14} = g_{41} = -g_{22} = -g_{33} = 1 , \qquad \text{other } g_{ab} = 0;$$

$$g^{ab} = g_{ab} .$$

To arrive at the connection 1-forms $\omega_{ab} = -\omega_{ba}$, we first express the exterior differentials $d\theta^a$ in terms of $\theta^c \wedge \theta^d$ as basis. From (48) we quickly find

$$d\theta^1 = -d[(m/r)du] = (m/r^2)dr \wedge du = (m/r^2)\, \theta^1 \wedge \theta^4 , \tag{49}$$

$$d\theta^2 = dr \wedge d\theta = r^{-1}\theta^1 \wedge \theta^2 - \tfrac{1}{2} r^{-1}(1 - 2m/r)\, \theta^4 \wedge \theta^2 , \tag{50}$$

$$d\theta^3 = \sin\theta\, dr \wedge d\phi + r\cos\theta\, d\theta \wedge d\phi$$

$$= r^{-1}[\theta^1 - \tfrac{1}{2}(1 - 2m/r)\theta^4] \wedge \theta^3 + r^{-1} \cot\theta\, \theta^2 \wedge \theta^3 , \tag{51}$$

$$d\theta^4 = 0 . \tag{52}$$

We now compare these results with the equations of structure, $d\theta^a = -\omega^a{}_b \wedge \theta^b$, and try to guess the solution for $\omega^a{}_b$. We know in advance that $\omega^1{}_4 = g^{14}\omega_{44} \equiv 0$, so the right side of (49) must arise from $\omega^1{}_1 \wedge \theta^1$. We therefore take as the solution of (49)

$$\omega^1{}_1 = (m/r^2)\theta^4 , \quad \omega^1{}_2 = A\theta^2 , \quad \omega^1{}_3 = B\theta^3 , \quad \omega^1{}_4 \equiv 0 , \tag{53a}$$

where the scalars A, B are as yet undetermined. (We could have added an arbitrary multiple of θ^1 to the expression for $\omega^1{}_1$. However, we shall verify at the end that the simple guess (53a) is correct.)

For the solution of (50), a simple guess is

$$\omega^2{}_1 = r^{-1}\theta^2 , \ \omega^2{}_2 \equiv 0, \ \omega^2{}_3 = C\theta^3, \ \omega^2{}_4 = -\tfrac{1}{2} r^{-1}(1 - 2m/r)\theta^2 . \tag{53b}$$

and for (51):

$$\omega^3{}_1 = r^{-1}\theta^3 , \ \omega^3{}_2 = r^{-1}\cot\theta\, \theta^3 , \ \omega^3{}_3 \equiv 0, \ \omega^3{}_4 = -\tfrac{1}{2} r^{-1}(1 - 2m/r)\theta^3 . \tag{53c}$$

The remaining 1-forms can now be inferred from the skew-symmetry of ω_{ab}; for instance

$$\omega^1{}_2 = g^{14}\omega_{42} = -\omega_{24} = \omega^2{}_4 = -\tfrac{1}{2} r^{-1}(1 - 2m/r)\theta^2$$

$$\omega^1{}_3 = \omega^3{}_4 = -\tfrac{1}{2} r^{-1}(1 - 2m/r)\theta^3 .$$

These results are compatible with (53a) and serve to fix the coefficients A, B. Also, $\omega^2{}_3 = -\omega^3{}_2$ fixes C. Similarly,

$$\omega^4{}_1 \equiv 0, \quad \omega^4{}_2 \equiv \omega^2{}_1 = -r^{-1}\theta^2 , \quad \omega^4{}_3 = \omega^3{}_1 = r^{-1}\theta^3 ,$$

$$\omega^4{}_4 = -\omega^1{}_1 = -(m/r^2)\,\theta^4 . \tag{53d}$$

This implies $\omega^4{}_b \wedge \theta^b = 0$, so our expressions for $\omega^4{}_b$ satisfy the last equation, (52).

We have verified that our tentative expressions for $\omega^a{}_b$ satisfy all equations of structure. Hence they furnish the unique correct solution.

The rest is straightforward. From the second equations of structure, $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$, we easily obtain

$$\begin{aligned}
\Omega^1{}_1 &= d\omega^1{}_1 + \omega^1{}_1 \wedge \omega^1{}_1 + \omega^1{}_2 \wedge \omega^2{}_1 + \omega^1{}_3 \wedge \omega^3{}_1 \\
&= d[(m/r^2)\theta^4] + 0 - \tfrac{1}{2}\, r^{-2}(1 - 2m/r)(\theta^2 \wedge \theta^2 + \theta^3 \wedge \theta^3) \\
&= -(2m/r^3)dr \wedge \theta^4 = -(2m/r^3)\,\theta^1 \wedge \theta^4 .
\end{aligned}$$

Comparison of this result with $\Omega^1{}_1 = \tfrac{1}{2}R^1{}_{1cd}\,\theta^c \wedge \theta^d$ immediately yields the following tetrad components of the Riemann tensor:

$$R^1{}_{141} = -R^1{}_{114} = 2m/r^3 ; \qquad \text{other } R^1{}_{1cd} = 0.$$

In similar fashion we find [$\dot{m} \equiv dm/du$] :

$$\begin{aligned}
\Omega^1{}_2 &= -(\dot{m}/r^2)\,\theta^2 \wedge \theta^4 - (m/r^3)\,\theta^1 \wedge \theta^2 , \\
\Omega^1{}_3 &= -(\dot{m}/r^2)\,\theta^3 \wedge \theta^4 - (m/r^3)\,\theta^1 \wedge \theta^3 , \\
\Omega^2{}_1 &= (m/r^3)\,\theta^2 \wedge \theta^4 , \\
\Omega^3{}_1 &= (m/r^3)\,\theta^3 \wedge \theta^4 , \\
\Omega^2{}_3 &= (2m/r^3)\,\theta^2 \wedge \theta^3 .
\end{aligned}$$

The skew-symmetry $\Omega_{ab} = -\Omega_{ba}$ determines the remaining ten curvature 2-forms.

By contraction we can derive the tetrad components of the Ricci tensor. For instance,

$$\begin{aligned}
R_{44} &= R^2{}_{442} + R^3{}_{443} \quad (\text{since } \Omega^1{}_4 = \Omega_{44} \equiv 0) \\
&= R^1{}_{242} + R^1{}_{343} \quad (\text{since } \Omega^2{}_4 = -\Omega^1{}_2 \text{, etc.}) \\
&= 2\dot{m}/r^2 .
\end{aligned}$$

The other R_{ab} are all found to vanish. Our final result can be written covariantly as

$$R_{\alpha\beta} = R_{ab}e^{(a)}{}_{\alpha}e^{(b)}{}_{\beta} = R_{44}e^{(4)}{}_{\alpha}e^{(4)}{}_{\beta} = 2(\dot{m}/r^{2})(\partial_{\alpha}u)(\partial_{\beta}u) .$$

Chapter III: Bivectors, Complex 3-Vectors and the Riemann Tensor of Space-Time

3.1. Preliminary Survey

In Riemannian geometry we repeatedly encounter objects skew-symmetric in one or more pairs of indices (for instance, the Riemann tensor or the connection 1-forms ω_{ab} for a null tetrad). Now, in space-time it happens that an interesting correspondence can be set up between "bivectors" (skew-symmetric tensors of order two) and the vectors of a complex Euclidean 3-space. This correspondence is quite remarkable, because it is 1-1 and Lorentz-invariant. We shall see in this chapter the great formal economies that result from exploiting it.

We devote this introductory section to a preliminary sketch of the basic ideas, which are very simple.

Our concern (all through the chapter) is with tensor algebra at a single point of space-time; accordingly, we fix our attention on the local Minkowskian four-dimensional tangent plane. Suppose, for the moment, that some definite Lorentz frame (local orthonormal tetrad) has been arbitrarily specified. Then the tetrad components F_{ab} of any bivector can be split into an "electric" and a "magnetic" triplet:†

$$F_{ab} \longleftrightarrow (H_m, E_m) \quad (m = 1,2,3) \tag{54a}$$

where

$$H_1 = F_{23}\text{, etc.;} \qquad E_m = F_{m4} \quad . \tag{54b}$$

In passing from F_{ab} to its *dual*

$$F^*_{ab} = \tfrac{1}{2}\sqrt{-g}\,\varepsilon_{abcd}\,F^{cd} \longleftrightarrow (-E_m, H_m)$$

the electric and magnetic parts get interchanged.

Only two algebraically independent Lorentz scalars can be formed from F_{ab} ; we

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† Indices a,b, ... , ℓ in the *first* half of the Latin alphabet are tetrad indices (range 1-4); indices m,n,p ... in the second half have range 1-3. The signature of the space-time metric is chosen so that g_{ab} = diag (-1, -1, -1, 1) in a local Lorentz frame.

nay take these to be

$$F_{ab} F^{ab} = - F^*_{ab} F^{ab}_* = 2(H^2 - E^2)$$

$$F^*_{ab} F^{ab} = - 4 \underset{\sim}{E}.\underset{\sim}{H}.$$

Consider now the effect of changing to a new Lorentz frame. The correspondence 54) imposed in the new frame, now associates a new pair of triplets H'_m, E'_m with the same ivector. Since $H'^2 - E'^2 = H^2 - E^2$ we could, if we wish, correlate each bivector orentz-invariantly to a cartesian 6-vector $(\underset{\sim}{E}, \underset{\sim}{H})$ in a pseudo-Euclidean space with metric iag (1,1,1,-1,-1,-1). Each Lorentz transformation would then set in gear some rotation f rectangular axes in the 6-space. But this idea would not be particularly useful, ecause it does not work backwards: not *every* transformation of the real orthogonal group $_3$(R) is the result of a Lorentz transformation - only that subset which preserves the alue of $E_m H_m$. (This is also obvious from the fact that it takes 15 parameters to pecify a 6-dimensional rotation, as against 6 for a homogeneous Lorentz transformation.)

However, there is a simple remedy for this defect. Introduce the *complex* vector

$$F^{(+)}_{ab} = F_{ab} - i F^*_{ab} \quad < \quad > \quad (F_m, - iF_m) \tag{55a}$$

ere

$$F_m = H_m + i E_m \quad . \tag{55b}$$

$^{+)}_{b}$ is *self-dual* in the sense that

$$F^{(+)*}_{ab} = i F^{(+)}_{ab} \quad .$$

stead of real bivectors F_{ab} , it is completely equivalent to work with the complex self-al combinations $F^{(+)}_{ab}$, since the correspondence between them is 1-1 (F_{ab} is recovered m $F^{(+)}_{ab}$ by taking its real part). The following interesting results now emerge:

(i) According to (55a), the 6-space of complex self-dual bivectors splits rentz-invariantly) into two essentially identical spaces $\mathcal{C}_3$ of complex 3-vectors F_m . h real bivector F_{ab} is thus linked in 1-1 fashion to a vector of $\mathcal{C}_3$.

(ii) The complex number

$$\tfrac{1}{4}F^{(+)}_{ab}\,F^{(+)ab} = \tfrac{1}{2}(F_{ab}F^{ab} - i\,F^{*}_{ab}F^{ab}) = (\underset{\sim}{H} + i\underset{\sim}{E})^2 = \delta^{mn}F_mF_n \qquad (56)$$

is essentially the only scalar which can be formed from $F^{(+)}_{ab}$. If we devise a Lorentz-invariant metric γ_{mn} for $\mathscr{C}_3$ such that the scalar (56) is equal to the squared norm $\gamma^{mn}F_mF_n$, then $\gamma^{mn} = \delta^{mn}$ and $\mathscr{C}_3$ is Euclidean. Each Lorentz frame in Minkowski space can now be linked to a cartesian frame in $\mathscr{C}_3$, each proper Lorentz transformation to an orthogonal transformation in $\mathscr{C}_3$.

(iii) *The converse is also true:* Given any orthogonal transformation in $\mathscr{C}_3$, there is exactly one proper Lorentz transformation which induces it. We shall verify this in detail later, but it can be made plausible by simple counting. To specify a rotation in $\mathscr{C}_3$ requires 3 complex or 6 real parameters, which is the number of parameters in the homogeneous Lorentz group.

This fundamental result allows us to rotate axes freely in $\mathscr{C}_3$ with the assurance that any such rotation is actually realizable by some rotation of the space-time tetrad. (For example, if F_{ab} is any real bivector for which (56) does not vanish, axes can always be rotated so that

$$F_1 = \tfrac{1}{2}(F^{(+)}_{ab}F^{(+)ab})^{\frac{1}{2}}, \qquad F_2 = F_3 = 0$$

for the associated complex 3-vector.) Later we shall make extensive use of this freedom.

Null bivectors, characterized by any of the equivalent conditions

$$\gamma^{mn}F_mF_n = 0, \qquad (57a)$$

$$F_{ab}F^{ab} = F^{*}_{ab}F^{ab} = 0, \qquad (57b)$$

$$E^2 = H^2, \qquad \underset{\sim}{E}\cdot\underset{\sim}{H} = 0, \qquad (57c)$$

have an especially significant role: they can be correlated to null directions in space-time. In fact, from the electromagnetic interpretation of (57c) as a plane wave, we know that any null bivector singles out a characteristic null direction (propagation vector). Conversely, to any given null-direction there corresponds a *class* of null electromagnetic fields all propagating along the given direction, but differing in amplitude and plane of polarization; this class is represented by a null ray λF_m in $\mathcal{C}_3$, where F_m satisfies (57a) and λ is an arbitrary complex coefficient. (To see this, note that the complex scale transformation

$$F_m \rightarrow F'_m = e^{i\theta} F_m ,$$

when written out, gives

$$H'_m + i E'_m = H_m \cos\theta - E_m \sin\theta + i(H_m \sin\theta + E_m \cos\theta),$$

which is just a rotation of the plane of polarization.)

Thus there is a 1-1 correspondence between null directions in space-time and null rays in $\mathcal{C}_3$. We shall next sketch how this can conveniently be used to pick out characteristic null vectors for the Weyl tensor C_{abcd} (trace-free part of the Riemann tensor). (In what follows, the duality operation applied to a skew-symmetric index-pair will be denoted by an asterisk placed over the pair.)

Instead of the Weyl tensor itself, it is equivalent and convenient to consider the complex combination

$$C^{(+)}_{abcd} = C_{abcd} - i\, C^{*}_{abcd} .$$

This shares all symmetry properties of the Riemann and Weyl tensors. In addition, as will be shown in the next section, it is self-dual in *both* index pairs:

$$C^{(+)}{}_{ab\overset{*}{cd}} = C^{(+)}{}_{\overset{*}{ab}cd} = i\, C^{(+)}{}_{abcd} .$$

Hence, if we apply the assignment (55) to each index pair independently, we see that $C^{(+)}{}_{abcd}$ (and therefore its real part, the Weyl tensor) is represented (Lorentz-covariantly) by a complex symmetric tensor C_{mn} in $\mathcal{C}_3$. For example,

$$C^{(+)}{}_{1423} = C^{(+)}{}_{14\overset{*}{14}} = i\, C^{(+)}{}_{1414} = -\, i\, C_{11} .$$

The cyclic identity

$$C^{(+)}{}_{1423} + C^{(+)}{}_{1342} + C^{(+)}{}_{1234} = 0$$

yields

$$i(C_{11} + C_{22} + C_{33}) = 0 ,$$

so that C_{mn} is trace-free: $\gamma^{mn} C_{mn} = 0$. Thus C_{mn} has 6 - 1 = 5 independent complex components, matching the 10 independent real components of the Weyl tensor.

To obtain null vectors k^a which are in some sense characteristic of the Weyl tensor, we look for the associated null 3-vectors F_m . We could, for instance, examine the null eigenvectors of C_{mn} in $\mathcal{C}_3$. It is somewhat more general to seek solutions F^m of

$$C_{mn} F^m F^n = 0, \quad \gamma_{mn} F^m F^n = 0 . \tag{58}$$

Since a complex scale factor which remains arbitrary in each solution of (58) does not affect the associated null direction k^a, it is simplest to think of (58) as a pair of *inhomogeneous* equations in a (projective) complex 2-space (e.g. by fixing $F_3 = 1$). Now, in a complex 2-space, two quadrics *always* intersect in 4 points (counting possible coincidences), each of which corresponds to a null ray in $\mathcal{C}_3$. We thus arrive at the theorem of Debever and Penrose, according to which a general Weyl tensor determines 4 null eigen-directions in space-time. An examination of the various possible coincidences in the solutions of (58) leads immediately to the Petrov-Pirani algebraic classification of the Weyl tensors.

The following sections will fill out this preliminary sketch with a more systematic presentation. In one inessential but noteworthy respect our detailed discussion will deviate from the foregoing: complex null tetrads rather than real orthonormal tetrads will be employed as basis vectors. The corresponding new bases in $\mathcal{C}_3$ will then include null 3-vectors. The imaginary unit i now enters the formalism from two quite different and independent sources. The slightly reduced transparency of the resulting formalism is more than compensated for by its greater adaptability and compactness.

3.2. Bivectors and their Duals

In this section we collect a number of basic properties of anti-symmetric tensors and the duality operation which hold in 4-dimensional space-time.

We define the *alternating pseudo-tensor* as

$$\eta_{\alpha\beta\gamma\delta} = (-g)^{\frac{1}{2}} \varepsilon_{\alpha\beta\gamma\delta} , \tag{59a}$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ ($= \varepsilon^{\alpha\beta\gamma\delta}$) is the Levi-Civita permutation symbol (skew in each pair of indices with $\varepsilon_{1234} = +1$). Raising indices with $g^{\mu\nu}$ yields

$$\eta^{\alpha\beta\gamma\delta} = -(-g)^{-\frac{1}{2}} \varepsilon^{\alpha\beta\gamma\delta} . \tag{59b}$$

If $Q^{\cdot\cdot\alpha\beta\cdot\cdot}$ is any tensor skew in the index pair α, β, we define its *dual* with respect to this pair by

$$Q^{\cdot\cdot}{}_{\alpha\beta}^{*}{}^{\cdot\cdot} = \tfrac{1}{2}\eta_{\alpha\beta\mu\nu}\, Q^{\cdot\cdot\mu\nu\cdot\cdot} . \tag{60}$$

From (59) we have

$$\eta_{\alpha\beta\mu\nu}\ \eta^{\mu\nu\rho\sigma} = -\,\varepsilon_{\alpha\beta\mu\nu}\ \varepsilon^{\mu\nu\rho\sigma} = -\,2\,\delta^{\rho\sigma}_{\alpha\beta} ,$$

where $\delta^{\rho\sigma}_{\alpha\beta} \equiv 2\,\delta_{\alpha}{}^{[\rho}\,\delta_{\beta}{}^{\sigma]}$ is a generalized Kronecker delta symbol. It follows that applying the duality operation twice to the *same* index-pair reproduces the original tensor, apart from a sign:

$$Q^{\cdot\cdot}{}_{\alpha\beta}^{**}{}^{\cdot\cdot} = -\,Q^{\cdot\cdot}{}_{\alpha\beta}{}^{\cdot\cdot} . \tag{61}$$

Next we consider any fourth-order tensor $Q_{\alpha\beta\gamma\delta}$, skew in each of the first and second pairs of indices:

$$Q_{\alpha\beta\gamma\delta} = Q_{[\alpha\beta]\gamma\delta} = Q_{\alpha\beta[\gamma\delta]} . \tag{62}$$

Let us evaluate the "double dual":

$${}^{*}Q^{*}{}_{\alpha\beta}{}^{\gamma\delta} = -\,1/4\ \varepsilon_{\alpha\beta\mu\nu}\ \varepsilon^{\gamma\delta\rho\sigma}\ Q^{\mu\nu}{}_{\rho\sigma} . \tag{63}$$

Now,

$$\varepsilon_{\alpha\beta\mu\nu}\ \varepsilon^{\gamma\delta\rho\sigma} = 4!\ \delta_{\alpha}{}^{[\gamma}\,\delta^{\delta}_{\beta}\,\delta^{\rho}_{\mu}\,\delta^{\sigma]}_{\nu} ,$$

and the 24 terms on the right can be split into 6 groups of 4 according to

$$\varepsilon_{\alpha\beta\mu\nu}\ \varepsilon^{\gamma\delta\rho\sigma} = \delta^{\gamma\delta}_{\alpha\beta}\ \delta^{\rho\sigma}_{\mu\nu} + \delta^{\rho\sigma}_{\alpha\beta}\ \delta^{\gamma\delta}_{\mu\nu} + $$
$$+\ 2\,\delta^{\rho[\gamma}_{\alpha\beta}\ \delta^{\delta]\sigma}_{\mu\nu} - 2\,\delta^{\sigma[\gamma}_{\alpha\beta}\ \delta^{\delta]\rho}_{\mu\nu} .$$

Substituting this into (63), we quickly obtain

$${}^{*}Q^{*}{}_{\alpha\beta}{}^{\gamma\delta} = -\,Q^{\gamma\delta}{}_{\alpha\beta} - \tfrac{1}{2}\,\delta^{\gamma\delta}_{\alpha\beta}\ Q^{\mu\nu}{}_{\mu\nu} + 4\,\delta^{[\gamma}_{[\alpha}\ Q^{\delta]\sigma}{}_{\beta]\sigma} . \tag{64}$$

Two special cases of this general identity are often quoted: if $F_{\alpha\beta}$, $G_{\alpha\beta}$ are any bivectors, then

$$F^{*}_{\alpha\rho}\ G^{\gamma\rho}_{*} - F^{\gamma\rho}\,G_{\alpha\rho} = -\,\tfrac{1}{2}\,\delta^{\gamma}_{\alpha}\,F_{\mu\nu}\,G^{\mu\nu} , \tag{65}$$

$$F^{*}_{\alpha\rho}\ F^{\gamma\rho} = 1/4\ \delta^{\gamma}_{\alpha}\ F^{*}_{\mu\nu}\,F^{\mu\nu} . \tag{66}$$

The last result shows that the dual of a bivector is also (apart from a scalar factor) the matrix inverse to it.

