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The Orthogonal and Symplectic Groups

BY

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P R E F A C E

The following lectures, given in 1957 at the School of Theoretical Physics of the Dublin Institute for Advanced Studies, are devoted to the theory of group representations. This theory is now better understood than it was when I wrote, some twenty years ago, my book "The Theory of Group Representations"; and its exposition in the present lectures is considerably simpler in many respects than that in my book. I may mention, for instance, the treatment of the modification rules for the rotation, symplectic and orthogonal groups, in which I have been able to use with great profit the ideas of Professor M. J. Newell. The treatment, in these lectures, of the analysis of the product of irreducible representations of the unitary group is more satisfactory than that given in my book. In the last lecture I have given a simple solution of the problem of analysing the representation $\{m\} \otimes \{\lambda_1, \lambda_2\}$ of the 2-dimensional unimodular unitary group. This problem is important in applications to nuclear physics and was not treated in my book.

I have, once again, the pleasure of ending a preface with the pious inscription:

Do chum Glóire Dé, agus Onóra na hEireann
To the Glory of God and Honour of Ireland.

December 1957 Francis D. Murnaghan

ERRATA

p. 13, last line: read

taking θ , α and σ as the parameters, ...

p. 32, line 8 from end: read

$$M_j = \sum_{p=1}^{r} N_p (C)^p_j \text{ , where ...}$$

line 6 from end: read

$$\sum_{p=1}^{r} \sum_{k=1}^{r} N_p (C)^p_k (A')^k_j = \sum_{p=1}^{r} A N_p A^{-1} (C)^p_j \text{ , } j = 1, \ldots, r,$$

p. 36, line 10: read

... rotation matrix $R_{12}(\phi)\, R_{23}(-\theta)\, R_{12}(\psi)$.

p. 44, line 9: read

... be the collection of $m \times m$

p. 46, lines 6 and 5 from end: read

... to assume the value 0 , it being under-

stood that when m_j , for example, ...

p. 49, line 7 from end: read

the jhl-row and kim-column is $a^j_k\, b^h_i\, c^l_m$ . ...

p. 56, line 2 from end: read

$= \sin\theta_j \cos\sigma_j$; $y_j = s_j \sin\sigma_j = \sin\theta_j \sin\sigma_j$; $j = 1, \ldots, N,$

p. 58, line 9 from end: read

$2N = n(n-1)$ in-class parameters ...

p. 64, line 10 from end: read

... basis defined by $X'_{[m]}$, where X' is

lines 6, 5 and 4 from end: read

Then $c'^{\{r\}} = \sum_{\alpha_1, \ldots, \alpha_m} a^{r_1}_{\alpha_1} \ldots a^{r_m}_{\alpha_m} c^{\{\alpha\}}$ so that $((p)\, c')^{\{r\}} =$

$$= \sum_{\alpha_1,\dots,\alpha_m} a_{\alpha_1}^{r_{p_1}} \dots a_{\alpha_m}^{r_{p_m}} c\,\{\alpha\} = \sum_{(\alpha)} a_{\alpha_{p_1}}^{r_{p_1}} \dots a_{\alpha_{p_m}}^{r_{p_m}} ((p)\,c)\,\{\alpha\} =$$

$$= \sum_{(\alpha)} a_{\alpha_1}^{r_1} \dots a_{\alpha_m}^{r_m} ((p)\,c)\,\{\alpha\}\ .$$ Thus $(p)\,c' = A_{[m]}\,((p)\,c)$ so that

p. 66, line 7 from end: read

average $w_{(p)}\{r\}$ of $v_{(p)}\{r\}$ over S_m is ...

line 4 from end: read

there are terms in the expansion of ...

p. 68, line 3: read

of the $D_{\_}$-dimensional representation ...

line 8 from end: read

... similarly, $\sum_{(\alpha)} m_{(\alpha)} \sigma_1^{\alpha_1} \dots \sigma_n^{\alpha_n}$ is

p. 123, line 2 from end: read

$$\sum_{(\lambda)} [\lambda]\,(z)\,\{\lambda\}\,(t)\,\Delta(t) = \dots$$

p. 124, line 4 from end: read

follows that $[\lambda_1, \dots, \lambda_{2k-1}, 0] = \dots$

p. 126, line 10 from end: read

... in the form $\begin{pmatrix} a_1 & a_2 & \dots & a_s \\ a_1 & a_2 & \dots & a_s \end{pmatrix}$ which is

p. 131, line 4 from end: read

$$= \sum_{(\alpha)} \frac{1}{\alpha_1!}\, s_1^{\alpha_1} \dots$$

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THE ORTHOGONAL AND SYMPLECTIC GROUPS

Lecture 1

1. Notation and terminology

We shall concern ourselves in these lectures with certain linear transformations in finite-dimensional linear vector spaces. The concept of a finite-dimensional linear vector space is quite abstract and our first task is to explain it as concretely as possible. To do this we must first decide upon the number field in which we shall work and we restrict our attention to two number fields, the field of complex numbers and the field of real numbers. Of these two the simpler is the field of complex numbers due to the fact that it is algebraically closed, every polynomial function, with complex coefficients, of a single complex variablo being zero for at least one value of the variable. On the other hand, the field of real numbers is not algebraically closed, the polynomial $x^2 + 1$, for example, not being zero for any real value of x. We suppose, then, that our underlying number field is the field of complex numbers and we shall indicate, when we wish to restrict ourselves to the real number field, the changes in the argument that are made necessary by the fact that this number field is not algebraically closed.

We denote by x a $n \times 1$ matrix i.e., a matrix of n rows and 1 column whose elements $x^1, \ldots, x^n$ are arbitrary complex numbers, n being any positive integer, and if $x_1, \ldots, x_n$ are n such $n \times 1$ matrices we denote by X the $n \times n$, or n-dimensional, matrix whose column matrices are $x_1, \ldots, x_n$. We suppose $x_1, \ldots, x_n$ so chosen that X has a reciprocal X^{-1}; then an arbitrary $n \times 1$

matrix may be written in the form Xc , c being the product of the given $n \times 1$ matrix by X^{-1} . The pair of matrices X , c defines an n-dimensional vector and the collection of vectors obtained by varying c , X being held fixed, is known as an n-dimensional linear vector space. Multiplication of a vector by a complex number and addition of any two vectors are defined by performing these operations upon the $n \times 1$ matrices c ; thus, if we denote that v is the vector which is defined by X , c by writing $c \rightarrow v$ and if m is any complex number, then $mc \rightarrow mv$ and, if $c_1 \rightarrow v_1$, $c_2 \rightarrow v_2$, then $c_1 + c_2 \rightarrow v_1 + v_2$. If $e_j \rightarrow v_j$, $j = 1, \ldots, n$, where e_j is the $n \times 1$ matrix all of whose elements are zero save the j^{th} which is 1 , then $c = c^1 e_1 + \ldots + c^n e_n$ so that $v = c^1 v_1 + \ldots + c^n v_n$. We term the set of n vectors $v_1, \ldots, v_n$ a basis in our n-dimensional linear vector space and we say that this basis is defined by X ; furthermore we term the elements of c the coordinates of v with respect to the basis which is defined by X .

If X' is any other n-dimensional matrix which possesses a reciprocal our arbitrarily given $n \times 1$ matrix may be written in the form $X'c'$ where $X'c' = Xc$ and the elements of c' furnish the coordinates, with respect to the basis which is defined by X', of the vector v whose coordinates, with respect to the basis which is defined by X, are furnished by the elements of c . Thus we may regard c and c' as different representations of the vector which is defined by the pair of matrices X, c or equivalently, by the pair of matrices X', c', the fundamental connection between the two representations being given by the relation $Xc = X'c'$ which it is convenient to write in the form

$$c' = Ac \; ; \quad A = X'^{-1}X \; .$$

We term A the matrix of the transformation from the basis which is determined by X to the basis which is determined by X' .

If B is any n-dimensional matrix which possesses a reciprocal the relationship $c \rightarrow d = Bc$ transforms the collection of all $n \times 1$ matrices into itself and, hence, the collection of all n-dimensional vectors into itself. If v is the vector defined by the pair

(X, c) and w is the vector defined by the pair (X, d) we write $v \to w = \beta v$. Since $B(mc) = m Bc$ and $B(c_1 + c_2) = Bc_1 + Bc_2$ we have

$$\beta(mv) = m\beta v \; ; \qquad \beta(v_1 + v_2) = \beta v_1 + \beta v_2$$

and we express these properties of the transformation β by the statement that β is linear. Since $d' = Ad = ABc = ABA^{-1}c'$ we have $B' = ABA^{-1}$; $A = X'^{-1}X$. We regard B and B' as different presentations, in the bases defined by X and X', respectively, of the same linear transformation β. The linear transformation β itself may be defined as the collection of all pairs (X, B), (X', B'), ... of non-singular n-dimensional matrices where

$$B' = ABA^{-1} \; ; \qquad A = X'^{-1}X$$

2. The n-dimensional unitary and orthogonal groups.

If c is any $n \times 1$ matrix we denote by c^t its transpose i.e., the $1 \times n$ matrix $(c^1, \ldots, c^n)$ and by $\bar{c}$ its conjugate i.e., the $n \times 1$ matrix whose elements are the conjugate complexes of the elements of c. We denote the result of combining these operations, in either order, i.e., the $1 \times n$ matrix $(\bar{c}^1, \ldots, \bar{c}^n)$, by c^* and we term c^* the star of c. Similarly, if C is any n-dimensional matrix whose column matrices are $c_1, \ldots, c_n$, C^* is the n-dimensional matrix whose row matrices are $c_1^*, \ldots, c_n^*$. If c_1 and c_2 are any two $n \times 1$ matrices we may associate with them either one of the two numbers $c_2^* c_1$ and $c_2^t c_1$. The first of these has the property, which the second does not have, of being real and non-negative when $c_2 = c_1$, it being positive save when c_1 is the zero $n \times 1$ matrix. If $c_1 \to v_1$, $c_2 \to v_2$ we term the two numbers $c_2^* c_1$ and $c_2^t c_1$ the first and second scalar products, respectively, of v_1 by v_2 with respect to the basis defined by X. The transpose of a 1×1 matrix, or number, is this number itself while the star of a 1×1 matrix is its conjugate and either the transpose, or star, of the product of any number of matrices is the product, in the reverse order, of their transposes, or stars, as the case may be. Thus the second scalar product of any two vectors, with respect to the basis defined by X, is a symmetric function of the two factor vectors while the first is not, in general, such a symmetric function, an interchange of the two factor vectors sending the scalar product into its conjugate.

It is clear that neither the first nor second scalar product of two vectors is, in general, independent of the basis used to define it. The first scalar product of v_1 by v_2 with respect to the basis defined by X' is $c_2'^* c_1' = c_2^* A^* A c_1$ and for this to be the same as $c_2^* c_1$, no matter what are the $n \times 1$ matrices c_1 and c_2, $A^* A$ must be the n-dimensional identity matrix E_n. Similarly, in order that the second scalar product of v_1 by v_2 be the same for the basis defined by X' as it is for the basis defined by X, no matter what are the vectors v_1 and v_2, $A^t A$ must be E_n. We term any n-dimensional matrix A which satisfies the relation $A^* A = E_n$ unitary and we term any n-dimensional matrix A which satisfies the relation $A^t A = E_n$ complex orthogonal. Since $\det A^*$ is the complex conjugate of $\det A$ the determinant of any unitary matrix is of modulus unity; similarly the determinant of any complex orthogonal matrix is either 1 or -1. Thus

1) An n-dimensional unitary matrix U is one whose reciprocal is its star, $\det U$ being a complex number of unit modulus.

2) An n-dimensional complex orthogonal matrix O is one whose reciprocal is its transpose, $\det O$ being either 1 or -1.

It is clear, from the manner in which they were introduced, that if U_1 and U_2 are any two n-dimensional unitary matrices so also is $U_1 U_2$ and that if O_1 and O_2 are any two n-dimensional complex orthogonal matrices so also is $O_1 O_2$. Thus the collection of all n-dimensional unitary matrices constitutes a group which is termed the n-dimensional unitary group and the collection of all n-dimensional complex orthogonal matrices constitutes a group which is termed the n-dimensional complex orthogonal group.

When we restrict ourselves to the real field there is no distinction between the unitary group and the orthogonal group since, when A is real, $A^* = A^t$. Thus the real unitary group is the same as the real complex orthogonal group. We shall be concerned in these lectures with the real complex orthogonal group rather than the complex orthogonal group and we shall use the adjective orthogonal, without the qualifying adjective complex, in the sense of real complex orthogonal. Thus the n-dimensional orthogonal group is the n-dimensional real complex orthogonal group or, equivalently, the n-dimensional real unitary group. An n-dimensional matrix O is orthogonal if

1) O is real
2) O^t is the reciprocal of O.

Det O is either $+1$ or -1 and those orthogonal matrices whose determinant is 1 constitute a group known as the n-dimensional rotation group. We term any matrix R of this group a rotation matrix so that an n-dimensional matrix R is a rotation matrix if

1) R is real
2) R^t is the reciprocal of R
3) det R is 1.

When $n = 2$ the reciprocal of $R = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, since $\det R = 1$, and since this must be the same as R^t we have $d = a$, $c = -b$ where $a^2 + b^2 = 1$ (since $\det R = 1$). Thus any 2-dimensional rotation matrix is of the form

$$R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}; \quad c = \cos\Theta, \quad s = \sin\Theta, \quad -\pi < \Theta \leq \pi .$$

Exercise. Show that the first scalar product of any two vectors with respect to a given basis is invariant under a linear transformation $v \to v' = \mathcal{B} v$ of the n-dimensional linear vector space if, and only if, the n-dimensional matrix B which presents $\mathcal{B}$ in the given basis is unitary and, similarly, that the second scalar product with respect to the given basis is invariant under $\mathcal{B}$ if, and only if, B is complex orthogonal.

Lecture 2

The Lorentz and symplectic groups

The concepts of the first and second scalar products of any two vectors with respect to a given basis may be generalised as follows. Let M be any n-dimensional matrix, not necessarily possessing a reciprocal; then we may associate with any two vectors v_1, v_2 and the basis defined by any non-singular n-dimensional matrix X one or other of the two numbers $c_2^*Mc_1$, $c_2^tMc_1$ and we term these numbers the first and second scalar products, respectively, relative to M, of v_1 by v_2, with respect to the basis defined by X. These scalar products, relative to M, will be the same with respect to the basis defined by X', for every pair of vectors v_1 and v_2 if, and only if, $A^*MA = M$ and $A^tMA = M$, respectively, where $A = X'^{-1}X$. The collection of all non-singular n-dimensional matrices A which satisfy the relation $A^*MA = M$ constitute a group as do also the collection of all non-singular n-dimensional matrices which satisfy the relation $A^tMA = M$. The relation $A^*MA = M$ is equivalent to the relation $A'^*M'A' = M'$ where $M' = U^*MU$, $A' = U^*AU$, U being any n-dimensional unitary matrix, and so the group defined by the first scalar product relative to M' is the same as the group defined by the first scalar product ~~with respect~~ relative to M. Similarly, if $M' = O^tMO$, O being any n-dimensional complex orthogonal matrix, the group defined by the second scalar product relative to M' is the same as the group defined by the second scalar product relative to M. Furthermore it is clear that M may be multiplied by any non-zero number without affecting the groups defined by either the first or second scalar products relative to it.

Example. When $n = 2$ and $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the group defined by the second scalar product relative to M is the collection of all 2-dimensional matrices L which satisfy the equation $L^tML = M$. This is known as the 2-dimensional complex Lorentz group. The real Lorentz group is the subgroup of this obtained by restricting L to be real. Writing $L = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ -b & -d \end{pmatrix} =$$

$$= \begin{pmatrix} a^2 - b^2 & ac - bd \\ ac - bd & c^2 - d^2 \end{pmatrix}$$

so that $a^2 - b^2 = 1$, $ac - bd = 0$, $c^2 - d^2 = -1$. Since M is non-singular, $\det L = \pm 1$ and we may restrict our attention to the Lorentz matrices for which $\det L = 1$, since any Lorentz matrix whose determinant is -1 may be obtained from one of these by changing the sign of either of its two column matrices. Writing $b = \sinh\theta$, $a = \pm\cosh\theta$, $c = k\sinh\theta$, $d = \pm k\cosh\theta$ where $k = 1$ since $\det L = 1$. Thus the 2-dimensional unimodular real Lorentz group is a 1-parameter group, the typical element of the group being

$$L(\theta) = \begin{pmatrix} \pm\cosh\theta & \sinh\theta \\ \sinh\theta & \pm\cosh\theta \end{pmatrix} ; \quad -\infty < \theta < \infty$$

There are two essential differences between this group and the 2-dimensional rotation group whose typical element is

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} ; \quad -\pi < \theta \leq \pi$$

Firstly the unimodular 2-dimensional Lorentz group is divided into two distinct pieces, a typical element of the first piece, which is a subgroup of the Lorentz group, being

$$L_1(\theta) = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} ; \quad -\infty < \theta < \infty$$

and a typical element of the second piece being

$$L_2(\theta) = \begin{pmatrix} -\cosh\theta & \sinh\theta \\ \sinh\theta & -\cosh\theta \end{pmatrix} ; \quad -\infty < \theta < \infty ,$$

while the rotation group has all its elements furnished by the single formula

$$R(\Theta) = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix} \quad ; \quad -\pi < \Theta \leq \pi$$

Secondly, the parametric space of the Lorentz group is unbounded while that of the rotation group is bounded. On setting $R = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ we have $M' = R^t M R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and so the 2-dimensional Lorentz group may be presented as the collection of matrices L' which are such that

$$(L')^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

L' being $R^t L R$. Thus $L_1'(\Theta) = \begin{pmatrix} \exp -\Theta & 0 \\ 0 & \exp\Theta \end{pmatrix}$ and $L_2'(\Theta) = -\begin{pmatrix} \exp\Theta & 0 \\ 0 & \exp -\Theta \end{pmatrix}$. For any value of n the Lorentz group is the collection of all n-dimensional matrices L which satisfy the equation $L^t M L = M$ where M is the diagonal n-dimensional matrix whose first $n-1$ diagonal elements are 1, the last one being -1. Since M is non-singular $\det L$ is either 1 or -1 and the matrices L whose determinants are 1 constitute a subgroup of the Lorentz group (any element of this subgroup being known as a proper Lorentz matrix). Any non-proper Lorentz matrix may be obtained from a proper Lorentz matrix by changing the sign of one (or any odd number) of its column natrices.

When $n = 2k$ is even and M is the 2k-dimensional matrix $I = \begin{pmatrix} 0 & -E_k \\ E_k & 0 \end{pmatrix}$ the group defined by means of the second scalar product relative to I is known as the 2k-dimensional symplectic group. Since I is non-singular, its reciprocal being $-I$, any 2k-dimensional matrix which satisfies the equation $S^t I S = I$ is non-singular and so the 2k-dimensional symplectic group is the collection of all 2k-dimensional matrices which satisfy the equation $S^t I S = I$. Writing S in the form $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$, where A, B, C, D are k-dimensional matrices, we have

$$S^t I S = \begin{pmatrix} B^t A - A^t B & B^t C - A^t D \\ D^t A - C^t B & D^t C - C^t D \end{pmatrix}$$

so that S is symplectic if, and only if,

1) A^tB and C^tD are symmetric

2) $D^tA - C^tB = E_k$

For example, I is itself a 2k-dimensional symplectic matrix. When $k = 1$, A, B, C and D are 1-dimensional matrices, i.e., ordinary complex numbers, and condition 1) becomes vacuous while condition 2) states that S is unimodular, i.e., of determinant 1. Thus the 2-dimensional symplectic group is the 2-dimensional unimodular group, i.e., the collection of all 2-dimensional matrices of determinant 1. The real 2-dimensional symplectic group is a 3-parameter group; if $S = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ we may take a, b and c as our 3 parameters, d being arbitrary when $a = 0$ and determined by the formula $(1 + bc) / a$ when $a \neq 0$. The complex 2-dimensional symplectic group is a 6-parameter group (we may take as our 6 parameters the real and imaginary parts of a, b and c).

Exercise. Show that the 2k-dimensional matrix $\begin{pmatrix} E_k & C \\ 0 & E_k \end{pmatrix}$ is unimodular and that it is symplectic if, and only if, the k-dimensional matrix C is symmetric. Show, further, that the 2k-dimensional symplectic matrices of this type constitute a group which is isomorphic with the additive group of k-dimensional symmetric matrices.

Exercise. Show that if S is symplectic so also is S^t. Hint. I is symplectic and $S^t = IS^{-1}I^{-1}$.

Exercise. Show that $\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix}$ is a 2k-dimensional symplectic matrix if A is an arbitrary k-dimensional matrix.

It is a remarkable fact that the 2k-dimensional symplectic group is unimodular and not, like the orthogonal group, composed of two pieces, one consisting of unimodular matrices and the other consisting of matrices of determinant -1. To prove this we first observe that $S = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is symplectic if, and only if,

$S^{-1} = \begin{pmatrix} D^t & -C^t \\ -B^t & A^t \end{pmatrix}$ so that $\det (A^t) = (\det A) / \det S$. Thus

S is unimodular if A is non-singular. If A is singular we consider the symplectic matrix $S'' = S'S =$
$\begin{pmatrix} E_k & C' \\ 0 & E_k \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix}$, C' being an arbitrary symmetric k-dimensional matrix. $A'' = A + C'B$ is not singular for every choice of C'; for, if it were, it would follow, on taking C' to be a diagonal matrix all of whose diagonal elements are zero save one, that each row matrix of B is a linear combination of the row matrices of A and this would imply, since A is singular, that S is singular. Hence S'' is unimodular and it follows, since S' is unimodular, that $S = S'^{-1}S''$ is unimodular.

