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Creators	Delgadillo-Blando, Rodrigo and O'Connor, Denjoe and Ydri, Badis
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Geometry in transition: A model of emergent geometry

Rodrigo Delgadillo-Blando^{a,b}, Denjoe O'Connor^a, Badis Ydri^c

^a*School of Theoretical Physics, DIAS, Dublin, Ireland.*

^b*Departamento de Física, CINVESTAV-IPN, México DF. México.*

^c*Institut für Physik, Humboldt-Universität zu Berlin, D-12489 Berlin, Germany.*

We study a three matrix model with global $SO(3)$ symmetry containing at most quartic powers of the matrices. We find an exotic line of discontinuous transitions with a jump in the entropy, characteristic of a 1st order transition, yet with divergent critical fluctuations and a divergent specific heat with critical exponent $\alpha = 1/2$. The low temperature phase is a geometrical one with gauge fields fluctuating on a round sphere. As the temperature increased the sphere evaporates in a transition to a pure matrix phase with no background geometrical structure. Both the geometry and gauge fields are determined dynamically. It is not difficult to invent higher dimensional models with essentially similar phenomenology. The model presents an appealing picture of a geometrical phase emerging as the system cools and suggests a scenario for the emergence of geometry in the early universe.

Our understanding of the fundamental laws of physics has evolved to a very geometrical one. However, we still have very little insight into the origins of geometry itself. This situation has been undergoing a significant evolution in recent years and it now seems possible to understand classical geometry as an emergent concept. The notion of geometry as an emergent concept is not new, see for example [1] for an inspiring discussion and [2, 3] for some recent ideas. We examine such a phenomenon in the context of noncommutative geometry [4] emerging from matrix models, by studying a surprisingly rich three matrix model [5, 6, 7]. The matrix geometry that emerges here has received attention as an alternative setting for the regularization of field theories [8, 9, 10, 11] and as the configurations of $D0$ branes in string theory [12, 13]. Here, however, the geometry emerges as the system cools, much as a Bose condensate or superfluid emerges as a collective phenomenon at low temperatures. There is no background geometry in the high temperature phase. The simplicity of the model, in this study, allows for a detailed examination of such an exotic transition. We suspect the asymmetrical nature of the transition may be generic to this phenomenon.

We consider the most general single trace Euclidean action (or energy) functional for a three matrix model invariant under global $SO(3)$ transformations containing no higher than the fourth power of the matrices. This model is surprisingly rich and in the infinite matrix limit can exhibit many phases as the parameters are tuned. We find that generically the model has two clearly distinct phases, one geometrical the other a matrix phase. Small fluctuations in the geometrical phase are those of a Yang-Mills theory and a scalar field around a ground state corresponding to a round two-sphere. In the matrix phase there is no background spacetime geometry and the fluctuations are those of the matrix entries around zero. In this note we focus on the subset of parameter space where, in the large matrix limit, the gauge group is Abelian.

For finite but large N , at low temperature, the model exhibits fluctuations around a fuzzy sphere [14]. In the infinite N limit the macroscopic geometry becomes classical. As the temperature is increased it undergoes a transition where the entropy jumps, yet the model has critical fluctuations and a divergent specific heat. As this critical coupling is approached the fuzzy sphere radius expands to a critical radius and the sphere evaporates. The neighbourhood of the critical point exhibits all the standard symptoms of a continuous 2nd order transition, such as large scale fluctuations, critical slowing down (of the Monte Carlo routine) and is characterized by a specific heat exponent which we argue is $\alpha = 1/2$, a value consistent with our numerical simulations. In the high temperature (strong coupling) phase the model is essentially a zero dimensional Yang-Mills theory.

By studying the eigenvalues of operators in the theory we establish that, in the matrix phase, the matrices D_a are characterised by continuous eigenvalue distributions which undergo a transition to a point spectrum characteristic of the fuzzy sphere phase. The point spectrum is consistent with $D_a = L_a/R$ where L_a are $su(2)$ angular momentum generators in the irreducible representation given by the matrix size and R is the radius of the fuzzy sphere. The full model received an initial study in [7] while a simpler version invariant under $D_a \rightarrow D_a + \Lambda \mathbf{1}$ arises naturally as the configuration of $D0$ branes in the large k limit of a boundary Wess-Zumino-Novikov-Witten model [13] and has been studied numerically in [5]. The interpretation here is novel, as are the results on the entropy and critical behaviour and the extension to the full model.

