

Title	The classical capacity of quantum channels with memory
Creators	Dorlas, T. C. and Morgan, Ciara
Date	2008
Citation	Dorlas, T. C. and Morgan, Ciara (2008) The classical capacity of quantum channels with memory. Physical Review A, 79 (3). ISSN 1050-2947
URL	https://dair.dias.ie/id/eprint/228/
DOI	DIAS-STP-08-17

The classical capacity of quantum channels with memory

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Abstract

We investigate the classical capacity of two quantum channels with memory: a periodic channel with depolarizing channel branches, and a convex combination of depolarizing channels. We prove that the capacity is additive in both cases. As a result, the channel capacity is achieved without the use of entangled input states. In the case of a convex combination of depolarizing channels the proof provided can be extended to other quantum channels whose classical capacity has been proved to be additive in the memoryless case.

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I. INTRODUCTION

The problem of determining the classical information-carrying capacity of a quantum channel is one which has not been fully resolved to date. In the case where the input to the channel is prepared in the form of non-entangled states, the classical capacity can be determined using a simple formula. However, if entanglement between multiple uses of the channel is permitted, then the channel capacity can only be determined asymptotically. We now discuss these concepts in detail.

Using product-state encoding, i.e. when a message is encoded into a tensor product of n quantum states on a finite-dimensional Hilbert space \mathcal{H} , this state can then be transmitted over a quantum channel given by a completely positive trace-preserving map (CPT) $\Phi^{(n)}$ on $\mathcal{B}(\mathcal{H}^{\otimes n})$. The associated capacity is known as the product-state capacity of the channel. If $\Phi^{(n)} = \Phi^{\otimes n}$ is a memoryless channel, then the product-state capacity is given by the supremum of the Holevo χ -quantity given by the right-hand side of (1), evaluated over all possible input state ensembles. This is also known as the Holevo capacity $\chi^*(\Phi)$ of the channel.

On the other hand, a block of input states could be permitted to be entangled over n channel uses. The classical capacity is defined as the limit of the capacity for such n -fold entangled states divided by n , as n tends to infinity. If the Holevo capacity of a memoryless channel is additive, then it is equal to the classical capacity of that channel and there is no advantage to using entangled input state codewords. The additivity conjecture for the Holevo capacity of most classes of memoryless channel remains open. However, the classical capacity of certain memoryless quantum channels have been shown to be additive: see [1], [2], [3], for example. On the other hand, there now exists an example of a memoryless channel for which the conjecture does not hold: see [4].

We remark that, Shor [5] (see also Fukuda [6]) proved that the additivity conjectures involving the entanglement of formation [7], the minimum output entropy [8], the strong superadditivity and the Holevo capacity [9], [10] are in fact equivalent.

In this article we consider the classical capacity of two particular channels with *memory* consisting of depolarizing channel branches, namely a periodic channel and a convex combination of memoryless channels.

In [11] Datta and Dorlas derived a general expression for the classical capacity of a quan-

tum channel with *arbitrary* Markovian correlated noise. We consider two special cases of this channel, that is, a periodic channel with depolarizing channel branches and a convex combination of memoryless channels, and we prove that the corresponding capacities are additive in the sense that they are equal to the product-state capacities. A convex combination of memoryless channels was discussed in [12] and can be described by a Markov chain which is aperiodic but not irreducible. Both channels are examples of a channel with long-term memory.

The article is organized as follows. The objectives as discussed above are formalized in Section I A. In Section 2 we introduce the periodic channel and investigate the product state capacity of the channel with depolarizing channel branches. We derive a result based on the invariance of the maximizing ensemble of the depolarizing channel, which enables us to prove that the capacity of such a periodic channel is additive. This is shown in Section 3. In Section 4 the additivity of the classical capacity of a convex combination of depolarizing channels is proved. This is done independently of the result derived in Section 2 and can therefore be generalized to a class of other quantum channels.