We conclude this lecture by an indication of the importance of the symplectic group in mechanics. The canonical equations of a conservative dynamical system with k degrees of freedom are

$$(p_j)_t = -H_{q^j}; \qquad (q^j)_t = H_{p_j}, \quad j = 1, 2, \ldots, k$$

where $q = (q^1, \ldots, q^j)$ and $p = (p_1, \ldots, p_j)$ are the coordinates and momenta, respectively, of the system, t is the time and $H = H(q, p)$ is the Hamiltonian function. On denoting by x the $2k \times 1$ matrix whose first k elements are those of p and whose last k elements are those of q, H_x is the $1 \times 2k$ matrix (H_p, H_q) and the canonical equations of the mechanical system appear in the form

$$x_t = I\,(H_x)^t$$

Under a differentiable transformation $x \to x'$ we have

$$x'_t = J\,x_t = J\,I\,(H_x)^t = J\,I\,(H_{x'}J)^t$$

$$= J\,I\,J^t(H_{x'})^t$$

where J is the 2k-dimensional Jacobian matrix x'_x, i.e., the matrix whose m^{th} row matrix is $(x'^m_{x^1}, \ldots, x'^m_{x^{2k}})$, $m = 1, \ldots, 2k$, and these will be of the canonical form $x'_t = I(H_{x'})^t$ if, and only if, $JIJ^t = I$, i.e., if, and only if J^t or, equivalently, J is symplectic. If f and g are any two differentiable functions of x, the Poisson bracket of f and g is $f_x\, I\, (g_x)^t$ and this will be unaffected by the differentiable transformation $x \longrightarrow x'$ if, and only if, the Jacobian matrix x'_x is symplectic.

Exercises.

1. Show that the 2-dimensional matrix $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$, whose elements are k-dimensional matrices, is both symplectic and unitary if, and only if, 1) $B^t A$ is symmetric, 2) $D = \bar{A}$, $C = -\bar{B}$, and 3) $A^*A + B^*B = E_k$. <u>Hint</u>. The reciprocal of $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ must be both $\begin{pmatrix} D^t & -C^t \\ -B^t & A^t \end{pmatrix}$ and $\begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix}$. <u>Note</u>. We term a 2k-dimensional matrix which is both symplectic and unitary U-symplectic and the collection of all such matrices is the 2k-dimensional U-symplectic group.

2. Show that if U is an arbitrary k-dimensional unitary matrix then $\begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix} = U \dotplus \bar{U}$ is a 2k-dimensional U-symplectic matrix.

3. Show that if M is alternating, i.e., if $M^t = -M$, then the second scalar product, relative to M, of any two vectors with respect to any basis, is an alternating function of the two factor vectors and deduce that the second scalar product, relative to M, of any vector by itself, with respect to any basis, is zero.

4. Show that if M is of the form N^*N, where N is any n-dimensional matrix which possesses a reciprocal, then the first scalar product, relative to M, of any vector by itself, with respect to any basis, is a non-negative real number which is zero only when the vector is the zero vector.

Lecture 3

1. The parametrisation of the unitary group.

The 1-dimensional unitary group is the collection of all complex numbers $\exp \Theta i$ of unit modulus. Thus it is a 1-parameter group whose parametric space $-\pi < \Theta \leq \pi$ is bounded. This parametric space is not closed but we close it by identifying its end points $-\pi$ and π ; thus, when we speak of a function $f(\Theta)$ of the group, rather than of Θ, we refer to a periodic function of period 2π, $f(-\pi)$ being the same as $f(\pi)$. Since $\exp \Theta i = c + si$, $c = \cos \Theta$, $s = \sin \Theta$, there is an isomorphism between the 1-dimensional unitary group, whose typical element is $\exp \Theta i$, and the 2-dimensional rotation group whose typical element is $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$. The 2-dimensional rotation group is a real representation, or realisation, of the 1-dimensional unitary group, the realisation of i being $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From this we deduce, on multiplication by $r = (a^2 + b^2)^{1/2}$, a realisation of the algebra of complex numbers $r \exp \Theta i = a + bi$, the realisation of $a + bi$ being $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

The 2-dimensional unitary group consists of the 2-dimensional matrices U which are such that $U^{-1} = U^*$. The determinant of U is a complex number $\exp \delta i$ of unit modulus and so U is the product of a unimodular 2-dimensional unitary matrix by $\exp \frac{\delta}{2} i$. If $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is a unimodular 2-dimensional matrix its reciprocal is $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ and so any unimodular 2-dimensional unitary matrix is of the form $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ where $a\bar{a} + b\bar{b} = 1$. Writing $a = a_1 + a_4 i$, $b = a_2 - a_3 i$, where a_1, a_2, a_3 and a_4 are real, we see that any unimodular 2-dimensional unitary matrix is of the form $a_1 \varepsilon_1 + a_2 \varepsilon_2 + a_3 \varepsilon_3 + a_4 \varepsilon_4$ where $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$ and

$$\varepsilon_1 = E_2, \quad \varepsilon_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \varepsilon_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Since $\varepsilon_2^{\,2} = \varepsilon_3^{\,2} = \varepsilon_4^{\,2} = - E_2$, $\varepsilon_3\varepsilon_4 = \varepsilon_2 = - \varepsilon_4\varepsilon_3$,
$\varepsilon_4\varepsilon_2 = \varepsilon_3 = - \varepsilon_2\varepsilon_4$,
$\varepsilon_2\varepsilon_3 = \varepsilon_4 = - \varepsilon_3\varepsilon_2$ we see that the unimodular 2-dimensional unitary group is a representation (in the field of complex numbers) of the group of unit real quaternions in which the representations of i, j and k are ε_2, ε_3 and ε_4, respectively. From this we obtain a 2-dimensional representation of the algebra of real quaternions in which $a + bi + cj + dk \rightarrow a\,\varepsilon_2 + b\,\varepsilon_2 + c\,\varepsilon_3 + d\,\varepsilon_4 =$

$= \begin{pmatrix} a + di & -b - ci \\ b - ci & a - di \end{pmatrix}$ and this furnishes, in turn, a 4-dimensional realisation of the algebra of real quaternions in which

$$a + bi + cj + dk \longrightarrow \begin{pmatrix} a & -d & -b & c \\ d & a & -c & -b \\ b & c & a & d \\ -c & b & -d & a \end{pmatrix}$$

the realisations of i , j , k being, respectively,

$$I = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Since, for any unimodular unitary 2-dimensional matrix $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, $a\bar{a} + b\bar{b} = 1$ we may write $|a| = c$, $|b| = s$ where $c = \cos\Theta$, $s = \sin\Theta$, $0 \leq \Theta \leq \pi/2$. When $\Theta = 0$, the matrix is of the form $D(\alpha) = \begin{pmatrix} \exp \alpha i & 0 \\ 0 & \exp -\alpha i \end{pmatrix}$ where $-\pi < \alpha \leq \pi$, α being the argument of a. In general the matrix is of the form $D(\alpha)\,U(\Theta, \sigma)$ where $\sigma = \beta + \alpha$, β being the argument of b, and

$$U(\Theta, \sigma) = \begin{pmatrix} c & -s \exp -\sigma i \\ s \exp \sigma i & c \end{pmatrix}; \quad -\pi < \sigma \leq \pi .$$

Thus the 2-dimensional unimodular unitary group is a 3-parameter group; taking Θ, α and σ as the parameters, the parametric space is

$0 \leq \theta \leq \pi/2$, $-\pi < \alpha \leq \pi$, $-\pi < \sigma \leq \pi$. Upon multiplication by $\exp \frac{\delta}{2} i$ and denoting by $D(\alpha_1, \alpha_2)$ the matrix $\begin{pmatrix} \exp \alpha_1 i & 0 \\ 0 & \exp \alpha_2 i \end{pmatrix}$ we see that the typical element of the 2-dimensional unitary group may be written in the form $D(\alpha_1, \alpha_2)\, U(\theta, \sigma)$ where $\alpha_1 = \delta/2 + \alpha$, $\alpha_2 = \delta/2 - \alpha$. Thus the 2-dimensional unitary group is a 4-parameter group; if the parameters are taken to be θ, σ, α_1 and α_2 the parametric space is $0 \leq \theta \leq \pi/2$, $-\pi < \sigma \leq \pi$, $-\pi < \alpha_1 \leq \pi$, $-\pi < \alpha_2 \leq \pi$.

When $n = 3$ we denote by $U_{12}(\theta, \sigma)$, $U_{13}(\theta, \sigma)$, $U_{23}(\theta, \sigma)$, respectively, the "plane" 3-dimensional unimodular unitary matrices

$$\begin{pmatrix} c & -s \exp -\sigma i & 0 \\ s \exp \sigma i & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} c & 0 & -s \exp -\sigma i \\ 0 & 1 & 0 \\ s \exp \sigma i & 0 & c \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \exp -\sigma i \\ 0 & s \exp \sigma i & c \end{pmatrix}$$

where $c = \cos\theta$, $s = \sin\theta$. If U is any given 3-dimensional unimodular unitary matrix we determine θ_1 and σ_1 so that $(U')^3_2$, the element in the third row and second column of $U' = U\, U^*_{23}(\theta_1, \sigma_1)$, is zero. Since $(U')^3_2 = c_1 (U)^3_2 - s_1 (\exp \sigma_1 i)(U)^3_3$ we achieve this by setting, if $(U)^3_3 = 0$, $\theta_1 = \pi/2$ (σ_1 being arbitrary) and, if $(U)^3_3 \neq 0$, $\sigma_1 = \arg (U)^3_2 - \arg (U)^3_3$, $\tan \theta_1 = |(U)^3_2| / |(U)^3_3|$, σ_1 being arbitrary if $(U)^3_2 = 0$. We next determine, in the same way, θ_2 and σ_2 so that $(U'')^3_1 = 0$, where $U'' = U'\, U^*_{13}(\theta_2, \sigma_2)$. Since $(U'')^3_2 = (U')^3_2 = 0$ the third row-matrix of U'' is a multiple of e_3^*, the multiolier being a complex number of unit modulus. Hence U'' is of the form $\begin{pmatrix} U_2 & 0 \\ 0 & \exp \alpha_3 i \end{pmatrix}$ where U_2 is a 2-dimensional unimodular unitary matrix. In other words

$$U'' = D(\alpha_1, \alpha_2, \alpha_3)\, U_{12}(\Theta_3, \sigma_3)$$

where $D(\alpha_1, \alpha_2, \alpha_3)$ is the diagonal 3-dimensional matrix whose diagonal elements are $\exp \alpha_1 i$, $\exp \alpha_2 i$, $\exp \alpha_3 i$, $\alpha_1 + \alpha_2 + \alpha_3 \equiv 0$, mod 2π. Thus

$$U = U'\, U_{23}(\Theta_1, \sigma_1) = U''\, U_{13}(\Theta_2, \sigma_2)\, U_{23}(\Theta_1, \sigma_1) =$$

$$= D(\alpha_1, \alpha_2, \alpha_3)\, U_{12}(\Theta_3, \sigma_3)\, U_{13}(\Theta_2, \sigma_2)\, U_{23}(\Theta_1, \sigma_1)$$

Upon multiplication by $\exp \frac{\delta}{3} i$, where $\exp \delta i$ is the determinant of any element, not necessarily unimodular, of the 3-dimensional unitary group, we see that the typical element of this group is of the form

$$D(\alpha_1, \alpha_2, \alpha_3)\, U_{12}(\Theta_3, \sigma_3)\, U_{13}(\Theta_2, \sigma_2)\, U_{23}(\Theta_1, \sigma_1)$$

the restriction $\alpha_1 + \alpha_2 + \alpha_3 \equiv 0$, mod 2π, being now withdrawn. Thus the 3-dimensional unitary group is a 9-parameter group and, if we use the angles Θ, σ and α as parameters, the three angles Θ vary over the first quadrant while the six angles σ and α are unrestricted over the interval $(-\pi, \pi]$.

When $n = 4$, the same argument shows that the typical element of the 4-dimensional unitary group may be written in the form

$$D(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\, U_{12}(\Theta_6, \sigma_6)\, U_{13}(\Theta_5, \sigma_5)\, U_{23}(\Theta_4, \sigma_4)\, U_{14}(\Theta_3, \sigma_3)\,.$$

$$U_{24}(\Theta_2, \sigma_2)\, U_{34}(\Theta_1, \sigma_1)$$

where $D(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the 4-dimensional diagonal matrix whose diagonal elements are $\exp \alpha_1 i$, $\exp \alpha_2 i$, $\exp \alpha_3 i$, $\exp \alpha_4 i$ and, for example,

$$U_{23}(\theta,\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s\exp -\sigma i & 0 \\ 0 & s\exp\sigma i & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \begin{array}{l} c = \cos\theta,\ s = \sin\theta \\ 0 \leq \theta \leq \pi/2, \\ -\pi < \sigma \leq \pi \end{array}$$

Thus the 4-dimensional unitary group is a 16-parameter group, there being 4 parameters α, $3+2+1 = 6$ parameters θ and 6 parameters σ. Similarly, the n-dimensional unitary group is an n^2-parameter group, there being n parameters α, $(n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}$ parameters θ and $\frac{n(n-1)}{2}$ parameters σ. In the parametric space the parameters θ vary over the interval $0 \leq \theta \leq \pi/2$ while the parameters σ and α vary over the interval $(-\pi, \pi]$. The first factor (on the left) in the factorisation of an n-dimensional unitary matrix is a diagonal n-dimensional matrix $D(\alpha_1, \dots, \alpha_n)$ whose diagonal elements are $\exp\alpha_1 i, \dots, \exp\alpha_n i$ and the remaining factors, each of which is a "plane" unimodular n-dimensional unitary matrix, may be grouped into n-1 sets of which the first, counting from the right, contains n-1 factors and is

$$U_{1n}(\theta_{n-1}, \sigma_{n-1})\, U_{2n}(\theta_{n-2}, \sigma_{n-2}) \dots U_{n-1\ n}(\theta_1, \sigma_1)$$

The next set, counting from the right, contains n-2 factors and is

$$U_{1\ n-1}(\theta_{2n-3}, \sigma_{2n-3}) \dots U_{n-2,n-1}(\theta_n, \sigma_n)$$

and so on to the last set which contains the single factor $U_{12}(\theta_N, \sigma_N)$ where $N = \frac{n(n-1)}{2}$. The n-dimensional unimodular unitary group is the (n^2-1)-parameter group obtained by subjecting the parameters α to the constraint $\alpha_1 + \alpha_2 + \dots + \alpha_n \equiv 0$, mod 2π.

2. <u>The parametrisation of the rotation group</u>

When we restrict ourselves to the real field the n-dimensional unitary group becomes the n-dimensional orthogonal (= real unitary) group. Since any orthogonal n-dimensional matrix of determinant -1 (= n-dimensional reflexion matrix) may be obtained by multiplying an n-dimensional rotation matrix (= orthogonal matrix of determinant 1) by the n-dimensional diagonal matrix whose diagonal elements are all 1 save the last, which is -1, we may, when we wish a parametrisation of the orthogonal group, confine our attention to the rotation group. The typical element of the 2-dimensional rotation group is

$$R(\Theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}; \quad c = \cos\Theta, \quad s = \sin\Theta; \quad -\pi < \Theta \leq \pi$$

so that this group is a 1-parameter group whose paramteric space is, when we adopt Θ as the parameter, the interval $-\pi < \Theta \leq \pi$. When $n = 3$ we denote by $R_{12}(\Theta)$, $R_{13}(\Theta)$, $R_{23}(\Theta)$, respectively, the "plane" 3-dimensional rotation matrices

$$\begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}, \quad c = \cos\Theta, \quad s = \sin\Theta.$$

If R is any 3-dimensional rotation matrix we first determine ϕ_1 so that $(R')^3_2 = 0$, $(R')^3_3 > 0$ where $R' = R\,R^t_{23}(\phi_1)$. Since $(R')^3_2 = c\,(R)^3_2 - s\,(R)^3_3$, $(R')^3_3 = s\,(R)^3_2 + c\,(R)^3_3$, we achieve this, if not both $(R)^3_2$ and $(R)^3_3$ are zero, by setting $c = k\,(R)^3_3$, $s = k\,(R)^3_2$ where $k^{-2} = \{(R)^3_2\}^2 + \{(R)^3_3\}^2$ and $k > 0$. If both $(R)^3_2$ and $(R)^3_3$ are zero we set $\phi_1 = 0$. We next determine Θ so that $(R'')^3_1 = 0$, $(R'')^3_3 > 0$, where $R'' = R'\,R^t_{13}(\Theta)$. Since $(R'')^3_1 = c\,(R')^3_1 - s\,(R')^3_3$, $(R'')^3_3 = s\,(R')^3_1 + c\,(R')^3_3$, Θ is determined by the equations $\cos\Theta = k\,(R')^3_3$, $\sin\Theta = k\,(R')^3_1$ where $k^{-2} = \{(R')^3_1\}^2 + \{(R')^3_3\}^2$ and $k > 0$ (not both $(R')^3_1$

and $(R')^3_3$ being zero since $(R')^3_2 = 0$ and R' possesses a reciprocal). Since $(R')^3_3 \geqslant 0$, Θ lies in the interval $-\pi/2 \leqq \Theta \leqq \pi/2$. Furthermore, since $(R')^3_2 = 0$, $k = 1$ (R' being a 3-dimensional rotation matrix) and $(R'')^3_3 = 1$. Hence the last column matrix of R'' is e_3 and R'' is a plane 3-dimensional rotation matrix, $R_{12}(\phi_2)$, so that

$$R = R' R_{23}(\phi_1) = R'' R_{13}(\Theta) R_{23}(\phi_1) = R_{12}(\phi_2) R_{13}(\Theta) R_{23}(\phi_2) .$$

Thus, the 3-dimensional rotation group is a 3-parameter group and, if the angles ϕ_1, ϕ_2, and Θ are taken as the parameters, the parametric space is $-\pi < \phi_1 \leqq \pi$, $-\pi < \phi_2 \leqq \pi$, $-\pi/2 \leqq \Theta \leqq \pi/2$. We refer to ϕ_1 and ϕ_2 as longitude angles and to Θ as a latitude angle.

The parameterisation of the 3-dimensional rotation group which we have just given is a modification of a parametrisation of this group which was given by Euler. If we write

$$R = R_{12}(\phi) R_{31}(\Theta) R_{12}(\psi)$$

where $R_{31}(\Theta) = R^t_{13}(\Theta)$, the angles ϕ, Θ, ψ, of which ϕ and ψ are longitude angles while Θ is a latitude angle, are the three Eulerian angles which serve to specify any 3-dimensional rotation. However, the repetition of the factor R_{12} has certain disadvantages and we adopt the factorisation $R_{12}(\phi_2) R_{13}(\Theta) R_{23}(\phi_1)$ in which none of the planes, in which the various plane rotations are performed, occurs twice.

When $n = 4$ we denote by $R_{12}(\Theta)$ the plane 4-dimensional rotation matrix

$$R_{12}(\Theta) = \begin{pmatrix} c & -s & 0 \\ s & c & \\ 0 & & E_2 \end{pmatrix} ; \quad c = \cos\Theta , \quad s = \sin\Theta$$

and so on. We first determine the longitude angle ϕ_1 so that $(R')^4_2 = 0$, $(R')^4_4 \geqslant 0$, where $R' = R R^t_{24}(\phi_1)$, and then the latitude angle Θ_1 so that $(R'')^4_1 = 0$, $(R'')^4_4 \geqslant 0$ where $R'' = R' R^t_{14}(\Theta_1)$. We next determine the longitude angle ϕ_2 and the latitude angle θ_2 so that, in turn, $(R''')_2^3 = 0$, where $R''' = R'' R_{23}^t(\phi_2)$, and $(R'''')_1^3 = 0$, where $R'''' = R''' R_{13}^t(\theta_2)$. Then, the elements in the last two rows and first two columns of R'''' are zero which implies that the 2×2 matrix of the elements in the last two rows and last two columns of R'''' is a 2-dimensional orthogonal matrix and, since the two diagonal elements of this 2×2 matrix do not have opposite signs, this 2-dimensional orthogonal matrix will be a 2-dimensional rotation matrix unless both of its diagonal elements are zero, the remaining two elements being both 1 or both -1. Thus, the elements in the first two rows and last two columns of R'''' are zero and R'''' is a diagonal 2-dimensional block matrix, which we denote by $D(\beta) = D(\beta_1, \beta_2)$, the diagonal blocks of $D(\beta)$ being 2-dimensional rotation matrices or 2-dimensional reflexion matrices. Hence

$$R = R' R_{24}(\phi_1) = R'' R_{14}(\theta_1) R_{24}(\phi_1) = R''' R_{23}(\phi_2) R_{14}(\theta_1) R_{24}(\phi_1) =$$

$$= R'''' R_{13}(\theta_2) R_{23}(\phi_2) R_{14}(\theta_1) R_{24}(\phi_1) =$$

$$= D(\beta) R_{13}(\theta_2) R_{23}(\phi_2) R_{14}(\theta_1) R_{24}(\phi_1) .$$

Thus, the 4-dimensional rotation group is a 6-parameter group; four of the six parameters, the ϕ's and the β's, are longitude angles and two of them, the θ's, are latitude angles.

When $n = 5$, we first determine the longitude angle ϕ_1 and the three latitude angles $\theta_1, \theta_2, \theta_3$ so that the first four elements of the last row matrix of $R' = R R_{45}^t(\phi_1) R_{35}^t(\theta_1) R_{25}^t(\theta_2) R_{15}^t(\theta_3)$ are zero, the last element being 1. Then the last row-matrix and column-matrix of R' are e_5^* and e_5, respectively, and the matrix of the first four rows and first four columns of R' is a 4-dimensional rotation matrix. Thus, the 5-dimensional rotation group is a 10-

parameter group, the typical element of the group being factorisable as follows:

$$R = D(\beta)\, R_{13}(\Theta_5)\, R_{23}(\phi_3)\, R_{14}(\Theta_4)\, R_{24}(\phi_2)\, R_{15}(\Theta_3)\, R_{25}(\Theta_2)\, R_{35}(\Theta_1)\, R_{45}(\phi_1);$$

five of the 10 parameters, the ϕ's and the β's, are longitude angles, the Θ's being latitude angles.

Continuing in this way we see that, when $n = 2k$ is even, the n-dimensional rotation group is a $\frac{1}{2}n(n-1) = k(2k-1)$-parameter group; of these parameters $3k-2$ are longitude angles and $2(k-1)^2$ are latitude angles. When $n = 2k+1$ is odd, the n-dimensional rotation group is a $\frac{1}{2}n(n-1) = k(2k+1)$-parameter group and of these parameters $3k-1$ are longitude angles and $2k^2 - 2k + 1$ are latitude angles. The first factor $D(\beta)$ on the left in the factorisation of an arbitrary n-dimensional rotation matrix is, whether n is even or odd, either $R_{12}(\beta_1) \ldots R_{2k-1,2k}(\beta_k)$ or the product of this by a diagonal block matrix whose diagonal elements are either E_2 or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the number of elements $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, if there are any such, being even.