Let D_a , $a = 1, 2, 3$, be three $N \times N$ Hermitian matrices and let us consider the action

$$\begin{aligned} S &= S_0 + V, \\ S_0 &= \frac{\tilde{\alpha}^4}{N} \left[-\frac{1}{4} \text{Tr}[D_a, D_b]^2 + \frac{2i}{3} \epsilon_{abc} \text{Tr} D_a D_b D_c \right] \\ V &= \frac{m^2 \tilde{\alpha}^4}{N} \left[-\text{Tr} D_a^2 + \frac{1}{2c_2} \text{Tr}(D_a^2)^2 \right], \end{aligned} \quad (1)$$

so that $\tilde{\alpha}^4 = \beta$ plays the rôle of inverse temperature and we fix $c_2 = \sum_a L_a^2 = (N^2 - 1)/4$ in this study [23]. The absolute minimum of the action is given by $D_a = L_a$. Expanding around this configuration, with $D_a = L_a + A_a$, yields a noncommutative Yang-Mills action with field strength

$$F_{ab} = i[L_a, A_b] - i[L_b, A_a] + \epsilon_{abc} A_c + i[A_a, A_b] \quad (2)$$

and gauge coupling $g^2 = 1/\tilde{\alpha}^4$. As written the gauge field includes a scalar field,

$$\Phi = \frac{1}{2} (x_a A_a + A_a x_a + \frac{A_a^2}{\sqrt{c_2}}), \quad (3)$$

as the component of the gauge field normal to the sphere when viewed as embedded in \mathbf{R}^3 where $x_a = L_a/\sqrt{c_2}$.

In the large N limit, taken with $\tilde{\alpha}$ and m held fixed the action for small fluctuations becomes that of a $U(1)$ gauge field coupled to a scalar field defined on a background commutative two-sphere [6]. For large m the scalar field is not excited. The model with $m = 0$ appears as a low energy limit of string theory [13].

One can see the background geometry as that of a fuzzy sphere [14] by noting that the x_a satisfy

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad [x_a, x_b] = \frac{i}{\sqrt{c_2}} \epsilon_{abc} x_c, \quad (4)$$

and the algebra generated by products of the x_a is the algebra of all $N \times N$ matrices with complex coefficients. The geometry enters through the Laplacian [9]

$$\hat{\mathcal{L}}^2 \cdot = [L_a, [L_a, \cdot]], \quad (5)$$

which has the same spectrum as the round Laplacian on the commutative sphere, but cut off at a maximum angular momentum $L = N - 1$. The fluctuations of the scalar have this Laplacian as kinetic term.

The ground state is found by considering the configuration $D_a = \phi L_a$ where ϕ plays the role of the inverse radius of the sphere. The effective potential [6, 15] obtained by integrating out fluctuations around this background is given, in the large N limit, by

$$\frac{V_{\text{eff}}}{2c_2} = \tilde{\alpha}^4 \left[\frac{\phi^4}{4} - \frac{\phi^3}{3} + m^2 \left(\frac{\phi^4}{4} - \frac{\phi^2}{2} \right) \right] + \log \phi^2 \quad (6)$$

It is unbounded from below at the origin, but, for sufficiently large $\tilde{\alpha}$, exhibits a local minimum near $\phi = 1$ which disappears for $\tilde{\alpha} < \tilde{\alpha}_*$. The critical curve $\tilde{\alpha}_*$ is determined from the point at which the real roots of $\partial V_{\text{eff}}/\partial \phi = 0$ merge and disappear. This occurs at

$$\phi_* = \frac{\tilde{\phi}_*}{1 + m^2} = \frac{3}{8(1 + m^2)} \left[1 + \sqrt{1 + \frac{32t}{9}} \right] \quad (7)$$

and gives the critical curve

$$2(1 + m^2)^3 \frac{(8/3)^3}{\tilde{\alpha}_*^4} = 1 + \frac{16t}{3} \left(1 + \frac{8t}{9} \right) + \left(1 + \frac{32t}{9} \right)^{3/2} \quad (8)$$

where $t = m^2(1 + m^2)$. This expression interpolates between $\tilde{\alpha}_* = (8/3)^{3/4}$ with $\phi_* = 3/4$ for $m = 0$ and $\tilde{\alpha}_* = (8/m^2)^{1/4}$, with $\phi_* = 1/\sqrt{2}$ for large m . Thus, as the system is heated, the radius, $R = \phi^{-1}$, expands from $R = 1$, at large $\tilde{\alpha}$ to $R_*^{-1} = \phi_*$ at $\tilde{\alpha}_*$. When $\tilde{\alpha} < \tilde{\alpha}_*$ the fuzzy sphere solution no longer exists, and, in effect, the fuzzy sphere evaporates as the radius approaches R_* .