For our purposes, the most important application of (64) is to the four-dimensional Riemann tensor. We find

$$R_{\alpha\beta}{}^{*}{}^{\gamma\delta}{}_{*} = - R_{\alpha\beta}{}^{\gamma\delta} - 4\,\delta^{[\gamma}_{[\alpha}\; S^{\delta]}_{\beta]} \;, \tag{67}$$

where S^{α}_{β} is the trace-free part of the Ricci tensor:

$$S^{\alpha}_{\beta} = R^{\alpha}_{\beta} - 1/4\;\delta^{\alpha}_{\beta}\,R, \qquad R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\beta\mu} \;. \tag{68}$$

In the case of the Weyl conformal tensor

$$C_{\alpha\beta}{}^{\gamma\delta} \equiv R_{\alpha\beta}{}^{\gamma\delta} + 2\,\delta^{[\gamma}_{[\alpha}\; R^{\delta]}_{\beta]} - 1/6\;\delta^{\gamma\delta}_{\alpha\beta}\; R \tag{69}$$

all contractions vanish, so we have simply

$$\overset{*}{C}{}^{*}_{\alpha\beta\gamma\delta} = - C_{\alpha\beta\gamma\delta} \;. \tag{70}$$

A complex tensor $P_{..\alpha\beta..}$ is called *self-dual* with respect to the skew index-pair α, β if

$$P_{..\overset{*}{\alpha\beta}..} = i\, P_{..\alpha\beta..} \;. \tag{71}$$

If $Q_{..\alpha\beta..} = Q_{..[\alpha\beta]..}$ is a *real* tensor then the complex combination

$$Q^{(+)}_{..\alpha\beta..} \equiv Q_{..\alpha\beta..} - i\, Q_{..\overset{*}{\alpha\beta}..} \tag{72}$$

is self-dual by virtue of (61). From (70) it follows that

$$C^{*}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\overset{*}{\gamma\delta}} \tag{73}$$

and hence that

$$C^{(+)}_{\alpha\beta\gamma\delta} \equiv C_{\alpha\beta\gamma\delta} - i\, C^{*}_{\alpha\beta\gamma\delta} \tag{74}$$

has all the algebraic symmetries of the Weyl tensor (including zero trace) and in addition is self-dual in *both* index pairs.

3.3. Simple Bivectors and their Associated 2-Flats

A bivector $F_{\alpha\beta}$ is called *simple* or *decomposable* if it can be expressed as a wedge product

$$F_{\alpha\beta} = q_{\alpha}\, E_{\beta} - q_{\beta}\, E_{\alpha} \;. \tag{75}$$

A necessary and sufficient condition for $F_{\alpha\beta}$ to be simple is

$$F_{[\alpha\beta} F_{\gamma]\delta} = 0 ,$$

a result valid in any number of dimensions. In four dimensions, the condition reduces to

$$F^*_{\mu\nu} F^{\mu\nu} = 0 \qquad (76)$$

by virtue of (66).

The right side of (75) is unaffected if we add to either $\underset{\sim}{q}$ or $\underset{\sim}{E}$ an arbitrary multiple of the other. Thus, $F_{\alpha\beta}$ does not fix $\underset{\sim}{q}$, $\underset{\sim}{E}$ uniquely but only the 2-flat Σ_2 containing them. Since every 2-flat contains space-like vectors, we may (with no loss of generality) assume that $\underset{\sim}{E}$ is an arbitrary space-like vector of Σ_2 ($\underset{\sim}{E}.\underset{\sim}{E} < 0$), and that $\underset{\sim}{q}$ is orthogonal to it. There are now three cases to be distinguished, according to whether $\underset{\sim}{q}$ is space-like, time-like or null.

If $F_{\alpha\beta} F^{\alpha\beta} < 0$, $\underset{\sim}{q}$ is time-like and Σ_2 is a "time-like 2-flat", i.e. a 2-flat containing time-like vectors. (The xt plane in Minkowski space-time is a typical example of such a 2-flat, and, indeed, can always be made to coincide with it by a suitable rotation of axes.) Physically, a simple bivector of this type can be thought of as a Maxwell field, reducible to a purely electric component by choosing a frame of reference whose 4-velocity is $\underset{\sim}{q}$ or any other time-like vector in Σ_2.

Similarly, $F_{\alpha\beta} F^{\alpha\beta} > 0$ makes $\underset{\sim}{q}$ and Σ_2 space-like - i.e. Σ_2 contains space-like vectors *only* (e.g. yz plane in Minkowski space-time). It can be thought of as a field reducible to a purely magnetic component.

The most interesting case is that of a *null bivector*, characterized by the vanishing of both invariants:

$$F^*_{\alpha\beta} F^{\alpha\beta} = 0, \qquad F_{\alpha\beta} F^{\alpha\beta} = 0 . \qquad (77)$$

In this case, $\underset{\sim}{q}$ is a null vector. The 2-flat Σ_2 touches the null cone along a null ray parallel (and simultaneously orthogonal!) to $\underset{\sim}{q}$; apart from this parallel congruence of null rays all directions in Σ_2 are space-like, and all orthogonal to $\underset{\sim}{q}$. To visualize such a *null 2-flat* we may think of the plane $x = ct$, $z = 0$ in Minkowski space-time. The electromagnetic analogue is, of course, a plane wave with Σ_2 defining the instantaneous plane of polarization.

If a bivector $F_{\alpha\beta}$ is simple, the same holds for its dual, according to (76) and (61). Thus, $F^*_{\alpha\beta}$ defines a "dual 2-flat" Σ^*_2. From (75) and the definition of the dual we see at once that $F^*_{\alpha\beta}\, q^\beta = F^*_{\alpha\beta}\, E^\beta = 0$; i.e. each vector in Σ^*_2 is orthogonal to every vector in Σ_2. If $F_{\alpha\beta}\, F^{\alpha\beta} \neq 0$, Σ_2 and Σ^*_2 can therefore have only the zero vector in common: they bear the same relation to each other as the xt and yz planes. If $F_{\alpha\beta}$ is null, the situation is different: Σ^*_2 must share the (self-orthogonal) null vector $\underset{\sim}{q}$ with Σ_2, since two null vectors in space-time are orthogonal if and only if they are parallel. To make this more concrete, suppose the axes rotated so that Σ_2 has the equations $z = ct$, $y = 0$. Then Σ^*_2 is given by $z = ct$, $x = 0$ and their intersection is the null ray $z = ct$, $x = y = 0$. We conclude that if a bivector satisfies (77), it and its dual can be written

$$F_{\alpha\beta} = 2\, k_{[\alpha} x_{\beta]} \quad , \qquad F^*_{\alpha\beta} = 2\, k_{[\alpha}\, y_{\beta]} \tag{78}$$

where k_α is a null vector (uniquely defined up to a real scale factor) and $\underset{\sim}{x}$, $\underset{\sim}{y}$ are a pair of space-like vectors of equal length, orthogonal to $\underset{\sim}{k}$ and to each other.

It will be useful to re-express (78) in complex form. The relations[†]

$$F_{\alpha\beta} = p_{\alpha\beta} + \bar{p}_{\alpha\beta} \iff p_{\alpha\beta} = F^{(+)}_{\alpha\beta} \equiv F_{\alpha\beta} - i\, F^*_{\alpha\beta} \tag{79a}$$

establish a 1-1 correspondence between *real* bivectors $F_{\alpha\beta}$ and *complex self-dual* bivectors $p_{\alpha\beta}$. Since $F^*_{\alpha\beta}\, F^{\alpha\beta}_* = -\, F_{\alpha\beta} F^{\alpha\beta}$ by (65), we have

$$\tfrac{1}{2}\, p_{\alpha\beta}\, p^{\alpha\beta} = F_{\alpha\beta}\, F^{\alpha\beta} - i\, F^*_{\alpha\beta}\, F^{\alpha\beta} \tag{79b}$$

showing that $F_{\alpha\beta}$ is null if $p_{\alpha\beta}$ is, and conversely. In that case, (78) gives

$$p_{\alpha\beta} = 2\, k_{[\alpha}\, \bar{t}_{\beta]} \quad , \quad t_\beta = x_\beta + i\, y_\beta \quad . \tag{80}$$

We thus reach the important conclusion that each complex null, self-dual bivector defines a unique *real* null direction. Conversely, any complex bivector of the form (80) with $\underset{\sim}{k}.\underset{\sim}{k} = \underset{\sim}{t}.\underset{\sim}{t} = \underset{\sim}{k}.\underset{\sim}{t} = 0$ is null and satisfies $p^*_{\alpha\beta} = \pm\, i\, p_{\alpha\beta}$.

3.4. Null Basis for 1-forms

As basis $\underset{\sim}{e}^{(a)}$ in the Minkowski tangent space at each point it is convenient to choose a *complex null tetrad,* following a procedure first extensively used by Sachs.

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† The bar denotes complex conjugation.

This is constructed as follows. We begin by selecting an arbitrary real null vector $\underset{\sim}{k}$. To be definite, let us suppose that $\underset{\sim}{k}$ is directed into the *future* half of the null cone. Next, we arbitrarily choose a second future-pointing real null vector $\underset{\sim}{m}$ (not parallel to $\underset{\sim}{k}$), and we normalize it by the condition $\underset{\sim}{k}.\underset{\sim}{m} = 1$. The pair $\underset{\sim}{k}$, $\underset{\sim}{m}$ span a time-like 2-flat. To complete our tetrad we require a pair of vectors spanning the orthogonal space-like 2-flat. These could be taken as two (real) unit space-like vectors $\underset{\sim}{x}$, $\underset{\sim}{y}$ orthogonal to $\underset{\sim}{k}$, $\underset{\sim}{m}$ and to each other. It is convenient to choose the equivalent linear combinations $\underset{\sim}{t} = 2^{-\frac{1}{2}}(\underset{\sim}{x} + i\underset{\sim}{y})$, $\bar{\underset{\sim}{t}} = 2^{-\frac{1}{2}}(\underset{\sim}{x} - i\underset{\sim}{y})$. Then $\underset{\sim}{t}.\bar{\underset{\sim}{t}} = -1$, $\underset{\sim}{t}.\underset{\sim}{t} = \bar{\underset{\sim}{t}}.\bar{\underset{\sim}{t}} = 0$, so that $\underset{\sim}{t}$, $\bar{\underset{\sim}{t}}$ are a conjugate pair of complex null vectors.

Our (dual) basis is now

$$\underset{\sim}{e}^{(1)} = \underset{\sim}{k} , \quad \underset{\sim}{e}^{(4)} = \underset{\sim}{m} , \quad \underset{\sim}{e}^{(2)} = \underset{\sim}{t} , \quad \underset{\sim}{e}^{(3)} = \bar{\underset{\sim}{t}} \tag{81a}$$

and the fundamental matrix $g^{ab} = \underset{\sim}{e}^{(a)}.\underset{\sim}{e}^{(b)}$ has components $g^{14} = g^{41} = 1$, $g^{23} = g^{32} = -1$, other $g^{ab} = 0$. For the inverse matrix g_{ab} and the basis vectors $\underset{\sim}{e}_{(a)} = g_{ab}\,\underset{\sim}{e}^{(b)}$ we immediately derive

$$g_{ab} = g^{ab} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} , \tag{82}$$

$$g = \det(g_{ab}) = +1 ,$$

$$\underset{\sim}{e}_{(1)} = \underset{\sim}{m} , \quad \underset{\sim}{e}_{(4)} = \underset{\sim}{k} , \quad \underset{\sim}{e}_{(2)} = -\bar{\underset{\sim}{t}} , \quad \underset{\sim}{e}_{(3)} = -\underset{\sim}{t} . \tag{81b}$$

The corresponding basis for 1-forms, $\theta^a = e^{(a)}{}_{\alpha}\, dx^{\alpha}$, satisfies the reality conditions

$$\bar{\theta}^1 = \theta^1 , \quad \bar{\theta}^4 = \theta^4 , \quad \bar{\theta}^2 = \theta^3 . \tag{83}$$

Inversely, we have

$$dx^{\alpha} = e_{(a)}{}^{\alpha}\, \theta^a = m^{\alpha}\theta^1 + k^{\alpha}\theta^4 - \bar{t}^{\alpha}\theta^2 - t^{\alpha}\theta^3 . \tag{84}$$

The expression for the metric in terms of the basic 1-forms is

$$ds^2 = g_{ab}\,\theta^a\,\theta^b = 2(\theta^1\theta^4 - \theta^2\theta^3) \quad . \tag{85}$$

This can be written equivalently as

$$g_{\alpha\beta} = g_{ab}\,e^{(a)}{}_\alpha\,e^{(b)}{}_\beta = k_\alpha\,m_\beta + k_\beta m_\alpha - t_\alpha\,\bar{t}_\beta - t_\beta\,\bar{t}_\alpha \ ,$$

in which form it is commonly known as the "completeness relation" for the tetrad.

One point is worth emphasizing. Since our basis is complex, a *real* vector $\underset{\sim}{A}$ (or 1-form $A_\alpha\,dx^\alpha = A_a\,\theta^a$) will have *complex* tetrad components $A_a = \underset{\sim}{A}\cdot\underset{\sim}{e}_{(a)}$. The condition for $\underset{\sim}{A}$ to be real is

$$A_a\,\theta^a = \overline{A_a\,\theta^a} \implies \bar{A}_1 = A_1\ ,\ \bar{A}_4 = A_4\ ,\ \bar{A}_2 = A_3 \quad . \tag{86}$$

This extends in an obvious way to tensors of higher order: *a tensor* $T_{\alpha\beta\ldots}$ *is real if and only if its tetrad components* $T_{ab\ldots} \equiv T_{\alpha\beta\ldots}\,e_{(a)}{}^\alpha e_{(b)}{}^{\beta\ldots}$ *go over into their complex conjugates on interchanging the indices* 2.3.

The tetrad components of the alternating pseudo-tensor $\eta_{\alpha\beta\gamma\delta} = (-\det g_{\mu\nu})^{\frac{1}{2}}\,\varepsilon_{\alpha\beta\gamma\delta}$ are

$$\begin{aligned}\eta_{abcd} &= (-\det g_{\mu\nu})^{\frac{1}{2}}\,\varepsilon_{\alpha\beta\gamma\delta}\,e_{(a)}{}^\alpha e_{(b)}{}^\beta e_{(c)}{}^\gamma e_{(d)}{}^\delta \\ &= (-\det g_{\mu\nu})^{\frac{1}{2}}\,\varepsilon_{abcd}\,\det\,(e_{(\ell)}{}^\lambda) \quad .\end{aligned}$$

Taking determinants of both sides of

$$e_{(a)}{}^\alpha\,g_{\alpha\beta}\,e_{(b)}{}^\beta = g_{ab}$$

we find

$$(\det g_{\alpha\beta})\,(\det e_{(\ell)}{}^\lambda)^2 = \det g_{ab} = 1 \ ,$$

so

$$(-\det g_{\mu\nu})^{\frac{1}{2}}\,(\det e_{(\ell)}{}^\lambda) = \pm\, i \quad . \tag{87}$$

Let us make the convention that the null tetrad is always to be so oriented with respect to the co-ordinate net that the sign in (87) is positive. Then

$$\eta_{abcd} = i\,\varepsilon_{abcd}\ ,\quad \eta^{abcd} = i\,\varepsilon^{abcd} \quad . \tag{88}$$

For future reference it will be useful to record the explicit form of the duality relations $P^*_{ab} = \frac{1}{2}\,\eta_{abcd}\,P^{cd}$. We note first the relation between the covariant and contravariant tetrad components:

$$p^{12} = -p_{43}, \quad p^{13} = -p_{42}, \quad p^{14} = p_{41}, \tag{89}$$
$$p^{23} = p_{32}, \quad p^{24} = -p_{31}, \quad p^{34} = -p_{21}.$$

We can now write down at once

$$\left.\begin{aligned} p^*_{12} &= i\,p_{12}, \quad p^*_{34} = i\,p_{34} \\ p^*_{13} &= -i\,p_{13}, \quad p^*_{24} = -i\,p_{24} \\ p^*_{23} &= i\,p_{41}, \quad p^*_{41} = i\,p_{23}. \end{aligned}\right\} \tag{90}$$

A *self-dual* complex bivector p_{ab} is characterized by

$$p_{13} = p_{24} = 0, \quad p_{23} = p_{41}, \tag{91}$$

and therefore has three independent complex components p_{12}, p_{34}, $p_{14} - p_{23}$; we find

$$\tfrac{1}{2}\, p_{ab}\, p^{ab} = 2\, p_{12}\, p_{34} - \tfrac{1}{2}(p_{14} - p_{23})^2 . \tag{92}$$

3.5. Complex Vectorial Basis for Real 2-forms

The six independent 2-forms $\theta^a \wedge \theta^b$ of course span the vector space of all bivectors ($2\, F_{\alpha\beta}\, dx^{[\alpha} dy^{\beta]} = F_{ab}\, \theta^a \wedge \theta^b$), but it is preferable to take as basis the three linear combinations

$$Z^1 \equiv \theta^3 \wedge \theta^4, \quad Z^2 \equiv \theta^1 \wedge \theta^2,$$
$$Z^3 \equiv \tfrac{1}{2}(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3) \tag{93a}$$

and their complex conjugates (cf. (83))

$$\bar{Z}^1 = \theta^2 \wedge \theta^4, \quad \bar{Z}^2 = \theta^1 \wedge \theta^3,$$
$$\bar{Z}^3 = \tfrac{1}{2}(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3). \tag{93b}$$

The tetrad components $Z^m{}_{ab}$ are:

$$Z^1{}_{ab} = 2\, \delta^3_{[a}\, \delta^4{}_{b]}, \quad Z^2{}_{ab} = 2\, \delta^1_{[a}\, \delta^2{}_{b]},$$
$$Z^3{}_{ab} = \delta^1_{[a}\, \delta^4{}_{b]} - \delta^2_{[a}\, \delta^3{}_{b]}.$$

From (91) it follows immediately that Z^m and $\bar{Z}^m$ are self-dual and anti-self-dual respectively:

$$Z^{m*}_{ab} = i\, Z^m_{ab} \; , \qquad \bar{Z}^{m*}_{ab} = -\, i\, \bar{Z}^m_{ab} \; . \tag{94}$$

Hence every self-dual bivector is expressible as a linear combination of the three Z^m alone. According to (79a) any *real* bivector F_{ab} can be written as the sum of a self-dual bivector and its complex conjugate. So there must be a relation of the form

$$\tfrac{1}{2}\, F_{ab}\, \theta^a \wedge \theta^b = F_m Z^m + \bar{F}_m \bar{Z}^m \; . \tag{95}$$

Reference to (93) shows that (95) is indeed an identity, with the numerical coefficients given by

$$\left.\begin{aligned} F_1 &\equiv F_{34} , & F_2 &\equiv F_{12} , & F_3 &\equiv F_{14} - F_{23} \\ \bar{F}_1 &= F_{24} , & \bar{F}_2 &= F_{13} , & \bar{F}_3 &= F_{14} + F_{23} \end{aligned}\right\} \tag{96}$$

(Compare the remarks following (86).)

By virtue of (94), the duality operation corresponds to multiplying F_m by i :

$$\tfrac{1}{2}\, F^*_{ab}\, \theta^a \wedge \theta^b = i\, F_m Z^m - i\, \bar{F}_m \bar{Z}^m \; . \tag{97}$$

Thus, the associated self-dual expression $F^{(+)}_{ab} = F_{ab} - i\, F^*_{ab}$ reduces, as expected, to

$$\tfrac{1}{4}\, F^{(+)}_{ab}\, \theta^a \wedge \theta^b = F_m Z^m \; , \tag{98}$$

with

$$\left.\begin{aligned} F_1 = \tfrac{1}{2} F^{(+)}_{34} , \quad F_2 = \tfrac{1}{2} F^{(+)}_{12} , \quad F_3 = \tfrac{1}{2}(F^{(+)}_{14} - F^{(+)}_{23}) \\ F^{(+)}_{13} = F^{(+)}_{24} = F^{(+)}_{14} + F^{(+)}_{23} = 0 \; . \end{aligned}\right\} \tag{99}$$

The complex invariant associated with F_{ab} is given by (cf. (92))

$$\begin{aligned} \tfrac{1}{4}\,(F_{ab} F^{ab} - i\, F^*_{ab} F^{ab}) &= \tfrac{1}{8}\, F^{(+)}_{ab} F^{ab}_{(+)} \\ &= 2\, F_1 F_2 - \tfrac{1}{2} F_3^{\,2} \\ &= \gamma^{mn} F_m F_n \; . \end{aligned} \tag{100}$$

The 3 x 3 matrix γ^{mn} and its inverse γ_{mn} are defined by

$$\gamma^{mn} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad ; \quad \gamma_{mn} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad . \tag{101}$$

At a given space-time event, we may thus relate each real bivector to a vector in a complex Euclidean 3-space $\mathcal{C}_3$ in which is defined a scalar product

$$\gamma^{mn} F_m G_n = \tfrac{1}{8} F^{(+)}_{\alpha\beta} G^{\alpha\beta}_{(+)} \quad , \tag{102}$$

corresponding to the (complex) "inner product" of the real bivectors $F_{\alpha\beta}$, $G_{\alpha\beta}$.