A. Preliminaries

A quantum state is described by a positive operator of unit trace $\rho \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ denotes the algebra of linear operators acting on a finite dimensional Hilbert space \mathcal{H} . The transmission of classical information over a quantum channel is achieved by encoding the information as quantum states. To accomplish this, a set of possible input states $\rho_j \in \mathcal{B}(\mathcal{H})$ with probabilities p_j are prepared, describing the ensemble $\{p_j, \rho_j\}$. The average input state to the channel is expressed as $\rho = \sum_j p_j \rho_j$. For a channel given by a completely positive trace preserving map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$, the average *output* state is $\tilde{\rho} = \sum_j p_j \Phi(\rho_j)$.

When a state is sent through a noisy quantum channel, the amount of information about the input state that can be inferred from the output state is called the *accessible information*. The Holevo bound, [13], provides an upper bound on the accessible information and is given by,

$$H(X : Y) \leq S \left(\sum_j p_j \rho_j \right) - \sum_j p_j S(\rho_j), \quad (1)$$

where $S(\rho) = -\text{tr}(\rho \log \rho)$ is the von Neumann entropy. Here X is the random variable

representing the classical input to the channel. The possible values x_j are mapped to states ρ_j which are transformed to $\Phi(\rho_j)$ by the channel. Then, a generalized measurement with corresponding POVM $\{E_j\}$ allows the determination of the output random variable Y with conditional probability distribution given by

$$\mathbb{P}(Y = x_k | X = x_j) = \text{tr}(\Phi(\rho_j)E_k). \quad (2)$$

The right hand side of equation (1) is called the Holevo- χ -quantity, usually denoted $\chi(\{p_j, \Phi(\rho_j)\})$. Holevo [9] and Schumacher and Westmoreland [10] proved independently that for a memoryless channel, the upper bound on $H(X : Y)$ is asymptotically achievable. Using product-state coding as described above, the input message to the channel is encoded into a product state codeword of length n and is transmitted over n copies of the channel. The Holevo Schumacher Westmoreland (HSW) Theorem states that the product state capacity of that channel is given by the supremum, over all input ensembles, of the Holevo quantity of that channel, where each input state is prepared as a product state codeword. In other words, the rate at which classical information can be sent over a quantum channel, where each input codeword is a product state comprised of states belonging to an ensemble is given by the following “single-letter” formula,

$$\chi^*(\Phi) = \sup_{\{p_j, \rho_j\}} \left[S \left(\Phi \left(\sum_j p_j \rho_j \right) \right) - \sum_j p_j S(\Phi(\rho_j)) \right] \quad (3)$$

where S is the von Neumann entropy. An ensemble which maximizes the Holevo quantity χ of a channel is known as a maximizing or optimal ensemble.

It was first shown in [14] that for some channels, it is possible to gain a higher rate of transmission by sending entangled states across multiple copies of a quantum channel. In general, allowing both entangled *input* states and output measurements and with an unlimited number of copies of the channel, the classical capacity of Φ is given by [15]

$$C(\Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \chi^*(\Phi^{(n)}), \quad (4)$$

where

$$\chi^*(\Phi^{(n)}) = \sup_{\{p_j^{(n)}, \rho_j^{(n)}\}} \left[S \left(\Phi^{(n)} \left(\sum_j p_j^{(n)} \rho_j^{(n)} \right) \right) - \sum_j p_j^{(n)} S \left(\Phi^{(n)} \left(\rho_j^{(n)} \right) \right) \right] \quad (5)$$

denotes the Holevo capacity of the channel $\Phi^{(n)}$ with an n -fold input state ensemble.

The Holevo capacity of a channel Φ is said to be *additive* if the following holds for an arbitrary channel Ψ :

$$\chi^*(\Phi \otimes \Psi) = \chi^*(\Phi) + \chi^*(\Psi). \quad (6)$$

In particular, if we can prove that the Holevo capacity of a particular channel is additive then

$$\chi^*(\Phi^{\otimes n}) = n \chi^*(\Phi), \quad (7)$$

which implies that the classical capacity of that channel is equal to the product state capacity, that is,

$$C(\Phi) = \chi^*(\Phi). \quad (8)$$

This will imply that the classical capacity of that channel cannot be increased by entangling inputs across two or more uses of the channel. Additivity has been proved for unital qubit channels [1], entanglement-breaking channels [2], and the depolarizing channel [3]. Here we use the latter result to prove equation (8) for a periodic channel with depolarizing channel branches and for a convex combination of depolarizing channels.