Lecture 4

1. <u>The class and in-class parameters of the n-dimensional unitary group.</u>

Let B be the n-dimensional matrix which presents, in the basis defined by any given non-singular n-dimensional matrix X, a linear transformation $v \to w$ in our n-dimensional linear vector space so that $d = Bc$, v being defined by X, c and w by X, d. Then the non-singular n-dimensional matrix $X' = XU$, where U is any n-dimensional unitary matrix, defines a basis in which the presentation of the linear transformation $v \to w$ is $B' = U^*BU$; for $Xd = X'd' = XUd'$ so that $d' = U^*d = U^*Bc = U^*BUc'$. We propose to show that U may be so determined, no matter what is the n-dimensional matrix B (not necessarily non-singular), so that B' is triangular (with zeros below the diagonal). To do this we first observe that any non-singular n-dimensional matrix X may be written in the form UT where U is an n-dimensional unitary matrix and T is a triangular n-dimensional matrix with zeros below the diagonal. To achieve this we first set u_1 = the quotient of x_1 by the magnitude $(x_1^*x_1)^{1/2} = m_1$ of x_1 (this magnitude not being zero since X is non-singular); we then set u_2 = the quotient of $x_2 - (u_1^*x_2)u_1$ by its magnitude m_2 (this magnitude not being zero since X is non-singular); we next set u_3 = the quotient of $x_3 - (u_1^*x_3)u_1 - (u_2^*x_3)u_2$ by its magnitude m_3 (this magnitude not being zero since X is non-singular) and so on. Then $x_1 = m_1u_1$; $x_2 = (u_1^*x_2)u_1 + m_2u_2$; $x_3 = (u_1^*x_3)u_1 + (u_2^*x_3)u_2 + m_3u_3$ and so on so that $X = UT$

where U is the n-dimensional matrix whose column matrices are $u_1, u_2, \ldots, u_n$ and T is the n-dimensional matrix (with zeros below the diagonal) whose column matrices are m_1e_1, $(u_1^*x_2)e_1 + m_2e_2$, $(u_1^*x_3)e_1 + (u_2^*x_3)e_2 + m_3e_3$ and so on. It is clear that $u_j^*u_j = 1$, $j = 1, \ldots, n$, $u_j^*u_k = 0$, $j < k$, and so $U^*U = E_n$ so that U is an n-dimensional unitary matrix. It follows that any $n \times 1$ matrix x_1 of unit magnitude (i.e., such that $x_1^*x_1 = 1$) may be taken to be the first column matrix of a unitary $n \times n$ matrix; all we have to do is to select any set of $n-1$ $n \times 1$ matrices $x_2, \ldots, x_n$ which are such that the n-dimensional matrix X whose column matrices are $x_1, \ldots, x_n$ is non-singular and write $X = UT$. Since $m_1 = 1$, the first column matrix of T is e_1 and $x_1 = u_1$.

Let, now, λ_1 be any characteristic number of B and x_1 any associated characteristic $n \times 1$ matrix of unit magnitude. If V_1 is a unitary $n \times n$ matrix whose first column matrix is x_1 we have $x_1 = V_1e_1$ and, since $Bx_1 = \lambda_1x_1$, $BV_1e_1 = \lambda_1V_1e_1$ so that $V_1^*BV_1e_1 = \lambda_1e_1$. Thus the first column matrix of $V_1^*BV_1$ is λ_1e_1 and $V_1^*BV_1$ is of the form $\begin{pmatrix} \lambda_1 & c_1^* \\ 0 & B' \end{pmatrix}$ where c_1 is some $(n-1) \times 1$ matrix. Applying the same argument to the $(n-1)$-dimensional matrix B' we see that $V'^*B'V'$ is of the form $\begin{pmatrix} \lambda_2 & c_2^* \\ 0 & B'' \end{pmatrix}$ where λ_2 is a characteristic number of B' and, hence, of B, c_2 is some $(n-2) \times 1$ matrix, B'' is some $(n-2) \times (n-2)$ matrix and V' is a unitary $(n-1) \times (n-1)$ matrix. On denoting the unitary $n \times n$ matrix $\begin{pmatrix} 1 & 0 \\ 0 & V' \end{pmatrix}$ by V_2 it follows that $V_2^*V_1^*BV_1V_2 = \begin{pmatrix} \lambda_1 & & c_1^* \\ 0 & \lambda_2 & c_2^* \\ 0 & 0 & B'' \end{pmatrix}$.

Continuing this argument we see that there exists an n-dimensional unitary matrix $V = V_1V_2 \ldots V_{n-1}$ such that V^*BV is a triangular n-dimensional matrix with zeros below the diagonal. If $B = U$ is unitary so also is V^*BV and so V^*UV is of the form $D(\alpha_1, \alpha_2, \dots, \alpha_n)$ where $D(\alpha_1, \dots, \alpha_n)$ is the diagonal $n \times n$ matrix whose column matrices are $(\exp \alpha_1 i)e_1, \dots, (\exp \alpha_n i)e_n$. The set of n numbers $\alpha_1, \dots \alpha_n$ is determined by the linear operator whose presentation in the basis defined by X is U, being the same for W^*UW as for U, where W is any n-dimensional unitary matrix. The collection of n-dimensional unitary matrices W^*UW obtained by letting W vary over the n-dimensional unitary group, U being held fixed, is termed a class of the group and the unordered collection of n numbers $\alpha_1, \dots, \alpha_n$ defines this class. This collection is unordered since, if W is the n-dimensional unitary matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \\ 0 & & E_{n-2} \end{pmatrix}$, for example,

$W^* D(\alpha_1, \alpha_2, \dots, \alpha_n) W = D(\alpha_2, \alpha_1, \alpha_3, \dots, \alpha_n)$. We may take $\alpha_1, \alpha_2, \dots, \alpha_n$, where $\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_n$, as n of the n^2 parameters of the n-dimensional unitary group the remaining $n^2 - n$ being obtained by writing V^* in the form $D(\beta_1, \dots, \beta_n) V_1$ where V_1 is the product of $\frac{1}{2} n(n-1)$ plane unitary matrices $U_{pq}(\theta, \sigma)$. Then $U = V D(\alpha_1, \dots, \alpha_n) V^* = V_1^* D(\alpha_1, \dots, \alpha_n) V_1$, since $D^*(\beta_1, \dots, \beta_n) D(\alpha_1, \dots, \alpha_n) D(\beta_1, \dots, \beta_n) = D(\alpha_1, \dots, \alpha_n)$. We term $\alpha_1, \dots, \alpha_n$ the class parameters and the θ's and σ's the in-class parameters (since they tell us where, in a given class $\alpha = (\alpha_1, \dots, \alpha_n)$, a particular element of the group is located).

2. The class and in-class parameters of the n-dimensional rotation group.

The 2-dimensional rotation group, whose typical element is

$R(\Theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$, $c = \cos\Theta$, $s = \sin\Theta$, $-\pi < \Theta \leq \pi$, is a commutative group and so, if $R(\emptyset)$ is a variable element of the group, $R^*(\emptyset)\, R(\Theta)\, R(\emptyset) = R(\Theta)$. Thus every class of the 2-dimensional rotation group consists of a single element and the single parameter Θ is a class parameter. To treat the case $n > 2$ we first observe that the argument given in the complex field shows that any non-singular real n-dimensional matrix may be written in the form $R\,T$ where R is an n-dimensional rotation matrix and T is a real ~~diagonal~~ triangular matrix with zeros below the diagonal. Secondly, the argument given in the complex field shows that, if λ_1 is a real characteristic number of B , then there exists an n-dimensional rotation matrix R such that $R^t B R$ is of the form $\begin{pmatrix} \lambda_1 & c^t \\ 0 & B' \end{pmatrix}$ where c is some real $(n-1) \times 1$ matrix and B' is some real $(n-1)$-dimensional matrix. Proceeding until we have exhausted the real characteristic numbers of B we may be confronted by a real matrix C which does not have any real characteristic number. This implies that C does not have a real characteristic $n \times 1$ matrix for, if x were real, the equation $Cx = \lambda x$ would force λ to be real. Hence the dimension of C is even and we denote this dimension by $2j$. If $x_1 + x_2 i$, where x_1 and x_2 are real, is a characteristic $2j \times 1$ matrix of C associated with a non-real characteristic number λ , so that $x_2 \neq 0$, x_1 is not a multiple of x_2 for, if it were, x_2 would be a characteristic $2j \times 1$ matrix of C associated with λ . If $x_3, \ldots, x_j$ is any set of $2j-2$ real $2j \times 1$ matrices which is such that the real $2j$-dimensional matrix X whose column matrices are $x_1, \ldots, x_{2j}$ is non-singular we may write $X = RT$ where R is a $2j$-dimensional rotation matrix and this implies that x_1 and x_2 are linear combinations of r_1 and r_2, the first and second column matrices of R, and that r_1 and r_2 are linear combinations of x_1 and x_2. Now $C x_1$ and $C x_2$, being the real and imaginary parts of λx, are linear combinations, with real coefficients, of x_1 and x_2 and, hence, of r_1 and r_2 and so $C r_1 = C R e_1$ and $C r_2 = C R e_2$ are linear combinations, with real coefficients, of $r_1 = Re_1$ and $r_2 = Re_2$. Thus the first and second column matrices R^tCRe_1 and R^tCRe_2 of R^tCR are linear combinations, with real coefficients, of e_1 and e_2 so that R^tCR is of the form $R^tCR = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ where A is some 2-dimensional matrix, B is some $2 \times 2j-2$ matrix, 0 is the zero $2j-2 \times 2$ matrix, and D is some $(2j-2)$-dimensional matrix. Continuing this argument we see that, if C is any $2j$-dimensional real matrix which does not possess a real characteristic number, there exists a $2j$-dimensional rotation matrix R such that R^tCR is a triangular j-dimensional block matrix, whose elements are 2-dimensional matrices, the elements below the diagonal being zero. Hence, B being any real n-dimensional matrix with m real and $2j$ non-real characteristic numbers, where $n = m + 2j$, there exists an n-dimensional rotation matrix R such that R^tBR is of the form

$$R^tBR = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$$

where B_1 is some triangular m-dimensional real matrix, with zeros below the diagonal, B_2 is some $m \times n-m$ real matrix, 0 is the zero $n-m \times m$ matrix, and B_3 is some triangular j-dimensional block matrix, whose elements are 2-dimensional matrices, the elements below the diagonal being zero. If B is an orthogonal (= real unitary) matrix the diagonal elements of B_1 (being the real characteristic numbers of B) are 1 or -1 and so, since R^tBR is orthogonal, B_1 is a diagonal matrix and B_2 is the zero $m \times n-m$ matrix. Similarly B_3 is a diagonal block matrix whose diagonal elements are 2-dimensional rotation matrices. If $n = 2k$ is even and B is a rotation matrix there will be an even number of -1's and an even number of $+1$'s in the diagonal of B_1 and, since $E_2 = R(0)$; $-E_2 = R(\pi)$, where $R(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$, $c = \cos\theta$, $s = \sin\theta$, we may, by transforming, if necessary, B_1 by means of a permutation matrix so as to have the $+1$'s and -1's together, write B_1 as a $(k-j)$-dimensional diagonal block matrix whose diagonal elements are 2-dimensional rotation matrices. Hence we have the following result:

If R is any $2k$-dimensional rotation matrix there exists a $2k$-dimensional rotation matrix R_1 such that $R_1^tRR_1$ is of the form $R(\alpha_1, \alpha_2, \ldots, \alpha_k)$ where $R(\alpha_1, \ldots, \alpha_k)$ is a k-dimensional diagonal block matrix whose diagonal elements are $\begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix}$, ..., $\begin{pmatrix} c_k & -s_k \\ s_k & c_k \end{pmatrix}$, $c_j = \cos\alpha_j$, $s_j = \sin\alpha_j$, $j = 1, \ldots, k$. We term the unordered set $\alpha_1, \ldots, \alpha_k$ the angles of R .

If R is any $(2k+1)$-dimensional rotation matrix there exists, similarly, a $(2k+1)$-dimensional rotation matrix R_1 such that $R_1^t R R_1$ is of the form $R(\alpha_1, \ldots, \alpha_k) = \begin{pmatrix} R_{2k} & 0 \\ 0 & 1 \end{pmatrix}$ where R_{2k} is a k-dimensional diagonal block matrix whose diagonal elements are 2-dimensional rotation matrices. In either case we may use the k angles $\alpha_1, \ldots, \alpha_k$, so ordered that $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_k$, as k of the parameters of the n-dimensional rotation group. When $n = 2k$ is even a change of sign of an even number of these class parameters does not change the class since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$

and $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, for example, is a 4-dimensional rotation matrix even though $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not a 2-dimensional rotation matrix. On the other hand, when $n = 2k+1$ is odd, a change of sign of any number, even or odd, of the class parameters does not change the class since $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, for example, is a 3-dimensional rotation matrix even though $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not a 2-dimensional rotation matrix. Thus a class function of the n-dimensional rotation group is, when $n = 2k+1$ is odd, an even symmetric function of the class parameters $\alpha_1, \ldots, \alpha_k$ while, when $n = 2k$ is even, it is a symmetric function of $\alpha_1, \ldots, \alpha_k$ which remains unchanged when two of these class parameters are changed in sign.

Since $R = R_1 R(\alpha_1, \ldots, \alpha_k) R_1^t$ and since $R(\alpha_1 \ldots, \alpha_k)$ commutes with the diagonal k-dimensional block matrix $D(\beta)$, where $R_1^t = D(\beta) R_1'^t$, $R_1'^t$ being a product of plane n-dimensional rotation matrices R_{pq}, the k angles β disappear from the product $R_1 R(\alpha_1, \ldots, \alpha_k) R_1^t$ and we may write $R = R_1' R(\alpha_1, \ldots, \alpha_k) R_1'^t$. When $n = 2k$ is even, R_1' involves $2k(k-1)$ parameters of which $2(k-1)$ are longitude angles and $2(k-1)^2$ are latitude angles. The k class parameters are $\alpha_1, \ldots, \alpha_k$ and $k-1$ of these vary over the interval $0 \leq \alpha \leq \pi$, the remaining one varying over the interval $-\pi \leq \alpha \leq \pi$. When $n = 2k+1$ is odd, R_1' involves $2k^2$ parameters of which $2k-1$ are longitude angles and $2k^2-2k+1$ are latitude angles. The k class parameters are $\alpha_1, \ldots, \alpha_k$ and each of these varies over the interval $0 \leq \alpha \leq \pi$.

Lecture 5

1. Representations of a matrix group.

If we have, associated with each element A of an n-dimensional matrix group, a linear transformation α, possessing a reciprocal, of an m-dimensional linear vector space, the association being such that if $A_1 \rightarrow \alpha_1$, $A_2 \rightarrow \alpha_2$, A_1 and A_2 being any two elements of the matrix group, then $A_2A_1 \rightarrow \alpha_2\alpha_1$, we term the collection of linear operators α an m-dimensional representation of the matrix group. To the identity element E_n of the matrix group corresponds the identity transformation $\mathcal{E}$ of the m-dimensional linear vector space since, if $E_n \rightarrow \mathcal{E}$, $\mathcal{E}^2 = \mathcal{E}$ which implies, since $\mathcal{E}$ possesses a reciprocal, that $\mathcal{E}$ is the identity transformation. It follows that, if $A \rightarrow \alpha$, then $A^{-1} \rightarrow \alpha^{-1}$. We shall, if A' is the m-dimensional matrix which represents, in the basis defined by any m-dimensional matrix X which possesses a reciprocal, the linear operator α, refer to the collection of matrices A' as a representation $A \rightarrow A'$ of our n-dimensional matrix group although, strictly speaking, this is a misnomer. The representation is the collection of linear operators α and the collection of m-dimensional matrices A' is merely the presentation of this representation in the basis defined by the non-singular m-dimensional matrix X. Thus, in our loose manner of speaking, the collection of matrices $A'' = CA'C^{-1}$, where C is any m-dimensional matrix which possesses a reciprocal, is the same representation as the collection of matrices A'.

Every matrix group possesses the following representations:

1) The identity representation; $m = 1$; $A' = 1$.

2) The determinant representation; $m = 1$; $A' = \det A$.

3) The modulus of the determinant representation; $m = 1$, $A' = |\det A|$.

4) The self-representation; $m = n$, $A' = A$.

5) The conjugate representation; $m = n$, $A'' = \bar{A}$.

If $A \to A'$ is any 1-dimensional representation of any matrix group, $(A')^p$, where p is any integer, belongs to the representation since A^p belongs to the group. Hence the representation is unbounded unless $|A'| = 1$. If $A \to A'$ is any m-dimensional representation of any matrix group, $A \to \det A'$ is a 1-dimensional representation of the group and so $|\det A'| = 1$ if the collection of numbers $\det A'$ is bounded. This will be the case if the n-dimensional matrix group is an r-parameter group whose parametric space is bounded and closed (in which case we shall refer to it as an n-dimensional compact group) provided the representation $A \to A'$ is continuous; for, then, the elements of A' are continuous functions of the points a of the parametric space so that $\det A'$ is bounded since the parametric space is bounded and closed.

2. The adjoint representation of an r-parameter matrix group.

Let Y be a typical element of an r-parameter matrix group and let y be the point of the parametric space to which Y corresponds. We term the transformation $Y \to Z = AY$ of the matrix group into itself, where A is any fixed element of the group, the left translation of the group which is induced by A and we indicate the corresponding transformation of the parametric space of the group by writing $z = ay$; similarly $Y \rightarrow Z' = YA$ is the right translation of the group which is induced by A , the corresponding transformation of the parametric space being indicated by writing $z' = ya$. We assume that $z = ay$ is a continuously differentiable function of a and y and that y^{-1} is a continuous function of y ; then, since $a = zy^{-1}$, the r-dimensional Jacobian matrix z_y , which is a continuous function of a and y , is a continuous function of y and z and we denote this continuous matrix function by $J(y, z)$. If we follow the left translation $Y \rightarrow Z = AY$ by the left translation $Z \rightarrow W = BZ$ we obtain the left translation $Y \rightarrow W = (BA)Y$ and the relation $w_y = w_z z_y$ tells us that $J(y, w) = J(z, w) J(y, z)$. The three points y , z and w may be taken arbitrarily, a being $z y^{-1}$ and b being $w z^{-1}$; regarding z as fixed and y and w as variable we see that $J(y, w)$ is the product of an r-dimensional matrix function of y alone by an r-dimensional matrix function of w alone. When $b = a^{-1}$, $w = y$ so that $J(y, w) = J(y, y) = E_r$ and $J(y, z)$ is the reciprocal of $J(z, w) = J(z, y)$. We denote $J(y, z)$ simply by $J(y)$ and have the relation $J(y, w) = J^{-1}(w) J(y)$. Similarly, for the right translation $Y \rightarrow Z' = YA$, we have, on denoting z'_y by $J'(y, z')$, $J'(y, w) = \{J'(w)\}^{-1} J'(y)$ where $J'(y)$ is an abbreviation for $J'(y, z')$, z' being any fixed point of the parametric space. We refer to z and z' as our base points, for left and right translations, respectively, and we shall usually take for the base points z and z' the point e of the parametric space which corresponds to the identity element E_n of our n-dimensional matrix group. Under a change of base point $z \rightarrow s$, $J(y)$ is multiplied by $J^{-1}(s) = J(z, s)$ and, under a change of base point

$z' \to s'$, $J'(y)$ is multiplied by $J'(z', s')$.

The collection of matrices $Y' = AYA^{-1}$, where A is any fixed element, and Y a variable element, of our n-dimensional matrix group, is a presentation of the self-representation of the group and the transformation $Y \to Y' = AYA^{-1}$ sends the identity element E_n of the group into itself, so that e is a fixed point of the corresponding transformation $y \to y' = aya^{-1}$ of the parametric space of the group. We may regard this transformation as the result of first performing the transformation $y \to w = ay$ and then performing the transformation $w \to y' = wa^{-1}$ and, since $w_y = J(y, w) = J^{-1}(w) J(y)$ and, $y'_w = J'(w,y') = \{J'(y')\}^{-1} J'(w)$, we have $y'_y =$ $\{J'(y')\}^{-1} J'(w) J^{-1}(w) J(y) = \{J'(y')\}^{-1} J'(ay) J^{-1}(ay) J(y)$ and upon evaluating this relation at $y = e$, we obtain $(y'_y)_{y=e} =$ $\{J'(e)\}^{-1} J'(a) J^{-1}(a) J(e)$. Taking e as our base point, for both left and right translations, so that $J(e)$ and $J'(e)$ are each the r-dimensional identity matrix E_r, this reduces to $(y'_y)_{y=e} =$ $J'(a) J^{-1}(a) = A'$, say, and it follows that the correspondence $A \to A'$ furnishes an r-dimensional representation of our matrix group; indeed, if we follow the transformation $Y \to Y' = AYA^{-1}$ of the matrix group by the transformation $Y' \to Y'' = BY'B^{-1}$ we obtain the transformation $Y \to Y'' = (BA) Y (BA)^{-1}$ and the relation $(y''_y)_{y=e} = (y''_{y'})_{y'=e} (y'_y)_{y=e}$ tells us that $(BA)' = B'A'$. This r-dimensional representation is known as the adjoint representation of the r-parameter matrix group; if the group is compact $|\det A'| = 1$ so that $|\det J'(a)| = |\det J(a)|$. It is clear that the adjoint representation is independent of the choice of parameters (only the presentation of this representation being affected by a differentiable transformation of parameters $y \to \tilde{y}$) ; for $A' \to \tilde{A}' = (\tilde{y}'_{\tilde{y}})_{\tilde{y}=\tilde{e}} = B^{-1}A'B$ where B is the Jacobian matrix of y with respect to $\tilde{y}$ evaluated at $\tilde{y} = \tilde{e}$.