Since $F_m = \delta^p_m$ when $F_{\alpha\beta} = Z^p_{\alpha\beta}$, (102) yields

$$\tfrac{1}{2} Z^p_{\alpha\beta} Z^{q\ \alpha\beta} = \gamma^{pq} \quad , \tag{103}$$

showing, as usual, that the metric coefficients are the inner products of the basis vectors Z^1, Z^2, Z^3 in $\mathcal{C}_3$.

This association of real bivectors with complex 3-vectors is an invariant one: a change of tetrad corresponds merely to a change of basis in $\mathcal{C}_3$. More explicitly, a Lorentz transformation which carries ($\underset{\sim}{k}$, $\underset{\sim}{m}$, $\underset{\sim}{t}$, $\underset{\sim}{\bar{t}}$) into another null tetrad satisfying (85) will turn Z^p into another self-dual triad satisfying (103). It will thus affect F_{ab} and F_p ; but since the 2-form $2 F^{(+)}_{\alpha\beta} dx^{[\alpha} dy^{\beta]} = F_p Z^p$ is left invariant, F_p and Z^p transform contragrediently and there is no effect on the vector representing $F_{\alpha\beta}$ in $\mathcal{C}_3$.

The really interesting result is the converse of this: an arbitrary rotation of Z^p into another self-dual triad satisfying (103) can always be brought about by (exactly one) Lorentz transformation of the null tetrad. (This means we can rotate axes freely in $\mathcal{C}_3$.)

To prove this, we have to show that the correlation between $\underset{\sim}{e}_{(a)}$ and Z^p is 1-1. That is, any given triad of self-dual bivectors $Z^p_{\alpha\beta}$ satisfying (103) should define a (unique) null tetrad (properly oriented - see (88)) in terms of which the Z^p are given by expressions of the form (93a).

In fact, it is easy to see that the required null tetrad is already fixed by Z^1, Z^2 alone. According to (103) and (101), the self-dual 2-forms Z^1 and Z^2 are null; by the remarks following (80), they define a pair of real null directions m^α and k^α respectively, such that

$$Z^1 = s \wedge m , \qquad Z^2 = k \wedge t$$

where $s \equiv s_\alpha dx^\alpha$, $t \equiv t_\alpha dx^\alpha$ are complex null 1-forms with $s_\alpha m^\alpha = t_\alpha k^\alpha = 0$. Two real scale factors at our disposal can be put to use by requiring $t_\alpha \bar{t}^\alpha = -1$, $k_\alpha m^\alpha = 1$. Arbitrary multiples of $\underset{\sim}{m}$, $\underset{\sim}{k}$ can still be added to $\underset{\sim}{s}$, $\underset{\sim}{t}$ respectively and we utilize this freedom to arrange $k_\alpha s^\alpha = m_\alpha t^\alpha = 0$. Then the (real) space-like 2-flat orthogonal to $\underset{\sim}{k}$ and $\underset{\sim}{m}$ contains both $\underset{\sim}{s}$ and $\underset{\sim}{t}$, whence $\underset{\sim}{s} = \lambda \underset{\sim}{t} + \mu \bar{\underset{\sim}{t}}$. From $\underset{\sim}{s}.\underset{\sim}{s} = 0$ we infer $\lambda\mu = 0$, while $Z^1{}_{\alpha\beta} Z^{2\alpha\beta} = 2$ yields $\mu = -\underset{\sim}{s}.\underset{\sim}{t} = 1$. Hence $\underset{\sim}{s} = \underset{\sim}{t}$ and we thus arrive at a null tetrad $\underset{\sim}{k}$, $\underset{\sim}{m}$, $\underset{\sim}{t}$, $\bar{\underset{\sim}{t}}$, in terms of which Z^1 and Z^2 are expressed by relations of the proper form (93a). The rest is routine.

The usefulness of this 1-1 correspondence between null tetrads and bivector bases Z^p in $\mathcal{C}_3$ is clear. If, for example, we wish to single out for special attention a pair of null bivectors $F_{\alpha\beta}$, $G_{\alpha\beta}$ (with $F^{(+)}_{\alpha\beta} G^{\alpha\beta}_{(+)} \neq 0$), we can point the two null legs Z^1, Z^2 of our basis along the directions of the corresponding 3-vectors F_p, G_p in $\mathcal{C}_3$, so that $F_p = (F_1, 0, 0)$, $G_p = (0, G_2, 0)$. That such a simplification can actually be achieved by a suitable rotation of the null tetrad $\underset{\sim}{k}$, $\underset{\sim}{m}$, $\underset{\sim}{t}$, $\bar{\underset{\sim}{t}}$ is a fact we have no need to verify explicitly; it is guaranteed by the result just proven.

To conclude this section, we record for later reference, some remarkable formulas for the inner products of the basic bivectors $Z^p_{\alpha\beta}$. We have, for example,

$$\begin{aligned} g^{\mu\nu} Z^1_{\alpha\mu} Z^2_{\beta\nu} &= 4\, g^{\mu\nu}\, \bar{t}_{[\alpha} m_{\mu]}\, k_{[\beta}\, t_{\nu]} \\ &= m_\alpha k_\beta - \bar{t}_\alpha t_\beta \\ &= \tfrac{1}{2} g_{\alpha\beta} + m_{[\alpha} k_{\beta]} - \bar{t}_{[\alpha} t_{\beta]} \qquad \text{by (85)} \\ &= \tfrac{1}{2} g_{\alpha\beta} - Z^3_{\alpha\beta} \quad . \end{aligned}$$

The general result is

$$\begin{aligned} 2\, g^{\mu\nu} Z^m_{\alpha\mu} Z^n_{\beta\nu} &= \gamma^{mn} g_{\alpha\beta} + \varepsilon^{mnp} Z_{p\alpha\beta} \\ g^{\mu\nu} Z^m_{\mu\ [\alpha} \bar{Z}^n_{\beta]\ \nu} &= 0 \end{aligned} \qquad (104)$$

3.6. CHARACTERISTIC NULL VECTORS OF THE WEYL TENSOR

The well-known algebraic classification of Weyl tensors initiated by Petrov can be recovered in a simple covariant form with the aid of the complex vectorial formalism.

We recall from (74) that the complex tensor

$$C^{(+)}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - i\, C^{*}_{\alpha\beta\gamma\delta} \tag{74}$$

is self-dual in both index pairs and therefore expressible in terms of the self-dual basis $Z^{m}{}_{\alpha\beta}$ as

$$C^{(+)}_{\alpha\beta\gamma\delta} = C_{mn}\, Z^{m}_{\alpha\beta}\, Z^{n}_{\gamma\delta} \quad . \tag{105}$$

The coefficients C_{mn} are the components of a complex tensor in $\mathcal{C}_3$. This tensor is symmetric ($C_{mn} = C_{nm}$, corresponding to $C^{(+)}_{abcd} = C^{(+)}_{cdab}$) and traceless:

$$\gamma^{mn}\, C_{mn} = 0 \,, \quad \text{i.e.} \quad C_{33} = 4\, C_{12} \quad . \tag{106}$$

[(106) follows directly from (104) and vanishing of all contractions of $C^{(+)}_{\alpha\beta\gamma\delta}$.]

All information about the Weyl tensor (10 independent real components) is thereby condensed into a 3 x 3 symmetric traceless matrix C_{mn} (5 independent complex components). To display the connection between these two objects explicitly, we take the real part of (105):

$$2\, C_{\alpha\beta\gamma\delta} = C_{mn}\, Z^{m}_{\alpha\beta}\, Z^{n}_{\gamma\delta} + \bar{C}_{mn}\, \bar{Z}^{m}_{\alpha\beta}\, \bar{Z}^{n}_{\gamma\delta} \quad . \tag{108a}$$

In terms of tensor products of the 2-forms $Z^{m}_{ab} = \frac{1}{2} Z^{m}_{ab}\, \theta^{a} \wedge \theta^{b}$ we can write equivalently

$$\tfrac{1}{2}\, C_{abcd}(\theta^{a} \wedge \theta^{b})(\theta^{c} \wedge \theta^{d}) = C_{mn}\, Z^{m}\, Z^{n} + \bar{C}_{mn}\, \bar{Z}^{m}\, \bar{Z}^{n} \;. \tag{108b}$$

We next seek the class of null 3-vectors F^{m} (arbitrary up to a complex scale-factor) satisfying:

$$C_{mn}\, F^{m}\, F^{n} = 0 \;, \tag{109a}$$

$$0 = \tfrac{1}{2}\, \gamma_{mn}\, F^{m} F^{n} \equiv F^{1}\, F^{2} - (F^{3})^{2} \;. \tag{109b}$$

(This class contains in particular all null, self-dual eigenbivectors of $C^{(+)}_{abcd}$.) Since each null complex 3-vector defines a unique null direction in space-time by (80), we shall arrive in this way at a set of characteristic null directions ("Debever-Penrose null vectors") associated with the Weyl tensor.

Let us set

$$F^1 = \xi^2 , \quad F^2 = \eta^2 , \quad F^3 = \xi\eta \tag{110}$$

so that (109b) is automatically satisfied. Substituting into (109a) and eliminating C_{33} by (106), we find

$$C_{11}\, \xi^4 + 2\, C_{13}\, \xi^3\eta + 6\, C_{12}\, \xi^2\eta^2 + 2\, C_{23}\, \xi\eta^3 + C_{22}\, \eta^4 = 0 \ . \tag{111}$$

The four roots of this quartic yield (ignoring possible coincidences for the moment) four Debever-Penrose null vectors.

We may always rotate axes in $\mathcal{L}_3$ so that the null axis Z^2 is parallel to a particular characteristic vector F^m, i.e.

$$F_m\, Z^m = \xi^2\, Z^2 \quad \text{or} \quad F^m = (\xi^2, 0, 0) \ .$$

This means $\eta = 0$ is a root of (111) and $C_{11} = 0$. Since $Z^2 = \theta^1 \wedge \theta^2 = k \wedge t$, we see that we have effectively rotated the null tetrad so that k^a is the Debever-Penrose vector corresponding to F_m. With this special choice of basis, only four complex coefficients appear in the tetrad expansion (108) of the Weyl tensor.

The relationship between the Weyl tensor and a principal null vector $\underset{\sim}{k}$ can also be expressed without bringing in the (largely arbitrary) vectors $\underset{\sim}{m}$, $\underset{\sim}{t}$, $\underset{\sim}{\bar{t}}$. We first note from (93a) that

$$Z^m{}_{ab}\, k^b = \delta^m_1\, \bar{t}_a + \tfrac{1}{2}\, \delta^m_3\, k_a \ , \tag{112}$$

and (105) then yields

$$-\, C^{(+)}_{abcd}\, k^b k^c = C_{11}\, \bar{t}_a\, \bar{t}_d + C_{13}\, k_{(a}\, \bar{t}_{d)} + C_{12}\, k_a\, k_d \ . \tag{113}$$

This holds for an *arbitrary* null tetrad. The necessary and sufficient condition that the null vector $\underset{\sim}{k}$ be a Debever-Penrose vector is that $C_{11} = 0$, i.e. that

$$C^{(+)}_{abcd}\, k^b\, k^c = k_a\, p_d + k_d\, p_a \tag{114}$$

where p_a is some complex vector. By (74) this complex condition, which can be written equivalently as $k_{[f}C^{(+)}_{a]bc[d}k_{e]}k^b k^c = 0$, is equivalent to the two real conditions

$$k_{[f}\, C_{a]bc\,[d}\, k_{e]}\, k^b\, k^c = 0 = k_{[f}\, C_{a]bc\,[\overset{*}{d}}\, k_{e]}\, k^b\, k^c \ . \tag{115}$$

Actually, it is easy to verify (take real and imaginary parts of (113)) that either one of the real conditions (115) is a consequence of the other.

We proceed now to consider the case ("algebraic degeneracy") where two or more of the Debever-Penrose vectors coincide. The characteristic equation (111) will have $\eta = 0$ as (at least) a double root if $C_{11} = C_{13} = 0$. The necessary and sufficient condition for algebraic degeneracy is therefore that

$$C^{(+)}_{abcd}\, k^b\, k^c \;=\; -\, C_{12}\, k_a\, k_d \quad . \tag{116}$$

We need not enter into details of the obvious algebraic classification which can be based on (111). However, two special cases are of interest:

(i) C_{abcd} is said to be Type N if k_a is a quadruple Debever-Penrose vector. The condition that $\eta = 0$ be a quadruple root of (111) is that $C_{11} = C_{12} = C_{13} = C_{23} = 0$. Hence a Type N Weyl tensor is expressible as

$$C^{(+)}_{abcd} \;=\; 4\, C_{22}\, k_{[a}\, t_{b]}\, k_{[c}\, t_{d]} \quad . \tag{117}$$

(ii) If C_{abcd} has two pairs of coincident Debever-Penrose vectors it is said to be Type D. We can then align the tetrad so that the two distinct null vectors are $\underset{\sim}{k}$, $\underset{\sim}{m}$. The associated complex 3-vectors are Z^2, Z^1 respectively; this means $\eta = 0$, $\xi = 0$ are double roots of (111), and $C_{12} \equiv \frac{1}{4}\, C_{33}$ is the only surviving component of C_{mn}. Thus, every Type D tensor satisfies, in addition to (116),

$$C^{(+)}_{abcd}\, m^b\, m^c \;=\; -\, C_{12}\, m_a\, m_d \quad . \tag{117a}$$

Schwarzschild space-time provides the best-known example of algebraic type D; in this case, $C_{12} = 2m/r^3$ and $\underset{\sim}{k}$, $\underset{\sim}{m}$ are the past and future directed radial null directions (cf. Section 4.6).

Chapter IV: Complex Vectorial Calculus

4.1. Introduction

In the previous chapter we saw how various relationships between null vectors, bivectors and vacuum Riemann tensors could be most economically unravelled in the Euclidean space $\mathcal{C}_3$ of complex 3-vectors, referring as little as possible to the original space-time. These were algebraic considerations, confined to a single space-time event. It is now natural to ask whether this approach can be extended so as to describe the point-to-point variation of geometrical structures in terms of objects in $\mathcal{C}_3$.

That this may lead to a rather neat formalism can be seen from the (in some ways misleading) analogy of a rigid field of frame vectors $\underset{\sim}{I}_{(p)}$ in ordinary Euclidean 3-space. Since a bivector in a 3-space can be correlated with a vector, we can replace

$$(\underset{\sim}{I}_{(p)} \cdot \nabla)\, \underset{\sim}{I}_{(q)} = -\gamma_{qrp}\, \underset{\sim}{I}_{(r)}$$

(involving the Ricci rotation coefficients $\gamma_{qrp} = -\gamma_{rqp}$) by the simpler equivalent relation

$$(\underset{\sim}{I}_{(p)} \cdot \nabla)\, \underset{\sim}{I}_{(q)} = -\underset{\sim}{\omega}_{(p)} \times \underset{\sim}{I}_{(q)}$$

involving three "Darboux vectors" $\underset{\sim}{\omega}_{(p)}$ ($\underset{\sim}{\omega}_{(3)}$ is the "angular velocity" of the rigid triad, for displacements along $\underset{\sim}{I}_{(3)}$). For an arbitrary displacement $d\underset{\sim}{x} = dx^p\, \underset{\sim}{I}_{(p)}$ the change of $\underset{\sim}{I}_{(q)}$ is

$$D\, \underset{\sim}{I}_{(q)} = -\underset{\sim}{\sigma} \times \underset{\sim}{I}_{(q)} \tag{118}$$

where $\underset{\sim}{\sigma} \equiv \underset{\sim}{\omega}_{(p)}\, dx^p$ is a vectorial symbol for a triad of 1-forms.

In a roughly similar manner, the exterior differentials of the basis Z^1, Z^2, Z^3 in $\mathcal{C}_3$ can be simply expressed in terms of three complex 1-forms $\sigma^p \equiv \sigma^p{}_\alpha\, dx^\alpha$. Since Z^p is rigidly geared to the space-time null basis θ^a, it is not surprising that there is a close relation between σ^p and the connection 1-forms $\omega^a{}_b$. In fact, $2\,\omega_{ab}$ is related to σ_p in precisely the way that a bivector F_{ab} is related to its associated complex 3-vector F_p (see (95) or (96)). The curvature 2-forms Ω_{ab} similarly have their counterpart Σ_p in $\mathcal{C}_3$. It turns out that the relationships between Z^p, σ_p and Σ_p can be written very simply. In this way we obtain a compact and nearly self-contained translation of Cartan's calculus of moving frames into a complex vectorial formalism. By proper choice of the basis, objects such as σ_p of course acquire a direct geometrical significance in terms of shear, rotation and expansion of principal null curves etc..

4.2. Connection 1-forms in $\mathcal{C}_3$

We consider a field of null tetrads $\underset{\sim}{e}^{(a)}$. The exterior differentials of the 1-forms $\theta^a = e^{(a)}{}_\alpha \, dx^\alpha$,

$$d\,\theta^a = -\,\omega^a{}_c \wedge \theta^c \tag{119}$$

completely define the connection 1-forms $\omega_{ab} = -\,\omega_{ba}$. These six 1-forms which, though not real, are equivalent to six real forms, can be linearly combined without loss of information into three complex 1-forms σ_p and their complex conjugates in exactly the same way that the six $\theta^a \wedge \theta^b$ were combined to form Z^p and $\bar{Z}^p$. We define σ_p by*

$$2\,\omega_{ab} = \sigma_p\, Z^p_{ab} + \bar{\sigma}_p\, \bar{Z}^p_{ab} \ . \tag{120a}$$

In terms of tensor products of forms, this reads more briefly

$$\omega_{ab}\, \theta^a \wedge \theta^b = \sigma_p\, Z^p + \bar{\sigma}_p\, \bar{Z}^p \tag{120b}$$

The explicit relations are formally the same as (96) apart from a common factor 2, i.e. $\sigma_3 = 2(\omega_{14} - \omega_{23})$ etc. Solving for ω_{ab} we find

$$\begin{aligned}
-\,\omega^1{}_1 &= \omega^4{}_4 = \tfrac{1}{4}(\sigma_3 + \bar{\sigma}_3), & -\,\omega^2{}_2 &= \omega^3{}_3 = \tfrac{1}{4}(\sigma_3 - \bar{\sigma}_3)\ ,\\
\omega^1{}_3 &= \omega^2{}_4 = -\,\tfrac{1}{2}\,\sigma_1\ , & \omega^1{}_2 &= \omega^3{}_4 = -\,\tfrac{1}{2}\,\bar{\sigma}_1\ ,\\
\omega^4{}_2 &= \omega^3{}_1 = \tfrac{1}{2}\,\sigma_2\ , & \omega^4{}_3 &= \omega^2{}_1 = \tfrac{1}{2}\,\bar{\sigma}_2\ .
\end{aligned} \tag{121}$$

Tetrad components of the 4-vector associated with the 1-form σ_p are defined by the expansion $\sigma_p = \sigma_{pa}\,\theta^a$, so that $\bar{\sigma}_p = \bar{\sigma}_{pa}\,\bar{\theta}^a$. Since $\bar{\theta}^2 = \theta^3$ some care is needed in handling the tetrad indices 2,3. For example,

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* The reason why this procedure works is that the 1-forms $\omega^a{}_b = -\underset{\sim}{e}_{(b)} \cdot D\,\underset{\sim}{e}^{(a)}$ formally satisfy the same "reality conditions" as the tetrad components of a real bivector [cf. eq. (86)].

$$\gamma^{1}{}_{2c}\ \theta^{c} = \omega^{1}{}_{2} = -\tfrac{1}{2}\,\bar{\sigma}_{1} = -\tfrac{1}{2}\,\bar{\sigma}_{1c}\,\bar{\theta}^{c}\ .$$

Hence

$$\gamma^{1}{}_{22} = -\tfrac{1}{2}\,\bar{\sigma}_{13}\ ,\quad \gamma^{1}{}_{23} = -\tfrac{1}{2}\,\bar{\sigma}_{12}\ . \tag{122}$$

Exactly the same scheme can be used to recast the six curvature 2-forms $\Omega^{a}{}_{b}$ as three complex 2-forms Σ_{p} :

$$\Omega_{ab}\ \theta^{a}\wedge\theta^{b} = \Sigma_{p}\,Z^{p} + \bar{\Sigma}_{p}\,\bar{Z}^{p}\ . \tag{123}$$

The explicit relations between $\Omega^{a}{}_{b}$ and Σ_{p} are the precise analogue of (121).