II. THE PERIODIC CHANNEL

A periodic channel acting on an n -fold density operator has the form

$$\Omega^{(n)}(\rho^{(n)}) = \frac{1}{L} \sum_{i=0}^{L-1} (\Omega_i \otimes \Omega_{i+1} \otimes \cdots \otimes \Omega_{i+n-1})(\rho^{(n)}), \quad (9)$$

where Ω_i are CPT maps and the index is cyclic modulo the period L .

We denote the Holevo quantity for the i -th branch of the channel by $\chi_i(\{p_j, \rho_j\})$, i.e.

$$\chi_i(\{p_j, \rho_j\}) = S\left(\sum_j p_j \Omega_i(\rho_j)\right) - \sum_j p_j S(\Omega_i(\rho_j)). \quad (10)$$

Since there is a correlation between the noise affecting successive input states to the periodic channel (9), the channel is considered to have memory and the product state capacity

of the channel is no longer given by the supremum of the Holevo quantity. Instead, the product state capacity of this channel is given by the following expression

$$C_p(\Omega) = \frac{1}{L} \sup_{\{p_j, \rho_j\}} \sum_{i=0}^{L-1} \chi_i(\{p_j, \rho_j\}). \quad (11)$$

Next, we introduce the depolarizing channel and investigate the product state capacity of a periodic channel with depolarizing channel branches.

A. A periodic channel with depolarizing channel branches

The quantum depolarizing channel can be written as follows

$$\Delta_\lambda(\rho) = \lambda\rho + \frac{1-\lambda}{d}I \quad (12)$$

where $\rho \in \mathcal{B}(\mathcal{H})$ and I is the $d \times d$ identity matrix. Note that in order for the channel to be completely positive the parameter λ must lie within the range

$$-\frac{1}{d^2-1} \leq \lambda \leq 1. \quad (13)$$

Output states from this channel have eigenvalues $(\lambda + \frac{1-\lambda}{d})$ with multiplicity 1 and $(\frac{1-\lambda}{d})$ with multiplicity $d-1$.

The minimum output entropy of a channel Φ is defined by

$$S_{min}(\Phi) = \inf_{\rho} S(\Phi(\rho)). \quad (14)$$

It is easy to see that the product capacity of the depolarizing channel is given by

$$\chi^*(\Delta_\lambda) = \log(d) - S_{min}(\Delta_\lambda), \quad (15)$$

where the minimum entropy is attained for any set of orthonormal vector states, and is given by

$$\begin{aligned} S_{min}(\Delta_\lambda) &= - \left(\lambda + \frac{1-\lambda}{d} \right) \log \left(\lambda + \frac{1-\lambda}{d} \right) \\ &\quad - (d-1) \left(\frac{1-\lambda}{d} \right) \log \left(\frac{1-\lambda}{d} \right). \end{aligned} \quad (16)$$

Next we show that the product state capacity of a periodic channel with L depolarizing channel branches is given by the sum of the maximum of the Holevo quantities of the

individual depolarizing channels, in other words we show that

$$\frac{1}{L} \sup_{\{p_j, \rho_j\}} \sum_{i=0}^{L-1} \chi_i(\{p_j, \rho_j\}) = \frac{1}{L} \sum_{i=0}^{L-1} \sup_{\{p_j, \rho_j\}} \chi_i(\{p_j, \rho_j\}). \quad (17)$$