Upon differentiating with respect to y the relation $Y' = AYA^{-1}$ and evaluating the result at $y = e$ we obtain

$$\left\{ \sum_{k=1}^{r} Y'_{y'^k} \, y'^k_{y^j} \right\}_{y=e} = A\,(Y_{y^j})_{y=e}\,A^{-1} \quad , \quad j = 1, \ldots, r$$

and we may write this relation in the form

$$\sum_{k=1}^{r} M_k\,(A')^k_j = A\,M_j\,A^{-1} \quad , \quad j = 1, \ldots, r$$

where $M_k = (Y_{y^k})_{y=e}$, $k = 1, \ldots, r$. We term the matrices $M_1, \ldots, M_r$ the characteristic matrices of our n-dimensional matrix group with respect to the parameters y and we see that, if the characteristic matrices are linearly independent in the real field, the adjoint representation may be determined by finding what linear combinations of $M_1, \ldots, M_r$ are $A M_1 A^{-1}, \ldots, A M_r A^{-1}$. If we set $M_j = \sum_{p=1}^{r} N_p (C)^p_j$, where the r-dimensional complex matrix C is non-singular, we have

$$\sum_{p=1}^{r} \sum_{k=1}^{r} N_p (C)^p_k \,(A')^k_j = \sum_{p=1}^{r} A\,N_p\,A^{-1}(C)^p_j \; , \quad j = 1, \ldots, r,$$

or, equivalently,

$$\sum_{p=1}^{r} \sum_{k=1}^{r} \sum_{j=1}^{r} N_p\,(CA'C^{-1})^p_q = A\,N_q\,A^{-1} \quad , \quad q = 1, \ldots, r \; .$$

Thus we may use any set of r linear combinations of the characteristic matrices which possesses a non-singular matrix instead of these matrices themselves, obtaining in this way merely another presentation $A \to CA'C^{-1}$

of the adjoint representation.

If U is any element of the n-dimensional unitary group we have $\overset{\tau}{U}{}^*U = E_n$ and, upon differentiating this relation with respect to the $n^2 \times 1$ parameter matrix and evaluating the result at the identity point e of the parametric space, we obtain $M_j^* + M_j = 0$, $j = 1, \ldots, n^2$. Thus each of the n^2 characteristic matrices is the product of an n-dimensional Hermitian matrix H, i.e. a matrix which is such that $\overset{\tau}{H}{}^* = H$, by i:

$$M_j = i H_j ; \qquad j = 1, \ldots, n^2 .$$

The derivative of $\det U$ with respect to y^j is the product of $\sum_{p,q} (U^*)^p_q (u^q_p)_{y^j}$ by $\det U$ so that $(\log \det U)_{y^j} = \mathrm{Tr}(U^* U_{y^j})$ where by $\mathrm{Tr} A$, where A is any square matrix, we mean the trace, i.e., the sum of the diagonal elements, of A. Upon evaluating this relation at the identity point e of the parametric space we see that

$$\mathrm{Tr}\, M_j = \left\{ (\log \det U)_{y^j} \right\}_{y=e} ; \qquad j = 1, \ldots, n^2 .$$

Thus the characteristic matrices of the n-dimensional unimodular unitary group, which is an n^2-1 parameter group, are of the form $i H$ where H is an n-dimensional Hermitian matrix whose trace is zero. Similarly, the characteristic matrices of the orthogonal group, or of the rotation group, are alternating n-dimensional real matrices, i.e., real matrices A such that $A^t + A = 0$ (the trace of any such matrix being necessarily zero) and the characteristic matrices of the 2k-dimensional symplectic group are complex 2k-dimensional matrices M such that $M^t I = - IM$ or, equivalently, since $I^t = - I$, such that IM is a symmetric 2k-dimensional matrix. Thus, if we write M in the form $\begin{pmatrix} N & Q \\ P & R \end{pmatrix}$, where N , P, Q and R are k-dimensional matrices so that $IM = \begin{pmatrix} -P & -R \\ N & Q \end{pmatrix}$, we see that P and Q are symmetric and $R = -N^t$ so that the trace of M is zero. For the U-symplectic group we have, in addition, the fact that iM is Hermitian so that $N^* = -N$, $R^* = -R$, $P^* = -Q$. Thus the characteristic matrices of the U-symplectic group are of the form $\begin{pmatrix} N & -P^* \\ P & \bar{N} \end{pmatrix}$ where $N^* = -N$ and P is symmetric.

We observe, in conclusion, that in the adjoint representation $A \longrightarrow A'$ of an n-dimensional r-parameter matrix group of which $-E_n$ is an element, the same matrix A' corresponds to the two elements $\pm A$ of the group. In particular, E_r corresponds to each of the two elements $\pm E_n$ of the matrix group and, if E_r does not correspond to any other element of the group, the only two elements of the group which correspond to A' are $\pm A$ (for if A' corresponds to A and to B , E_r corresponds to AB^{-1} so that $AB^{-1} = \pm E_n$ or, equivalently, $B = \pm A$). When this is the case we say that the matrix group $\{A\}$, or any representation of it, is a two-valued, or spin, representation of the matrix group $\{A'\}$.

Lecture 6

1. Spin representations of the 3-dimensional rotation group.

The typical element of the 2-dimensional unimodular unitary group is of the form $a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3 + a_4\varepsilon_4$ where $\varepsilon_1 = E_2$,

$$\varepsilon_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \varepsilon_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and a_1, a_2, a_3, a_4 are any four real numbers the sum of whose squares is 1. We take $a_2 = y^1$, $a_3 = y^2$, $a_4 = y^3$ as our three parameters and observe that to each point y of the parametric space there correspond two elements of the group with equal and opposite values of a_1 (both E_2 and $-E_2$, for example, corresponding to $y = 0$). The derivative of a_1 with respect to y is the zero 1×3 matrix at $y = 0$ and so the 3 characteristic matrices are $M_1 = \varepsilon_2$, $M_2 = \varepsilon_3$, $M_3 = \varepsilon_4$. Since $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix} =$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix} = C\varepsilon_3 + S\varepsilon_4,$$

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix} = -S\varepsilon_3 + C\varepsilon_4,$$ where $c = \cos\theta$, $s = \sin\theta$, $C = \cos 2\theta$, $S = \sin 2\theta$, the matrix which corresponds, in the adjoint representation of the 2-dimensional unimodular unitary group, to $R(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ is the 3-dimensional rotation matrix

$$R_{23}(2\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C & -S \\ 0 & S & C \end{pmatrix}.$$ Similarly, the matrix which corresponds

to $D(\phi) = \begin{pmatrix} \exp \phi i & 0 \\ 0 & \exp - \phi i \end{pmatrix}$ is the 3-dimensional rotation matrix $R_{12}(2\phi) = \begin{pmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $C = \cos 2\phi$, $S = \sin 2\phi$.
Now, any 2-dimensional unimodular unitary matrix may be written in the form $D(\alpha)\, U(\theta,\sigma)$ where $U(\theta,\sigma) = \begin{pmatrix} c & -s \exp -\sigma i \\ s \exp \sigma i & c \end{pmatrix} =$
$= U(-\theta, \sigma')$, $\sigma' = \sigma - \pi$, $0 \leq \theta \leq \pi/2$, $-2\pi < \sigma' \leq 0$
and, since $U(-\theta, \sigma') = D(\sigma'/2)\, R(-\theta)\, D(-\sigma'/2)$, it follows that every 2-dimensional unimodular unitary matrix may be written in the form $U = D(\phi/2)\, R(-\theta/2)\, D(\psi/2)$ where $0 \leq \phi < 4\pi$, $0 \leq \theta \leq \pi$, $0 \leq \psi < 2\pi$, the matrix which corresponds to U in the adjoint representation being the 3-dimensional rotation matrix $R_{12}(\phi)\, R_{23}(-\theta)\, R_{12}(\psi)$.
A slight modification of the argument given for the Euler factorisation of any three dimensional rotation matrix shows that every 3-dimensional rotation matrix is of the form $R_{12}(\phi)\, R_{23}(-\theta)\, R_{12}(\psi)$, $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \psi < 2\pi$ and so the adjoint representation of the 2-dimensional unimodular unitary group is the 3-dimensional rotation group. The 3-dimensional rotation group is covered twice in this representation, an increase of ϕ by 2π changing U into $-U$ but not affecting the 3-dimensional rotation matrix which corresponds to U. We term any representation of the 2-dimensional unimodular unitary group a spin representation of the 3-dimensional rotation group.

The typical element U of the 2-dimensional unitary group is of the form $(\exp \tau i)\, V$ where V is a unimodular unitary 2-dimensional matrix. Taking as our parameters the three parameters y^1, y^2, y^3 of the 2-dimensional unimodular unitary group and τ, so that the origin of the parametric space furnishes the two elements $\pm E_2$ of the group, we see that the four characteristic matrices of the group are $M_1 = \mathcal{E}_2$,

$M_2 = \mathcal{E}_3$, $M_3 = \mathcal{E}_3$, $M_4 = \mathcal{E}_1 = E_2$. Since $U M U^* = V M V^*$, where M is any 2-dimensional matrix, each matrix of the adjoint representation of the 2-dimensional unitary group is of the form $\begin{pmatrix} A'_3 & 0 \\ 0 & 1 \end{pmatrix}$, where A'_3 is an arbitrary 3-dimensional rotation matrix. We express this result by the statement that the adjoint representation of the 2-dimensional unitary group is reducible, being the sum of the 3-dimensional rotation group (which is a representation of the 2-dimensional unitary group) and the identity representation of the 2-dimensional rotation group.

2. <u>The element of volume of an r-parameter group</u>.

Under the left translation $Y \longrightarrow W = AY$ of an n-dimensional matrix group the parametric space undergoes the transformation $y \longrightarrow w = ay$ and the r-dimensional Jacobian matrix $w_y = J(y, w)$ is of the form $J^{-1}(w)\, J(y)$ where $J(y) = J(y, z)$, z being any fixed point of the parametric space. We obtain, as A varies over the matrix group, from any continuous function $\phi(y)$ a whole class of continuous functions $\phi_A(y)$ defined by the relation $\phi_A(y) = \phi(ay) = \phi(w)$ and it is easy to see that the integral $\int \phi_A(y) |J(y)|\, d(y)$ is independent of the element A of the matrix group; indeed $\int \phi(w)\, |J(y)|\, d(y) = \int \phi(w)\, |J(y)|\, |y_w|\, d(w)$ and $|y_w| = |J^{-1}(y)|\, |J(w)|$ so that

$$\int \phi_A(y)\, |J(y)|\, d(y) = \int \phi(w)\, |J(w)|\, d(w) = \int \phi(y)|\, J(y)|\, d(y).$$

We term $|J(y)|\, d(y)$ a volume element of the r-parameter group and we denote this volume element by dV_y ; the integral $\int \phi(y)|\, J(y)|\, d(y) = \int \phi(y)\, dV_y$ is termed the integral of the continuous function $\phi(y)$ over the group and we see that each member of the class of continuous functions $\phi_A(y)$ (each of which is derivable from any other by means of a left translation of the group, the left translation which is induced by BA^{-1} sending $\phi_A(y)$ into $\phi_B(y)$) has the same integral over the group. We may repeat the argument for right translations of the group, obtaining an element of volume $d'V_y = |J'(y)|\, d(y)$ but, if the r-parameter matrix group is compact, $|J'(y)|$ is the product of $|J(y)|$ by the positive constant $|F'(e)|\,|F^{-1}(e)|$ so that $d'V_y$ is the product of dV_y by a positive constant. Under a change of base point $z \to \tilde{z}$, $J(y)$ is multiplied by the constant matrix $J(z, \tilde{z})$ so that dV_y is indeterminate to the extent of a multiplicative positive constant; thus the two elements of volume dV_y and $d'V_y$ are, for a compact r-parameter group, essentially the same and, when the base points z and z' are so chosen that $d'V_y = dV_y$, each member of the whole class of continuous functions $\phi'(y) = \phi(ya)$ has the same integral over the group as does each member of the whole class of continuous functions $\phi_A(y) = \phi(ay)$. We shall, generally, normalise dV_y by dividing it by the integral of the constant function $\phi(y) = 1$ over this group and, when this is done, we term $\int \phi(y)\, dV_y$ the average of the continuous function $\phi(y)$ over the group. Thus the whole class of continuous functions $\phi_A(y) = \phi(ay)$ and the whole class of continuous functions $\phi'_A(y) = \phi(ya)$ have the same averages over the compact r-parameter matrix group.

Upon taking the differential of the relation $W = AY$ we obtain $dW = A\, dY = W Y^{-1} dY$ and this relation may be written in the form $\delta W = \delta Y$, where $\delta Y = Y^{-1} dY$ and $\delta W = W^{-1} dW$. Evaluating this relation at $w = z$, where z is our base point for left translations, and denoting the value of $W^{-1} W_{wp}$ at $w = z$ by N_p, $p = 1, \ldots, r$, we obtain

$$\delta Y = \sum_{p=1}^{r} N_p \sum_{s=1}^{r} (w^p_{y^s})_{w=z} \, dy^s$$

$$= \sum_{p=1}^{r} N_p \sum_{s=1}^{r} \{J(y)\}^p_s \, dy^s \quad .$$

Hence we may obtain $J(y)$ conveniently by expressing δY as a linear combination of $N_1, \ldots, N_r$, the coefficients being r linear forms in $dy^1, \ldots, dy^r$ and the matrix of these linear forms being $J(y)$. Thus for the 1-dimensional unitary group, whose typical element is $\exp yi$, $-\pi < y \leq \pi$, we have only one matrix N which is i and, since $\delta Y = i\,dy$, $J(y) = 1$ and $dV_y = dy$, the normalised element of volume being $\frac{1}{2\pi}\,dy$. Similarly, for the 2-dimensional rotation group, whose typical element is $R(y) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$, $c = \cos y$, $s = \sin y$, $-\pi < y \leq \pi$, we have only one matrix N which is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and, since $\delta Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} dy$, $J(y) = 1$ and $dV_y = dy$, the normalised element of volume being $\frac{1}{2\pi}\,dy$. The r matrices $N_1, \ldots, N_r$ may be replaced, if convenient, by any set of r linearly independent linear combinations of them, with constant coefficients, the effect of this being merely to replace $J(y)$ by the product of it by a constant non-singular matrix.

Example. The element of volume of the 3-dimensional rotation group.

The typical element of the 3-dimensional rotation group is of the form $R = R_{12}(\phi)\, R_{31}(\theta)\, R_{12}(\psi)$, where ϕ and ψ are longitude angles, θ is a latitude angle and $R_{31}(\theta) = R_{13}(-\theta)$. Taking $\phi = y^1$, $\theta = y^2$, $\psi = y^3$ as parameters, the identity point e of the parametric space is a singular point of the coordinate system y (in much the same way as the origin is a singular point of a system of polar coordinates) since it is furnished by the two relations $y^2 = 0$, $y^1 + y^3 \equiv 0 \pmod{2\pi}$ and not by three independent relations. For this reason we take $z^1 = 0$, $z^2 = \pi/2$, $z^3 = 0$ as our base point and our first step is to calculate $N_1 = (R^{-1}R_\phi)_{y=z}$, $N_2 = (R^{-1}R_\Theta)_{y=z}$, $N_3 = (R^{-1}R_\psi)_{y=z}$. We have, since, at z;

$$R_{12}(\phi) = E_3, \quad R_{31}(\Theta) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad R_{12}(\psi) = E_3 \text{ and}$$

$$\{R_{12}(\phi)\}_\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = M_{12}, \text{ say,} \quad \{R_{31}(\Theta)\}_\Theta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = -M_{23}, \text{ say,}$$

$$N_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = M_{31}$$

$$N_3 = M_{12}$$

and we have merely to express $R^{-1}dR$ as a linear combination of the three matrices M_{23}, M_{31}, M_{12}. Since $R_{12}(-\phi)\{R_{12}(\phi)\}_\phi = M_{12}$, $R_{31}(-\Theta)\{R_{31}(\Theta)\}_\Theta = M_{31}$, we have

$$R^{-1}R_\phi = R_{12}(-\psi)\, R_{31}(-\Theta)\, M_{12} R_{31}(\Theta)\, R_{12}(\psi) = -\sin\Theta\cos\psi\, M_{23} + \sin\Theta\sin\psi\, M_{31}$$

$$R^{-1}R_\Theta = R_{12}(-\psi)\, M_{31}\, R_{12}(\psi) = \sin\psi\, M_{23} + \cos\psi\, M_{31}$$

$$R^{-1}R_\psi = M_{12}$$

so that
$$R^{-1}dR = (-\sin\Theta\cos\psi\, d\phi + \sin\psi\, d\Theta)\, M_{23} + (\sin\Theta\sin\psi\, d\phi + \cos\psi\, d\Theta)\, M_{31} + M_{12}\,d\psi$$
and

$dV_y = \sin\Theta\, d(\phi, \Theta, \psi)$. The normalised element of volume is the quotient of this by $8\pi^2$.

Exercise 1. Show that an element of volume of the 2-dimensional unimodular unitary group, whose typical element is of the form $U =$ $= D(\phi/2)\, R(\theta/2)\, D(\psi/2)$, $0 \leq \phi < 4\pi$, $0 \leq \theta \leq \pi$, $0 \leq \psi < 2\pi$, is $\sin\theta\, d(\phi, \theta, \psi)$ and that the normalised element of volume is the quotient of this by $16\pi^2$. Hint. Take $\phi = 0$, $\theta = \pi$, $\psi = 0$ as the base point and express $U^{-1}dU$ as a linear combination of the three matrices $N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $N_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $N_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Exercise 2. Show that $|\det A|^{-n} d(a_1^1, \ldots, a_n^n)$ is an element of volume of the n-dimensional real linear group, i.e., the n^2-parameter group which consists of all real n-dimensional matrices A which possess reciprocals.

Exercise 3. Show that $|\det A|^{-2n} d(a_1^1, b_1^1, \ldots, a_n^n, b_n^n)$ is an element of volume of the n-dimensional complex linear group, i.e., the $2n^2$-parameter group which consists of all non-singular complex matrices A, with elements $a_k^j + b_k^j i$, where a_k^j, b_k^j, $j = 1, \ldots, n$, $k = 1, \ldots, n$, are real.

Lecture 7

1. The unitary nature of continuous representations of a compact r-parameter matrix group.

If v is the n-dimensional vector whose coordinates, with respect to the basis which is defined by an arbitrary n-dimensional matrix X which possesses a reciprocal, are furnished by the elements of an arbitrary $n \times 1$ matrix c, the first scalar product, with respect to the basis which is defined by X and relative to an arbitrary n-dimensional matrix M, of v by itself is the complex number c^*Mc. Since the conjugate complex of this complex number is its star c^*M^*c the first scalar product of v by itself, with respect to the basis defined by X and relative to M, will be real, no matter what is the n-dimensional vector v, if, and only if, $c^*M^*c = c^*Mc$, no matter what is the $n \times 1$ matrix c. Taking $c = e_j$, we see that $\bar{m}_j^j = m_j^j$, $j = 1, \ldots, n$ and, taking $c = e_j \pm e_k$, we see that $\bar{m}_j^k \pm \bar{m}_k^j = m_k^j \pm m_j^k$, so that $\bar{m}_j^k = m_k^j$, $j = 1, \ldots, n$, $k = 1, \ldots, n$. Thus the first scalar product of v by itself, with respect to the basis defined by X and relative to M, will be real, no matter what is the n-dimensional vector v, if, and only if, $M^* = M$, i.e., if, and only if, M is Hermitian. If this scalar product is not only real but positive, save when v is the zero n-dimensional vector, we say that the n-dimensional matrix M is positively definite. It is clear that the n-dimensional identity matrix is positively definite and this implies that the Hermitian matrix N^*N, where N is any n-dimensional matrix which possesses a reciprocal, is positively definite; for $c^*N^*Nc = d^*d$, where $d = Nc$ and d is the zero $n \times 1$ matrix only when $c = N^{-1}d$ is the zero $n \times 1$ matrix. Conversely, any positively definite matrix P is of the form N^*N; indeed, there exists a unitary matrix U such that U^*PU, which is Hermitian since P is Hermitian, is triangular with zeros below the diagonal. Being Hermitian, U^*PU is diagonal and, being positively definite, its diagonal elements are positive real numbers. Hence U^*PU is of the form D^*D where D is a diagonal matrix (whose diagonal elements are indeterminate to the extent of multiplying complex numbers of unit modulus) and $P =$ $= UD^*DU^* = N^*N$ where $N = UDU^*$.

Let, now, $Y \rightarrow Y'$ be any continuous m-dimensional representation of a compact n-dimensional r-parameter matrix group so that the elements of Y' are continuous functions of the $r \times 1$ parameter matrix y. If c is an arbitrary $m \times 1$ matrix, other than the zero $m \times 1$ matrix, $c^*Y'^*(y)Y'(y)c$ is positive for every point y of the parametric space and so the average of $c^*Y'^*(y)Y'(y)c$ over the given compact n-dimensional r-parameter matrix group is positive; this average is c^*Pc, where the m-dimensional Hermitian matrix P is the average of $Y'^*(y)Y'(y)$ over the group and so we know that P is positively definite and, hence, of the form N^*N where N is an m-dimensional matrix which possesses a reciprocal. If $Y \rightarrow W = YA$ is any right translation of our matrix group the average of $Y'^*(ya)Y'(ya) = W'^*(y)\, W'(y) = A'^*\, Y'^*(y)\, Y'(y)\, A'$ over the group is the same as that of $Y'^*(y)\, Y'(y)$ and so $A'^*N^*NA' =$ $= N^*N$, no matter what is the matrix A' of our m-dimensional representation. Hence $NA'N^{-1}$ is a unitary m-dimensional matrix and we have the following important result:

If $A \rightarrow A'$ is any continuous m-dimensional representation Γ of a compact r-parameter matrix group there exists a basis for the m-dimensional linear vector space in which the linear transformations which constitute Γ operate or, as we shall say, for the carrier space of Γ, with respect to which the m-dimensional matrices which present Γ are all unitary. We express this result by the statement that all continuous representations of a compact r-parameter matrix group are unitary.

2. Reducible representations of a compact r-parameter matrix group.

Let $\{\alpha\}$ be any collection of linear transformations of an m-dimensional linear vector space and let $\{A\}$ be the collection of $m \times m$ matrices which present these linear transformations with respect to the basis defined by any $m \times m$ matrix X which possesses a reciprocal. Furthermore, let $v_1, \ldots, v_p$ be $p < m$ linearly independent vectors of the carrier space S of the linear transformations $\{\alpha\}$ and let $c_1, \ldots, c_p$ be the $m \times 1$ matrices which furnish the coordinates of $v_1, \ldots, v_p$, respectively, with respect to the basis defined by the m-dimensional matrix X. Denoting by C any m-dimensional matrix, possessing a reciprocal, whose first p column $m \times 1$ matrices are $c_1, \ldots, c_p$, the coordinates of $v_1, \ldots, v_p$ with respect to the basis defined by $X' = XC$ are furnished, since $Xc_1 = X'e_1, \ldots, Xc_p = X'e_p$, by the matrices $e_1, \ldots, e_p$. If each linear transformation of the set of linear transformations $\{\alpha\}$ sends each of the p vectors $v_1, \ldots, v_p$ into a linear combination of $v_1, \ldots, v_p$, we say that the p-dimensional subspace S_1 of S which is spanned by the p linearly independent m-dimensional vectors $v_1, \ldots, v_p$, i.e., which consists of all linear combinations of these vectors, is invariant under the collection of linear transformations $\{\alpha\}$. When this is the case the matrices of the collection of matrices $\{A'\}$, which present the collection of linear transformations $\{\alpha\}$ with respect to the basis defined by X', are all of the form $\begin{pmatrix} A'^1_1 & A'^1_2 \\ 0 & A'^2_2 \end{pmatrix}$ where A'^1_1 is a p-dimensional matrix, 0 is the zero $(m-p) \times p$ matrix, A'^1_2 is a $p \times (m-p)$ matrix and A'^2_2 is a $(m-p)$-dimensional matrix. We say that the collection of linear transformations $\{\alpha\}$ is reducible and we term the presentation $\{A'\}$ of this collection a presentation in reduced form of the collection.