4.3. Equations of Structure in $\mathcal{C}_3$

We take exterior differentials of the basic 2-forms

$$Z^{p} = \tfrac{1}{2}\,Z^{p}{}_{ab}\ \theta^{a}\wedge\theta^{b} \tag{124}$$

recalling that the coefficients $Z^{p}{}_{ab}$ are constants [see (93)]. We find

$$d\,Z^{p} = \tfrac{1}{2}\,Z^{p}{}_{ab}(d\theta^{a}\wedge\theta^{b} - \theta^{a}\wedge d\theta^{b})$$
$$= Z^{p}{}_{ab}\ d\theta^{a}\wedge\theta^{b}$$

by virtue of the skew-symmetry of $Z^{p}{}_{ab}$ and the fact that the 2-form $d\theta^{b}$ commutes with the 1-form θ^{a} [equation (3)]. Using (119) and (120a) to express $d\theta^{a}$ in terms of σ_{p} , we obtain

$$dZ^{p} = -\tfrac{1}{2}\,Z^{p}{}_{ab}(Z^{ma}{}_{c}\,\sigma_{m} + \bar{Z}^{ma}{}_{c}\,\bar{\sigma}_{m})\wedge\theta^{c}\wedge\theta^{b}\ .$$

According to (104), this reduces to

$$dZ^{p} = -\tfrac{1}{4}\,\epsilon^{pmn}\,Z_{nbc}\,\sigma_{m}\wedge\theta^{c}\wedge\theta^{b}\ ,$$

i.e.

$$dZ^{p} = \tfrac{1}{2}\,\epsilon^{pmn}\,\sigma_{m}\wedge Z_{n}\ . \tag{125}$$

These relations are the transcription of Cartan's *first* equations of structure into $\mathcal{C}_3$.

Since the g_{ab} are constant for our null tetrad, the *second* equations of structure (43) may be written

$$\Omega_{ab} = d\omega_{ab} + \omega_{ac}\wedge\omega^{c}{}_{b}\ . \tag{126}$$

Through the relations (123) and (120a) we can immediately express Ω_{ab} in terms of Σ_{p} and $d\omega_{ab}$ in terms of $d\sigma_{p}$. For the last term of (126) we have

$$4\,\omega_{ac} \wedge \omega^{c}{}_{b} = (\sigma_m Z^{m}{}_{ac} + \text{c.c.}) \wedge (\sigma_n Z^{nc}{}_{b} + \text{c.c.})$$

$$= -\tfrac{1}{2}\,\varepsilon^{mnp} Z_{pab}\,\sigma_m \wedge \sigma_n + \text{c.c.}$$

by (104). Because of the linear independence of the six bivectors Z^p, $\bar{Z}^p$, we may infer

$$\Sigma_p = d\sigma_p - \tfrac{1}{2}\,\varepsilon_{pmn}\,\sigma^m \wedge \sigma^n \qquad (127)$$

as the complex vectorial equivalent of (126).

4.4. Bianchi Identities

The Bianchi identities express the identical vanishing of $d^2\sigma_p$, i.e. they are the compatibility conditions of (127):

$$d\,\Sigma_p = -\,\varepsilon_{pmn}\,d\sigma^m \wedge \sigma^n \quad .$$

Substituting for $d\sigma^m$ from (127) and simplifying yields

$$d\,\Sigma_p + \varepsilon_{pmn}\,\Sigma^m \wedge \sigma^n = 0 \quad . \qquad (128)$$

One quickly checks that the compatibility conditions of (128) ($d^2\,\Sigma_p \equiv 0$) do not give anything new.

An algebraic identity for the curvature 2-forms Σ_p (equivalent to the cyclic identity) follows from (125) by taking the exterior differential:

$$0 = d^2\,Z^p = \varepsilon^{pmn}\,(d\sigma_m \wedge Z_n - \sigma_m \wedge dZ_n) \quad .$$

Substituting for $d\sigma_m$ and dZ_n leads to

$$\Sigma_{[m} \wedge Z_{n]} = 0 \,, \qquad (129)$$

which should be compared with (46).

4.5. Decomposition of the Riemann Tensor

We now come to the important question: how does the decomposition of the Riemann tensor into its irreducible components (Weyl tensor, traceless part of the Ricci tensor and curvature invariant) appear in $\mathcal{C}_3$? In particular, can we write Einstein's vacuum equations $R_{ab} = 0$ conveniently as conditions on the 2-forms Σ_p?

Since the six Z^p, $\bar{Z}^p$ form a basis for all 2-forms, we may expand Σ_p as

$$\Sigma_p = (C_{pq} - \frac{1}{6} R \gamma_{pq}) Z^q + E_{p\bar{q}} \bar{Z}^q , \tag{130}$$

where C_{pq} , R and $E_{p\bar{q}}$ are complex numerical coefficients whose meaning we shall elucidate. Because of the way we have split the coefficient of Z^q, we may take C_{pq} to be traceless: $\gamma^{pq} C_{pq} = 0$.

We next form the wedge product of (130) with Z^r , and note the identities

$$Z^m \wedge Z^n = - i \eta \gamma^{mn} , \qquad Z^m \wedge \bar{Z}^n = 0 \tag{131}$$

which can be read off directly from the defining relations (93). Here, the 4-form

$$\eta \equiv i \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = \eta_{\alpha\beta\gamma\delta} dx^\alpha dy^\beta dz^\gamma dw^\delta \tag{132}$$

represents the volume of a 4-cell (cf. eq. (88)). We thus find from (129) that $C_{qp} = C_{pq}$. (It is also easy to show, e.g. by taking real parts of (136) below and noting the symmetry $R_{abcd} = R_{cdab}$, that

$$E_{q\bar{p}} = \bar{E}_{p\bar{q}} , \tag{133}$$

so $E_{p\bar{q}}$ is hermitian.) We have explicitly

$$i \eta E_{p\bar{q}} = \Sigma_p \wedge \bar{Z}_q , \qquad \tfrac{1}{2} i \eta R = \Sigma_p \wedge Z^p . \tag{134}$$

We proceed to consider the relationship between the numbers C_{pq}, R, $E_{p\bar{q}}$ and the tetrad components of the curvature tensor. From the expressions (123) for Ω_{ab} in terms of Σ_p , we find immediately for the corresponding self-dual combination [cf. eq. (98)]:

$$\Omega_{ab} - i \Omega^*_{ab} = \Sigma_p Z^p_{ab} . \tag{135}$$

Taking tetrad components of these 2-forms and using (130) yields

$$\begin{aligned} R_{abcd} - i R^*_{abcd} &= (C_{pq} - \frac{1}{6} R \gamma_{pq}) Z^p_{ab} Z^q_{cd} \\ &\quad + E_{p\bar{q}} Z^p{}_{ab} \bar{Z}^q_{cd} . \end{aligned} \tag{136}$$

We contract this equation, bearing in mind eq. (104), also that

$$R^{*}{}_{ab}{}^{bd} \equiv \tfrac{1}{2} \eta_{abef} R^{efbd} = 0$$

by the cyclic identity. The result is

$$R_{ad} = \tfrac{1}{4} R g_{ad} + \gamma^{p\bar{q}}{}_{ad} E_{p\bar{q}} \tag{137}$$

where we have defined the complex scalars

$$Y^{p\bar{q}}{}_{ab} \equiv g^{cd}\, Z^p{}_{ac}\, \bar{Z}^q{}_{db} \ . \tag{138}$$

Now, it is easy to prove [e.g. from (65)] that a self-dual and an anti-self-dual bivector are always orthogonal. Hence

$$Z^p{}_{ab}\, \bar{Z}^{qab} = 0 \ , \tag{139}$$

and consequently $g^{ab}\, Y^{p\bar{q}}{}_{ab} = 0$. Contraction of (137) then shows that the coefficient R is real and may be identified with the curvature invariant, as already anticipated by our notation. We thus see that *the hermitian matrix* $E_{p\bar{q}}$ (involving nine independent real numbers) *corresponds to the trace-free part* $R_{ab} - \frac{1}{4}\, g_{ab}\, R$ *of the Ricci tensor.* For some purposes it is useful to have the explicit relation between the two. A straightforward calculation gives for the coefficients in (137):

$$Y^{p\bar{q}}{}_{\alpha\beta} = \begin{bmatrix} m_\alpha m_\beta & \bar{t}_\alpha \bar{t}_\beta & m_{(\alpha}\bar{t}_{\beta)} \\ t_\alpha t_\beta & k_\alpha k_\beta & k_{(\alpha}t_{\beta)} \\ m_{(\alpha}t_{\beta)} & k_{(\alpha}\bar{t}_{\beta)} & \tfrac{1}{2}(k_{(\alpha}m_{\beta)} + t_{(\alpha}\bar{t}_{\beta)}) \end{bmatrix} \begin{matrix} p=1 \\ p=2 \\ p=3 \end{matrix}$$

$$\qquad q=1 \qquad\qquad q=2 \qquad\qquad q=3 \tag{140}$$

It is clear that $Y^{p\bar{q}}{}_{\alpha\beta}$ is hermitian in p, q, symmetric and trace-free in α, β. The *same* matrix gives the inverse relations: since

$$Y^{p\bar{q}}{}_{\alpha\beta}\, Y_{m\bar{n}}{}^{\alpha\beta} = \delta^p_m\, \delta^q_n \tag{141}$$

(as we may easily verify from (138) and (104)) we obtain

$$E_{p\bar{q}} = Y_{p\bar{q}}{}^{\alpha\beta}\, (R_{\alpha\beta} - \tfrac{1}{4}\, g_{\alpha\beta}\, R) \ . \tag{142}$$

The important particular result of this analysis is the form assumed by Einstein's vacuum equations in $\mathcal{B}_3$. By (134),

$$R_{\alpha\beta} = 0 \Longleftrightarrow \Sigma_p \wedge \bar{Z}_p = 0 = \Sigma_p \wedge Z^p \ . \tag{143}$$

The remaining term $C_{pq}\, Z^p{}_{ab}\, Z^q{}_{cd}$ in the expansion (136) has the property that all its contractions vanish. It therefore corresponds to the *Weyl tensor* in the manner described in Section 3.6.

Finally, it is worth recording how the conditions for algebraic degeneracy (Section 3.6) appear in the present context. The Weyl tensor has k^α as a characteristic null vector iff the basis in $\mathcal{b}_3$ is oriented so that $C_{11} = 0$, i.e.

$$\Sigma_1 \wedge Z^2 = 0 \quad . \tag{144}$$

C_{abcd} is algebraically degenerate, with k^α as a double (at least) characteristic null vector, iff $C_{11} = C_{13} = 0$, i.e.

$$\Sigma_1 \wedge Z^2 = \Sigma_1 \wedge Z^3 = 0 \quad . \tag{145}$$

It is of Petrov Type D, with k^α, m^α as the two double null vectors, iff $C_{11} = C_{13} = C_{22} = C_{23} = 0$, $C_{12} \neq 0$, i.e. if

$$\left.\begin{aligned} \Sigma_2 \wedge Z^1 &= \Sigma_2 \wedge Z^3 = 0 \\ (\Sigma_3 - \tfrac{1}{3} R\, Z_3) \wedge Z^3 &\neq 0 \end{aligned}\right\} \tag{146}$$

holds in addition to (145).

4.6. Example: The Kerr Metric

As an illustration, we consider the Kerr metric

$$\begin{aligned} ds^2 = {} & dv^2 - 2\, dr dv - 2\, a \sin^2\theta\, dr d\phi - (r^2 + a^2)\sin^2\theta\, d\phi^2 \\ & - F\, d\theta^2 - (2mr/F)(dv + a \sin^2\theta\, d\phi)^2 \end{aligned} \tag{147}$$

with $F(r,\theta) \equiv r^2 + a^2\cos^2\theta$. This reduces to the canonical form

$$ds^2 = 2\, \theta^1 \theta^4 - 2\, \theta^2 \theta^3 \tag{148}$$

in terms of the null tetrad associated with the 1-forms

$$\left.\begin{aligned} \sqrt{2}\, \theta^1 &= - dv - a \sin^2\theta\, d\phi \\ \sqrt{2}\, \theta^4 &= 2\, dr + (f/F)\, \sqrt{2}\, \theta^1 \\ \sqrt{2}\, \theta^2 &= F^{\frac{1}{2}}\, (d\theta + i \sin\theta\, d\phi) - i\, a\, F^{-\frac{1}{2}} \sin\theta \,.\, \sqrt{2}\, \theta^1 \\ \theta^3 &= \bar{\theta}^2 \end{aligned}\right\} \tag{149}$$

where $f(r) \equiv r^2 - 2\, mr + a^2$.

Taking exterior differentials yields:

$$2^{-\frac{1}{2}}\, d\theta^1 = - a^2 F^{-\frac{3}{2}} \sin\theta \cos\theta\, (\theta^2 + \theta^3) \wedge \theta^1 - $$
$$- i\, a\, F \cos\theta\, (\theta^2 \wedge \theta^3)\ , \tag{150a}$$

$$2^{\frac{1}{2}}\, d\theta^2 = - 2\, i\, a\, F^{-\frac{3}{2}}\, r \sin\theta\, (\theta^1 \wedge \theta^4) - $$
$$- F^{-\frac{3}{2}}\, (r^2 + a^2) \cot\theta\, (\theta^2 \wedge \theta^3) - $$
$$- F^{-1}\, r(\theta^2 \wedge \theta^4) - f\, F^{-2}\, r(\theta^1 \wedge \theta^2)\ , \tag{150b}$$

$$2^{-\frac{1}{2}}\, d\theta^4 = - F^{-2}(mr^2 - a^2 r \sin^2\theta - ma^2 \cos^2\theta)(\theta^1 \wedge \theta^4) - $$
$$- i\, a\, f\, F^{-2} \cos\theta\, (\theta^2 \wedge \theta^3)\ . \tag{150c}$$

It is now a simple matter to write out the 3-forms dZ^m in terms of $\theta^a \wedge Z^n$ and compare the results with $dZ^m = \frac{1}{2}\, \varepsilon^{mpq}\, \sigma_p \wedge Z_q$ to obtain σ_p. (It is easiest to start with dZ^3.) In this manner we find (uniquely):

$$2^{-\frac{1}{2}}\, \sigma_1 = - (r + i\, a \cos\theta)(i\, a\, F^{-\frac{3}{2}} \sin\theta .\ \theta^1 + F^{-1}\theta^2)\ , \tag{151a}$$

$$2^{-\frac{1}{2}}\, \sigma_2 = - (r + i\, a \cos\theta)(fF^{-2}\theta^3 - i\, a\, F^{-\frac{3}{2}} \sin\theta\ .\ \theta^4)\ , \tag{151b}$$

$$2^{-\frac{1}{2}}\, \sigma_3 = [2F^{-2}(mr^2 - a^2 r \sin^2\theta - ma^2 \cos^2\theta) - iafF^{-2} \cos\theta]\, \theta^1 - $$
$$- (F^{-\frac{1}{2}} \cot\theta - iarF^{-\frac{3}{2}} \sin\theta)(\theta^2 - \theta^3) - iaF^{-1} \cos\theta\ .\ \theta^4\ . \tag{151c}$$

The absence of terms involving θ^3 and θ^4 from (151a) is highly significant. According to (121), $\sigma_1 = - 2\omega^1{}_3$ so we have

$$\sigma_{14} = 0 \implies 0 = - \gamma^1{}_{34} = k^\alpha{}_{|\beta}\, t_\alpha\, k^\beta$$
$$\implies k^\alpha{}_{|\beta}\, k^\beta = \lambda\, k^\alpha\ , \tag{152}$$

i.e. the null vector k^α is tangent to *geodesics*.* Furthermore, these null geodesics are

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* This could, of course, also have been inferred from the absence of $\theta^3 \wedge \theta^4$ in (150a). We can strengthen this result somewhat by noting from (151c) that σ_{34} is pure imaginary. Hence

$$0 = \tfrac{1}{4}\, (\sigma_{34} + \bar{\sigma}_{34}) = - \gamma^1{}_{14} = k^\alpha{}_{|\beta}\, m_\alpha\, k^\beta\ ,$$

so we actually have $k^\alpha{}_{|\beta}\, k^\beta = 0$. Since the contravariant form of k^α is

$$k^\alpha = (k^v,\ k^r,\ k^\theta,\ k^\phi) = (0,\ 1,\ 0,\ 0) = \partial x^\alpha / \partial r$$

we conclude that r is an affine parameter along the null geodesics.

"shear-free" (Section 5.1), because

$$\sigma_{13} = 0 \implies 0 = -\gamma^{1}{}_{33} = k_{\alpha|\beta}\, t^{\alpha}\, t^{\beta} . \tag{153}$$

Similarly, the absence of θ^1, θ^2 from (151b) implies that m^{α} is tangent to a second congruence of shear-free null geodesics.

Now, according to the Goldberg-Sachs theorem (to be discussed in the next chapter), a null vector field in empty space-time is tangent to shear-free geodesics if and only if it is (at least) a double Debever-Penrose vector. Hence $\underset{\sim}{k}$, $\underset{\sim}{m}$ as defined by (149) should each be of this type, and the Kerr metric should be algebraically degenerate Type D.

We now check this directly by computing the curvature 2-forms $\Sigma_m = d\sigma_m - \frac{1}{2}\epsilon_{mpq}\, \sigma^p \wedge \sigma^q$. A straightforward but rather lengthy calculation leads eventually to the simple results

$$\Sigma_1 = C_{12}\, Z^2 , \qquad \Sigma_2 = C_{12}\, Z^1 , \qquad \Sigma_3 = 4\, C_{12}\, Z^3 \tag{154}$$

with

$$C_{12} = 2m/(r - i\, a \cos\theta)^3 . \tag{155}$$

Since terms in $\bar{Z}^n$ are absent from (154), and $\Sigma_m \wedge Z^m = 0$, this confirms that the Kerr space-time is empty [cf. eq. (143)]. According to (146), the form of equations (154) guarantees that the Weyl tensor is Type D with $\underset{\sim}{k}$, $\underset{\sim}{m}$ as principal null directions.

The foregoing results and particularly the intermediate calculations would have been very much simpler for the *Schwarzschild metric* (obtainable by setting a = 0 in the preceding formulae), which would for that reason have served much better for purely illustrative purposes. We have chosen to be more elaborate both because of the intrinsic importance of the Kerr solution, and because (as shown by Kinnersley) it virtually exhausts the class of "interesting" (i.e. asymptotically flat) vacuum metrics of Type D.

Our formulae can be used to illustrate a curious conservation law for Type D vacuum fields discovered some years ago by Ehlers and Sachs. Its physical significance is not at present understood.

Keeping things as simple as possible for a moment, we consider the Schwarzschild field. From (150) with a = 0, we can read off

$$d\theta^1 = 0 , \quad d(r^{-1}\,\theta^1) = r^{-2}\,\theta^1 \wedge \theta^4 ,$$

$$d(r^{-1}\cot\theta \,.\, \theta^2) = - d(r^{-1}\cot\theta \,.\, \theta^3)$$

$$= 2^{-\frac{1}{2}}\, r^{-2}\, \theta^2 \wedge \theta^3 ,$$

so that

$$d[r^{-1}(\theta^1 - 2^{\frac{1}{2}}\cot\theta \,.\, \theta^2)] = r^{-2}\, Z^3 , \tag{156}$$

which therefore has a vanishing exterior differential. Since $C_{12} = 2m/r^3$ for the Schwarzschild field, our result can be written

$$d(C_{12}^{\,2/3}\, Z^3) = 0 . \tag{157}$$

In classical tensor language, (157) expresses the vanishing of the cyclic divergence of the bivector $C_{12}^{\,2/3}\, Z^3{}_{\alpha\beta}$, and (156) shows how this bivector can be written as the curl of a vector potential.