Let $\Delta_{\lambda_1}, \Delta_{\lambda_2}, \dots, \Delta_{\lambda_L}$ denote d -dimensional depolarizing channels with respective error parameters $\lambda_1, \lambda_2, \dots, \lambda_L$. Using the capacity given by equation (15) and since every depolarizing channel can be maximized using a single ensemble of orthogonal pure states independently of the error parameter, the right-hand side of equation (17) can be written as

$$\begin{aligned} \frac{1}{L} \sum_{i=0}^{L-1} \sup_{\{p_j, \rho_j\}} \chi_i(\{p_j, \rho_j\}) &= 1 - \frac{1}{L} S_{min}(\Delta_{\lambda_1}) + \dots \\ &+ \frac{1}{L} S_{min}(\Delta_{\lambda_L}). \end{aligned} \quad (18)$$

Clearly, the left-hand side of equation (17) is bounded above by the right-hand side. On the other hand, choosing the ensemble to be an orthogonal basis of states with uniform probabilities, we have

$$\frac{1}{L} \sum_{i=0}^{L-1} \chi_i(\{p_j, \rho_j\}) = 1 - \frac{1}{L} \sum_{i=0}^{L-1} S_{min}(\Delta_{\lambda_i}). \quad (19)$$

We can now conclude that equation (17) holds for a periodic channel with L depolarizing branches of arbitrary dimension.

B. The classical capacity of a periodic channel

We now consider the classical capacity of the periodic channel, Ω_{per} given by equation (9), where $\Omega_i = \Delta_{\lambda_i}$ are depolarizing channels with dimension d . Denote by $\Psi_0^{(n)}, \dots, \Psi_{L-1}^{(n)}$ the following product channels

$$\Psi_i^{(n)} = \Delta_{\lambda_i} \otimes \dots \otimes \Delta_{\lambda_{i+n-1}}, \quad (20)$$

where the index is taken modulo L .

We define a *single use* of the periodic channel, Ω_{per} , to be the application of one of the depolarizing maps Δ_{λ_i} . If n copies of the channel are available, then with probability $\frac{1}{L}$ one of the product branches $\Psi_i^{(n)}$ will be applied to an n -fold input state.

We aim to prove the following theorem.

Theorem 1. *The classical capacity of the periodic channel Ω_{per} with depolarizing channel branches is equal to its product state capacity,*

$$C(\Omega_{per}) = C_p(\Omega_{per}) = 1 - \frac{1}{L} \sum_{i=0}^{L-1} S_{min}(\Delta_{\lambda_i}).$$

To prove Theorem 1 we first need a relationship between the supremum of the Holevo quantity χ^* and the channel branches $\Psi_i^{(n)}$. King [3] proved that the supremum of the Holevo quantity of the product channel $\Delta_\lambda \otimes \Psi$ is additive, where Δ_λ is a depolarizing channel and Ψ is a completely arbitrary channel, i.e.,

$$\chi^*(\Delta_\lambda \otimes \Psi) = \chi^*(\Delta_\lambda) + \chi^*(\Psi). \quad (21)$$

It follows immediately that

$$\begin{aligned} \chi^*(\Psi_i^{(n)}) &= \chi^*(\Delta_{\lambda_i}) + \chi^*(\Psi_{i+1}^{(n-1)}) \\ &= \sum_{i=0}^{L-1} \chi^*(\Delta_{\lambda_i}) + \chi^*(\Psi_i^{(n-L)}). \end{aligned} \quad (22)$$

Next, we use this result to prove Theorem 1.

Proof. The classical capacity of an arbitrary quantum channel Ω is given by

$$C(\Omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{\{p_j^{(n)}, \rho_j^{(n)}\}} \chi(\{p_j, \Omega^{(n)}(\rho_j^{(n)})\}). \quad (23)$$

In Section II A we showed that the product state capacity of the periodic channel Ω_{per} can be written as

$$C_p(\Omega_{per}) = \frac{1}{L} \sum_{i=0}^{L-1} \chi^*(\Delta_{\lambda_i}). \quad (24)$$

Using the product channels $\Psi_i^{(n)}(\rho_j^{(n)})$ defined by equations (20), the periodic channel Ω_{per} can be written as

$$\Omega_{per}^{(n)}(\rho_j^{(n)}) = \frac{1}{L} \sum_{i=0}^{L-1} \Psi_i^{(n)}(\rho_j^{(n)}). \quad (25)$$

Since it is clear that

$$C(\Omega_{per}) \geq C_p(\Omega_{per}), \quad (26)$$

we concentrate on proving the inequality in the other direction.