Let us now suppose that the collection of linear transformations $\{\alpha\}$ constitutes a continuous m-dimensional representation Γ of a compact n-dimensional r-parameter matrix group. When the carrier space S of Γ possesses an invariant subspace of dimension $p < m$ we say that Γ is reducible and we term a presentation of Γ which is of the form $A' = \begin{pmatrix} A'^1_1 & A'^1_2 \\ 0 & A'^2_2 \end{pmatrix}$ a presentation in reduced form (A' being the m-dimensional matrix of Γ which corresponds to an arbitrary matrix A of our n-dimensional matrix group). Since Γ is unitary there exists an m-dimensional matrix B, possessing a reciprocal, such that $BA'B^{-1}$ is unitary, for every A', and we see, on writing $B = UT$ that this implies the existence of a triangular matrix T, possessing a reciprocal, such that $TA'T^{-1}$ is unitary for every A'. T is of the form $\begin{pmatrix} T^1_1 & T^1_2 \\ 0 & T^2_2 \end{pmatrix}$ as is also T^{-} and so $TA'T^{-1}$ is of the form $\begin{pmatrix} A''^1_1 & A''^1_2 \\ 0 & A''^2_2 \end{pmatrix}$ as is also its reciprocal. On the other hand, the star of $TA'T^{-1}$ is $\begin{pmatrix} (A''^1_1)^* & 0 \\ (A''^1_2)^* & (A''^2_2)^* \end{pmatrix}$, where 0 is the zero $p \times (m-p)$ matrix, and it follows that $A''^1_2 = 0$.

Hence Γ may be presented in the form $A' = \begin{pmatrix} A'^1_1 & 0 \\ 0 & A'^2_2 \end{pmatrix}$ where A'^1_1 is a p-dimensional matrix which possesses a reciprocal and A'^2_2 is a (m-p)-dimensional matrix which possesses a reciprocal. The correspondences $A \to A'^1_1$, $A \to A'^2_2$ are representations Γ_1 and Γ_2, of dimensions p and m-p, respectively, of our n-dimensional matrix group and we say that Γ is the sum $\Gamma_1 + \Gamma_2$ of Γ_1 and Γ_2. Since $\begin{pmatrix} A'^2_2 & 0 \\ 0 & A'^1_1 \end{pmatrix}$ is the transform of $\begin{pmatrix} A'^1_1 & 0 \\ 0 & A'^2_2 \end{pmatrix}$ by an m-dimensional permutation matrix, addition of representations of a compact r-parameter matrix group is commutative, $\Gamma_2 + \Gamma_1$ being the same as $\Gamma_1 + \Gamma_2$. When $\Gamma_2 = \Gamma_1 = \Gamma'$, say, we write $2\Gamma'$ instead of $\Gamma' + \Gamma'$.

If either Γ_1 or Γ_2 is reducible we reduce it in the same way and continue, if necessary, this process until all the representations we encounter are irreducible (every 1-dimensional representation being irreducible) and so we see that every m-dimensional representation Γ of a compact r-parameter matrix group is of the form $m_1\Gamma_1 + m_2\Gamma_2 + \ldots + m_k\Gamma_k$, where $m_1, \ldots, m_k$ are positive integers and $\Gamma_1, \ldots, \Gamma_k$ are irreducible representations of the group. We term $m_1\Gamma_1 + \ldots + m_k\Gamma_k$ an analysis of Γ into its irreducible components. It is convenient to permit the numerical coefficients to assume the value 0, it being understood that when m_j, for example, is 0 the irreducible representation Γ_j does not appear in the analysis of Γ. We shall see in the next lecture that the analysis of Γ into irreducible components is unique.

3. <u>The irreducibility criterion.</u>

If a continuous representation Γ, of dimension m, of a given matrix group is reducible there exist matrices, other than scalar matrices, i.e., multiples of the identity $m \times m$ matrix, which commute with each of the matrices which present Γ in any basis. Indeed, if the basis is so chosen that the matrices A' of Γ appear in the reduced form $\begin{pmatrix} A'^1_1 & 0 \\ 0 & A'^2_2 \end{pmatrix} = A'^1_1 \dotplus A'^2_2$, the non-scalar matrix

$m_1 E_p \dotplus m_2 E_{m-p}$, where m_1 and m_2 are any two unequal complex numbers and p is the dimension of A'^1_1, commutes with each of the matrices A' and the commutability of two matrices and the scalar quality of a matrix are independent of the basis adopted; for $BC_1B^{-1} \cdot BC_2B^{-1} = BC_2B^{-1} \cdot BC_1B^{-1}$ if $C_1C_2 = C_2C_1$ and $BE_mB^{-1} = E_m$, B being any m-dimensional matrix which possesses a reciprocal and C_1 and C_2 being any two m-dimensional matrices. On the other hand, if Γ is irreducible the only matrices which commute with all the matrices A' of any presentation of it are scalar matrices. Indeed the relation $A'B = BA'$ says that $A'b_j$, $j = 1, \ldots, m$, is the linear combination $a'^1_j b_1 + \ldots + a'^m_j b_m$ of the column $m \times 1$ matrices $b_1, \ldots, b_m$ of B and hence, since Γ is irreducible, B is either the zero $m \times m$ matrix or else possesses a reciprocal. If B possesses a reciprocal there exists at least one complex number λ such that $B' = B - \lambda E_m$ does not possess a reciprocal and $A'B' = B'A'$. Hence B' is the zero $m \times m$ matrix or, in other words, B is a scalar matrix. Thus we have the following useful result:

A continuous representation Γ of a compact r-parameter matrix group is irreducible if, and only if, the only matrices which commute with all the matrices of any presentation of Γ are scalar matrices. It follows that all continuous irreducible representations of any commutative compact r-parameter matrix group are 1-dimensional. For, if B' is any matrix of any presentation of any continuous irreducible representation Γ of such a group, B' is scalar since it commutes with all the matrices A' of this presentation of Γ; hence all the matrices of this presentation are scalar and this implies, since Γ is irreducible, that Γ is 1-dimensional. For example, the plane rotation group, whose typical element is $R(\Theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$, $c = \cos\Theta$, $s = \sin\Theta$, $-\pi < \Theta \leq \pi$, possesses only 1-dimensional continuous irreducible representations. Thus the self-representation of the plane rotation group is reducible so that there must exist a constant 2-dimensional matrix B such that $B^{-1} R(\Theta) B = \begin{pmatrix} \exp \Theta i & 0 \\ 0 & \exp - \Theta i \end{pmatrix}$, $\exp \Theta i$ and $\exp - \Theta i$ being the characteristic numbers of $R(\Theta)$. $B = 2^{-1/2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ is such a constant matrix.

Lecture 8

1. The Kronecker product of two representations of a matrix group.

If a^j_k is the element in the j^{th} row and k^{th} column of a $p \times r$ matrix A and b^h_i is the element in the h^{th} row and i^{th} column of a $q \times s$ matrix B, $a^j_k b^h_i$ is the element in the jh-row and ki-column of a $pq \times rs$ matrix which we term the Kronecker product of B by A and denote by $A \times B$, it being understood that the rows of $A \times B$ are arranged in the order 11, 12, ... 1q, 21, ..., pq and the columns in the order 11, 12, ..., 1s, 21, ..., rs. It is clear that $A \times B$ is a linear function of its two factor matrices A and B, i.e., that $mA \times B = m(A \times B)$, $A \times mB = m(A \times B)$, $(A_1 + A_2) \times B = (A_1 \times B) + (A_2 \times B)$, $A \times (B_1 + B_2) = (A \times B_1) + (A \times B_2)$, m being any complex number. Furthermore, Kronecker multiplication of matrices is associative, $A \times (B \times C)$ being the same as $(A \times B) \times C$; indeed, if C is any $t \times u$ matrix, each of the matrices $A \times (B \times C)$ and $(A \times B) \times C$ is the $pqt \times rsu$ matrix of which the element in the jhl-row and kim-column is $a^j_k b^h_i c^l_m$. The $pq \times rs$ matrix $A \times B$ may be written as the $p \times r$ block matrix

$$\begin{pmatrix} a^1_1 B & a^1_2 B & \dots & a^1_r B \\ \vdots & & & \\ a^p_1 B & \dots & \dots & a^p_r B \end{pmatrix}$$

whose elements are $q \times s$ matrices and it follows that, if C and D are any two matrices which are such that the products AC and BD may be formed, then $(A \times B)(C \times D) = (AC \times BD)$. In particular, if A' and B' are square matrices of dimension p and A'' and B'' are square matrices of dimension q then

$(A' \times A'')(B' \times B'') = (A'B' \times A''B'')$. $E_p \times E_q$ is E_{pq} and it follows that, if A' and A'' possess reciprocals, then $A' \times A''$ possesses the reciprocal $A'^{-1} \times A''^{-1}$. It is clear that $(A \times B)^* = A^* \times B^*$ so that, if U' and U'' are any two unitary matrices, of dimensions p and q , respectively, then $U' \times U''$ is a pq-dimensional unitary matrix.

If A and B are square matrices, of dimensions p and q , respectively, the relations $c' = Ac$, $d' = Bd$, where c and d are arbitrary $p \times 1$ and $q \times 1$ matrices, respectively, imply the relation $c' \times d' = (A \times B)(c \times d)$. If c furnishes the coordinates, with respect to the basis defined by any p-dimensional matrix X , which possesses a reciprocal, of a p-dimensional vector v and d furnishes the coordinates, with respect to the basis defined by any q-dimensional matrix X' , which possesses a reciprocal, of a q-dimensional vector w, we denote by $v \times w$ the pq-dimensional vector whose coordinates, with respect to the basis defined by the pq-dimensional matrix $X \times X'$, which possesses the reciprocal $X^{-1} \times X'^{-1}$, are furnished by $c \times d$ and we term $v \times w$ the Kronecker product of w by v . If $A \rightarrow A'$ is the presentation, with respect to the basis defined by X , of a p-dimensional representation Γ_1 of an n-dimensional matrix group and $A \rightarrow A''$ is the presentation, with respect to the basis defined by X' , of a q-dimensional representation Γ_2 of the same matrix group, then $A \rightarrow A' \times A''$ is the presentation, with respect to the basis defined by $X \times X'$, of a pq-dimensional representation of the n-dimensional matrix group. We term this representation of the matrix group the Kronecker product of Γ_2 by Γ_1 and we denote it by the symbol $\Gamma_1 \Gamma_2$; when $\Gamma_2 = \Gamma_1 = \Gamma$, say, we write Γ^2 instead of $\Gamma\Gamma$ and we term Γ^2 the Kronecker square (or, simply, the square) of Γ so that the dimension of Γ^2 is the square of the dimension of Γ.

Let, now, c be any $pq \times 1$ matrix whose elements c^{jk}, $j = 1, \ldots, p$, $k = 1, \ldots, q$ are arranged in the order $c^{11}, c^{12}, \ldots, c^{1q}, c^{21}, \ldots, c^{pq}$ and denote by d the $pq \times 1$ matrix whose elements d^{kj} are defined by the relations $d^{kj} = c^{jk}$, these elements being arranged in the order $d^{11}, d^{12}, \ldots, d^{1p}, d^{21}, \ldots, d^{qp}$. Then $d = Pc$, where P is a pq-dimensional permutation matrix. The relation $c' = (A \times B)\, c$, yields $d' = Pc' = P\,(A \times B)\, c = P\,(A \times B)\, P^{-1} d$ and, since $d'^{kj} = c'^{jk} = \sum_{\alpha,\beta} a^j_\alpha b^k_\beta c^{\alpha\beta} = \sum_{\beta,\alpha} b^k_\beta a^j_\alpha d^{\beta\alpha}$, $d' = (B \times A)\, d$. Since d is an arbitrary $pq \times 1$ matrix, P possessing a reciprocal, it follows that $B \times A = P\,(A \times B)\, P^{-1}$ so that Kronecker multiplication of representations of a matrix group is commutative, $\Gamma_2\Gamma_1$ being the same as $\Gamma_1\Gamma_2$.

2. <u>The orthogonality relations.</u>

The relation $c' = (A \times B)\, c$, where A and B are any p-dimensional and q-dimensional matrices, respectively, and c is any $pq \times 1$ matrix,is equivalent to the pq relations

$$c'^{jk} = \sum_{\alpha,\beta} a^j_\alpha b^k_\beta c^{\alpha\beta}, \quad j = 1, \ldots, p\,; \quad k = 1, \ldots, q\,.$$

If, then, we denote by C and C' the $p \times q$ matrices of which the elements in the j^{th} row and k^{th} column are c^{jk} and c'^{jk}, respectively, we have

$$C' = A\,C\,B^t$$

If $c' = c$, in which case we say that c is an invariant $pq \times 1$ matrix of the pq-dimensional matrix $A \times B$, we have $C = A\,C\,B^{t}$. Similarly, if c is an invariant $pq \times 1$ matrix of the pq-dimensional matrix $A \times \bar{B}$ we have $C = A\,C\,B^{*}$ and this relation may be written, if B is unitary, in the form $AC = CB$. If $A \to A'$ is the presentation, with respect to the basis defined by any p-dimensional matrix X which possesses a reciprocal, of a p-dimensional irreducible continuous representation Γ_1 of a compact n-dimensional r-parameter group and $A \to A''$ is a unitary presentation, with respect to the basis defined by a q-dimensional matrix X' which possesses a reciprocal, of a q-dimensional irreducible continuous representation Γ_2 of the same matrix group, then the correspondence $A \to A' \times \overline{A''}$ presents, with respect to the basis defined by the pq-dimensional matrix $X \times X'$, which possesses the reciprocal $X^{-1} \times X'^{-1}$, the continuous representation $\Gamma_1 \bar{\Gamma}_2$ of the matrix group (where $\bar{\Gamma}_2$ is the representation which is presented, with respect to the basis defined by X, by the correspondence $A \to \overline{A''}$). A vector v of the carrier space of $\Gamma_1 \bar{\Gamma}_2$ will be an invariant vector of $\Gamma_1 \bar{\Gamma}_2$ if, and only if, $c = (A' \times \overline{A''})\,c$, where c is the $pq \times 1$ matrix which furnishes the coordinates of v with respect to the basis defined by $X \times X'$, this relation being valid for all matrices A of our n-dimensional matrix group. Since the presentation $A \to A''$ of Γ_2 is, by hypothesis, unitary we see that v is an invariant vector of $\Gamma_1 \bar{\Gamma}_2$ if, and only if, $A'\,C = C\,A''$ for every element A of our matrix group. If $c_1, \ldots, c_q$ are the q $p \times 1$ column matrices of C the relation $A'\,C = C\,A''$ says that $A'\,c_j$ is the linear combination $\sum_{\alpha} c_{\alpha}\, a''^{\alpha}_{j}$ of $c_1, \ldots, c_q$, $j = 1, \ldots, q$, and this implies, since Γ_1 is irreducible, that either C is the zero $p \times q$ matrix or else $q \geq p$ and p of the q $p \times 1$ matrices $c_1, \ldots, c_q$ are linearly independent.

Furthermore, the relation $A'C = CA''$ is equivalent to the relation $A''^*C^* = C^*A'^*$ and the collection of matrices $\{A''^*\}$ is irreducible, Γ_2 being irreducible. Indeed, if there existed a q-dimensional matrix B, possessing a reciprocal, such that $B^{-1}A''^*B = \begin{pmatrix} M_1^1 & M_2^1 \\ 0 & M_2^2 \end{pmatrix}$, we would have $B^*A''B^{-1*} = \begin{pmatrix} M_1^{1*} & 0 \\ M_2^{1*} & M_2^{2*} \end{pmatrix}$ from which it would follow that, s being the dimension of M_1^1, $A'' v_k$, where $v_k = B^{-1*}e_k$, $k = s+1, \ldots, q$, is a linear combination of $v_{s+1}, \ldots, v_q$ and this cannot happen since Γ_2 is irreducible. Hence, either C^* is the zero $q \times p$ matrix or $p \geqslant q$, q of the p column $q \times 1$ matrices of C^* being linearly independent. Thus either $C = 0$ or $q = p$, the p dimensional matrix C possessing a reciprocal so that $A'' = C^{-1}A'C$. In other words, C is the zero $p \times q$ matrix unless $\Gamma_2 = \Gamma_1 = \Gamma$, say. In this latter case we may set $A' = A''$ and it follows from the relation $A''C = CA''$, since Γ is irreducible, that C is a scalar matrix so that the corresponding $p^2 \times 1$ matrix c has all its elements c^{jk}, $j = 1, \ldots, p$, $k = 1, \ldots, p$, zero save those for which $k = j$ and these are all equal. It is clear that, if d is any $pq \times 1$ matrix, the average of $(Y' \times \overline{Y''})\, d$ over our group furnishes the coordinates, with respect to the basis defined by $X \times X'$, of an invariant vector of $\Gamma_1 \overline{\Gamma}_2$; indeed $(A' \times \overline{A''})(Y' \times \overline{Y''})\, d = (Z' \times \overline{Z''})\, d$, where $Z = AY$ so that $Z' = A'Y'$, $Z'' = A''Y''$, and the average of $(Z' \times \overline{Z''})\, d$ over our matrix group is the same as the average of $(Y' \times \overline{Y''})\, d$ over this group. Taking for d, in turn, the pq $pq \times 1$ matrices $e_{11}, e_{12}, \ldots, e_{pq}$ we see that, if Γ_2 is different from Γ_1, the average of $Y' \times \overline{Y''}$ over the group is zero. On the other hand, if $\Gamma_2 = \Gamma_1 = \Gamma$, say, the average of $(Y')^j_k\, (\overline{Y'})^h_i$ over the group, where $Y \longrightarrow Y'$ is any unitary presentation of the continuous p-dimensional irreducible representation Γ of our compact r-parameter group, is zero unless $j = h$, in which case it is a complex number which is independent of h . Thus the average of $(Y')^j_k \, (\overline{Y'})^h_i =$ $= (Y')^j_k \, (Y'*)^i_h = m^i_k \, \delta^j_h$ where δ^j_h is the element in the j^{th} row and h^{th} column of the p-dimensional identity matrix. On setting $j = h$ and summing with respect to h we see, since Y' is unitary, that $m^i_k =$ $= \frac{1}{p} \delta^i_k$. Hence the average of $(Y')^j_k \, (Y'*)^i_h$ over the group $= \frac{1}{p} \delta^j_h \, \delta^i_k$. This and the fact that the average of $(Y')^j_k \, (\overline{Y''})^h_i \equiv (Y')^j_k \, (Y''*)^i_h$ over the group is zero constitute what are known as the orthogonality relations connecting irreducible continuous representations of a compact r-parameter matrix group. On setting $k = j$, $h = i$ and summing with respect to j and i we see that the average of $|\mathrm{Tr}\, Y'|^2$ over the group is 1 and that the average of $\mathrm{Tr}\, Y' \, \overline{\mathrm{Tr}\, Y''}$ over the group is zero. It follows that, if Γ_2 is different from Γ_1 , $\mathrm{Tr}\, Y''$ cannot be the same as $\mathrm{Tr}\, Y'$ for every element Y of our compact r-parameter matrix group. Thus the collection of numbers $\mathrm{Tr}\, Y'$ characterises any irreducible continuous representation Γ of the group and we term this collection of numbers the character of Γ . Denoting this character of Γ by $\mathrm{ch}\,\Gamma$ we express the fact that the average of $|\mathrm{Tr}\, Y'|^2$ over the group is 1 by saying that the squared magnitude of $\mathrm{ch}\,\Gamma$ is 1 and we express the fact that the average of $(\mathrm{Tr}\, Y')\,(\overline{\mathrm{Tr}\, Y''})$ over the group is zero by saying that the scalar product of $\mathrm{ch}\,\Gamma_1$ by $\mathrm{ch}\,\Gamma_2$ is zero. It follows at once that the analysis $\Gamma = c_1\Gamma_1 + \dots + c_m\Gamma_m$ of any reducible continuous representation of a compact r-parameter matrix group into its irreducible components is unique. Indeed $\mathrm{ch}\,\Gamma = c_1 \, \mathrm{ch}\,\Gamma_1 + \dots + c_m \, \mathrm{ch}\,\Gamma_m$ so that c_j = the scalar product of $\mathrm{ch}\,\Gamma$ by $\mathrm{ch}\,\Gamma_j$, $j = 1, \dots, m$, i.e., the average of $\overline{ch \Gamma_j} \, ch \Gamma'$ over the group.

For 1-dimensional representations the character of a representation is the collection of numbers which constitute the representation. For the 1-dimensional unitary group, or the 2-dimensional rotation group, all irreducible continuous representations are 1-dimensional (since the groups are commutative) and the typical element of any irreducible continuous representation is of the form $\exp m \Theta i$, $-\pi < \Theta \leq \pi$, where m is an integer, since $\exp m \Theta i$ is periodic with period 2π . For these groups the orthogonality relations state that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\exp m \Theta i|^2 \, d\Theta = 1 , \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |\exp (m-n) i \Theta| \, d\Theta = 0 , \quad m \neq n ;$$

these are the basic relations which underlie the theory of Fourier series.

Lecture 9

The class factor of the element of volume of the n-dimensional unitary group.