Equation (157) remains valid for the Kerr metric, as one can verify straightforwardly from the formulae (155) and (151) for C_{12} and dZ^3 . To appreciate its role as a conservation law, we integrate over any regular three-dimensional region V_3 to obtain

$$\int_{\partial V_3} C_{12}^{\,2/3}\, Z^3 = \int_{V_3} d(C_{12}^{\,2/3}\, Z^3) = 0 \tag{158}$$

by the generalized Stokes theorem (12). Thus $\int_{S_2} C_{12}^{\,2/3}\, Z^3$, taken over any regular closed 2-space S_2 , is invariant under continuous deformation of S_2 . We must now distinguish two cases, according to whether or not S_2 encloses the singularity of C_{12} at $r = 0$, $\theta = \frac{1}{2}\pi$. In the latter case, it can be contracted continuously to a point, and the integral vanishes. On the other hand, if S_2 does enclose the singularity, the integral is a nonvanishing constant. To obtain its value, we may take a simple limiting form for S_2: $v = \text{const.}$, $r = R = \text{const.}$, with $R \to \infty$. We decompose S_2 into cells $(d\underset{\sim}{x},\, d\underset{\sim}{y})$ defined by the co-ordinate net: $dx^\alpha = \delta_2^\alpha\, d\theta$, $dy^\alpha = \delta_3^\alpha\, d\phi$. Equations (149) give the values of the 1-forms θ^a for a *general* displacement $(dv, dr, d\theta, d\phi)$; from there we can read off

$$\theta^2(d\underset{\sim}{x}) \approx t_\alpha\, dx^\alpha \approx 2^{-\frac{1}{2}}\, R\, d\theta , \qquad \theta^2(d\underset{\sim}{y}) \approx 2^{-\frac{1}{2}}\, i\, R \sin\theta\, d\phi ,$$

while $\theta^3 = \bar{\theta}^2$, and θ^1, θ^4 are of order a for $d\underset{\sim}{x}$ or $d\underset{\sim}{y}$ as arguments. Hence

$$Z^3(d\underset{\sim}{x},\ d\underset{\sim}{y}) \approx -t_{[\alpha}\,\bar{t}_{\beta]}\,dx^\alpha\,dy^\beta$$
$$\approx \tfrac{1}{2}\, i\, R^2 \sin\Theta\, d\theta\, d\phi \qquad (R >> a) ,$$

and, by (155),

$$\int_{S_2} C_{12}^{2/3}\, Z^3 = \lim_{R\to\infty} \iint \frac{(2m)^{2/3}}{R^2}\ \tfrac{1}{2}\, i\, R^2 \sin\theta\, d\theta\, d\phi$$
$$= 2\,\pi\, i\,(2m)^{2/3} \qquad (159)$$

for any S_2 enclosing the Kerr singularity.

4.7. The Ehlers-Sachs Conservation Law

Equation (157) is a general identity for any Type D vacuum metric (the null tetrad is supposed to be oriented so that the two degenerate characteristic null vectors are $\underset{\sim}{k}$ and $\underset{\sim}{m}$). We shall present the proof here, as it provides a good illustration of the power and simplicity of the complex vectorial formalism.

The Bianchi identities (128) for $d\Sigma_3$ read

$$d\Sigma_3 - \Sigma_1 \wedge \sigma_2 + \Sigma_2 \wedge \sigma_1 = 0 . \qquad (160)$$

Now, according to (146) and (130), we have

$$\Sigma_1 = C_{12}\, Z^2 , \qquad \Sigma_2 = C_{12}\, Z^1 , \qquad \Sigma_3 = 4\, C_{12}\, Z^3$$

for a Type D vacuum metric. Also, the equations of structure (125) yield

$$d\, Z^3 = \tfrac{1}{2}\, Z^1 \wedge \sigma_1 - \tfrac{1}{2}\, Z^2 \wedge \sigma_2 .$$

Substituting into (160), we find

$$0 = 4\, d(C_{12}\, Z^3) + 2\, C_{12}\, dZ^3$$
$$= 6\, C_{12}^{1/3}\ d(C_{12}^{2/3}\, Z^3)$$

which is the required result.

The theorem can be extended to any vacuum field of "Type II" - i.e. having two (but not three) coincident Debever-Penrose vectors - though both the statement and proof are now a little more involved. In this case (157) holds if the tetrad is chosen so that $\underset{\sim}{k}$ is the double Debever-Penrose vector (hence $C_{11} = C_{13} = 0$) and so that $C_{23} = 0$ (it is not hard to show that this can always be done for Type II). We now have

$$\Sigma_1 = C_{12} Z^2 , \quad \Sigma_2 = C_{12} Z^1 + C_{22} Z^2 , \quad \Sigma_3 = 4 C_{12} Z^3 .$$

When we substitute into (160), the only difference from the previous case is the appearance of an extra term

$$C_{22} Z^2 \wedge \sigma_1 = C_{22} \sigma_1 \wedge \theta^1 \wedge \theta^2 .$$

But this vanishes for an algebraically degenerate vacuum field by the Goldberg-Sachs theorem ($\sigma_{13} = \sigma_{14} = 0$, see (153)). So our previous conclusion remains valid.

Chapter V: Null Rays and the Goldberg-Sachs Theorem

5.1. Null Geodesic Congruences

We begin with a review (in conventional tensor notation) of the well-known "ray optics" for a congruence of null geodesics developed by Sachs.

Let the congruence be characterized by the equations $x^\alpha = x^\alpha(y^1, y^2, y^3, v)$, where the y's are constant along each ray, and v is an affine parameter, arbitrary up to a linear transformation

$$v \to v' \equiv A(y)v + B(y) \quad . \tag{161}$$

The tangent vector $k^\alpha \equiv \partial x^\alpha(y,v)/\partial v$ satisfies

$$g_{\alpha\beta}\, k^\alpha k^\beta = 0 \,, \qquad \delta k^\alpha/\delta v \equiv k^\alpha{}_{|\beta}\, k^\beta = 0 \,. \tag{162}$$

Let

$$\xi^\alpha \equiv (\partial x^\alpha/\partial y^s)\, dy^s \qquad\qquad (dy^s = \text{const.})$$

be an infinitesimal connection vector joining two given adjacent rays. Then

$$\begin{aligned} \frac{\delta\xi^\alpha}{\delta v} &= \left(\frac{\partial^2 x^\alpha}{\partial v \partial y^s} + \Gamma^\alpha_{\beta\gamma} \frac{\partial x^\beta}{\partial y^s} \frac{\partial x^\gamma}{\partial v}\right) dy^s \\ &= k^\alpha{}_{|\beta}\, \xi^\beta \quad , \end{aligned} \tag{163}$$

which means that a connection vector undergoes Lie-transport along the rays. From (162) and (163)

$$(\delta/\delta v)\,(\xi^\alpha k_\alpha) = 0 \,, \tag{164}$$

so that if we choose $\underset{\sim}{\xi}$ orthogonal to $\underset{\sim}{k}$ at one point, this condition will be maintained along the whole length of the ray.

The propagation equation (163) enables us to relate deformations of the spaces orthogonal to the rays to various derivatives of $\underset{\sim}{k}$. As an example (relevant to the concept of a "trapped surface" in gravitational collapse), the divergence of $\underset{\sim}{k}$ may be interpreted as a rate of dilatation of 2-area in the following manner. At a given point of null ray R_1 construct two infinitesimal space-like vectors $\underset{\sim}{\xi}$, $\underset{\sim}{\eta}$ orthogonal to R_1, and let $A = \frac{1}{2}\, \xi\, \eta \sin\theta$ be the area of the triangle formed by them. Suppose R_2 and R_3 are the null rays which pass through the tips of $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$. As we slide $\underset{\sim}{\xi}$, $\underset{\sim}{\eta}$ forwards along R_1 with their tips remaining on R_2 and R_3 the area A(v) changes according to

$$A^{-1}\ dA/dv \ = \ k^{\mu}{}_{|\mu} \quad . \tag{165}$$

The proof of (165) is particularly easy if we assume that $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ are momentarily orthogonal. (This does not involve any real restriction of generality.) Then

$$(\xi\ \eta\ \sin\ \theta)^{-1}\ (d/dv)(\xi\ \eta\ \sin\ \theta) = \xi^{-1}\ d\xi/dv + \eta^{-1}\ d\eta/dv$$

where we have set $\theta = \frac{\pi}{2}$ on the right. From (163),

$$d(-\ \xi^2)\ /\ dv \ = \ k_{\alpha|\beta}\ \xi^{\alpha}\ \xi^{\beta}$$

and thus

$$A^{-1}\ dA/dv \ = -k_{\alpha|\beta}\ (x^{\alpha}x^{\beta} + \ y^{\alpha}y^{\beta})$$

where $\underset{\sim}{x}$, $\underset{\sim}{y}$ are unit space-like vectors in the directions of $\underset{\sim}{\xi}$, $\underset{\sim}{\eta}$. If we introduce a null tetrad ($\underset{\sim}{k}$, $\underset{\sim}{m}$, $\underset{\sim}{t}$, $\underset{\sim}{\bar{t}}$) with $\underset{\sim}{t}$, $\underset{\sim}{\bar{t}}$ in the plane of $\underset{\sim}{x}$, $\underset{\sim}{y}$, then we can write

$$\begin{aligned} A^{-1}\ dA/dv \ &= \ -\ 2\ k_{\alpha|\beta}\ \bar{t}^{(\alpha}\ t^{\beta)} \\ &= \ k^{\mu}{}_{|\mu} \end{aligned}$$

by virtue of (162) and the completeness relation $g^{\alpha\beta} = 2\ k^{(\alpha}m^{\beta)} - 2\ \bar{t}^{(\alpha}\ t^{\beta)}$.

Following this line of thought, let us decompose (163) with respect to a quite arbitrary field of null tetrads ($\underset{\sim}{k}$, $\underset{\sim}{m}$, $\underset{\sim}{t}$, $\underset{\sim}{\bar{t}}$), subject only to $\underset{\sim}{k}$ pointing along the rays:

$$\begin{aligned} \delta\xi_{\alpha}/\delta v \ = \ k_{[\alpha|\beta]}\ \xi^{\beta} + (k_{(\alpha|\beta)} + k^{\mu}{}_{|\mu}\ \bar{t}_{(\alpha}\ t_{\beta)})\ \xi^{\beta}\ - \\ -\ k^{\mu}{}_{|\mu}\ \bar{t}_{(\alpha}\ t_{\beta)}\ \xi^{\beta} \quad . \end{aligned} \tag{166}$$

The interpretation of the last term as a dilatation has already been considered. The first term on the right is a displacement orthogonal to $\underset{\sim}{\xi}$ which does not affect scalar products of connection vectors and therefore represents a rigid rotation. The second term has a trace-free coefficient and corresponds to a shear.

Now, $\delta\xi^{\alpha}/\delta v$ is orthogonal to k^{α} and therefore lies in the local tangent null 3-flat generated by $\underset{\sim}{k}$, i.e. it has no component parallel to $\underset{\sim}{m}$. Its component parallel to $\underset{\sim}{k}$ could be removed at *one* point of any ray by a change of parameter $v \rightarrow A(y)v$ (i.e., we could arrange that $m^{\alpha}\ k_{\alpha|\beta}\ \xi^{\beta} = 0$ at the point in question). We shall be concerned in the first place with invariant information that can be extracted from (166) considering only *first* derivatives of $\underset{\sim}{k}$ at a *single* point, and for this it will be sufficient to focus on the projection of (166) onto the 2-space of $\underset{\sim}{t}$ and $\bar{\underset{\sim}{t}}$. In two dimensions a skew matrix is completely determined by a single element, and a trace-free symmetric matrix by a pair of real elements or a single complex element. So we can expand the coefficients of (166) in the form

$$k_{[\alpha|\beta]} = -2\, i\, \omega\, t_{[\alpha}\, \bar{t}_{\beta]} + (\ldots)_{[\alpha}\, k_{\beta]} \tag{167a}$$

$$k_{(\alpha|\beta)} + 2\,\theta\, \bar{t}_{(\alpha}\, t_{\beta)} = \sigma\, t_\alpha\, t_\beta + \bar{\sigma}\, \bar{t}_\alpha\, \bar{t}_\beta + (\ldots)_{(\alpha}\, k_{\beta)} \tag{167b}$$

where we have set $\theta \equiv \frac{1}{2} k^\mu{}_{|\mu}$ and the vectors represented by dots are linear combinations of $\underset{\sim}{t}$, $\bar{\underset{\sim}{t}}$ and $\underset{\sim}{k}$ which are not of primary interest at present. The "complex shear" σ is given by

$$\sigma = k_{\alpha|\beta}\, \bar{t}^\alpha\, \bar{t}^\beta \quad . \tag{168a}$$

Also, since

$$\theta = -\, k_{(\alpha|\beta)}\, \bar{t}^\alpha\, t^\beta \quad , \quad i\,\omega = -\, k_{[\alpha|\beta]}\, \bar{t}^\alpha\, t^\beta \tag{168b}$$

it is convenient to introduce a "complex dilatation"

$$\rho \equiv \theta + i\,\omega = -\, k_{\alpha|\beta}\, \bar{t}^\alpha\, t^\beta \quad . \tag{168c}$$

We can then rewrite (166) as

$$\delta\xi^\alpha\, \delta v = [-\,2\,\rho\, t^\alpha\, \bar{t}^\beta + \sigma\, t^\alpha\, t^\beta + \bar{\sigma}\, \bar{t}^\alpha\, \bar{t}^\beta + k^{(\alpha}(\ldots)^{\beta)}]\, \xi_\beta \quad . \tag{171}$$

We now have to consider to what extent the numbers ρ, σ depend on our choice of null tetrad. Tetrad transformations which preserve scalar products of the tetrad vectors and the direction of $\underset{\sim}{k}$ (ray direction) are:

(i) *Real scale transformations* (which can be related to (161)):

$$\underset{\sim}{k} \rightarrow A^{-1}\, \underset{\sim}{k} \,, \qquad \underset{\sim}{m} \rightarrow A\, \underset{\sim}{m} \;, \qquad \underset{\sim}{t} \rightarrow \underset{\sim}{t} \,. \tag{172}$$

According to (168) the effect of these is to multiply ρ and σ by a nonvanishing real vector which can be considered constant along each ray:

$$\rho \rightarrow A\, \rho \,, \qquad \sigma \rightarrow A\, \sigma \quad . \tag{173}$$

(ii) *"Null rotations"*, which turn the 2-flat of $\underset{\sim}{t}$, $\bar{\underset{\sim}{t}}$ into a different space-like 2-flat within the null 3-flat orthogonal to $\underset{\sim}{k}$:

$$\underset{\sim}{k} \to \underset{\sim}{k}\,, \qquad \underset{\sim}{t} \to \underset{\sim}{t} + \lambda\,\underset{\sim}{k} \qquad (\lambda \text{ complex}), \tag{174}$$

($\underset{\sim}{m}$ undergoes a more complicated transformation, not needed for our present purpose). From (162) and (168) we find that ρ and σ are invariant under null rotations.

(iii) *Spatial rotations* of $\underset{\sim}{t}$, $\bar{\underset{\sim}{t}}$ in their own plane:

$$\underset{\sim}{k} \to \underset{\sim}{k}\,, \qquad \underset{\sim}{m} \to \underset{\sim}{m}\,, \qquad \underset{\sim}{t} \to e^{i\phi}\,\underset{\sim}{t}\,, \tag{175}$$

where ϕ is any real scalar field. These transformations leave ρ invariant, but change σ by a phase factor, $\sigma \to \sigma\, e^{-2i\phi}$.

Thus, ρ and σ are arbitrary to the extent of multiplicative real and complex factors A and $A\,e^{-2i\phi}$ respectively ($A \neq 0$). Of course, this freedom could be reduced by imposing further restrictions to pin down the tetrads (e.g. requiring that the tetrad vectors be parallel-propagated along the rays). The important thing to notice is that the *vanishing* of any of the numbers σ, θ or ω expresses an invariant property of the rays. This can also be seen more directly from the formulae

$$\left.\begin{aligned} \theta &= \tfrac{1}{2}\, k^{\mu}{}_{|\mu} \\ \omega^2 = \tfrac{1}{2}\, k_{[\alpha|\beta]}\, k^{[\alpha|\beta]}\,, &\qquad \sigma\bar{\sigma} = \tfrac{1}{2}\, k_{(\alpha|\beta)}\, k^{(\alpha|\beta)} - \theta^2 \end{aligned}\right\} \tag{176}$$

(take squares of (167)) which involve only the vector tangent to the rays. We note in particular that $\omega = 0$ is the necessary and sufficient condition that the rays be orthogonal to null hypersurfaces, since it is equivalent to $k_{[\alpha|\beta}\, k_{\gamma]} = 0$ by (167a).

5.2. Complex Vectorial Ray Analysis

We now reintroduce our numerical notation for the tetrad vectors:

$$\underset{\sim}{e}^{(1)} = \underset{\sim}{e}_{(4)} = \underset{\sim}{k}\,, \quad \underset{\sim}{e}^{(2)} = -\,\underset{\sim}{e}_{(3)} = \underset{\sim}{t}\,, \quad \underset{\sim}{e}^{(3)} = -\,\underset{\sim}{e}_{(2)} = \bar{\underset{\sim}{t}}\,,$$
$$\underset{\sim}{e}^{(4)} = \underset{\sim}{e}_{(1)} = \underset{\sim}{m}\,.$$

The absolute change of $\underset{\sim}{e}^{(a)}$ in an arbitrary displacement $d\underset{\sim}{x}$ is

$$D\,\underset{\sim}{e}^{(a)} = -\,\omega^{a}{}_{b}\,\underset{\sim}{e}^{(b)} = \underset{\sim}{e}^{(b)}\,\gamma^{a}{}_{bc}\,\theta^{c}(d\underset{\sim}{x})\,.$$

For a displacement along the rays, $\underset{\sim}{dx} = \underset{\sim}{e}^{(1)}\ dv$, and

$$\theta^c(\underset{\sim}{dx}) = \underset{\sim}{e}^{(c)}.\underset{\sim}{dx} = g^{c1}\ dv = \delta^{c4}\ dv \ ,$$

$$D\ \underset{\sim}{e}^{(a)} = -\ \underset{\sim}{e}^{(b)}\ \gamma^a{}_{b4}\ dv \qquad (\underset{\sim}{dx} = \underset{\sim}{k}\ dv). \tag{177}$$

The rays are *geodesic* if and only if $D\ \underset{\sim}{e}^{(1)} \propto \underset{\sim}{e}^{(1)}$, i.e.

$$\gamma^1{}_{24} = \gamma^1{}_{34} = 0 \Longleftrightarrow \sigma_{14} = 0 \tag{178}$$

by (121). Further, v is an affine parameter along the geodesic rays if

$$\gamma_{14} = 0 \Longleftrightarrow \sigma_{34} + \bar{\sigma}_{34} = 0 \ . \tag{179}$$

The complex dilatation and shear now appear as two of the Ricci rotation coefficients:

$$\rho \equiv -\ k_{\alpha|\beta}\ \bar{t}^\alpha\ t^\beta = -\gamma^1{}_{23} = \tfrac{1}{2}\ \bar{\sigma}_{12} \ , \tag{180}$$

$$\sigma \equiv k_{\alpha|\beta}\ \bar{t}^\alpha\ \bar{t}^\beta = \gamma^1{}_{22} = -\ \tfrac{1}{2}\ \bar{\sigma}_{13} \ , \tag{181}$$

by (122).

In Section (4.6) we considered a congruence of null rays in Kerr's space-time: these were integral curves of one of the two distinct Debever-Penrose vectors. We can now easily read off the complex dilatation and shear of this congruence. We found [see (151)]

$$2^{-\frac{1}{2}}\sigma_1 = -\ (r - i\ a\cos\theta)^{-1}[i\ a\ F^{-\frac{1}{2}}\sin\theta.\theta^1 + \theta^2] \ .$$

Hence the congruence is geodesic and

$$\rho = 2^{-\frac{1}{2}}\ (r + i\ a\cos\theta)^{-1} \ , \qquad \sigma = 0 \ . \tag{182}$$

Since ρ has an imaginary part, the congruence is not hypersurface-orthogonal unless $a = 0$ (Schwarzschild).

We conclude by stating our main conclusion somewhat more compactly. The 3-form $\sigma_1 \wedge \theta^1 \wedge \theta^2$ vanishes if and only if $\sigma_1 = \sigma_{1a}\ \theta^a$ does *not* involve θ^3 or θ^4. Hence: *the null congruence with tangent* $\underset{\sim}{e}^{(1)}$ *is geodesic and shear-free iff*

$$\sigma_{14} = \sigma_{13} = 0 \Longleftrightarrow \sigma_1 \wedge Z^2 = 0 \ . \tag{183}$$

This condition can also be given a seemingly different geometrical interpretation. The vector fields $\underset{\sim}{k}$, $\underset{\sim}{t}$ are orthogonal to 2-spaces ψ = const., χ = const. (i.e. $\underset{\sim}{k} = \alpha\ \nabla\ \psi + \beta\ \nabla\ \chi$, $\underset{\sim}{t} = \gamma\ \nabla\ \psi + \delta\ \nabla\ \chi$) if and only if

$$k_{[\alpha|\beta}\, k_{\gamma}\, t_{\delta]} = 0 , \qquad t_{[\alpha|\beta}\, t_{\gamma}\, k_{\delta]} = 0 ,$$

or

$$d\theta^1 \wedge \theta^1 \wedge \theta^2 = 0 = d\theta^2 \wedge \theta^1 \wedge \theta^2 . \tag{184}$$

Now,

$$\begin{aligned} d\theta^1 &= -\omega^1{}_3 \wedge \theta^3 + (\text{terms involving} \wedge \theta^1, \wedge \theta^2) \\ &= -\tfrac{1}{2}\,\sigma_1 \wedge \theta^3 + (\qquad \text{ditto} \qquad) , \\ d\theta^2 &= -\tfrac{1}{2}\,\sigma_1 \wedge \theta^4 + (\qquad \text{ditto} \qquad) , \end{aligned}$$

so that (184) is equivalent to $\sigma_1 \wedge Z^2 = 0$. Thus, (183) is the necessary and sufficient condition that $\underset{\sim}{k}$, $\underset{\sim}{t}$ be orthogonal to 2-spaces. For the Kerr example just mentioned, we have from (149),

$$\begin{aligned} 2^{\frac{1}{2}}\, \theta^1 &= -\,d(v - i\, a \cos\theta) + i\, a \sin^2\theta\, d\psi , \\ \theta^2 + i\, a\, F^{-\frac{1}{2}} \sin\theta . \theta^1 &= 2^{-\frac{1}{2}}\, F^{\frac{1}{2}} \sin\theta\, d\psi , \end{aligned}$$

where $\psi = \ln \tan \tfrac{1}{2}\theta + i\phi$. In this case the orthogonal 2-spaces are the complex surfaces ψ = const., $v - i\, a \cos\theta$ = const.