Furthermore, defining $\mathcal{S} = \langle S \rangle / N^2$, we have

$$\mathcal{S} = \frac{3}{4} + \frac{i\tilde{\alpha}^4}{12N^2} \epsilon_{abc} \text{Tr}(D_a D_b D_c) - \frac{m^2 \tilde{\alpha}^4}{2N^2} \text{Tr}(D_a D_a) \quad (9)$$

which for the fuzzy sphere phase becomes

$$\mathcal{S} = \frac{3}{4} - \tilde{\alpha}^4 \left(\frac{1}{24} \phi^3 - \frac{m^2}{8} \phi^2 \right) \quad (10)$$

with ϕ given by the local minimum of V_{eff} . Expanding in the neighbourhood of the critical point gives

$$\mathcal{S} = \mathcal{S}_* - \frac{1}{4} \frac{(\tilde{\phi}_*^2 + 2t\tilde{\phi}_*)}{\sqrt{3\tilde{\phi}_* + 4t}} (\beta - \beta_c)^{1/2} + \dots \quad (11)$$

and predicts that the transition has a divergent specific heat with exponent $\alpha = 1/2$. For $m = 0$ we find $\mathcal{S}_* = 5/12$ while in the matrix phase it takes the constant value $\mathcal{S} = 3/4$ hence $\Delta\mathcal{S} = 1/3$, i.e. the entropy [24] has a discrete jump of $1/9$ per degree of freedom (there are 3 matrices) across the transition. These fractions are in excellent agreement with the values obtained numerically suggesting that a more complete exact solution to the model, at least with $m = 0$, may exist.

Monte Carlo Simulations: The model (1) with $m = 0$ was studied in [5] and an initial study of the full model was reported in [7]. In Monte Carlo simulations we use the Metropolis algorithm and the action (1). The errors were estimated using the jackknife method.

We observe that \mathcal{S} collapses well for different N . Defining $\tilde{\alpha}_s$ as the value of $\tilde{\alpha}$ at which curves of the average value of the action, $\langle S \rangle$, for different N cross, gives a good estimate of the location of the transition.

For $m = 0$ we observe (figure 1) a divergence in the specific heat, $C_v := \langle (S - \langle S \rangle)^2 \rangle / N^2$, as the critical value $\tilde{\alpha}_s$ is approached from above, i.e. from the low temperature phase. For $\tilde{\alpha} > \tilde{\alpha}_s$ the model is in the fuzzy sphere phase and C_v rapidly approaches $C_v = 1$ as $\tilde{\alpha}$ is increased. For $\tilde{\alpha} < \tilde{\alpha}_s$ the model is in a matrix phase and $C_v = 0.75$, retaining this constant value right from the transition at $\tilde{\alpha}_s$ to $\tilde{\alpha} = 0$. For large N we observe that the location $\tilde{\alpha}_{max}$ of the peak in C_v and the minimum $\tilde{\alpha}_{min}$ coincide to the numerical accuracy we have explored and agree well with the coupling, $\tilde{\alpha}_s$, at which $\langle S \rangle$ intersect for different values of N . The critical coupling determined either as $\tilde{\alpha}_{max}$ or $\tilde{\alpha}_{min}$ from C_v , or as $\tilde{\alpha}_s$ gives good agreement with (8) and the numerical value found in [5].

The behaviour when $m \neq 0$ is qualitatively similar to $m = 0$. We observe again that, for large $\tilde{\alpha}$, the model is in the fuzzy sphere phase where $C_v = 1$.