Any n-dimensional unitary matrix U may be written in the form $U = D\,U_{n-1} \dots U_1$ where D is a diagonal unitary matrix (so that its diagonal elements are complex numbers of unit modulus) and $U_1, \dots, U_{n-1}$ are products of plane n-dimensional unimodular unitary matrices of the type

$$U_{1,2}(\Theta,\sigma) = \begin{pmatrix} c & -s\exp -\sigma i & 0 \\ s\exp\sigma i & c & \\ & 0 & E_{n-2} \end{pmatrix}, \quad c = \cos\Theta,\ s = \sin\Theta,\quad 0 \leq \Theta \leq \pi/2,\ -\pi < \sigma \leq \pi,$$

U_1 being $U_{1,n}(\Theta_{n-1}, \sigma_{n-1})\, U_{2,n}(\Theta_{n-2}, \sigma_{n-2}) \dots U_{n-1,n}(\Theta_1, \sigma_1)$, U_2 being $U_{1,n-1}(\Theta_{2n-2}, \sigma_{2n-2})\, U_{2,n-1}(\Theta_{2n-3}, \sigma_{2n-3}) \dots U_{n-2,n-1}(\Theta_n, \sigma_n)$ and so on to U_{n-1} which is $U_{1,2}(\Theta_N, \sigma_N)$, $N = \frac{1}{2}n(n-1)$. Writing $s\cos\sigma = x$, $s\sin\sigma = y$, so that $c = (1 - x^2 - y^2)^{1/2}$, $U_{1,2}(\Theta,\sigma)$ becomes a function of x, y which reduces to E_n at $x = 0$, $y = 0$, its derivatives with respect to x and y reducing to $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & \\ & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & i & 0 \\ i & 0 & \\ & 0 & 0 \end{pmatrix}$, respectively, at $x = 0$ and $y = 0$. If we take as our n^2 parameters the n arguments of the diagonal elements of D, the N x's and the N y's which are furnished by the formulas $x_j = s_j \cos\sigma_j = \sin\Theta_j \cos\sigma_j$; $y_j = s_j \sin\sigma_j = \sin\Theta_j \sin\sigma_j$; $j = 1, \dots, N$, the origin is the identity point of the parametric space and the n^2

characteristic matrices $M_1, \ldots, M_{n^2}$ are as follows:

1) n of the type iH_j, $j = 1, \ldots, n$, where H_j is the n-dimensional diagonal matrix all of whose diagonal elements are zero save the j^{th} which is 1; these correspond to the n parameters which are the arguments of the diagonal elements of D.

2) N of the type $M_{p,q}$, $p < q = 2, \ldots, n$, where $M_{p,q}$ is the n-dimensional matrix all of whose elements are zero save those in the p^{th} column and q^{th} row, and in the q^{th} column and p^{th} row, which are 1 and -1, respectively; these correspond to the N x's.

3) N of the type $i\,H_{p,q}$, $p < q = 2, \ldots, n$, where $H_{p,q}$ is the n-dimensional matrix all of whose elements are zero save those in the p^{th} column and q^{th} row, and in the q^{th} column and p^{th} row, which are both 1; these correspond to the N y's.

If, then, we denote by E_k^j, $j = 1, \ldots, n$, $k = 1, \ldots, n$, the n-dimensional matrix all of whose elements are zero save the element in the j^{th} column and k^{th} row, which is 1, each of the n^2 matrices E_k^j is a linear combination of the n^2 characteristic matrices $M_1, \ldots, M_{n^2}$, the $n^2 \times n^2$ matrix of the n^2 linear combinations so obtained possessing a reciprocal. Hence, in order to obtain the element of volume of the n-dimensional unitary group, we have merely to express $\delta U = U^*dU$ as a linear combination of the n^2 matrices E_k^j and to determine the modulus of the determinant of the $n^2 \times n^2$ matrix of the n^2 coefficients of this linear combination (those coefficients being linear combinations of the differentials of the n^2 parameters). Since each of the coefficients in question is an element of the $n \times n$ matrix δU our procedure is as follows: We determine the modulus of the determinant of the $n^2 \times n^2$ matrix of the n^2 elements of δU, each element of δU being a linear combination of the differentials of the n^2 parameters.

The group functions we shall have to integrate over the group will all be class functions, i.e., functions $\phi(z)$ defined over the parametric space which are such that $\phi(aza^{-1}) = \phi(z)$, where a is an arbitrary fixed point of the parametric space and z varies over the parametric space. For this reason it is convenient to use the class and in-class parameters which are defined as follows. If Z is any n-dimensional unitary matrix there exists an n-dimensional unitary matrix A such that A^*ZA is a diagonal n-dimensional unitary matrix $D(z)$, whose diagonal elements $z_1, \ldots, z_n$ are complex numbers of unit modulus which are the characteristic numbers of Z, and we say that $D(z)$ is a diagonal representative of the class of the unitary group to which Z belongs. Writing A^* in the form $D\,U_{n-1} \ldots U_1 = DV^*$, say, we have $Z = A\,D(z)\,A^* = V\,D(z)\,V^*$, the diagonal factor D of A^* disappearing since $D^*\,D(z)\,D = D(z)$ owing to the commutativity of diagonal matrices. The n class parameters are the n arguments of $z_1, \ldots, z_n$ and the $2N = n(n-1)$ in-class parameters are the N x's and N y's which appear in the N plane unitary n-dimensional matrices whose product $U_{n-1} \ldots U_1$ is V^*. Since $Z\,V = V\,D(z)$, we have $(dZ)\,V + Z\,dV = (dV)\,D(z) + V\,dD(z)$ so that $(dZ)\,V = (dV)\,D(z) + V\,dD(z) - Z\,dV$ and $(\delta Z)\,V = V\,D^{-1}(z)\,V^*\,(dV)\,D(z) + V\,D^{-1}(z)\,dD(z) - dV = V\,D^{-1}(z)\,\delta V\,D(z) + V\,\delta D(z) - dV$ so that $V^*\,\delta Z\,V = D^{-1}(z)\,\delta V\,D(z) + \delta D(z) - \delta V$ or, equivalently, $\delta Z = V\left\{D^{-1}(z)\,\delta V\,D(z) + \delta D(z) - \delta V\right\}V^*$. Since the n-dimensional unitary group is compact the determinant of the $n^2 \times n^2$ matrix C which is defined by the relations

$$V M_j V^* = \sum_\alpha M_\alpha (C)^\alpha_j , \quad j = 1, \ldots, n^2 ,$$

where the M_j , $j = 1, \ldots, n^2$, are the characteristic matrices of the group, has modulus unity (the correspondence $V \longrightarrow C$ being the adjoint representation of the group). $V^*\delta Z\, V = D^{-1}(z)\ \delta V\, D(z) + \delta D(z) - \delta V$ is of the form $\sum_\beta M_\beta F^\beta$, where the F^j , $j = 1, \ldots, n^2$, are n^2 linear combinations of the differentials of the parameters; hence, $\delta Z = \sum_{\alpha,\beta} M_\alpha (C)^\alpha_\beta F^\beta$ and the modulus of the determinant of the matrix of the n^2 linear combinations $\sum_\beta (C)^j_\beta F^\beta$, $j = 1, \ldots, n^2$, of the differentials of the parameters is the same as the modulus of the determinant of the matrix of the n^2 linear combinations F^j , $j = 1, \ldots, n^2$, of these differentials since it is the product of this latter modulus by $|\det C| = 1$. Thus all we have to do, in order to determine the element of volume of the n-dimensional unitary group, is to calculate the modulus of the determinant of the $n^2 \times n^2$ matrix of the elements of $V^*\delta Z\, V = D^{-1}(z)\ \delta V\, D(z) + \delta D(z) - \delta V$, each element of $V^*\delta Z\, V$ being a linear combination of the differentials of the n^2 parameters. Since the diagonal elements of $D^{-1}(z)\ \delta V\, D(z)$ are the same as the corresponding diagonal elements of δV the linear form furnished by the p^{th} diagonal element of $V^*\delta Z\, V$ is $i\, d\Theta_p$ where $z = \exp \Theta_p i$, $p = 1, \ldots, n$, and so we have only to determine the modulus of the determinant of the $(n^2 - n)$-dimensional matrix of the $n^2 - n$ linear combinations of the differentials of the parameters which are furnished by the non-diagonal elements of $V^*\delta Z\, V$. The element in the first column and second row, for example, of $V^*\delta Z\, V$ is $((z_1/z_2) - 1)\ (\delta V)^1_2$ and, since the coefficients of the linear combination $(\delta V)^1_2$ of the differentials of the parameters are functions of the in-class parameters, we see that the element of volume dV_z of the n-dimensional unitary group may be written as the product of $\left| \prod_{p<q} ((z_p/z_q)-1)((z_q/z_p)-1) \right| d(\theta_1, \ldots, \theta_n)$ by a factor which does not involve the class parameters $\theta_1, \ldots, \theta_n$. In the process of averaging a class function over the group this second factor concels out and we term the first factor (which is merely the class factor of the element of volume) the element of volume of the group. Since $|z_p| = 1$, $p = 1, \ldots, n$, this class factor is

$$\prod_{p<q} |z_p - z_q|^2 \, d(\theta_1, \ldots, \theta_n) .$$

The product $\prod_{p<q}(z_p - z_q)$ may be written as the Vandermonde determinant of $z_1, \ldots, z_n$, i.e., the determinant of the n-dimensional matrix whose p^{th} row matrix is $(z_1^{n-p}, \ldots, z_n^{n-p})$, $p = 1, \ldots, n$, and so it is $\sum \pm z_1^{p_1} \ldots z_n^{p_n}$ where $p_1, \ldots, p_n$ is a permutation of the n numbers $n-1, \ldots, 1, 0$, the $+$ sign being used when the permutation is even and the $-$ sign when it is odd. Since $\int_{-\pi}^{\pi} z_j^p \, d\theta_j = 0$ unless $p = 0$, in which case it is 2π, only those $n!$ of the $(n!)^2$ terms which occur in the product $(\sum \pm z_1^{p_1} \ldots z_n^{p_n})(\sum \pm \bar{z}_1^{q_1} \ldots \bar{z}_n^{q_n})$ for which $q_1 = p_1, \ldots, q_n = p_n$ contribute anything to the integral of $\prod_{p<q} |z_p - z_q|^2$ over the class parameter space and each of these terms contributes $(2\pi)^n$. Thus the normalised element of volume or, more precisely, the normalised class factor of the element of volume of the n-dimensional unitary group is

$$\frac{1}{n!}(2\pi)^{-n} \prod_{p<q} |z_p - z_q|^2 \, d(\theta_1, \ldots, \theta_n)$$

If m is any non-negative integer we say that $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition of m, containing not more than n parts, if $\lambda_1, \ldots, \lambda_n$ are non-negative integers, arranged in order of non-ascending magnitude, whose sum is m and we term the number of non-zero λ's the number of parts in (λ). Setting $\ell_1 = \lambda_1 + n - 1$, $\ell_2 = \lambda_2 + n - 2$, ..., $\ell_n = \lambda_n$ the non-negative integers $\ell_1, \ldots, \ell_n$ are all different and are arranged in decreasing order of magnitude and we introduce the function $A(\ell)$ of the n variables $z_1, \ldots, z_n$ which is defined as follows: $A(\ell)$ is the determinant of the n-dimensional matrix whose p^{th} row matrix is $z_1^{\ell_p}, \ldots, z_n^{\ell_p}$, $p = 1, \ldots, n$. Thus, $A(n-1, \ldots, 1, 0)$ is the Vandermonde determinant of $z_1, \ldots, z_n$ and, like this Vandermonde determinant, which we shall denote by Δ, $A(\ell)$ is an alternating function of $z_1, \ldots, z_n$. Thus the quotient of $A(\ell)$ by Δ is a symmetric function of $z_1, \ldots, z_n$ which we shall denote by the symbol $\{\lambda\}$. We write only the non-zero parts of (λ); thus if $\lambda_k > 0$ while $\lambda_{k+1} = 0$ (so that $\lambda_j = 0$ if $j > k$, we denote $\{\lambda_1, \ldots, \lambda_k, 0, \ldots 0\}$ by $\{\lambda_1, \ldots, \lambda_k\}$ and when all the λ's are 0 we denote $\{0, 0, \ldots, 0\}$ simply by $\{0\}$. For example, $\{0\} = 1$ and $\{1\} = z_1 + \ldots + z_n = s_1$, say. It is easy to see that the average of the squared modulus of $\{\lambda\}$ over the n-dimensional unitary group is 1. Indeed, since the normalised element of volume is $\frac{1}{n!}(2\pi)^{-n}\,\overline{\Delta}\Delta\, d(\Theta_1, \ldots, \Theta_n)$, this average is $\frac{1}{n!}(2\pi)^{-n}\int_{-\pi}^{\pi}\cdots\int \overline{A(\ell)}\,A(\ell)\, d(\Theta_1, \ldots, \Theta_n)$ and $A(\ell) = \sum \pm z_1^{p_1}\cdots z_n^{p_n}$, where $(p_1, \ldots, p_n)$ is a permutation of the numbers $\ell_1, \ldots, \ell_n$, so that

$$\int_{-\pi}^{\pi}\cdots\int \overline{A(\ell)}\,A(\ell)\, d(\Theta_1, \ldots, \Theta_n)$$

is independent of (ℓ), having the same value $n!\,(2\pi)^n$ as it had when $(\ell) = (n-1, \ldots, 1, 0)$. Similarly, if $(\lambda') = (\lambda_1', \ldots, \lambda_n')$ is different from $(\lambda) = (\lambda_1, \ldots, \lambda_n)$, the average of $\overline{\{\lambda'\}}\{\lambda\}$ over the group is zero; indeed $(\ell') =$

$= (\ell'_1, \ldots, \ell'_n)$ is different from $(\ell) = (\ell_1, \ldots, \ell_n)$ so that no term in the product of $\sum \pm z_1^{p_1} \ldots z_n^{p_n}$ by $\sum \pm \bar{z}_1^{p'_1} \ldots \bar{z}_n^{p'_n}$, where $p'_1, \ldots, p'_n$ is a permutation of the n numbers $\ell'_1, \ldots, \ell'_n$, reduces to unity.

Exercise 1. Show that, if $\lambda_n > 0$, $\{\lambda\}$ is the product of $\{\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n, 0\}$ by $(\det Z)^{\lambda_n}$.

Hint. $\det Z = z_1 \ldots z_m$.

Note. The result of this exercise shows us that, over the unimodular subgroup of the n-dimensional unitary group, $\{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_n > 0$, $= \{\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_m, 0\}$.

Exercise 2. Show that $\overline{\{\lambda\}} = (\det Z)^{-\lambda_1} \{\lambda_1 - \lambda_m, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2, 0\}$

Note. This exercise shows that, if $\{\lambda\}$ is real, $\{\lambda_1, \ldots \lambda_n\} = \{\lambda_1 - \lambda_m, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n, 0\} = \{\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2, 0\}$ over the unimodular subgroup of the n-dimensional unitary group.

Lecture 10

The symmetrized and anti-symmetrized powers of a representation.

If A' is any d-dimensional matrix the Kronecker product $A' \times \dots \times A'$, involving m factors each equal to A', is a d^m-dimensional matrix which is known as the Kronecker m^{th} power of A' and we denote it by $A'_{[m]}$. If $A \rightarrow A'$ is a representation Γ of a given matrix group then $A \rightarrow A'_{[m]}$ is the representation $\Gamma \dots \Gamma$ (there being m factors each equal to Γ) which we term the m^{th} power of Γ and denote by Γ^m. If the given matrix group is the n-dimensional unitary group we may take Z' to be unitary for every element Z of the group and this implies that $Z'_{[m]}$ is unitary for every Z and, if X is a d-dimensional matrix, possessing a reciprocal, which defines a basis in which Z' is a diagonal d-dimensional matrix, whose diagonal elements $z_1', \dots, z_d'$ are the characteristic numbers of Z', then $Z'_{[m]}$ is a diagonal d^m-dimensional unitary matrix whose $(r_1 r_2 \dots r_m)$-th diagonal element is $z'_{r_1} \dots z'_{r_m}$ (for the element in the $(s_1 s_2 \dots s_m)$-th row and $(r_1 r_2 \dots r_m)$-th column of $Z'_{[m]}$ is $(Z')^{s_1}_{r_1} \dots (Z')^{s_m}_{r_m}$). The character of Γ^m is $(\mathrm{ch}\,\Gamma)^m = S_1^m$ where $S_1 = z_1' + \dots + z_d'$ is the sum of the characteristic numbers of Z'. In particular, when Γ is the self-representatation $Z \rightarrow Z$, so that $d = n$, the character of Γ^m is s_1^m where $s_1 = z_1 + \dots + z_n$ is the sum of the characteristic numbers of Z. The character of the self-representation Γ itself is $s_1 = \{1\}$ and it follows, since the average of $|\{1\}|^2$ over the group is 1, that the self-representation Γ of the n-dimensional unitary group is irreducible. Indeed, if $\Gamma = m_1\Gamma_1 + \dots + m_k\Gamma_k$

is an analysis of Γ into irreducible components, $\{1\} = \text{ch}\,\Gamma =$ $= m_1\,\text{ch}\,\Gamma_1 + \dots + m_k\,\text{ch}\,\Gamma_k$ so that the average of $|\{1\}|^2$ over the group $= m_1^2 + \dots + m_k^2$. Since this average is 1 and since the coefficients $m_1, \dots, m_k$ are non-negative integers, all of them must be 0 save one which is 1 ; in other words Γ is irreducible.

If X is any d-dimensional matrix which possesses a reciprocal, $X_{[m]}$ defines a basis in the carrier space of Γ^m. Let c be the $d^m \times 1$ matrix whose elements furnish, with respect to the basis defined by $X_{[m]}$, the coordinates of an arbitrary vector v of the carrier space of Γ^m and denote by $(p)c$ the $d^m \times 1$ matrix whose $(r_1 r_2 \dots r_m)$-coordinate is $c^{r_{p_1} \dots r_{p_m}}$ where $(p) = \uparrow\begin{pmatrix} p_1 & \dots & p_m \\ 1 & \dots & m \end{pmatrix}$ is any permutation of the m symbols $1, \dots, m$, i.e., any element of the symmetric group S_m on m symbols. If c' is the $d^m \times 1$ matrix which furnishes, with respect to the basis defined by $X'_{[m]}$, where X' is any d-dimensional matrix which possesses a reciprocal, the coordinates of v, we have $c' = A_{[m]}c$ where $A = X'^{-1}X$. The $(r_1 r_2 \dots r_m)$-th element of c is $c^{r_1 \dots r_m}$ and we denote this, for brevity, by $c^{\{r\}}$. Then $c'^{\{r\}} = \sum_{\alpha_1, \dots, \alpha_m} a^{r_1}_{\alpha_1} \dots a^{r_m}_{\alpha_m} c^{\{\alpha\}}$ so that $((p)\,c')^{\{r\}} =$

$$= \sum_{\alpha_1, \dots, \alpha_m} a^{r_{p_1}}_{\alpha_1} \dots a^{r_{p_m}}_{\alpha_m} c^{\{\alpha\}} = \sum_{(\alpha)} a^{r_{p_1}}_{\alpha_{p_1}} \dots a^{r_{p_m}}_{\alpha_{p_m}} ((p)\,c)^{\{\alpha\}} =$$

$$= \sum_{(\alpha)} a^{r_1}_{\alpha_1} \dots a^{r_m}_{\alpha_m} ((p)\,c)^{\{\alpha\}}.$$ Thus $(p)\,c' = A_{[m]}((p)c)$ so that

the $d^m \times 1$ matrix $(p)\,c'$ furnishes, with respect to the basis defined by $X'_{[m]}$, the coordinates of the vector of the carrier space of Γ^m whose coordinates, with respect to the basis defined by $X_{[m]}$, are furnished by (p) c . We restrict ourselves to bases of the carrier space of Γ^{m} which are of the form $X_{[m]}$ and we denote by (p) v the vector of this space whose coordinates, with respect to the basis defined by $X_{[m]}$, are furnished by the $d^m \times 1$ matrix (p) c , where c is the $d^m \times 1$ matrix which furnishes, with respect to the basis defined by $X_{[m]}$, the coordinates of an arbitrary vector v of the carrier space of Γ^{m} .

The collection of vectors v of the carrier space of Γ^{m} which are such that (p) v = v , for every element (p) of the symmetric group S_m on m symbols, constitutes, when $m > 1$, a linear vector space S' of dimension $< d^m$ i.e., a proper subspace of the carrier space of Γ^{m} . S' is invariant under each of the linear transformations which constitute Γ^{m} ; indeed, if $v \to v'$ under any of these linear transformations,

$$((p)\ c')^{\{r\}} = \sum_{(\alpha)} (Z')^{r_{p_1}}_{\alpha_1} \cdots (Z')^{r_{p_m}}_{\alpha_m} c^{\{\alpha\}}$$

$$= \sum_{(\alpha)} (Z')^{r_{p_1}}_{\alpha_{p_1}} \cdots (Z')^{r_{p_m}}_{\alpha_{p_m}} ((p)\ c)^{\{\alpha\}}$$

so that $(p)\ v' = ((p)\ v)'$. If, then, $(p)\ v = \pm v$, $(p)\ v' = \pm v'$, the same sign being used in each of the two equations, which proves the invariance under Γ^{m} of the linear subspace S' of the carrier space S of Γ^{m} and also the invariance under Γ^{m} of the linear subspace S" of S which consists of the vectors v of S which are such that $(p)\ v = \pm v$, the + sign being used when the permutation (p) is even and the - sign when it is odd. We term the representation of the n-dimensional unitary group which is induced in S' by Γ^{m} the symmetrized m^{th} power of Γ and we denote this symmetrized m^{th} power of Γ by $\Gamma \otimes \{m\}$; we term, similarly, the representation of the n-dimensional unitary group which is induced in S'' by Γ^m the anti-symmetrized m^{th} power of Γ and we denote this anti-symmetrized m^{th} power by $\Gamma \otimes \{1^m\}$, where $(1^m) = (1, 1, \ldots, 1)$ denotes the partition of m into m parts each equal to 1. When Γ is the self-representation we denote $\Gamma \otimes \{m\}$ and $\Gamma \otimes \{1^m\}$ simply by $\{m\}$ and $\{1^m\}$, respectively.