5.3. Robinson's Theorem

There is yet another way of saying that space-time admits a null geodesic shear-free congruence with tangent $\underset{\sim}{k}$. An equivalent condition is this: space-time admits a field of null bivectors which satisfy Maxwell's source-free equations and which have $\underset{\sim}{k}$ as propagation vector.

Let us begin by showing that the first condition is a consequence of the second. We are given a real bivector field $F^{\alpha\beta}$ satisfying

$$\begin{aligned} F_{\alpha\beta}\, F^{\alpha\beta} &= F_{\alpha\beta}\, \overset{*}{F}{}^{\alpha\beta} = 0 , \\ F_{[\alpha\beta|\gamma]} &= F^{\alpha\beta}{}_{|\beta} = 0 . \end{aligned}$$

The associated self-dual bivector satisfies the equivalent conditions

$$F^{(+)}_{\alpha\beta}\, F^{\alpha\beta}_{(+)} = 0 , \qquad F^{(+)}_{[\alpha\beta|\gamma]} = 0 . \tag{185}$$

By a rotation and rescaling of axes in $\mathcal{L}_3$ we can bring $Z^2{}_{\alpha\beta}$ into coincidence with $F^{(+)}_{\alpha\beta}$

(cf. p. 33), and Maxwell's equations (185) now reduce to

$$dZ^2 = 0 \ .$$

According to the equations of structure (125),

$$dZ^2 = \sigma_1 \wedge Z^3 + \tfrac{1}{2}\,\sigma_3 \wedge Z^2 \ . \tag{186}$$

Taking wedge products of both sides with θ^1 and θ^2 to eliminate the last term, we find

$$\sigma_1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^1 = 0 = \sigma_1 \wedge \theta^1 \wedge \theta^4 \wedge \theta^2 \ ,$$

$$\text{i.e.} \quad \sigma_{14} = \sigma_{13} = 0 \ , \tag{187}$$

which are the conditions that $\underset{\sim}{k}$ be tangent to shear-free null geodesics.

To prove the reverse implication, assume that (187) is given. We then obtain from (186),

$$\begin{aligned} dZ^2 &= \sigma_{11}\,\theta^1 \wedge (-\tfrac{1}{2}\,\theta^2 \wedge \theta^3) + \sigma_{12}\,\theta^2 \wedge (\tfrac{1}{2}\,\theta^1 \wedge \theta^4) \\ &\quad + \tfrac{1}{2}\,\sigma_{33}\,\theta^3 \wedge Z^2 + \tfrac{1}{2}\,\sigma_{34}\,\theta^4 \wedge Z^2 \\ &= \psi \wedge Z^2 \ , \end{aligned} \tag{188}$$

where we have defined the 1-form $\psi \equiv \psi_a\,\theta^a \equiv \tfrac{1}{2}(\sigma_{33} - \sigma_{11})\,\theta^3 + \tfrac{1}{2}(\sigma_{34} - \sigma_{12})\,\theta^4$. Taking exterior differentials of (188),

$$\begin{aligned} 0 &= d\psi \wedge Z^2 - \psi \wedge (\psi \wedge Z^2) \\ &= d\psi \wedge Z^2 \ . \end{aligned} \tag{189}$$

Now,

$$\begin{aligned} d\psi &= d\psi_a \wedge \theta^a + \psi_a\,d\theta^a \\ &= \psi_{a|b}\,\theta^b \wedge \theta^a - \psi_a\,\omega^a{}_b \wedge \theta^b \\ &= (\psi_{4|3} - \psi_{3|4})\,\theta^3 \wedge \theta^4 - \tfrac{1}{4}\,\psi_3(\sigma_3 - \bar{\sigma}_3) \wedge \theta^3 - \\ &\quad - \tfrac{1}{2}\,\psi_4\,\bar{\sigma}_2 \wedge \theta^3 - \tfrac{1}{4}\,\psi_4(\sigma_3 + \bar{\sigma}_3) \wedge \theta^4 + \\ &\quad + (\text{terms involving } \wedge\,\theta^1, \wedge\,\theta^2) \ . \end{aligned}$$

Hence (189) yields

$$\psi_{4|3} - \psi_{3|4} + \tfrac{1}{4}\,\psi_3(\sigma_{34} - \bar{\sigma}_{34}) + \tfrac{1}{2}\,\psi_4\,\bar{\sigma}_{24} - \tfrac{1}{4}\,\psi_4(\sigma_{33} + \bar{\sigma}_{33}) = 0 \ . \tag{190}$$

We have written $\phi_{|a}$ for the directional derivative of the scalar ϕ in the tetrad direction $\underset{\sim}{e}_{(a)}$:

$$\phi_{|a} \equiv e_{(a)}{}^{\alpha} (\partial_\alpha \phi) \ . \tag{191}$$

Notice the commutation rule for this operation:

$$\begin{aligned} \phi_{|ab} - \phi_{|ba} &= 2 (\partial_\alpha \phi \cdot e_{[(a)}{}^{\alpha})_{|b]} \\ &= 2 \phi_{|c} \ \gamma^{c}{}_{[ab]} \ . \end{aligned} \tag{192}$$

Returning now to (188), if we can rewrite this as

$$d(e^{-\phi} Z^2) = 0$$

with an appropriate scalar ϕ, our proof will be complete, since the bivector $e^{-\phi} Z^2_{\alpha\beta}$ would then satisfy (185). All that remains is to show that a function ϕ exists satisfying

$$\phi_{|3} = \psi_3 \qquad\qquad \phi_{|4} = \psi_4 \ .$$

The integrability condition of this pair of equations can be read off from

$$\begin{aligned} \phi_{|34} - \phi_{|43} &= 2 \phi_{|a} \gamma^{a}{}_{[43]} \\ &= - \tfrac{1}{4} \phi_{|3} (\sigma_{34} - \bar{\sigma}_{34}) - \tfrac{1}{2} \phi_{|4} \sigma_{24} + \\ &\quad + \tfrac{1}{2} \phi_{|3} \bar{\sigma}_{13} + \tfrac{1}{4} \phi_{|4} (\sigma_{33} + \bar{\sigma}_{33}) \ , \end{aligned}$$

and is satisfied by virtue of (190) and (187).

5.4. Principal Null Congruence in Algebraically Special Vacuum Field

We proceed next to a proof of the first (and easier) half of the Goldberg-Sachs theorem. This states: if a (non-flat) vacuum space-time is algebraically special, with $\underset{\sim}{k}$ as the repeated principal null vector, then the null curves tangent to k are geodesic and shear-free.

According to (145) and (130), we have

$$\Sigma_1 = C_{12} Z^2 \ , \qquad \Sigma_3 = C_{32} Z^2 + 4 C_{12} Z^3$$

for an algebraically special vacuum field with repeated null vector $\underset{\sim}{k}$. From the Bianchi identity (128) for $d\Sigma_1$

$$d(C_{12} Z^2) = - \tfrac{1}{2} \sigma_1 \wedge \Sigma_3 + \tfrac{1}{2} \sigma_3 \wedge \Sigma_1 \ ,$$

and simplifying this with the aid of expression (186) for dZ^2, we quickly find

$$d\, C_{12} \wedge Z^2 = -3\, C_{12}\, \sigma_1 \wedge Z^3 - \tfrac{1}{2}\, C_{23}\, \sigma_1 \wedge Z^2 \quad . \tag{193}$$

Since $Z^2 \wedge \theta^1 = Z^2 \wedge \theta^2 = 0$, two terms in this equation can be eliminated by taking wedge products with θ^1 or θ^2. This yields

$$C_{12}\, \sigma_1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^1 = 0 = C_{12}\, \sigma_1 \wedge \theta^1 \wedge \theta^4 \wedge \theta^2 \quad .$$

If $C_{12} \neq 0$, we can immediately deduce the geodesic shear-free condition $\sigma_{14} = \sigma_{13} = 0$. If $C_{12} = 0$, but $C_{23} \neq 0$, we can infer $\sigma_1 \wedge Z^2 = 0$ from (193), and this is again the geodesic shear-free condition. The remaining case $C_{12} = C_{23} = 0$, $C_{22} \neq 0$ is easily dealt with by examining the Bianchi identity for $d\Sigma_2$.

5.5. Shear-free Null Rays in Vacuum

Let us now suppose that a vacuum space-time admits a geodesic shear-free null congruence with tangent $\underset{\sim}{k}$, so we are given

$$R_{\alpha\beta} = 0 \ , \qquad \sigma_1 \wedge Z^2 = 0 \quad . \tag{194}$$

We wish to show that $C_{11} = C_{13} = 0$, i.e. that space-time is algebraically special with $\underset{\sim}{k}$ as repeated principal null vector. This is the second half of the Goldberg-Sachs theorem.

From the second equations of structure (127),

$$\Sigma_1 = d\,\sigma_1 - \tfrac{1}{2}\,\sigma_3 \wedge \sigma_1 \quad .$$

Taking wedge products with Z^2, and noting the second of (194),

$$\begin{aligned} \Sigma_1 \wedge Z^2 &= d\,\sigma_1 \wedge Z^2 \\ &= d\,(\sigma_1 \wedge Z^2) + \sigma_1 \wedge d\, Z^2 \ , \end{aligned}$$

$$\text{i.e.} \quad C_{1p}\, Z^p \wedge Z^2 = 0 \quad \text{by (186)} \quad .$$

This means $C_{11} = 0$, i.e. $\underset{\sim}{k}$ is a principal null vector of the Weyl tensor. (So far, we have made no use of the vacuum condition, cf. (130) and (131).)

We consider next the Bianchi identity for Σ_1 :

$$d(C_{12}\, Z^2 + C_{13}\, Z^3) = -\tfrac{1}{2}\,\sigma_1 \wedge \Sigma_3 + \tfrac{1}{2}\,\sigma_3 \wedge \Sigma_1 \quad .$$

Since we wish to concentrate on C_{13}, we eliminate the term involving dC_{12} by taking wedge products with θ^1 and with θ^2. This completely removes the term $d(C_{12}\, Z^2)$, because

$$d\, Z^2 \wedge \theta^1 = 0 = d\, Z^2 \wedge \theta^2$$

by virtue of (186) and (194) [cf. (184)]. In this way we arrive at the two equations

$$C_{13|4} = C_{13}\,(2\,\sigma_{12} + \tfrac{1}{2}\,\sigma_{34}) \,, \qquad (195a)$$

$$C_{13|3} = C_{13}\,(2\,\sigma_{11} + \tfrac{1}{2}\,\sigma_{33}) \,. \qquad (195b)$$

It is now a straightforward matter to show that the only solution of these equations is the trivial one. Using (192), we can in fact reduce the compatibility condition of (195) precisely to $C_{13} = 0$, and this completes the proof.

CHAPTER VI: SOME FURTHER DEVELOPMENTS

6.1. TENSOR-VALUED DIFFERENTIAL FORMS

While of unrivalled compactness and elegance, Cartan's calculus is essentially limited to skew objects and cannot altogether displace tensor analysis. However, it is possible to create an attractive synthesis which combines the advantages of the two systems of notation in the framework of a calculus of tensor-valued differential forms.

For example, with a tensor $\phi^{\alpha}{}_{\mu\nu}$, which is (effectively) skew in its two covariant indices, we can associate a "vector-valued 2-form"

$$\phi^{\alpha} \quad = \quad 2! \ \phi^{\alpha}{}_{\mu\nu} \, dx^{[\mu} dy^{\nu]} \quad . \tag{196}$$

Since $\partial_{[\lambda} \ \phi^{\alpha}{}_{\mu\nu]}$ is not a tensor, the 3-form $d\phi^{\alpha}$ will not transform like a vector. We therefore introduce a "covariant differential" $D\phi^{\alpha}$ defined as follows. Let us adopt a co-ordinate basis (Section 2.4) in which

$$\theta^{\alpha} \quad = \quad dx^{\alpha} \ , \qquad \omega^{\alpha}{}_{\beta} \quad = \quad \Gamma^{\alpha}{}_{\beta\gamma} \ dx^{\gamma} \ . \tag{197}$$

Then we define

$$D \ \phi^{\alpha} \quad = \quad d\phi^{\alpha} + \omega^{\alpha}{}_{\beta} \wedge \phi^{\beta} \quad . \tag{198}$$

This gives (adding a term which vanishes for a symmetric affine connexion)

$$\begin{aligned} D \ \phi^{\alpha} \quad &= \quad 3! \ (\partial_{\lambda} \ \phi^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\beta\lambda} \ \phi^{\beta}{}_{\mu\nu} \ - \\ &\qquad\qquad - 2 \ \Gamma^{\beta}{}_{\lambda[\nu} \ \phi^{\alpha}{}_{\mu]\beta}) \ dx^{[\lambda} \ dy^{\mu} \ dz^{\nu]} \\ &= \quad 3! \ \phi^{\alpha}{}_{\mu\nu|\lambda} \ dx^{[\lambda} \ dy^{\mu} \ dz^{\nu]} \end{aligned} \tag{199}$$

which is plainly a vector. Similarly, the covariant differential of a covector-valued n-form ϕ_{α} is

$$D \ \phi_{\alpha} \quad = \quad d\phi_{\alpha} - \omega^{\beta}{}_{\alpha} \wedge \phi_{\beta} \quad . \tag{200}$$

The expressions for arbitrary tensor-valued forms are built up from these in an obvious way. A tensor-valued 0-form $A^{\alpha}{}_{\beta}$ is just an ordinary tensor, and $DA^{\alpha}{}_{\beta} = A^{\alpha}{}_{\beta|\gamma} \ dx^{\gamma}$ is its absolute differential.

D is a linear differential operator which satisfies Leibnitz rules analogous to (6) and (7). However, it differs from the exterior derivative in that D^2 is not null. For a vector-valued n-form ϕ^{α}, we have

$$D^2\,\phi^\alpha = d(D\,\phi^\alpha) + \omega^\alpha{}_\beta \wedge D\phi^\beta$$

$$= d^2\phi^\alpha + d\omega^\alpha{}_\beta \wedge \phi^\beta - \omega^\alpha{}_\beta \wedge d\phi^\beta + \omega^\alpha{}_\beta \wedge D\phi^\beta$$

$$= \Omega^\alpha{}_\beta \wedge \phi^\beta \qquad (201)$$

where

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\mu \wedge \omega^\mu{}_\beta \qquad (202)$$

is the tensor-valued curvature 2-form. For a 0-form A^α , the left side of (201) is

$$D^2\,A^\alpha = 2!\,A^\alpha{}_{|\mu\lambda}\,dx^{[\lambda}\,dy^{\mu]}$$

$$= R^\alpha{}_{\beta\lambda\mu}\,A^\beta\,dx^{[\lambda}\,dy^{\mu]} \quad . \qquad (203)$$

Thus (201) is an elegant generalization of the Ricci commutation relations.

Since $d\theta^\alpha = d^2x^\alpha = 0$, we see that $D\theta^\alpha$ represents the *torsion*, and vanishes for a symmetric affine connexion:

$$D\theta^\alpha = \omega^\alpha{}_\beta \wedge \theta^\beta = \Gamma^\alpha{}_{\beta\gamma}\,\theta^\gamma \wedge \theta^\beta = 0 \,. \qquad (204)$$

Hence

$$0 = D^2\theta^\alpha = \Omega^\alpha{}_\beta \wedge \theta^\beta \,, \qquad (205)$$

which is the cyclic identity for the curvature (Section 2.7). The Bianchi identity (45), derivable from (202) and $d^2\omega^\alpha{}_\beta \equiv 0$, now takes the remarkably simple form

$$D\,\Omega^\alpha{}_\beta = 0 \,. \qquad (206)$$

Further details, including the generalization to spaces with torsion, are in Trautman's paper cited in the bibliography.

6.2. Lie Differentiation

In the absence of an affine connexion, there are just two invariant differential operations which generate new forms from old: exterior differentiation and Lie differentiation. Lie derivatives of forms play a central role in the theory of systems of differential equations, Hamiltonian mechanics and the geometry of congruences and null spaces. We shall begin with a few words about the definition of the Lie derivative, because the text-book treatments do not always bring out the simple geometrical meaning of this concept.

Let us consider any "geometrical object" $\Phi(x)$ (indices suppressed), for example a (relative) tensor, a (tensor-valued) differential form or an affine connexion. The defining property of a geometrical object is that, under each co-ordinate transformation $x \to x'$ it transforms *linearly* (this could be generalized, cf. ref. 31):

$$\Phi'(x') = L(x',x)\,\Phi(x) \tag{207}$$

and the transformations L form a group:

$$L(x'',x')\,L(x',x) = L(x'',x); \quad L(x',x)\,L(x,x') = 1\,. \tag{208}$$

Although it is not actually necessary, let us suppose, to be perfectly definite, that a single co-ordinate chart x covers the domain of interest to us, and let us *fix* this chart. Let a smooth transformation $x \to \bar{x}(x)$ be given which associates with each point x in a domain D a *new point* $\bar{x}$ in a domain $\bar{D}$. With each geometrical object Φ at x we can associate a *new object* $\bar{\Phi}$ at $\bar{x}$ by the rule

$$\bar{\Phi}(\bar{x}) = L(\bar{x},x)\,\Phi(x) \tag{209}$$

The essential property of $\bar{\Phi}$ is that it transforms in the same way as Φ under co-ordinate transformations.

[To prove this, we simply note that the rule for forming $\bar{\Phi}$ in a new co-ordinate system x' is

$$\bar{\Phi}'(\bar{x}') = L(\bar{x}',x')\,\Phi'(x')\,. \tag{210}$$

Using the transformation law (207) of the original object Φ, and then (208), (209), we obtain at once

$$\bar{\Phi}'(\bar{x}') = L(\bar{x}',\bar{x})\,\bar{\Phi}(\bar{x}) \tag{211}$$

as the transformation law of the new object.]

Since $\bar{\Phi}(\bar{x})$ and $\Phi(\bar{x})$ are objects of the same type (relative tensors, tensor-valued differential forms or affine connexions) their difference, evaluated at the *same* point,

$$\Delta\,\Phi(\bar{x}) = \Phi(\bar{x}) - \bar{\Phi}(\bar{x}) \tag{212}$$

(called the *Lie difference*) has tensorial character.

If ϕ is a scalar then $L = 1$ and $\bar{\phi}(\bar{x}) = \phi(x)$, i.e. the new scalar $\bar{\phi}$ inherits at $\bar{x}$ the value ϕ had at x. The Lie difference $\Delta\,\varphi(\bar{x}) = \phi(\bar{x}) - \phi(x)$ is just the increment of the original scalar field in the displacement $x \to \bar{x}$.

The infinitesimal vector dx^α connecting two points x and x + dx transforms into

$$\overline{dx^\alpha} = (\partial\bar{x}^\alpha/\partial x^\beta)\, dx^\beta = d(\bar{x}^\alpha) \quad , \tag{213}$$

which is the vector connecting the transformed points $\bar{x}$ and $\overline{x + dx}$.

If the definition of the original object Φ is extended from x to $\bar{x}$ in such a way that $\Phi(\bar{x}) = \bar{\Phi}(\bar{x})$, i.e. $\Delta\,\Phi = 0$, then Φ is said to have been *Lie-transported* from x to $\bar{x}$. Thus, a vector which connects x and x + dx will, if Lie-transported to $\bar{x}$, connect the transformed points $\bar{x}$ and $\overline{x + dx}$. (As the mathematician would put it, every point transformation induces a natural mapping of tangent vectors.)

Suppose a vector field $\xi^\alpha(x)$ is given. Then translations along the integral curves

$$\frac{dx^\alpha}{dt} = \xi^\alpha \tag{214}$$

generate a continuous group of point transformations $x^\alpha(t_1) \rightarrow x^\alpha(t_2)$. We define the *Lie derivative* of a field Φ with respect to $\vec{\xi}$ by

$$L_{\vec{\xi}}\,\Phi(x) = \lim_{\Delta t \rightarrow 0} \Delta\,\Phi(\bar{x})/\Delta t \tag{215}$$

where

$$x^\alpha = x^\alpha(t), \quad \bar{x}^\alpha = x^\alpha(t + \Delta t) \ . \tag{216}$$

If $\Phi(x)$ is an affine connexion or a (relative) tensor, then $L_{\vec{\xi}}\,\Phi$ is a tensor or relative tensor of the same weight. The Lie operator L_ξ is a "derivation", i.e. a linear operator which satisfies Leibnitz's rule. For a scalar (or scalar-valued form) ϕ, $L_{\vec{\xi}}\,\phi$ is just the ordinary derivative along the curves:

$$L_{\vec{\xi}}\,\phi = \xi^\lambda \partial_\lambda \phi = d\phi/dt \ . \tag{217}$$

If $L_{\vec{\xi}}\,\Phi = 0$, Φ is Lie-transported along the congruence. A Lie-transported scalar stays constant on each curve. A Lie-transported connecting vector dx^α slides along two neighbouring curves of the congruence as if on rails, its two end-points moving equal parameter-distances along the two curves. A co-vector A_α is Lie-transported if $A_\alpha dx^\alpha$ stays constant for all Lie-transported connecting vectors dx^α . And so on.