As $\tilde{\alpha}$ is decreased for fixed N , C_v goes through a peak at $\tilde{\alpha}_{max}$, with the peak value decreasing as m^2 is increased. As $\tilde{\alpha}$ is further decreased from $\tilde{\alpha}_{max}$, C_v goes through a minimum at $\tilde{\alpha}_{min}$ and then increases again, while asymptoting to the value 0.75 at $\tilde{\alpha} = 0$. As the value of m^2 is increased, our numerical study confirms that the fuzzy sphere-matrix model transition is shifted to lower values of $\tilde{\alpha}$. Extrapolating $m^2 \rightarrow \infty$ we infer that the critical coupling goes to zero. This limiting model should be related to the fuzzy Yang-Mills model without scalar field [16]. By extrapolating the measured values of $\tilde{\alpha}_{max}$ and $\tilde{\alpha}_{min}$ to $N = \infty$ we obtain the critical value $\tilde{\alpha}_c$. This agrees with $\tilde{\alpha}_s$ and hence we infer both are detecting the matrix-to-sphere phase transition. However, $\tilde{\alpha}_c$ deviates measurably from (8); e.g., for $m^2 = 200$ $\tilde{\alpha}_c = 0.403 \pm 0.014$ while (8) gives $\tilde{\alpha}_* = 0.446982$. Thus, though (8) gives a good indicative value for the transition it is not precise. Our results are summarised in a phase diagram in figure 2.

We further observe that the jump in entropy decreases as m is increased and the nature of the transition seems to change. Our theoretical analysis (11) still indicates a divergent specific heat with exponent $\alpha = 1/2$ but with a narrowing critical regime as m is increased. However, we find no direct evidence for such a regime in the numerical data for large m . We have not been able to determine with any precision where the transition becomes continuous; however, the entropy jump seems to disappear at $m^2 \sim 20$. Also the data for $\tilde{\alpha}_s$ and $\tilde{\alpha}_c$ separate in this region (see figure 2) indicating a possible multi-critical point. It is also possible that beyond this point the transition is in fact 3rd order, a behaviour typical of many matrix models; however, the persistence of the critical line as determined by $\tilde{\alpha}_s$ the crossing point of the $\langle S \rangle$ curves suggests the transition is 2nd order continuous, consistent with the theoretical analysis. We know of no other model that exhibits transitions of the type presented here and further numerical and theoretical study is warranted.

More detail on the structure of the phases can be obtained from the distribution of eigenvalues of observables. Here we focus on D_3 (by rotational symmetry all matrices have the same eigenvalue distribution). The characteristic behaviour of the distributions of eigenvalues in the fuzzy sphere and matrix phases is illustrated in figures 3 and 4 respectively. Thus we see that, as one crosses the critical curve in figure 2, the eigenvalue distribution of D_3 undergoes

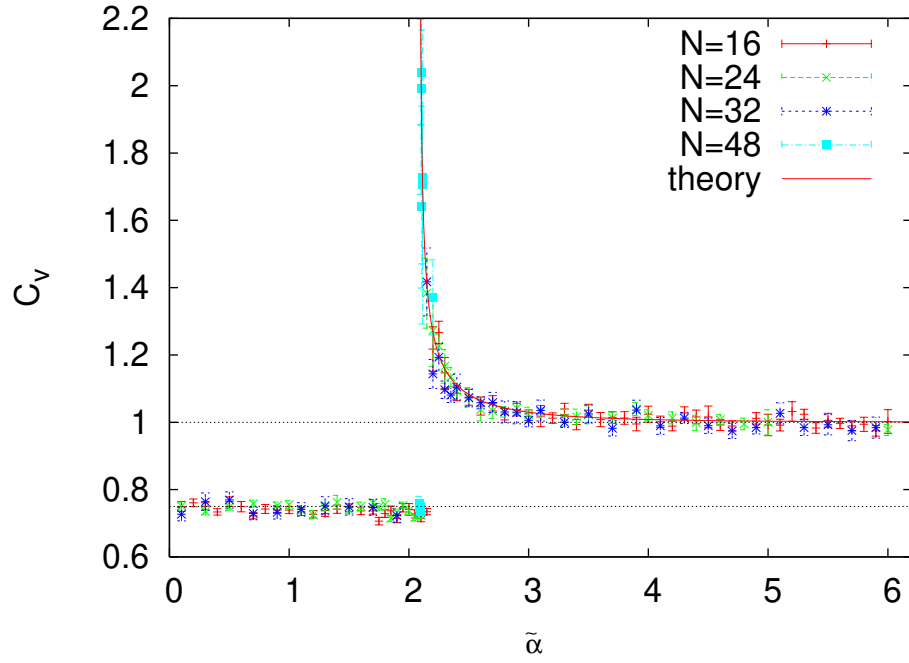


FIG. 1: The specific heat for the model S_0 as a function of coupling or inverse temperature $\beta = \tilde{\alpha}^4$. The theoretical curve is obtained from the local minimum of the effective potential Eq. (6)