First suppose that

$$C(\Omega_{per}) \geq \frac{1}{L} \sum_{i=0}^{L-1} \chi^*(\Delta_{\lambda_i}) + \epsilon, \quad (27)$$

for some $\epsilon > 0$. Then $\exists n_0$ such that if $n \geq n_0$, then

$$\frac{1}{n} \sup_{\{p_j^{(n)}, \rho_j^{(n)}\}} \chi\left(\Omega_{per}^{(n)}\left(\rho_j^{(n)}\right)\right) \geq \frac{1}{L} \sum_{i=0}^{L-1} \chi^*(\Delta_{\lambda_i}) + \frac{\epsilon}{2}. \quad (28)$$

The supremum in equation (23) is taken over all possible input ensembles $\{\rho_j^{(n)}, p_j^{(n)}\}$. Therefore, for $n \geq n_0$, there exists an ensemble $\{\rho_j^{(n)}, p_j^{(n)}\}$ such that

$$\frac{1}{n} \chi\left(\left\{p_j, \Omega_{per}^{(n)}\left(\rho_j^{(n)}\right)\right\}\right) \geq \frac{1}{L} \sum_{i=0}^{L-1} \chi^*(\Delta_{\lambda_i}) + \frac{\epsilon}{2}. \quad (29)$$

The Holevo quantity can be expressed as the average of the relative entropy of the average ensemble state with respect to members of the ensemble

$$\chi(\{p_k, \rho_k\}) = \sum_k p_k S\left(\rho_k, \left\| \sum_k p_k \rho_k \right.\right), \quad (30)$$

where, $S(A \| B) = \text{tr}(A \log A) - \text{tr}(A \log B)$, represents the relative entropy of A with respect to B . (Vedral [16] has argued that the distinguishability of quantum states can be measured by the quantum relative entropy.) Since the relative entropy is jointly convex in its arguments [17], it follows that the Holevo quantity of the periodic channel Ω_{Dep} is also convex.

Therefore, by (25),

$$\chi\left(\left\{p_j^{(n)}, \Omega_{per}^{(n)}\left(\rho_j^{(n)}\right)\right\}\right) \leq \frac{1}{L} \sum_{i=0}^{L-1} \chi\left(\left\{p_j^{(n)}, \Psi_i^{(n)}\left(\rho_j^{(n)}\right)\right\}\right). \quad (31)$$

Using equation (29) we thus have

$$\frac{1}{L} \sum_{i=0}^{L-1} \chi^*(\Delta_{\lambda_i}) + \frac{\epsilon}{2} \leq \frac{1}{nL} \sum_{i=0}^{L-1} \chi\left(\left\{p_j^{(n)}, \Psi_i^{(n)}\left(\rho_j^{(n)}\right)\right\}\right). \quad (32)$$

It follows that there is an index i such that

$$\frac{1}{L} \sum_{i=0}^{L-1} \chi^*(\Delta_{\lambda_i}) + \frac{\epsilon}{2} \leq \frac{1}{n} \chi\left(\left\{p_j^{(n)}, \Psi_i^{(n)}\left(\rho_j^{(n)}\right)\right\}\right). \quad (33)$$

But equation (22) implies that

$$\chi\left(\left\{p_j^{(n)}, \Psi_i^{(n)}\left(\rho_j^{(n)}\right)\right\}\right) \leq \frac{n}{L} \sum_{i=0}^{L-1} \chi^*(\Delta_{\lambda_i}). \quad (34)$$