If v is any vector of S the average, $\frac{1}{m!} \sum_{(p)} (p)\, v = w$, of v over the symmetric group S_m on m symbols is a vector of S'; for $(p')\, w$, where p' is any element of S_m, $= \frac{1}{m!} \sum_{(p)} (p'p)\, v = w$, and, on taking v to be a vector of S', so that $(p)\, v = v$ for every element (p) of S_m, we see that every vector of S' is of the type $w = \frac{1}{m!} \sum_{(p)} (p)\, v$. Similarly $\frac{1}{m!} \sum_{(p)} \pm (p)\, v$, the + or minus sign being used according as (p) is even or odd, respectively, is a vector of S'' and every vector of S'' is of this type. On denoting by $v_{\{r\}}$ the vector $v_{r_1 \ldots r_m}$ of S, all of whose coordinates, with respect to the basis defined by $X_{[m]}$, where X is any d-dimensional matrix which possesses a reciprocal, are zero save the $(r_1 \ldots r_m)$-th, which is 1, $(p)\, v_{\{r\}} = v_{r_{p_1} \ldots r_{p_m}} = v_{(p)\{r\}}$ and so the average $w_{(p)\{r\}}$ of $v_{(p)\{r\}}$ over S_m is the same as the average $w_{\{r\}}$ of $v_{\{r\}}$ over S_m. We agree, then, that $r_1 \leqq r_2 \leqq \cdots \leqq r_m$ and consider the vectors $w_{\{r\}}$ of S', there being as many of these as there are terms in the expansion of $(x_1 + \ldots + x_d)^m$; for example, when $m = 2$ there are $\frac{d(d+1)}{2}$ vectors $w_{\{r\}}$ and, generally, there are $\frac{(d+m-1)!}{(d-1)!\, m!}$ of the vectors $w_{\{r\}}$. These vectors of S' are linearly independent since all of the coordinates of $w_{\{r\}}$, with respect to the basis defined by $X_{[m]}$, are zero save the $(r_{p_1} r_{p_2} \dots r_{p_m})$-th co-ordinates, where (p) is any element of S_m, and these have the same non-zero value. Since every vector of S' is of the form $\frac{1}{m!}\sum_{(p)} (p)\, v$, which is a linear combination of the vectors $w_{\{r\}}$, $r_1 \leq r_2 \leq \dots \leq r_m$, the $\frac{(d+m-1)!}{(d-1)!\, m!}$ vectors $w_{\{r\}}$ constitute a basis for S' whose dimension D is $\frac{(d+m-1)!}{(d-1)!\, m!}$. If X is so chosen that Z' is diagonal, $Z'_{[m]}$ is diagonal, its $(r_1 \dots r_m)$-th diagonal element being $z'_{r_1} \dots z'_{r_m}$, and, since $z'_{r_{p_1}} \dots z'_{r_{p_m}} = z'_{r_1} \dots z'_{r_m}$, $w'_{\{r\}} = z'_{r_1} \dots z'_{r_m} w_{\{r\}}$, $r_1 \leq r_2 \leq \dots \leq r_m$. Thus, when we introduce a basis in S whose first D vectors are the vectors $w_{\{r\}}$, the matrix formed by the first D columns and first D rows of the presentation of Z' in this basis is diagonal, its $(r_1 r_2 \dots r_m)$-th diagonal element, where $r_1 \leq r_2 \leq \dots \leq r_m$, being $z'_{r_1} \dots z'_{r_m}$ and the matrix formed by the first D columns and last $d^m - D$ rows of this presentation of $Z'_{[m]}$ is the zero $(d^m - D) \times D$ matrix. Thus the character of $\Gamma \otimes \{m\}$ is the complete symmetric function $h'_m = \sum_{(r)} z'_{r_1} \dots z'_{r_m}$ of degree m of the p characteristic numbers $z'_1, \dots, z'_p$ of Z'. For example, $h'_2 = \sum_j (z'_j)^2 + \sum_{j<k} z'_j z'_k$,

$h'_3 = \sum_j (z'_j)^3 + \sum_{j<k} z'_j z'_k + \sum_{j<k<\ell} z'_j z'_k z'_\ell$ and so on.

It is clear that, if $w_- = \frac{1}{m!}\sum_{(p)} \pm (p)\, v$, the + or − sign being used according as (p) is even or odd, then $w_{-\{r\}} = \frac{1}{m!}\sum_{(p)} \pm (p)\, v_{\{r\}}$ is the zero vector unless $r_1, r_2, \dots, r_m$ are all different. Since $w_{-(p)\{r\}} = w_{-\{r\}}$ we take $r_1 < r_2 < \dots < r_m$ and the argument given above shows that the $D_{-} = \binom{d}{m} = \frac{d!}{(d-m)!\, m!}$ vectors $w_{-\{r\}}$, $r_1 < r_2 < \dots < r_m$, constitute a basis in S'', and that the character of the D_{-}-dimensional representation $\Gamma' \otimes \{1^m\}$ of the n-dimensional unitary group is the elementary symmetric function $\sigma'_m = \sum_{j_1 < j_2 < \dots < j_m} z'_{j_1} \dots z'_{j_m}$ of degree m of the p characteristic numbers $z'_1, \dots, z'_p$ of Z'.

The character of the representation $\{m\}$ of the n-dimensional unitary group is the complete symmetric function h_m of the n characteristic numbers $z_1, \dots, z_n$ of a typical element Z of the group and the character of the representation $\{1^m\}$ of the n-dimensional unitary group is the elementary symmetric function σ_m of $z_1, \dots, z_n$. Then $h^{(\alpha)} = h_1^{\alpha_1} h_2^{\alpha_2} \dots h_n^{\alpha_n}$, where $\alpha_1, \dots, \alpha_n$ are non-negative integers such that $\alpha_1 + 2\alpha_2 + \dots + n\alpha_n = m$, is the character of the representation $\{1\}^{\alpha_1}\{2\}^{\alpha_2} \dots \{n\}^{\alpha_n}$ of the n-dimensional unitary group and, if $m_{(\alpha)} = m_{\alpha_1 \alpha_2 \dots \alpha_n}$ is any set of non-negative integers, $\sum_{(\alpha)} m_{(\alpha)} h^{(\alpha)} = \sum_{\alpha_1, \dots, \alpha_n} m_{\alpha_1 \dots \alpha_n} h_1^{\alpha_1} \dots h_n^{\alpha_n}$ is the character of a representation of the n-dimensional unitary group; similarly, $\sum_{(\alpha)} m_{(\alpha)} \sigma_1^{\alpha_1} \dots \sigma_n^{\alpha_n}$ is the character of a representation of this group. If the coefficients $m_{(\alpha)}$ are allowed to assume negative, as well as non-negative, integral values we term the expression $\sum_{(\alpha)} m_{(\alpha)} h^{(\alpha)}$, or the expression $\sum_{(\alpha)} m_{(\alpha)} \sigma^{(\alpha)}$, a generalised character, of degree m, of the n-dimensional unitary group. On separating the negative coefficients $m_{(\alpha)}$, if any such exist, from the positive coefficients we see that any generalised character, of degree m, is the difference between two actual characters and, hence, is a linear combination, with integral coefficients, of characters of irreducible representations of the n-dimensional unitary group. If the average of the squared modulus of a generalised character of degree m, over the group, is 1 all of the coefficients (of characters of irreducible representations) must be 0 save one, which is 1 or - 1. If the value of the generalised character at the identity element of the group is positive this single non-zero coefficient must be 1 (since the character of any representation at the identity element of the group is positive, being the dimension of the representation) and the generalised character is the character of an irreducible representation of the group.

Lecture 11.

1. The irreducible continuous representations of the n-dimensional unitary group.

The symmetric function $\{\lambda\}$ of the n characteristic numbers $z_1, \dots, z_n$ of a typical element Z of the n-dimensional unitary group may be expressed in a form which makes it clear that $\{\lambda\}$ is a generalised character of this group. To show this we consider the function $f(t) = (1 - z_1 t) \dots (1 - z_n t) = \sigma_0 - \sigma_1 t + \dots + (-1)^n \sigma_n t^n$, where $\sigma_0 = 1$, $\sigma_1 = \sum_j z_j$, $\sigma_2 = \sum_{j<k} z_j z_k$ and so on and t is an indeterminate.

Since $(1 - z_j t)^{-1} = 1 + z_j t + z_j^2 t^2 + \dots$, $|z_j t| < 1$,

$\{f(t)\}^{-1} = h_0 + h_1 t + h_2 t^2 + \dots$ where $h_0 = 1$, $h_1 = \sum_j z_j$,

$h_2 = \sum_j z_j^2 + \sum_{j<k} z_j z_k$ and so on are the complete symmetric functions of $z_1, \dots, z_n$. On writing $\sigma'_j = (-1)^j \sigma_j$, $j = 0, 1, \dots, n$, $\sigma'_j = 0$, $j > n$, we may write $f(t)$ in the form $\sum_0^\infty \sigma'_j t^j$ and the fact that $\left\{\sum_0^\infty \sigma'_j t^j\right\}\left\{\sum_0^\infty h_k t^k\right\} = 1$ may be expressed by the statement

that, no matter what is the positive integer m, the following two m-dimensional triangular matrices are reciprocals:

$$\Sigma_m = \begin{pmatrix} \sigma'_0 & \sigma'_1 & \cdots & \sigma'_{m-1} \\ 0 & \sigma'_0 & \cdots & \sigma'_{m-2} \\ \vdots & & & \\ 0 & \cdots & 0 & \sigma'_0 \end{pmatrix}; \qquad H_m = \begin{pmatrix} h_0 & h_1 & \cdots & h_{m-1} \\ 0 & h_0 & \cdots & h_{m-2} \\ \vdots & & & \\ 0 & \cdots & 0 & h_0 \end{pmatrix}$$

If ℓ is any non-negative integer we denote by (h_ℓ) the $1 \times n$ matrix $(h_{\ell-n+1}, h_{\ell-n+2}, \ldots, h_\ell)$, where $h_j = 0$ if $j < 0$, so that, if ℓ is one of the n integers $0, 1, \ldots, n-1$, (h_ℓ) is a row matrix of H_n which implies that $(h_\ell)\Sigma_n$ is a row matrix of the n-dimensional identity matrix E_n. Hence, if ℓ is one of the n integers $0, 1, \ldots, n-1$,

$$z^\ell = (h_\ell)\Sigma_n \begin{pmatrix} z^{n-1} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$ no matter what is the complex number z.

This relation remains valid for any integer $\ell \geqslant n$ if z is one of the n numbers $z_1, \ldots, z_n$. Indeed, m being any positive integer, $\{f(t)\}^{-1}$ is the sum of a polynomial $h_0 + h_1t + \ldots, + h_{m-1}t^{m-1}$, of degree $< m$, and $t^m\{h_m + h_{m+1}t + \ldots\}$ and so $f(t)\{h_m + h_{m+1}t + \ldots\}$ is a polynomial $r_0 + r_1t + \ldots + r_{n-1}t^{n-1}$ of degree $< n$ whose coefficients vary with m. Since $f(t)\{h_0 + h_1t + \ldots + h_{m-1}t^{m-1}\} + t^m(r_0 + r_1t + \ldots + r_{n-1}t^{n-1}) = 1$ and, since $f(t) = 0$ if $t = z_k^{-1}$, $k = 1, \ldots, m$, we have $z_k^{m+n-1} = r_0z_k^{n-1} + \ldots + r_{n-1}$, $k = 1, \ldots, n$; $m = 1, 2, \ldots$.

The relation $f(t)\{h_m + h_{m+1}t + \ldots\} = r_0 + r_1t + \ldots + r_{n-1}t^{n-1}$ yields the relations $h_m\sigma'_0 = r_0$, $h_m\sigma'_1 + h_{m+1}\sigma'_0 = r_1$, $\ldots$, $h_m\sigma'_{n-1} + \ldots + h_{m+n-1}\sigma'_0 = r_{n-1}$ or, equivalently, $(h_\ell)\Sigma_n = (r_{n-1})$, where (r_{n-1}) denotes the $1 \times n$ matrix $(r_0, r_1, \ldots, r_{n-1})$ and $\ell = m+n-1$. Hence

$$z_k^\ell = (r_{n-1})\begin{pmatrix} z_k^{n-1} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} = (h_\ell)\Sigma_n\begin{pmatrix} z_k^{n-1} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix},$$

ℓ any non-negative integer. If, then, $\ell_1, \ldots, \ell_n$ is any set of unequal non-negative integers, arranged in descending order of magnitude, the n-dimensional matrix whose row matrices are $(z_1^{\ell_j}, \ldots, z_n^{\ell_j})$, $j = 1, \ldots, n$, may be written as the product of three n-dimensional matrices of which the first is the n-dimensional matrix whose row matrices are $(z_1^{n-j}, \ldots, z_n^{n-j})$, $j = 1, \ldots, n$, the second is $\sum_n$ and the third is the n-dimensional matrix whose row matrices are (h_{ℓ_j}), $j = 1, \ldots, n$. Since $\{\lambda\}$ is the quotient of $A(\ell)$ by $\Delta = A(n-1, \ldots, 1, 0)$ and since $\sum_n$, being a triangular matrix all of whose diagonal elements $= 1$, is unimodular it follows that $\{\lambda\}$ is the determinant of the n-dimensional matrix whose row matrices are $(h_{\ell_1}), \ldots, (h_{\ell_n})$. The diagonal elements of this n-dimensional matrix are $h_{\lambda_1}, \ldots, h_{\lambda_n}$, where $\ell_1 = \lambda_1 + n-1$, $\ell_2 = \lambda_2 + n-2, \ldots, \ell_n = \lambda_n$, and so, if (λ) has k **parts**, so that $\lambda_j = 0$ if $j > k$, $\{\lambda\}$ is the determinant of the k-dimensional matrix whose row matrices are (h_{ℓ_1}), $(h_{\ell_2}), \ldots, (h_{\ell_k})$, where $\ell_1 = \lambda_1 + k-1, \ldots, \ell_k = \lambda_k$. For example, $\{m\} = h_m$ so that $\{m\}$ is the character of the symmetrized m^{th} power of the self-representation of the n-dimensional unitary group. Since the average of $|\{m\}|^2$ over the group is 1 it follows that the symmetrized m^{th} power of the self-representation of the n-dimensional unitary group is irreducible. We denote this irreducible representation of the n-dimensional unitary group either by $\Gamma_{(m)}$ or by the symbol $\{m\}$ for its character. When $(\lambda) = (1^m)$ is the partition of m into m parts, each $= 1$, $\{1^m\}$ is $(-1)^m$ times the cofactor of the element in the first column and last row of H_{m+1}, and hence, since H_{m+1} is unimodular, $\{1^m\} = (-1)^m$ times the element σ'_m in the first row and last column of $\sum_{m+1}$. Thus $\{1^m\} = \sigma_m$ is the character of the anti-symmetrized m^{th} power of the self-representation of the n-dimensional unitary group. Since the average of $|\{1^m\}|^2$ over the group is 1 it follows that the anti-symmetrized m^{th} power of the self-representation of the n-dimensional unitary group is irreducible. We denote this irreducible representation of the n-dimensional unitary group either by $\Gamma_{(1^m)}$ or by the symbol $\{1^m\}$ for its character.

Since $\{\lambda\}$ is a polynomial function of degree m in $z_1, \ldots, z_n$ and since it is the determinant of the n-dimensional matrix whose row matrices are $(h_{\ell_1}), \ldots (h_{\ell_n})$ it is a linear combination, with integral coefficients, of the products $h^{(\alpha)} = h_1^{\alpha_1} \ldots h_m^{\alpha_m}$ where the α's are non-negative integers which are connected by the relation $\alpha_1 + 2\alpha_2 + \ldots + m\alpha_m = m$. Hence $\{\lambda\}$ is a generalized character of the n-dimensional unitary group and, since the average of $|\{\lambda\}|^2$ over the group is 1, it is either the character of an irreducible representation of the group or the negative of such a character. To settle this question we determine the value of $\{\lambda\}$ at the identity element of the group, i.e., at $z_1 = z_2 = \ldots = z_m = 1$. Writing $z_1 = 1 + \varepsilon_1$, $z_2 = 1 + \varepsilon_2$, ..., $z_n = 1 + \varepsilon_n$, where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are infinitesimals, the lowest order terms in $A(\ell)$ are the product of the determinant of the n-dimensional matrix whose p^{th} row matrix is $(1, \ell_p, \frac{\ell_p(\ell_p-1)}{2!}, \ldots \frac{\ell_p(\ell_p-1)\ldots(\ell_p-n+2)}{(n-1)!})$ by $\sum_{(p)} \pm \varepsilon_1^{p_1} \ldots \varepsilon_n^{p_n}$ where $(p) = (p_1, \ldots, p_n)$ is a permutation of the n symbols $0, 1, \ldots, n-1$, the + or − sign being used according as (p) is even or odd, respectively. Since the determinant of the n-dimensional matrix whose p^{th} row matrix is $(1, \ell_p, \frac{\ell_p(\ell_p-1)}{2!}, \ldots \frac{\ell_p(\ell_p-1)\ldots(\ell_p-n+2)}{(n-1)!})$ is the quotient of the difference product $\Delta(\ell) =$

$= (\ell_1 - \ell_2) \cdots (\ell_{n-1} - \ell_n)$ by $2!\, 3! \ldots (n-1)!$ it follows that the limit of $\{\lambda\} = \frac{A(\ell)}{\Delta} = \frac{A(\ell_1, \ldots, \ell_n)}{A(n-1, \ldots, 0)}$ at $\varepsilon_1 = 0, \ldots, \varepsilon_n = 0$, is the quotient of the difference product $\Delta(\ell)$ of the n integers $\ell_1, \ldots, \ell_n$ by the difference product $(n-1)!\,(n-2)! \ldots 2!\,1!$ of the n integers $n-1, \ldots, 1, 0$ and, hence, since $\{\lambda\}$ is a continuous function of $z_1, \ldots, z_n$, the value of $\{\lambda\}$ at $z_1 = 1, \ldots, z_n = 1$ is the positive integer

$$d_{(\lambda)} = \Delta(\ell) \div (n-1)!\,(n-2)! \ldots 1!$$

Thus $\{\lambda\}$ is the character of an irreducible continuous representation, of dimension $d_{(\lambda)}$, of the n-dimensional unitary group; we denote this irreducible representation by $\Gamma_{(\lambda)}$ or by the symbol $\{\lambda\}$ for its character. The various irreducible representations of the n-dimensional unitary group which we obtain in this way, by considering all partitions, with not more than n parts, of all non-negative integers are all distinct.

We now proceed to show that any continuous irreducible representation of the n-dimensional unitary group is the product of one of the irreducible representations $\Gamma_{(\lambda)}$ by a power (possibly zero) of the 1-dimensional representation $Z \to \det \bar{Z} = (\det Z)^{-1}$. We first observe that the diagonal elements of the n-dimensional unitary group constitute a commutative subgroup of the unitary group and, so, if Γ is any continuous representation of the unitary group, there exists a basis for the carrier space of Γ with respect to which the matrices Z' of Γ which correspond to the diagonal elements Z of the unitary group are all diagonal, since all irreducible representations of a commutative group are 1-dimensional.

We denote by $\beta_1, \ldots, \beta_d$, where d is the dimension of Γ, the arguments of the diagonal elements of Z' and by $\theta_1, \ldots, \theta_n$ the arguments of the diagonal elements of Z. Then each of the β's is a continuous function of the n real variables $\theta_1, \ldots, \theta_m$ and $\beta_j(\theta_1, \ldots, \theta_n) + \beta_j(\theta'_1, \ldots, \theta'_n) = \beta_j(\theta_1+\theta'_1, \ldots, \theta_m+\theta'_n)$, $j = 1, \ldots, d$. In particular,

$$\beta_j(\theta_1, \ldots, \theta_n) = \beta_j^1(\theta_1) + \beta_j^2(\theta_2) + \ldots + \beta_j^n(\theta_m),$$
$$j = 1, \ldots, d,$$

where $\beta_j^1(\theta) = \beta_j(\theta, 0, \ldots, 0)$, $\beta_j^2(\theta) = \beta_j(0, \theta, 0, \ldots, 0)$, $\ldots$, $\beta_j^n(\theta) = \beta_j(0, \ldots, 0, \theta)$.

Each of the continuous functions $\beta_j^k(\theta)$, $j = 1, \ldots, d$, $k = 1, \ldots, n$, of the single real variable θ satisfies the equation $f(\theta) + f(\theta') = f(\theta + \theta')$ and so $\beta_j^k(\theta) = m_j^k \theta$, where m_j^k is a constant which must be an integer since $\beta_j(\theta_1, \ldots, \theta_n) = \beta_j^1(\theta_1) + \ldots + \beta_j^n(\theta_n)$ increases by an integral multiple of 2π when any one of the n variables $\theta_1, \ldots, \theta_m$ increases by 2π. Thus $\beta_j(\theta_1, \ldots, \theta_n) = \sum_k m_j^k \theta_k$ so that $z'_j = \exp(\beta_j i) = z_1^{m_j^1} \ldots z_n^{m_j^n}$. If none of the integers m_j^k is negative, TrZ' is a polynomial function of $z_1, \ldots, z_n$ and, in any event, the product of TrZ' by a suitable non-negative integral power of $\det Z = z_1 \ldots z_n$ is a polynomial function of $z_1, \ldots z_n$. Since every class of the unitary group has a diagonal representative it follows that the product of $ch\,\Gamma$ by a suitable non-negative integral power of $\det Z$ is a polynomial function of $z_1, \ldots, z_n$. Now, $ch\,\Gamma$ is a symmetric function of $z_1, \ldots, z_n$ and so $(ch\,\Gamma)\Delta$

is an alternating function of $z_1, \ldots, z_n$ as is also the product of $(\mathrm{ch}\,\Gamma)\Delta$ by any non-negative integral power of $\det Z = z_1 \ldots z_n$. If $c_{(s)} z_1^{s_1} \ldots z_n^{s_n}$ is any term of this product so also is $\pm\, c_{(s)} z_1^{s_{p_1}} \ldots z_n^{s_{p_n}}$, where $(p) = \uparrow\begin{pmatrix} p_1 & \ldots & p_n \\ 1 & \ldots & n \end{pmatrix}$ is any element of S_n, the + or - sign being used according as (p) is even or odd, respectively. Hence the product of $\mathrm{ch}\,\Gamma$ by a suitable non-negative integral power of $\det Z$ is a linear combination of the functions $\{\lambda\}$. The coefficients of this linear combination must be non-negative integers since the coefficient of $\{\lambda\}$ tells us how often $\Gamma_{(\lambda)}$ occurs in the analysis of the product of Γ by a power of the 1-dimensional representation $Z \longrightarrow \det Z$ into its irreducible components. If Γ is irreducible so also is the product of Γ by a power of the 1-dimensional representation $Z \rightarrow \det Z$ and so all the coefficients must be zero save one which is 1. In other words, every irreducible continuous representation of the n-dimensional unitary group is one of the representations $\Gamma_{(\lambda)}$ or the product of one of these by a power of the 1-dimensional representation $Z \rightarrow \det \bar{Z} = (\det Z)^{-1}$. For example, the representation defined by the correspondence $Z \rightarrow \bar{Z}$, whose character is σ_{n-1}/σ_n, is the product of $\Gamma_{(1^{n-1})}$ by the representation $Z \rightarrow \det \bar{Z}$.