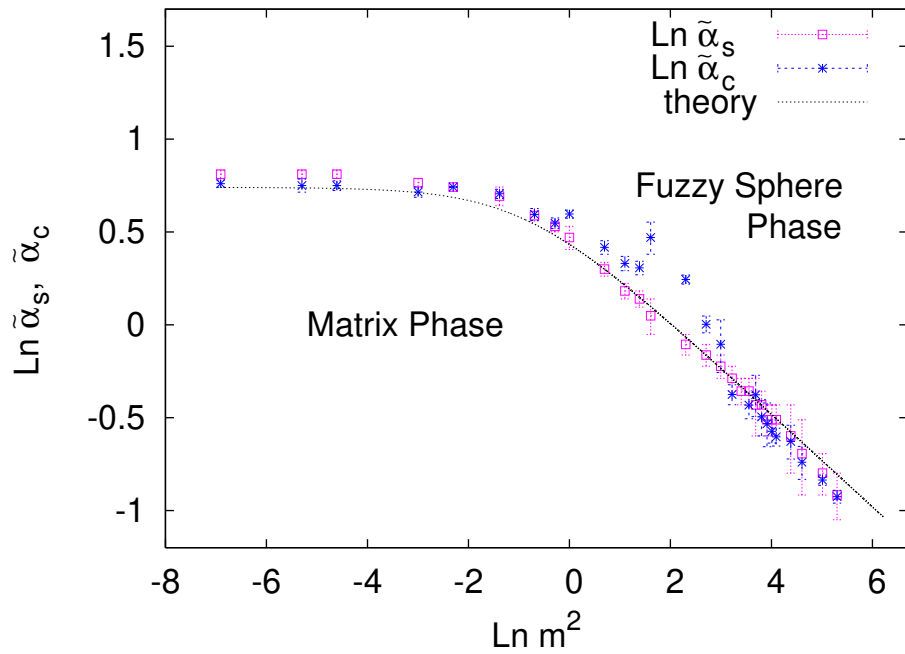


FIG. 2: The phase diagram. The theoretical curve is given by Eq. (8). The specific heat exhibits a divergence for small m , with the transition becoming smooth around $\ln m^2 \sim 2$ where the curves $\tilde{\alpha}_s$ and $\tilde{\alpha}_c$ separate.

a transition from a point spectrum to a continuous distribution symmetric around zero, the latter can be fit by the one-cut distribution [20] for a single matrix in a quartic potential.

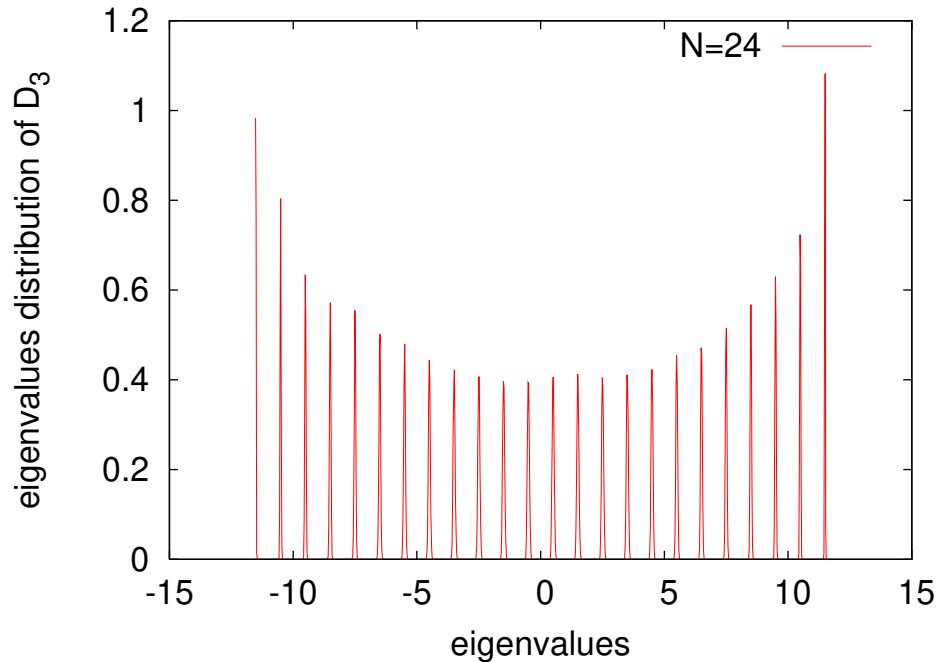


FIG. 3: The eigenvalue distribution for D_3 , with $N = 24$, in the fuzzy sphere for $\tilde{\alpha} = 5$ and $m^2 = 200$. It corresponds to $D_3 = L_3$.