For a relative tensor (or tensor-valued form) $\phi^\alpha{}_\beta$ of weight w, the transformation law

$$\bar{\phi}^\alpha{}_\beta(\bar{x}) = \phi^a{}_b(x)\, \frac{\partial\bar{x}^\alpha}{\partial x^a}\, \frac{\partial x^b}{\partial\bar{x}^\beta} \left| \frac{\partial x}{\partial\bar{x}} \right|^w \tag{218}$$

yields

$$\bar{\phi}^{\alpha}{}_{\beta}(\bar{x}) = \phi^{\alpha}{}_{\beta}(x) + \Delta t(\phi^{\mu}{}_{\beta}\xi^{\alpha}_{,\mu} - \phi^{\alpha}{}_{\mu}\xi^{\mu}_{,\beta} - w\,\phi^{\alpha}{}_{\beta}\xi^{\mu}_{,\mu}) \tag{219}$$

for the infinitesimal transformation $\bar{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}\Delta t$, where the commas denote partial differentiation. Hence

$$L_{\vec{\xi}}\,\phi^{\alpha}{}_{\beta} = \phi^{\alpha}{}_{\beta,\mu}\,\xi^{\mu} - \phi^{\mu}{}_{\beta}\xi^{\alpha}_{,\mu} + \phi^{\alpha}{}_{\mu}\xi^{\mu}_{,\beta} + w\,\phi^{\alpha}{}_{\beta}\xi^{\mu}_{,\mu}\,. \tag{220}$$

The expression on the right-hand side must be a relative tensor, although that is not obvious from its appearance. If the manifold has a symmetric affine connexion, the partial derivatives can be replaced by covariant derivatives. (This plainly makes no difference at the origin of a geodesic co-ordinate system, and must therefore be valid generally, since the objects concerned are tensors.) From (220), the general rule for constructing Lie derivatives will be apparent. Note the particular results

$$L_{\vec{\xi}}\,\xi^{\alpha} = 0\,, \quad L_{\vec{\xi}}\,g_{\alpha\beta} = 2\,\xi_{(\alpha|\beta)}\,, \quad L_{\vec{\xi}}\,\xi_{\alpha} = 2\,\xi_{[\alpha|\beta]}\,\xi^{\beta}\,. \tag{221}$$

For a connexion, the transformation law

$$\bar{\Gamma}^{\alpha}_{\beta\gamma}(\bar{x}) = \frac{\partial x^{b}}{\partial \bar{x}^{\beta}}\,\frac{\partial x^{c}}{\partial \bar{x}^{\gamma}}\left(\frac{\partial \bar{x}^{\alpha}}{\partial x^{a}}\,\Gamma^{a}_{bc} + \frac{\partial^{2}\bar{x}^{\alpha}}{\partial x^{b}\partial x^{c}}\right) \tag{222}$$

yields, after some work (assuming Γ symmetric),

$$\begin{aligned} L_{\vec{\xi}}\,\Gamma^{\alpha}_{\beta\gamma} &= \xi^{\alpha}{}_{|\beta\gamma} + \xi^{\mu}\,R^{\alpha}{}_{\mu\beta\gamma} \\ &= \xi^{\alpha}{}_{|(\beta\gamma)}\,. \end{aligned} \tag{223}$$

For any vector field $\xi^{\alpha} = dx^{\alpha}/dt$ we can always find "adapted" or "co-moving" co-ordinates $(x^{0}, x^{1}, \ldots)$ such that $x^{0} = t$ and the remaining co-ordinates are constant along each integral curve, i.e. $\xi^{\lambda}\partial_{\lambda}x^{\mu} = \delta^{\mu}_{0}$. In these co-ordinates the components $\xi^{\mu} = \delta^{\mu}_{0}$ are numerical constants and the Lie derivative of any relative tensor reduces to its partial derivative along the curves:

$$L_{\vec{\xi}}\,\Phi = \partial\Phi|\partial x^{0} \qquad \text{if} \quad \xi^{\mu} = \delta^{\mu}_{0}\,. \tag{224}$$

The Lie derivative of a relative tensor is in fact completely defined by the two statements (i) that it is a relative tensor of the same type and weight, and (ii) that it reduces to (224) in adapted co-ordinates.

We finally mention some simple illustrations of these ideas.

Let us express the condition that a metric $g_{\alpha\beta}$ is conformally invariant along a congruence (e.g. the Robertson-Walker metrics). This means that the squared length $g_{\alpha\beta}\, dx^\alpha dx^\beta$ of a Lie-transported connecting vector changes by a factor $(1 + \lambda(x)\Delta t)$ (independent of the direction of dx^α) under a displacement Δt along the curves, i.e.

$$L_{\vec{\xi}}\,(g_{\alpha\beta}\; dx^\alpha\; dx^\beta) \;=\; \lambda\; g_{\alpha\beta}\; dx^\alpha\; dx^\beta$$

if $L_{\vec{\xi}}\,(dx^\alpha) = 0$. Hence

$$L_{\vec{\xi}}\; g_{\alpha\beta} \;=\; \lambda\; g_{\alpha\beta} \tag{225a}$$

or

$$\xi_{(\alpha|\beta)} \;=\; \tfrac{1}{2}\,\lambda\; g_{\alpha\beta}\;. \tag{225b}$$

A vector ξ^α satisfying (225) is called a conformal Killing vector.

The condition that a hydrodynamical flow with 4-velocity u^α be rigid is that the orthogonal distance $(g_{\alpha\beta} - u_\alpha u_\beta)\; dx^\alpha dx^\beta$ between adjacent world-lines stays constant. This yields

$$L_u\;(g_{\alpha\beta} - u_\alpha u_\beta) \;=\; 0 \tag{226a}$$

or

$$u_{(\alpha|\beta)} - \dot{u}_{(\alpha}\; u_{\beta)} \;=\; 0 \tag{226b}$$

where $\dot{u}_\alpha = u_{\alpha|\mu}\; u^\mu$ is the acceleration vector. This is the familiar condition for a Born-type rigid motion.

In an n-dimensional manifold, let V be an n-dimensional region whose points are carried by Lie transport along a congruence with tangent $\vec{\xi}$ (e.g. Hamiltonian evolution of a set of points in phase space). Let

$$\omega \;=\; \sqrt{\gamma}\; dx' \wedge \ldots \wedge dx^n \tag{227}$$

be a volume measure on V. The condition that the flow be volume preserving is

$$L_{\vec{\xi}}\;\sqrt{\gamma} \;=\; 0\;, \tag{228a}$$

or, by (220), since $\sqrt{\gamma}$ is a scalar density (w = 1),

$$\partial_\mu(\sqrt{\gamma}\;\xi^\mu) \;=\; 0. \tag{228b}$$

6.3. Differential Forms, Exterior Derivatives and Lie Derivatives

We consider a general manifold (without affine connexion). The interaction of the operations of Lie and exterior differentiation with inner and exterior multiplication leads to a number of general identities which have found wide application in mechanics, gauge theories and general relativity.

We begin by defining the *inner product* $\vec{\xi}.\alpha$ (also written $\vec{\xi} \lrcorner \alpha$) of a contravariant vector ξ^λ and a differential form α. If α is the m-form

$$\alpha \;=\; m!\; \alpha_{\lambda_1 \ldots \lambda_m}\; dx^{\lambda_1} \ldots dz^{\lambda_m} \qquad (229)$$

associated with the completely skew tensor $\alpha_{\lambda_1 \ldots \lambda_m}$, then $\vec{\xi}.\alpha$ is the (m-1) form

$$\vec{\xi}.\alpha \;=\; m!\; \xi^{\lambda_1}\; \alpha_{\lambda_1 \ldots \lambda_m}\; dy^{[\lambda_2} \ldots dz^{\lambda_m]} \qquad (230)$$

which is m times the (m-1) form associated with the skew tensor $\xi^{\lambda_1}\; \alpha_{\lambda_1 \ldots \lambda_m}$.

Since

$$(m+n)\; \alpha_{[\lambda_1 \ldots \lambda_m} \beta_{\mu_1 \ldots \mu_n]} \;=\; m\; \alpha_{\lambda_1 [\lambda_2 \ldots \lambda_m} \beta_{\mu_1 \ldots \mu_n]} \;+$$

$$+\; n(-1)^m \alpha_{[\lambda_2 \ldots \lambda_m \mu_1} \beta_{|\lambda_1| \mu_2 \ldots \mu_n]} \qquad (231)$$

(the bars around λ_1 indicate that it is excluded from the skew-symmetrization) we see at once that

$$\vec{\xi}.\;(\alpha \wedge \beta) \;=\; (\vec{\xi}.\alpha) \wedge \beta \;+\; (-1)^m\; \alpha \wedge (\vec{\xi}.\beta)\;. \qquad (232)$$

In particular, if f is a 0-form (scalar) then $\vec{\xi}.df$ is again a scalar, given by

$$\vec{\xi}.df \;=\; \xi^\lambda\; \partial_\lambda f, \qquad (233)$$

so that

$$L_{\vec{\xi}}\, f \;=\; \vec{\xi}.df\;. \qquad (234)$$

The extension of (234) to a form of degree ≥ 1 is the very important "Cartan identity"

$$L_{\vec{\xi}}\, \alpha \;=\; \vec{\xi}.d\alpha + d(\vec{\xi}.\alpha)\;. \qquad (235)$$

Here, and in future, it is to be understood that the vector arguments of the forms are Lie-transported.

[To prove this result, we note that, in adapted co-ordinates ($\xi^\mu = \delta^\mu_o$),

$$\frac{1}{m}\,\vec{\xi}.\alpha \;=\; (m-1)!\;\; \alpha_{o\lambda_2\ldots\lambda_m}\, dy^{\lambda_2} \ldots dz^{\lambda_m} \;. \tag{236}$$

The exterior derivative of this (m - 1) form is the m-form

$$\frac{1}{m}\, d(\vec{\xi}.\alpha) \;=\; m!\;\partial_{[\lambda_1}\alpha_{|o|\lambda_2\ldots\lambda_m]}\, dx^{\lambda_1} \ldots dz^{\lambda_m} \;. \tag{237}$$

On the other hand,

$$\begin{aligned}\vec{\xi}.d\alpha &= (m+1)!\;\;\partial_{[o}\,\alpha_{\lambda_1\ldots\lambda_m]}\, dx^{\lambda_1} \ldots dz^{\lambda_m} \\ &= m!\;\{\partial_o\,\alpha_{\lambda_1\ldots\lambda_m} - m\,\partial_{[\lambda_1}\alpha_{|o|\lambda_2\ldots\lambda_m]}\}\, dx^{\lambda_1} \ldots dz^{\lambda_m} \;.\end{aligned} \tag{238}$$

The first term is $L_\xi\,\alpha$ in adapted co-ordinates; the second term cancels with (237).]

It is easy to check in adapted co-ordinates that Lie differentiation and exterior differentiation commute:

$$L_{\vec{\xi}}\,(d\alpha) \;=\; d(L_{\vec{\xi}}\,\alpha) \;. \tag{239}$$

Further results easily proved along similar lines are

$$L_{\vec{\xi}}\,(\alpha \wedge \beta) \;=\; (L_{\vec{\xi}}\,\alpha) \wedge \beta \;+\; \alpha \wedge (L_{\vec{\xi}}\,\beta) \;; \tag{240}$$

$$L_{f\vec{\xi}}\,\alpha \;=\; f\,L_\xi\,\alpha \;+\; df \wedge (\vec{\xi}.\alpha) \;; \tag{241}$$

$$L_{\vec{\xi}}\,(\vec{\eta}.\alpha) \;=\; [\vec{\xi},\vec{\eta}]\,.\,\alpha + \vec{\eta}\cdot(L_{\vec{\xi}}\,\alpha) \;. \tag{242}$$

Here, the *Lie bracket* $[\vec{\xi},\vec{\eta}]$ is the contravariant vector

$$\begin{aligned}[\vec{\xi},\vec{\eta}]^\alpha &= \xi^\beta\,\partial_\beta\,\eta^\alpha - \eta^\beta\,\partial_\beta\,\xi^\alpha \\ &= L_{\vec{\xi}}\,\eta^\alpha \;=\; -\,L_{\vec{\eta}}\,\xi^\alpha \;.\end{aligned} \tag{243}$$

Finally, we note the simple commutation property: the commutator of two Lie derivatives $L_{\vec{\eta}}$, $L_{\vec{\xi}}$ (operating on any geometrical object) is equal to the Lie derivative with respect to $[\vec{\xi},\vec{\eta}]$:

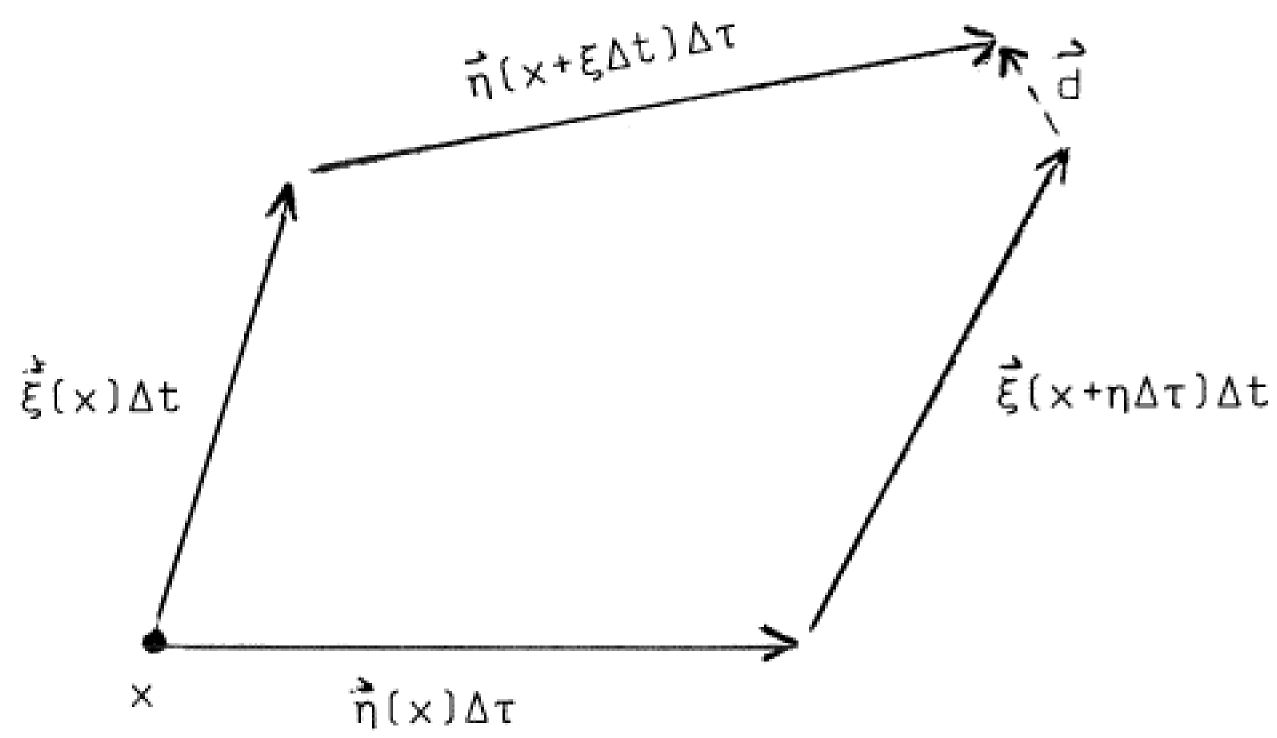
$$L_{\vec{\xi}}\;L_{\vec{\eta}} \;-\; L_{\vec{\eta}}\;L_{\vec{\xi}} \;=\; L_{[\vec{\xi},\vec{\eta}]} \;. \tag{244}$$

This is easily established for scalars and for vectors $\vec{\zeta}$, where it merely expresses the Jacobi identity for the Lie bracket:

$$[\vec{\xi}, [\vec{\eta},\vec{\zeta}]] + [\vec{\eta}, [\vec{\zeta},\vec{\xi}]] + [\vec{\zeta}, [\vec{\xi},\vec{\eta}]] = 0 \ . \tag{245}$$

Its validity for more general objects then follows by Leibnitz's rule. The geometrical meaning of (244) can be understood from the diagram in which the extra little displacement $\vec{d}$ needed to close the four-sided figure is

$$\vec{d} = [\vec{\xi},\vec{\eta}]\ \Delta t\ \Delta\tau \ . \tag{246}$$



6.4. Symplectic Mechanics

As a simple illustration of how the preceding formulas are applied, we give a quick sketch of Hamiltonian mechanics from the geometrical point of view.

Hamilton's equations are

$$\frac{dq^a}{dt} \overset{*}{=} \frac{\partial H}{\partial p_a} \ , \qquad \frac{dp_a}{dt} \overset{*}{=} -\frac{\partial H}{\partial q^a} \quad (a = 1, \ldots , n) \ . \tag{247}$$

(Equations valid only in canonical co-ordinates are adorned with a star.) We relabel the co-ordinates as

$$(x^1, \ldots , x^{2n}) \overset{*}{=} (q^1, p_1, \ldots , q^n, p_n) \ . \tag{248}$$

Then (247) can be rewritten

$$\xi^\alpha = \frac{dx^\alpha}{dt} = \varepsilon^{\alpha\beta} \frac{\partial H}{\partial x^\beta} \quad (\alpha, \beta = 1, \ldots , 2n) \tag{249}$$

where the skew matrix $\varepsilon^{\alpha\beta}$ has the form (in canonical co-ordinates) of a diagonal sequence of 2 x 2 matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

i.e. the only nonvanishing components are

$$\varepsilon^{j,j+1} = -\varepsilon^{j+1,j} \stackrel{*}{=} 1 . \tag{250}$$

Hamilton's equations in the form (249) are evidently valid in *arbitrary* co-ordinates x^α provided $\varepsilon^{\alpha\beta}$ is considered to transform as a tensor.

The Poisson bracket of two functions f(q,p), g(q,p) is defined to be

$$[f,g] = \varepsilon^{\alpha\beta}(\partial_\alpha f)(\partial_\beta g) \tag{251}$$

This is manifestly invariant under *arbitrary* co-ordinate transformations. The normally tedious problem of proving invariance of [f,g] under canonical transformations is here reduced to a triviality. In any set of canonical co-ordinates, (251) of course assumes the standard form

$$[f,g] \stackrel{*}{=} \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a} \tag{252}$$

Define $-\varepsilon_{\alpha\beta}$ as the skew matrix inverse to $\varepsilon^{\alpha\beta}$:

$$\varepsilon^{\alpha\gamma}\varepsilon_{\beta\gamma} = \delta^\alpha_\beta . \tag{253}$$

In canonical co-ordinates,

$$\varepsilon_{\alpha\beta} \stackrel{*}{=} \varepsilon^{\alpha\beta} . \tag{254}$$

The 2-form $\varepsilon_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$ has a vanishing exterior derivative, a result immediately evident in canonical co-ordinates, where $d\varepsilon_{\alpha\beta} \stackrel{*}{=} 0$. Hence it can be written as the exterior differential of a 1-form ω:

$$d\omega = \varepsilon_{\alpha\beta}\, dx^\alpha \wedge dx^\beta . \tag{255}$$

The forms $d\omega$ and ω are called the symplectic 2-form and presymplectic 1-form of the dynamical system. In canonical co-ordinates (255) is immediately integrable:

$$\omega \stackrel{*}{=} \varepsilon_{\alpha\beta}\, x^\alpha dx^\beta + df \tag{256}$$

where f is an arbitrary scalar.

Hamilton's equations (249) can be re-expressed as

$$\xi^\alpha \varepsilon_{\alpha\beta} = \partial H/\partial x^\beta , \tag{257}$$

which is equivalent to

$$\vec{\xi}.d\omega = dH . \tag{258}$$

From (234) and (257),

$$\frac{dH}{dt} = \mathcal{L}_{\vec{\xi}} H = \vec{\xi}.dH = 0 \tag{259}$$

With the aid of the Cartan identity (235), we find

$$\begin{aligned} \mathcal{L}_{\vec{\xi}}\, d\omega &= \vec{\xi}.d^2\omega + d(\vec{\xi}.d\omega) \\ &= d^2H = 0\ . \end{aligned} \tag{260}$$

It should be noted that this result (unlike (259)) is not immediately obvious, since arbitrary canonical co-ordinates, in which $\varepsilon_{\alpha\beta}$ are constants, do not necessarily coincide with adapted co-ordinates, in which $\mathcal{L}_{\vec{\xi}}$ reduces to $\partial/\partial t$.