Therefore the inequalities (33) and hence the assumption made in equation (27) cannot hold, and

$$C(\Omega_{per}) \leq C_p(\Omega_{per}). \quad (35)$$

Putting the above equation together with equation (26) yields the required result. \square

III. THE CLASSICAL CAPACITY OF A CONVEX COMBINATION OF MEMORYLESS CHANNELS

In [12] the product state capacity of a convex combination of memoryless channels was determined. Given a finite collection of memoryless channels Φ_1, \dots, Φ_M with common input Hilbert space \mathcal{H} and output Hilbert space \mathcal{K} , a convex combination of these channels is defined by the map

$$\Phi^{(n)}(\rho^{(n)}) = \sum_{i=1}^M \gamma_i \Phi_i^{\otimes n}(\rho^{(n)}), \quad (36)$$

where γ_i , ($i = 1, \dots, M$) is a probability distribution over the channels Φ_1, \dots, Φ_M . Thus, a given input state $\rho^{(n)} \in \mathcal{B}(\mathcal{H}^{\otimes n})$ is sent down one of the memoryless channels with probability γ_i . This introduces long-term memory, and as a result the (product-state) capacity of the channel $\Phi^{(n)}$ is no longer given by the supremum of the Holevo quantity. Instead, it was proved in [12] that the product-state capacity is given by

$$C_p(\Phi) = \sup_{\{p_j, \rho_j\}} \left[\bigwedge_{i=1}^M \chi(\{p_j, \Phi_i(\rho_j)\}) \right]. \quad (37)$$

Let Δ_{λ_i} be depolarizing channels with parameters λ_i as above, and Φ_{rand} denote the channel whose memoryless channel branches are given by $\Lambda_i^{(n)}$ where

$$\Lambda_i^{(n)} = \Delta_{\lambda_i}^{\otimes n}. \quad (38)$$

Since the capacity of the depolarizing channel decreases with the error parameter the product state capacity of Φ_{rand} is given by

$$C_p(\Phi_{rand}) = \bigwedge_{i=1}^M \chi^*(\Delta_{\lambda_i}) = \chi^* \left(\bigvee_{i=1}^M \lambda_i \right). \quad (39)$$

We aim to prove the following theorem.

Theorem 2. *The classical capacity of a convex combination of depolarizing channels is equal to its product state capacity*

$$C(\Phi_{rand}) = C_p(\Phi_{rand}).$$

Proof. According to [12] the classical capacity of this channel can be written as follows

$$C(\Phi_{rand}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sup_{\{p_j^{(n)}, \rho_j^{(n)}\}} \bigwedge_{i=1}^M \chi\left(\left\{p_j^{(n)}, \Lambda_i^{(n)}\left(\rho_j^{(n)}\right)\right\}\right). \quad (40)$$

Suppose that

$$C(\Phi_{rand}) \geq \bigwedge_{i=1}^M \chi^*(\Delta_{\lambda_i}) + \epsilon, \quad (41)$$

for some $\epsilon > 0$.

Then $\exists n_0$, such that if $n \geq n_0$, then

$$\begin{aligned} \frac{1}{n} \sup_{\{p_j^{(n)}, \rho_j^{(n)}\}} \bigwedge_{i=1}^M \chi\left(\left\{p_j^{(n)}, \Lambda_i^{(n)}\left(\rho_j^{(n)}\right)\right\}\right) &\geq \bigwedge_{i=1}^M \chi^*(\Delta_{\lambda_i}) \\ &+ \epsilon. \end{aligned} \quad (42)$$

Hence, for $n \geq n_0$ there exists an ensemble $\{p_j^{(n)}, \rho_j^{(n)}\}$ such that

$$\frac{1}{n} \bigwedge_{i=1}^M \chi\left(\left\{p_j^{(n)}, \Lambda_i^{(n)}\left(\rho_j^{(n)}\right)\right\}\right) \geq \bigwedge_{i=1}^M \chi^*(\Delta_{\lambda_i}) + \epsilon. \quad (43)$$