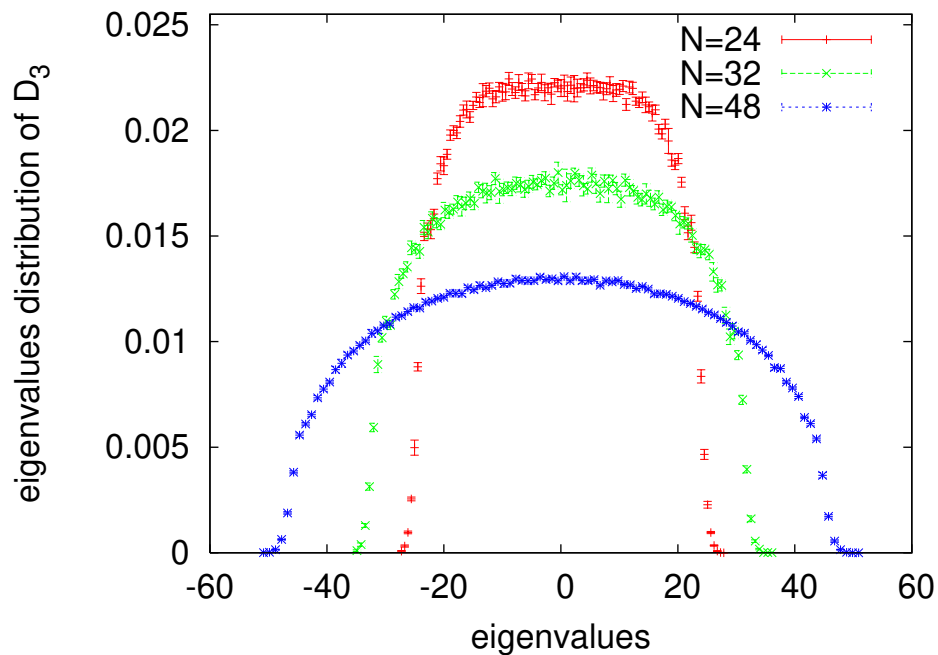


FIG. 4: The eigenvalue distributions for D_3 in the matrix phase for $N = 24, 32, 48$, $\tilde{\alpha} = 0.20$ and $m^2 = 200$.

Conclusions: We have found an exotic transition in a simple three matrix model. The nature of the transition is very different if approached from the high or low temperature side. The high temperature phase is described

by Yang-Mills in zero dimensions so there is no background spacetime geometry. The system in this phase is well approximated by 3 decoupled matrices each with the same quartic potential. The value $C_v = 0.75$ coincides with the specific heat of 3 independent matrix models with quartic potential in the high temperature limit and is therefore consistent with this interpretation.

As the system cools a geometrical phase condenses and at sufficiently low temperatures the system is described by small fluctuations on a two sphere of a $U(1)$ gauge field coupled to a massive scalar field. The critical temperature is pushed upwards as the scalar field mass is increased (see figure 2). We also find evidence that the UV/IR mixing [22] typical of many non-commutative field theories is linked to the geometrical instability and disappears as the scalar mass is increased. Once the geometrical phase is well established the specific heat takes the value 1 with the gauge and scalar fields each contributing $1/2$.

The model of emergent geometry described here, though reminiscent of the random matrix approach to two dimensional gravity [3] is in fact very different. The manner in which spacetime emerges is also different from that envisaged in string pictures where continuous eigenvalue distributions [2] or a Liouville mode [21] give rise to extra dimensions. It is closely connected to the $D0$ brane scenario described in [12] and our model with $m = 0$ is a dimensionally reduced version of a boundary WZNW models in the large k limit [13]. It is not difficult to invent higher dimensional models with essentially similar phenomenology to that presented here (see [17], [19] and [18]).

We believe that the model presented above gives an appealing picture of how a geometrical phase might emerge as the system cools and suggests a very novel scenario for the emergence of geometry in the early universe. In such a scenario the temperature can be viewed as an effect of other degrees of freedom present in a more realistic model but not directly participating in the transition we describe.

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[24] For the entropy, $\mathcal{I} = N^2 S + \ln Z$, the contribution from $\ln Z$ is continuous and $S = \beta U/N^2$, with U the internal energy, contains the full discontinuity.