From (260) it follows that the various integral invariants of Poincaré,

$$I_2 = \tfrac{1}{2} \int_{\Sigma_2} d\omega \overset{*}{=} \int dq^a \wedge dp_a \tag{261}$$

$$I_4 = \tfrac{1}{4} \int_{\Sigma_4} d\omega \wedge d\omega \tag{262}$$

$$\begin{aligned} I_{2n} &= \frac{1}{2^n} \int_{\Sigma_{2n}} (\wedge\ d\omega)^n \\ &\overset{*}{=} dq^1 \wedge dp_1 \wedge \ldots \wedge dq^n \wedge dp_n \end{aligned} \tag{263}$$

are not only invariant under canonical transformations, but also constants of the motion:

$$\frac{dI_{2k}}{dt} = \mathcal{L}_{\vec{\xi}}\ I_{2k} = 0\ . \tag{264}$$

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Summary of Basic Formulae

Wedge product and exterior differential

If $\theta \equiv A_\alpha dx^\alpha$, $\phi \equiv 2!\ F_{\alpha\beta}\ dx^{[\alpha} dy^{\beta]}$,

then

$$\phi \wedge \theta \equiv 3!\ F_{\alpha\beta}\ A_\gamma\ dx^{[\alpha} dy^\beta dz^{\gamma]} ,$$

$$d\phi \equiv 3!\ \partial_\alpha\ F_{\beta\gamma}\ dx^{[\alpha} dy^\beta dz^{\gamma]} .$$

If α, β are forms of degree a, b:

$$\alpha \wedge \beta = (-1)^{ab}\ \beta \wedge \alpha , \tag{3}$$

$$d\ (\alpha \wedge \beta) = d\ \alpha \wedge \beta + (-1)^a\ \alpha \wedge d\ \beta . \tag{7}$$

For any form Ω, $d^2\Omega = 0$. (8)

Riemannian Geometry

$$\text{Basis } \underset{\sim}{e}_{(a)} , \quad g_{ab} \equiv \underset{\sim}{e}_{(a)} \cdot \underset{\sim}{e}_{(b)} . \tag{14}$$

Ricci rotation coefficients:

$$\gamma^a{}_{bc} \equiv -\ e^{(a)}{}_{\beta|\gamma}\ e_{(b)}{}^\beta\ e_{(c)}{}^\gamma = +\ e_{(b)}{}^\beta{}_{|\gamma}\ e^{(a)}{}_\beta\ e_{(c)}{}^\gamma .$$

Basic 1-forms:

$$\theta^a \equiv e^{(a)}{}_\alpha\ dx^\alpha . \tag{18}$$

Connection 1-forms:

$$\omega^a{}_b = \gamma^a{}_{bc}\ \theta^c . \tag{27}$$

Covariant differentials of basis vectors:

$$D\ \underset{\sim}{e}^{(a)} = -\ \omega^a{}_b\ \underset{\sim}{e}^{(b)} , \qquad D\ \underset{\sim}{e}_{(b)} = \underset{\sim}{e}_{(a)}\ \omega^a{}_b . \tag{28}$$

Riemann tensor defined by

$$A_{\beta|\gamma\delta} - A_{\beta|\delta\gamma} = A_\alpha\ R^\alpha{}_{\beta\gamma\delta} .$$

Ricci tensor

$$R_{\alpha\beta} \equiv R^\mu{}_{\alpha\beta\mu} .$$

Curvature 2-forms:

$$\Omega^a{}_b \equiv \tfrac{1}{2}\ R^a{}_{bcd}\ \theta^c \wedge \theta^d . \tag{42}$$

Cartan's equations;

$$ds^2 = g_{ab}\,\theta^a\,\theta^b\,, \tag{21}$$

$$d\,g_{ab} = \omega_{ab} + \omega_{ba}\,, \tag{30}$$

$$d\,\theta^a = -\,\omega^a{}_b \wedge \theta^b\,, \tag{32}$$

$$\Omega^a{}_b = d\,\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\,. \tag{43}$$

Cyclic identity:

$$\Omega^a{}_b \wedge \theta^b = 0\,. \tag{46}$$

Bianchi identity:

$$d\,\Omega^a{}_b = \Omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \Omega^c{}_b\,. \tag{45}$$

Complex Vectorial Formalism

Conventions: Greek indices (range 1-4) are co-ordinate indices;

Latin a,b, ... ℓ (range 1-4) are tetrad indices;

Latin m,n,p, ... (range 1-3) are complex vectorial indices.

Signature of space-time metric (-,-,-,+) .

Null tetrad:

$$\underset{\sim}{e}^{(1)} = \underset{\sim}{e}_{(4)} = \underset{\sim}{k}\,,\quad \underset{\sim}{e}^{(4)} = \underset{\sim}{e}_{(1)} = \underset{\sim}{m}\,,\quad \underset{\sim}{e}^{(2)} = -\,\underset{\sim}{e}_{(3)} = \underset{\sim}{t}\,,$$

$$\underset{\sim}{e}^{(3)} = -\underset{\sim}{e}_{(2)} = \bar{\underset{\sim}{t}};\quad \underset{\sim}{k}\cdot\underset{\sim}{m} = -\,\underset{\sim}{t}\cdot\bar{\underset{\sim}{t}} = 1 \quad \text{(other products zero).}$$

$$ds^2 = g_{ab}\,\theta^a\,\theta^b = 2(\theta^1\,\theta^4 - \theta^2\,\theta^3)\,, \tag{85}$$

$$[g_{ab}] = [g^{ab}] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\,. \tag{82}$$

N.B. If $T_{ab\ldots}$ are tetrad components of any (real or complex) tensor, then $\bar{T}_{ab\ldots}$ means: take complex conjugates *and* interchange tetrad indices 2, 3.

Alternating pseudo-tensor $\eta_{\alpha\beta\gamma\delta} \equiv (-g)^{\frac{1}{2}}\,\varepsilon_{\alpha\beta\gamma\delta}$; tetrad components $\eta_{abcd} = \eta^{abcd} = i\,\varepsilon_{abcd}$. Dual: $F^*_{ab} \equiv \frac{1}{2}\,\eta_{abcd}\,F^{cd}$.

Self-dual basis for 2-forms:

$$\left.\begin{aligned} Z^1 &\equiv \theta^3 \wedge \theta^4 , \quad Z^2 \equiv \theta^1 \wedge \theta^2 , \quad Z^3 \equiv \tfrac{1}{2}(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3) \\ \bar{Z}^1 &= \theta^2 \wedge \theta^4 , \quad \bar{Z}^2 = \theta^1 \wedge \theta^3 , \quad \bar{Z}^3 = \tfrac{1}{2}(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) \end{aligned}\right\} \qquad (93)$$

so that $Z^2{}_{\alpha\beta} = 2\, k_{[\alpha}\, t_{\beta]}$, etc.

Metric tensor of complex 3-space:

$$\gamma^{mn} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\tfrac{1}{2} \end{bmatrix} \; ; \qquad \gamma_{mn} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad (101)$$

Complex Vectorial Algebra

$$\tfrac{1}{2} Z^m{}_{\alpha\beta}\, Z^{n\alpha\beta} = \gamma^{mn} , \qquad Z^m{}_{\alpha\beta}\, \bar{Z}^{n\alpha\beta} = 0 , \qquad (103)$$

$$2\, g^{\mu\nu} Z^m{}_{\alpha\mu}\, Z^n{}_{\beta\nu} = \gamma^{mn} g_{\alpha\beta} + \varepsilon^{mnp} Z_{p\alpha\beta} , \qquad (104)$$

$$g^{\mu\nu} Z^m{}_{\alpha\mu}\, \bar{Z}^n{}_{\beta\nu} \equiv Y^{m\bar{n}}{}_{\alpha\beta} , \qquad (138)$$

where $Y^{m\bar{n}}{}_{\alpha\beta}$ is hermitian in m, n, symmetric and trace-free in α, β [see (140) and (141)].

$$Z^m \wedge Z^n = - i\, \gamma^{mn}\, \eta , \qquad Z^m \wedge \bar{Z}^n = 0 , \qquad (131)$$

where

$$\eta \equiv (-g)^{\frac{1}{2}}\, \varepsilon_{\alpha\beta\gamma\delta}\, dx^\alpha dy^\beta dz^\gamma dw^\delta = i\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 . \qquad (132)$$

Expansion of arbitrary real bivector F_{ab} in terms of basis Z^m:

$$\tfrac{1}{2} F_{ab}\, \theta^a \wedge \theta^b = F_m Z^m + \bar{F}_m \bar{Z}^m , \qquad (95)$$

where $F_m = \tfrac{1}{2} F_{ab} Z_m{}^{ab}$; explicitly:

$$\left.\begin{aligned} F_1 &= F_{34} , \quad F_2 = F_{12} , \quad F_3 = F_{14} - F_{23} \\ \bar{F}_1 &= F_{24} , \quad \bar{F}_2 = F_{13} , \quad \bar{F}_3 = F_{14} + F_{23} \end{aligned}\right\} . \qquad (96)$$

$$\gamma^{mn} F_m G_n = \tfrac{1}{8} F^{(+)}_{\alpha\beta}\, G^{\alpha\beta}_{(+)} = \tfrac{1}{4} (F_{\alpha\beta} G^{\alpha\beta} - i\, F^*_{\alpha\beta} G^{\alpha\beta}) . \qquad (102)$$

If $F_{ab} \longleftrightarrow F_m$ then $F^*_{ab} \longleftrightarrow i\,F_m$. (97)

Complex Vectorial Calculus

Connection 1-forms $\sigma_p = \sigma_{pa}\,\theta^a$ and curvature 2-forms Σ_p in $\mathcal{C}_3$:

$$\omega_{ab}\,\theta^a \wedge \theta^b = \sigma_p\, Z^p + \bar{\sigma}_p\, \bar{Z}^p \qquad (120)$$

$$\Omega_{ab}\,\theta^a \wedge \theta^b = \Sigma_p\, Z^p + \bar{\Sigma}_p\, \bar{Z}^p \ . \qquad (123)$$

Explicit formulae $\sigma_p = \omega_{ab}\, Z_p{}^{ab}$, $\Sigma_p = \Omega_{ab}\, Z_p{}^{ab}$ are the same as (96) apart from a factor 2, i.e. $\sigma_1 = 2\,\omega_{34}$, $\Sigma_2 = 2\,\Omega_{12}$ etc. The inverse relations are

$$-\omega^1{}_1 = \omega^4{}_4 = \tfrac{1}{4}\,(\sigma_3 + \bar{\sigma}_3)\ , \qquad -\omega^2{}_2 = \omega^3{}_3 = \tfrac{1}{4}\,(\sigma_3 - \bar{\sigma}_3)\ ,$$

$$\omega^1{}_2 = \omega^3{}_4 = -\tfrac{1}{2}\,\bar{\sigma}_1\ , \qquad \omega^1{}_3 = \omega^2{}_4 = -\tfrac{1}{2}\,\sigma_1\ ,$$

$$\omega^4{}_2 = \omega^3{}_1 = \tfrac{1}{2}\,\sigma_2\ , \qquad \omega^4{}_3 = \omega^2{}_1 = \tfrac{1}{2}\,\bar{\sigma}_2\ ,$$

$$\omega^1{}_4 = \omega^4{}_1 = \omega^2{}_3 = \omega^3{}_2 = 0\ , \qquad (121)$$

with analogous expressions for $\Omega^a{}_b$ in terms of Σ_p .

N.B.: $\bar{\sigma}_{pa}$ is defined by $\overline{\sigma_p} = \overline{\sigma_{pa}}\,\bar{\theta}^a$ (see (122)).

Decomposition of Σ_p into Weyl tensor, trace-free Ricci tensor and curvature invariant:

$$\Sigma_p = (C_{pq} - \tfrac{1}{6}\,R\gamma_{pq})\, Z^q + E_{p\bar{q}}\,\bar{Z}^q \ , \qquad (130)$$

where

$$C_{abcd} - i\,C^*_{abcd} = C_{mn}\, Z^m{}_{ab}\, Z^n{}_{cd}\ , \qquad (105)$$

$$C_{abcd} + i\,C^*_{abcd} = \bar{C}_{mn}\, \bar{Z}^m{}_{ab}\, \bar{Z}^n{}_{cd}\ ,$$

$$R_{ab} - \tfrac{1}{4}\,g_{ab}\,R = \gamma^{p\bar{q}}{}_{ab}\, E_{p\bar{q}}\ , \qquad (137)$$

$$E_{p\bar{q}} = \bar{E}_{q\bar{p}} = \gamma_{p\bar{q}}{}^{ab}(R_{ab} - \tfrac{1}{4}\,g_{ab}\,R)\ . \qquad (142)$$

$$\gamma^{pq}\, C_{pq} = 0 \iff C_{33} = 4\,C_{12}\ . \qquad (106)$$

First equations of structure:

$$dZ^p = \tfrac{1}{2}\, \varepsilon^{pmn}\, \sigma_m \wedge Z_n \;; \tag{125}$$

explicitly:

$$dZ^1 = -\tfrac{1}{2}\, \sigma_3 \wedge Z^1 - \sigma_2 \wedge Z^3 \;; \quad dZ^2 = \tfrac{1}{2}\, \sigma_3 \wedge Z^2 + \sigma_1 \wedge Z^3 \;;$$

$$dZ^3 = \tfrac{1}{2}\, \sigma_1 \wedge Z^1 - \tfrac{1}{2}\, \sigma_2 \wedge Z^2 \quad .$$

Second equations of structure:

$$\Sigma_p = d\, \sigma_p - \tfrac{1}{2}\, \varepsilon_{pmn}\, \sigma^m \wedge \sigma^n \;; \tag{127}$$

explicitly:

$$\Sigma_1 = d\sigma_1 - \tfrac{1}{2}\, \sigma_3 \wedge \sigma_1 \;; \qquad \Sigma_2 = d\sigma_2 + \tfrac{1}{2}\, \sigma_3 \wedge \sigma_2 \;;$$

$$\Sigma_3 = d\sigma_3 + \sigma_1 \wedge \sigma_2 \quad .$$

Bianchi identities:

$$d\, \Sigma_p = \varepsilon_{pmn}\, \sigma^m \wedge \Sigma^n \;; \tag{128}$$

explicitly:

$$d\Sigma_1 = \tfrac{1}{2}\, \sigma_3 \wedge \Sigma_1 - \tfrac{1}{2}\, \sigma_1 \wedge \Sigma_3 \;,$$

$$d\Sigma_2 = \tfrac{1}{2}\, \sigma_2 \wedge \Sigma_3 - \tfrac{1}{2}\, \sigma_3 \wedge \Sigma_2 \;,$$

$$d\Sigma_3 = \sigma_2 \wedge \Sigma_1 - \sigma_1 \wedge \Sigma_2 \quad .$$

Cyclic identity:

$$\Sigma_m \wedge Z_n = \Sigma_n \wedge Z_m \quad . \tag{129}$$

Commutation rule for directional derivatives

$\phi_{|a} \equiv e^{\alpha}_{(a)}(\partial_\alpha \phi)$:

$$\phi_{|ab} - \phi_{|ba} = 2\, \phi_{|c}\, \gamma^c{}_{[ab]} \quad . \tag{192}$$

Complex Vectorial Dictionary

The null curves with tangent $\underset{\sim}{k}$ are *geodesics* iff

$$k_{\alpha|\beta}\, t^\alpha\, k^\beta = 0 \iff \sigma_{14} = 0 \quad . \tag{178}$$

The parameter v defined by $k^\alpha = \partial x^\alpha/\partial v$ is an *affine* parameter along the $\underset{\sim}{k}$-geodesics iff

$$k_{\alpha|\beta}\, m^\alpha\, k^\beta = 0 \Longleftrightarrow \sigma_{34} + \bar{\sigma}_{34} = 0 \; . \tag{179}$$

The tetrad vectors are parallel-transported along the $\underset{\sim}{k}$-geodesics iff

$$e^{(a)}_{\alpha|\beta}\, k^\beta = 0 \Longleftrightarrow \sigma_{p4} = 0 \; .$$

Complex shear:

$$\sigma \equiv k_{\alpha|\beta}\, \bar{t}^\alpha\, \bar{t}^\beta = -\tfrac{1}{2}\,\bar{\sigma}_{13} \; . \tag{181}$$

Complex dilatation:

$$\rho \equiv -k_{\alpha|\beta}\, \bar{t}^\alpha\, t^\beta = \tfrac{1}{2}\,\bar{\sigma}_{12} \; , \tag{180}$$

$$\rho = \theta + i\omega \; , \quad \theta = \tfrac{1}{2}\, k^\mu{}_{|\mu} \; , \quad \omega^2 = \tfrac{1}{2}\, k_{[\alpha|\beta]}\, k^{[\alpha|\beta]} \; . \tag{176}$$

Tetrad vector $\underset{\sim}{k}$ is a principal null vector of the Riemann tensor iff

$$C_{11} = 0 \Longleftrightarrow \Sigma_1 \wedge Z^2 = 0 \; . \tag{144}$$

The Weyl tensor is algebraically special, with $\underset{\sim}{k}$ as repeated principal null vector iff

$$C_{11} = C_{13} = 0 \Longleftrightarrow \Sigma_1 \wedge Z^2 = \Sigma_1 \wedge Z^3 = 0 \; . \tag{145}$$

Vacuum field equations:

$$R_{\alpha\beta} = 0 \Longleftrightarrow \Sigma_p = C_{pq}\, Z^q \Longleftrightarrow \Sigma_p \wedge \bar{Z}^q = 0 = \Sigma_p \wedge Z^p \; . \tag{143}$$

Relationship to Newman-Penrose Formalism

Newman-Penrose spin coefficients:

$$\kappa \equiv k_{\mu|\nu}\, t^\mu\, k^\nu = \tfrac{1}{2}\,\sigma_{14} \; ; \quad \pi \equiv -m_{\mu|\nu}\, \bar{t}^\mu\, k^\nu = \tfrac{1}{2}\,\sigma_{24} \; ;$$

$$\varepsilon \equiv \tfrac{1}{2}(k_{\mu|\nu}\, m^\mu - t_{\mu|\nu}\, \bar{t}^\mu) k^\nu = -\tfrac{1}{4}\,\sigma_{34} \; ;$$

$$\rho \equiv k_{\mu|\nu}\, t^\mu\, \bar{t}^\nu = -\tfrac{1}{2}\,\sigma_{12} \; ; \quad \lambda \equiv -m_{\mu|\nu}\, \bar{t}^\mu\, \bar{t}^\nu = -\tfrac{1}{2}\,\sigma_{22} \; ;$$

$$\alpha \equiv \tfrac{1}{2}(k_{\mu|\nu}\, m^\mu - t_{\mu|\nu}\, \bar{t}^\mu)\bar{t}^\nu = \tfrac{1}{4}\,\sigma_{32} \; ;$$

$$\sigma \equiv k_{\mu|\nu}\, t^\mu\, t^\nu = -\tfrac{1}{2}\,\sigma_{13} \; ; \quad \mu \equiv -m_{\mu|\nu}\, \bar{t}^\mu\, t^\nu = -\tfrac{1}{2}\,\sigma_{23} \; ;$$

$$\beta \equiv \tfrac{1}{2}(k_{\mu|\nu}\, m^{\mu} - t_{\mu|\nu}\, \bar{t}^{\mu})t^{\nu} = \tfrac{1}{4}\, \sigma_{33} \;;$$

$$\tau \equiv k_{\mu|\nu}\, t^{\mu}\, m^{\nu} = \tfrac{1}{2}\, \sigma_{11} \;; \qquad \nu \equiv - m_{\mu|\nu}\, \bar{t}^{\mu}\, m^{\nu} = \tfrac{1}{2}\, \sigma_{21} \;;$$

$$\gamma \equiv \tfrac{1}{2}(k_{\mu|\nu}\, m^{\mu} - t_{\mu|\nu}\, \bar{t}^{\mu})t^{\nu} = - \tfrac{1}{4}\, \sigma_{31} \;.$$

Note: ρ, σ as defined by Newman and Penrose are the conjugates of the definitions (180), (181).

Physical components of Weyl tensor (Newman-Penrose components Ψ_A in terms of C_{mn}):

$$\Psi_0 \equiv - C_{\alpha\beta\gamma\delta}\, k^{\alpha}\, t^{\beta}\, k^{\gamma}\, t^{\delta} = \tfrac{1}{2}\, C_{11}$$

$$\Psi_1 \equiv - C_{\alpha\beta\gamma\delta}\, k^{\alpha}\, m^{\beta}\, k^{\gamma}\, t^{\delta} = - \tfrac{1}{4}\, C_{13}$$

$$\Psi_2 \equiv - C_{\alpha\beta\gamma\delta}\, \bar{t}^{\alpha}\, m^{\beta}\, k^{\gamma}\, t^{\delta} = - \tfrac{1}{2}\, C_{12}$$

$$\Psi_3 \equiv - C_{\alpha\beta\gamma\delta}\, \bar{t}^{\alpha}\, m^{\beta}\, k^{\gamma}\, m^{\delta} = \tfrac{1}{4}\, C_{23}$$

$$\Psi_4 \equiv - C_{\alpha\beta\gamma\delta}\, \bar{t}^{\alpha}\, m^{\beta}\, \bar{t}^{\gamma}\, m^{\delta} = - \tfrac{1}{2}\, C_{22} \;.$$