But King [3] proved that the product state capacity of the depolarizing channel is equal to its classical capacity, therefore

$$\chi^*\left(\Lambda_i^{(n)}\right) = n \chi^*(\Delta_{\lambda_i}). \quad (44)$$

In other words, $\chi\left(\left\{p_j^{(n)}, \Lambda_i^{(n)}\left(\rho_j^{(n)}\right)\right\}\right)$ is bounded above by $\chi^*(\Delta_{\lambda_i})$. Now, if i_0 is such that

$$\bigwedge_{i=1}^M \chi^*(\Delta_{\lambda_i}) = \chi^*(\Delta_{\lambda_{i_0}}), \quad (45)$$

then

$$\begin{aligned} \frac{1}{n} \bigwedge_{i=1}^M \chi\left(\left\{p_j^{(n)}, \Lambda_i^{(n)}\left(\rho_j^{(n)}\right)\right\}\right) &\leq \chi\left(\left\{p_j^{(n)}, \Lambda_{i_0}^{(n)}\left(\rho_j^{(n)}\right)\right\}\right) \\ &\leq \chi^*(\Delta_{\lambda_{i_0}}). \end{aligned} \quad (46)$$

Therefore

$$\frac{1}{n} \bigwedge_{i=1}^M \chi \left(\left\{ p_j^{(n)}, \Lambda_i^{(n)} \left(\rho_j^{(n)} \right) \right\} \right) \leq \bigwedge_{i=1}^M \chi^* (\Delta_{\lambda_i}). \quad (47)$$

This contradicts the assumption made by equation (41) and therefore

$$C(\Phi_{rand}) \leq \bigwedge_{i=1}^M \chi^* (\Delta_{\lambda_i}) = C_p(\Phi_{rand}). \quad (48)$$

On the other hand, it is clear that $C(\Phi_{rand}) \geq C_p(\Phi_{rand})$, and therefore $C(\Phi_{rand}) = C_p(\Phi_{rand})$. \square

Remark. *Note that, in contrast to the proof of Theorem 1, the proof above does not rely on the invariance of the maximizing ensemble of the depolarizing channel. The proof uses the additivity of the Holevo quantity of the depolarizing channel (see Eqn. (44)) and the result can therefore be generalized to channels for which the additivity of the Holevo capacity has been proved.*

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- [1] C. King, J. Math. Phys. **43**, 4641 (2002).
 - [2] P. W. Shor, J. Math. Phys. **43**, 4334 (2002).
 - [3] C. King, IEEE Trans. Info. Theory **49**, 229 (2003).
 - [4] M.B. Hastings, arXiv:quant-ph/08093972.
 - [5] P.W. Shor, Commun. Math. Phys. **246**, 453 (2004).
 - [6] M. Fukuda, Quantum Inf. Process **6**, 179 (2007).
 - [7] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin and W.K. Wootters, Phys. Rev. A **54**, 3824 (1996).
 - [8] C. King and M.B. Ruskai, IEEE Trans. Info. Theory **47**, 192 (2001).
 - [9] A.S. Holevo, IEEE Trans. Info. Theory **44**, 269 (1998).
 - [10] B. Schumacher and M.D. Westmoreland, Phys. Rev. A **56**, 131 (1997).
 - [11] N. Datta and T.C. Dorlas, J. Phys. A. **40**, 8147 (2007).
 - [12] N. Datta and T.C. Dorlas, arXiv:quant-ph/07120722.
 - [13] A.S. Holevo, Statistical problems in quantum physics, *Proceedings of the Second Japan-USSR Symposium on Probability Theory*, Springer Berlin/ Heidelberg, **330**, (1973).
 - [14] C. Fuchs, C. Bennett and J. Smolin, arXiv:quant-ph/9611006.

- [15] A. S. Holevo, *Probl. Inf. Transm.* **8**, 62 (1972).
- [16] V. Vedral, *Rev. Mod. Phys.* **74**, 197 (2002).
- [17] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* , Cambridge University Press, (2000).