2. The irreducible continuous representations of the n-dimensional unimodular unitary group.

Each of the irreducible representations $\Gamma_{(\lambda)}$ of the n-dimensional unitary group furnishes a representation of the unimodular subgroup of the unitary group. If this representation were reducible certain homogeneous polynomial functions, of degree m, of the characteristic numbers $z_1, \dots, z_n$ of a typical element Z of the n-dimensional unitary group would vanish if these characteristic numbers are subjected to the constraint $z_1 \dots z_n = 1$ (for each element of $Z = V\, D(z)\, V^*$ is a homogeneous linear function of the n variables $z_1, \dots, z_n$). But this cannot be since it would imply that these homogeneous polynomial functions of $z_1, \dots, z_n$ would vanish identically so that the representation $\Gamma(\lambda)$ of the n-dimensional unitary group would be reducible. Indeed, if $f(z_1, \dots, z_n)$ is one of these functions, we would have $\sum_{k=1}^{n} f_{z^k}\, dz^k = 0$ provided that $\sum_{k=1}^{n} \frac{dz^k}{z^k} = 0$, so that $f_{z^k} = \lambda / z^k$ where λ, which is independent of k, is an undetermined multiplier. λ must be zero since $n\lambda = \sum_{k=1}^{n} z^k f_{z^k} = mf = 0$. Continuing this argument we see that all the derivatives of f, of any order, with respect to the variables $z_1, \dots, z_n$ are zero so that $f(z_1, \dots, z_n)$ vanishes identically. The same argument as in the case of the unitary group shows that the irreducible representations $\Gamma(\lambda)$ of the unimodular subgroup exhaust the continuous irreducible representations of this subgroup (the representation $Z \rightarrow \det \bar{Z}$, which we had to introduce in the case of the unitary group, being the identity representation of the unimodular subgroup of the unitary group).

Exercise 1. Show that the adjoint representation of the n-dimensional unitary group is $\Gamma \bar{\Gamma}$ where Γ is the self-representation of the group. Hint. If E_j^k is the n-dimensional matrix all of whose elements are zero save the element in the j^{th} row and k^{th} column, which is 1, the element in the j^{th} row and k^{th} column of $Z\,E_j^k\,Z*$ is $(Z)_j^j\;(Z*)_k^k$; hence the character of the adjoint representation is $|ch\,\Gamma|^2$.

Exercise 2. Show that $\Gamma_{(1)}\,\Gamma_{(1^m)} = \Gamma_{(21^{m-1})} + \Gamma_{(1^{m+1})}$.

Hint. Develop the determinant of the $(m+1)$-dimensional matrix

$$\begin{pmatrix} h_1 & \cdots & & h_{m+1} \\ h_0 & h_1 & \cdots & h_m \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & \cdots & h_0 & h_1 \end{pmatrix}$$

in terms of the first column.

Exercise 3. Show that the adjoint representation of the n-dimensional unitary group, $n > 1$, is reducible, being the sum of the identity representation $\Gamma_{(o)}$ and the irreducible representation which is the product of $\Gamma_{(21^{n-2})}$ by the 1-dimensional representation $Z \to \det \bar{Z}$.

Exercise 4. Show that the adjoint representation of the n-dimensional unimodular unitary group, $n > 1$, is irreducible, being $\Gamma_{(21^{n-2})}$.

Lecture 12

1. <u>The class factor of the element of volume of the 2k-dimensional rotation group.</u>

In discussing the n-dimensional rotation group we must treat separately the case where n is even and the case where n is odd and we take up first the case where $n = 2k$ is even. Any 2k-dimensional rotation matrix R may be written in the form $R = D R_{k-1} \cdots R_1$ where D is a k-dimensional diagonal block matrix, whose diagonal elements are 2-dimensional rotation or reflexion matrices, and $R_1, \ldots, R_{k-1}$ are products of plane 2k-dimensional rotation matrices of the type

$$R_{1,2}(\beta) = \begin{pmatrix} c & -s & \\ s & c & 0 \\ 0 & & E_{2k-2} \end{pmatrix}, \quad c = \cos\beta, \quad s = \sin\beta,$$

R_1 being $R_{1,2k-1}(\Theta_{4k-6}) \cdots R_{2k-3,2k-1}(\Theta_{2k-2})\, R_{2k-2,2k-1}(\phi_2)\, R_{1,2k}(\Theta_{2k-3}) \cdots$ $R_{2k-3,2k}(\Theta_1)\, R_{2k-2,2k}(\phi_1)$, R_2 being $R_{1,2k-3}(\Theta_{8k-16}) \cdots$ $R_{2k-5,2k-3}(\Theta_{6k-10})\, R_{2k-4,2k-3}(\phi_4)\, R_{1,2k-2}(\Theta_{6k-11}) \cdots R_{2k-5,2k-2}(\Theta_{4k-5}) \times$ $R_{2k-4,2k-2}(\phi_3)$ and so on to R_{k-1} which is $R_{1,3}(\Theta_p)\, R_{2,3}(\phi_{2k-2}) \times$ $R_{1,4}(\Theta_{p-1})\, R_{2,4}(\phi_{2k-3})$ where $p = 2\,(k-1)^2$. The p Θ's are latitude angles, varying over the interval $-\pi/2 \leq \Theta \leq \pi/2$, the $2(k-1)$ ϕ's are longitude angles, varying over the interval $-\pi < \phi \leq \pi$, and $k-1$ of the angles β, namely $\beta_2, \ldots, \beta_k$ are latitude angles, varying over the interval $-\pi/2 \leq \beta \leq \pi/2$ while the remaining one, β_1, is a longitude angle, varying over the interval $-\pi < \beta_1 \leq \pi$. Taking as our $N = \frac{n(n-1)}{2} = k(2k-1)$ parameters for the 2k-dimensional rotation group the k β's and $2k(k-1)$ angles Θ and ϕ, the origin is the identity point of the parametric space and the N characteristic matrices are of the type $M_{p,q}$, $p < q = 2, \ldots, 2k$, where $M_{p,q}$ is the $2k$-dimensional matrix all of whose elements are zero save those in the p^{th} column and q^{th} row, and in the q^{th} column and p^{th} row which are 1 and -1, respectively. In order to obtain the element of volume of the $2k$-dimensional rotation group we have to express $\delta R = R^t dR$ as a linear combination of the N characteristic matrices $M_{p,q}$ and to determine the modulus of the determinant of the $N \times N$ matrix of the N coefficients of this linear combination (these coefficients being linear combinations of the differentials of the N parameters). Since the group functions we shall have to integrate over the group will all be class functions we introduce the class and in-class parameters which are defined as follows. If Z is any $2k$-dimensional rotation matrix there exists a $2k$-dimensional rotation matrix B such that $B^t Z B$ is a diagonal k-dimensional block matrix $D(\alpha_1, \ldots, \alpha_k) = D(\alpha)$, whose diagonal elements are $\begin{pmatrix} c_j & -s_j \\ s_j & c_j \end{pmatrix}$, $c_j = \cos\alpha_j$, $s_j = \sin\alpha_j$, $j = 1, \ldots, k$, and we term $D(\alpha)$ a diagonal block representative of the class of the $2k$-dimensional rotation group to which Z belongs. Writing B^t in the form $D R_{k-1} \ldots R_1 = D S^t$, say, we have $Z = B D(\alpha) B^t = S D(\alpha) S^t$, the diagonal factor D of B^t disappearing. The k class parameters are the angles $\alpha_1, \ldots, \alpha_k$ and the $2k(k-1)$ in-class parameters are the angles Θ and ϕ which occur in the $2k(k-1)$ plane $2k$-dimensional rotation matrices whose product is S^t. Since $ZS = S D(\alpha)$ we find, as in the discussion of the n-dimensional unitary group, that $S^t\ \delta Z\ S = D^{-1}(\alpha)\ \delta S\ D(\alpha) + \delta D(\alpha) - \delta S$ and we are confronted with the problem of expressing $D^{-1}(\alpha)\ \delta S \cdot D(\alpha) + \delta D(\alpha) - \delta S$ as a linear combination of the N characteristic matrices $M_{p,q}$ or, equivalently, of expressing $U \{ D^{-1}(\alpha)\ \delta S\ D(\alpha) + \delta S \} U^*$ as a linear combination of the N matrices $U\ M_{p,q}\ U^*$, where U is any 2k-dimensional unitary matrix. Taking U to be the matrix which appears, when written as a k-dimensional block matrix, as a diagonal matrix each of whose diagonal elements is $2^{-1/2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ we find that $U\ D(\alpha)\ U^*$ is the 2k-dimensional diagonal matrix whose diagonal elements are $\exp \alpha_1 i = z_1$, $\exp -\alpha_1 i = 1/z_1$, ..., $\exp \alpha_k i = z_k$, $\exp -\alpha_k i = 1/z_k$ and that $U\ M_{1,2}\ U^*$, for example, is the 2k-dimensional diagonal matrix $N_{1,2}$ all of whose diagonal elements are zero save the first and second, which are i and − i, respectively. $U\ \delta S\ U^*$ is a linear combination of the N matrices $U\ M_{p,q}\ U^*$ and the diagonal elements of all these matrices are zero save when p is one of the k numbers 1, 3, ..., 2k−1 and q = p+1. Thus the diagonal elements of $U\ D^{-1}(\alpha)\ \delta S\ D(\alpha)\ U^* = U\ D^{-1}(\alpha)\ U^* \cdot U\ \delta S\ U^* \cdot U\ D(\alpha)\ U^*$ are the same as the diagonal elements of δS and, since $U\ \delta D(\alpha)\ U^* = d\alpha_1\ M_{1,2} + \dots + d\alpha_k\ M_{2k-1,2k}$ we see that the coefficient of $U\ M_{2j-1,2j}\ U^*$ in $U \{ D^{-1}(\alpha)\ \delta S\ D(\alpha) + \delta D(\alpha) - \delta S \} U^*$ is $d\alpha_j$, $j = 1, \dots, k$. When $U\ M_{1,3}\ U^*$, $U\ M_{1,4}\ U^*$, $U\ M_{2,3}\ U^*$ and $U\ M_{1,4}\ U^*$ are written as k-dimensional block matrices all of their elements are zero save those in the first row and second column and in the second row and first column and a simple calculation shows that $\frac{1}{2} U \{ M_{1,3} + M_{2,4} - i\ M_{2,3} + i\ M_{1,4} \} U^*$ is the k-dimensional block matrix all of whose elements are zero save the element in the first row and second column, which is $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, and the element in the second row and first column which is $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It follows that, if $N_{1,3} = M_{1,3} + M_{2,4} - i\,M_{2,3} + i\,M_{1,4}$, then $U\,D^{-1}(\alpha)\,N_{1,3}\,D(\alpha)\,U^* = U\,D^{-1}(\alpha)\,U^* \cdot U\,N_{1,3}\,U^* \cdot U\,D(\alpha)\,U^* =$ $= (z_2/z_1)\,N_{1,3}$. Similarly, on writing $N_{2,4} = M_{1,3} + M_{2,4} + i\,M_{2,3} - i\,M_{1,4}$, $U\,D^{-1}(\alpha)\,N_{2,4}\,D(\alpha)\,U^* = (z_1/z_2)\,N_{2,4}$; on writing $N_{1,4} = M_{1,4} + M_{2,3} + i\,M_{1,3} - i\,M_{2,4}$, $U\,D^{-1}(\alpha)\,N_{1,4}\,D(\alpha)\,U^* = (z_1 z_2)^{-1} N_{1,4}$ and, on writing $N_{2,3} = M_{1,4} + M_{2,3} - i\,M_{1,3} + i\,M_{2,4}$, $U\,D^{-1}(\alpha)\,N_{2,3}\,D(\alpha)\,U^* = z_1 z_2 N_{2,3}$. $U\,\delta S\,U^*$ is a linear combination of the N matrices $N_{p,q}$, $p < q$, and it follows, as in the case of the unitary group, that the element of volume of the 2k-dimensional rotation group may be written as the product of two factors, one of which, the class factor, involves only the class parameters $\alpha_1, \ldots, \alpha_k$ and the other of which, the in-class factor, involves only the in-class parameters Θ and ϕ. Since the only functions we shall have to integrate over the group will be class functions we shall be concerned only with the class factor which we shall term simply the element of volume of the 2k-dimensional rotation group. This element of volume is $\prod_{p<q} \left| (1 - z_p/z_q)(1 - z_q/z_p) \times (1 - z_p z_q)(1 - 1/(z_p z_q)) \right| d(\alpha_1, \ldots, \alpha_k) = \prod_{p<q} \left| (z_p - z_q) \times (1 - z_p z_q) \right|^2 d(\alpha_1, \ldots, \alpha_k)$. The expression

$\prod_{p<q} |(z_p - z_q)(1 - z_p z_q)|$ is the modulus of the difference product of the k numbers $z_p + \bar{z}_p$ since $z_p + \bar{z}_p - z_q - \bar{z}_q = (z_p - z_q)(1 - \bar{z}_p \bar{z}_q)$, the z's being complex numbers of unit modulus. Since $(z_p + \bar{z}_p)^j$, where j is any positive integer, $= z_p^j + \bar{z}_p^j$ + a linear combination of terms $z_p^q + \bar{z}_p^q$, $q = 0, 1, \ldots, j-1$, it follows, on writing the difference product of the k numbers $z_p + \bar{z}_p$ as a Vandermonde determinant, that $\prod_{p<q} |(z_p - z_q)(1 - z_p z_q)|^2$ is the square of the determinant $C(k-1, \ldots, 1, 0)$ of the k-dimensional matrix whose j^{th} row matrix is $(c_{k-j}(\alpha_1), \ldots, c_{k-j}(\alpha_k))$ where $c_p(\alpha) = 2\cos(p\alpha)$ if $p > 0$ while $c_0(\alpha) = 1$. $C(k-1, \ldots, 1, 0) = \sum_{(p)} \pm c_{p_1}(\alpha_1) \ldots c_{p_k}(\alpha_k)$, where $(p) = \begin{pmatrix} p_1 & \ldots & p_k \\ 1 & \ldots & k \end{pmatrix}$ is an arbitrary element of S_k, the + or − sign being used according as (p) is even or odd, and only those $k!$ of the $(k!)^2$ terms $(\pm c_{p_1}(\alpha_1) \ldots c_{p_k}(\alpha_k))(\pm c_{q_1}(\alpha_1) \ldots c_{q_k}(\alpha_k))$, which occur in $\{C(k-1, \ldots, 1, 0)\}^2$, for which $(q) = (p)$ contribute anything to the integral of 1 over the group, each of these terms contributing $2^{k-1}\pi^k$ (since $k-1$ of the angles $\alpha_1, \ldots, \alpha_k$ are latitude angles while one is a longitude angle). Thus the normalised element of volume of the 2k-dimensional rotation group is

$$dV_{(\alpha)} = (2^{k-1}\pi^k k!)^{-1} \{C(k-1, \ldots, 1, 0)\}^2 \, d(\alpha_1, \ldots, \alpha_k)$$

If (λ) is any partition, involving not more than k parts, of any non-negative integer m, we set $\ell_1 = \lambda_1 + k - 1, \ldots, \ell_k = \lambda_k$ and introduce the class function $[\lambda] = \dfrac{C(\ell_1, \ldots, \ell_k)}{C(k-1, \ldots, 1, 0)}$, where $C(\ell_1, \ldots, \ell_k)$ is the determinant of the k-dimensional matrix whose

p^{th} row matrix is $(c_{\ell_p}(\alpha_1), \ldots, c_{\ell_p}(\alpha_k))$, $p = 1, \ldots, k$, and the same argument as in the case of the unitary group shows that, if $\ell_p = 0$, the average of $[\lambda]^2$ over the group is 1 while, if $\ell_p > 0$, this average is 2 since $c_{\ell_p}(\alpha) = 1$ if $\ell_p = 0$ while $c_{\ell_p}(\alpha) = 2\cos \ell_p \alpha$ if $\ell_p > 0$. Similarly, if $(\lambda) \neq (\lambda')$, the average of $[\lambda].[\lambda']$ over the group is zero.

2. <u>The element of volume of the (2k+1)-dimensional rotation group.</u>

When $n = 2k+1$ is odd we have, in addition to the $k(2k-1)$ characteristic matrices of the 2k-dimensional rotation group, 2k characteristic matrices $M_{1,2k+1}, \ldots, M_{2k,2k+1}$. The number of class parameters (all of which are latitude angles) is the same, namely k, as when $n = 2k$, there being 2k additional in-class parameters, but, now, all the class parameters are latitude angles. We take as our transforming unitary matrix U the (2k+1)-dimensional matrix obtained by adding e^*_{2k+1} as a last row matrix to the U we used when discussing the 2k-dimensional rotation group (the last column matrix of the new U being e_{2k+1}). On setting $M_{1,2k+1} + i\, M_{2,2k+1} = N_{1,2k+1}$, $M_{1,2k+1} - i\, M_{2,2k+1} = N_{2,2k+1}$ we see that $U D^{-1}(\alpha)\, N_{1,2k+1}\, D(\alpha)\, U^* = z_1\, N_{1,2k+1}$ and that $U D^{-1}(\alpha)\, N_{2,2k+1}\, D(\alpha)\, U^* = (1/z_1)\, N_{2,2k+1}$. Thus the class factor of the element of volume of the (2k+1)-dimensional rotation group is

$$\left|(1 - z_1) \ldots (1 - z_k) \prod_{p<q} (z_p - z_q)(1 - z_p z_q)\right|^2 d(\alpha_1, \ldots, \alpha_k)$$

Since $|1 - z_p|^2 = 4\sin^2(\alpha_p/2)$ we see, on multiplying the p^{th} column matrix of the k-dimensional matrix whose determinant is

$C(k-1, \ldots, 1, 0)$ by $2 \sin \alpha_p/2$, that $\left| (1-z_1) \ldots (1-z_k) \prod_{p<q} \prod_{p<q} (z_p - z_q)(1 - z_p z_q) \right|^2$ is the square of the determinant $S(k-\frac{1}{2}, \ldots, \frac{3}{2}, \frac{1}{2})$ of the k-dimensional matrix whose p^{th} row matrix is $(s_{k+\frac{1}{2}-p}(\alpha_1), \ldots, s_{k+\frac{1}{2}-p}(\alpha_k))$, $p = 1, \ldots, k$, where $s_{\ell+\frac{1}{2}}(\alpha) = 2i \sin (\ell + \frac{1}{2})\alpha$, ℓ any non-negative integer. The same argument as in the case of the 2k-dimensional rotation group shows that the volume of the $(2k+1)$-dimensional rotation group is $k!\,(2\pi)^k$, all the angles $\alpha_1, \ldots, \alpha_k$ being latitude angles. Thus the normalised element of volume of the $(2k+1)$-dimensional rotation group is

$$dV_{(\alpha)} = \left\{ (2\pi)^k k! \right\}^{-1} \left\{ S(k-\tfrac{1}{2}, \ldots, \tfrac{1}{2}) \right\}^2 d(\alpha_1, \ldots, \alpha_k) .$$

For the 3-dimensional rotation group this reduces to $\frac{2}{\pi} \sin^2 \frac{\alpha}{2}\, d\alpha$.

If (λ) is any partition, involving not more than k parts, of any non-negative integer m, we set, as before, $\ell_1 = \lambda_1 + k-1, \ldots, \ell_k = \lambda_k$ and introduce the function

$$[\lambda] = \frac{S(\ell_1 + \frac{1}{2}, \ldots, \ell_k + \frac{1}{2})}{S(k - \frac{1}{2}, \ldots, \frac{1}{2})},$$

where $S(\ell_1 + \frac{1}{2}, \ldots, \ell_k + \frac{1}{2})$ is the determinant of the k-dimensional matrix whose p^{th} row matrix is $(s_{\ell_p+\frac{1}{2}}(\alpha_1), \ldots, s_{\ell_p+\frac{1}{2}}(\alpha_k))$, $p = 1, \ldots, k$. Then the average of $[\lambda]^2$ over the group is 1 and the average of $[\lambda][\lambda']$, where $(\lambda') \neq (\lambda)$, over the group is zero. For the 3-dimensional rotation group (λ) has only one part, m, so that $[\lambda] = \sin (m + \frac{1}{2})\alpha / \sin \frac{1}{2}\alpha$, the value of $[\lambda]$ at the identity point $\alpha = 0$ of the parametric space being $2m + 1$. We shall see in the next lecture that the functions $[\lambda]$ are the characters of irreducible representations $\Gamma_{[\lambda]}$ of the $(2k + 1)$-dimensional rotation group and that these representations $\Gamma_{[\lambda]}$ exhaust the continuous irreducible representations of this group. For example, the irreducible representations of the 3-dimensional rotation group are of dimensions 1 (the identity representation), 3 (the self-representation), 5 , 7 , 9 , 11 , ... and so on.

Lecture 13

1. The irreducible continuous representations of the 2k-dimensional rotation group.

The characteristic numbers of a typical element Z of the $2k$-dimensional rotation group are $z_j = \exp\alpha_j i$ and $\bar{z}_j = \exp -\alpha_j i$, $j = 1, \ldots, k$. Hence the polynomial function $f(t) = \sigma_0 - \sigma_1 t + \ldots + \sigma_{2k} t^{2k} = \sigma'_0 + \sigma'_1 t + \ldots + \sigma'_{2k} t^{2k}$ of the indeterminate t is $\prod_{j=1}^{k} (1 - tz_j)(1 - t\bar{z}_j) = \prod_{j=1}^{k} (z_j - t)(\bar{z}_j - t)$, since $\bar{z}_j z_j = 1$. Thus $t^{2k} f(1/t) = f(t)$ so that $\sigma'_{2k} + \sigma'_{2k-1} t + \ldots + \sigma'_0 t^{2k} = \sigma'_0 + \sigma'_1 t + \ldots + \sigma'_{2k} t^{2k}$ which implies that

$\sigma'_{2k-j} = \sigma'_j$, $j = 0, 1, \ldots, k-1$. Since $f(z_j) = 0$ we have $\sigma'_0 + \sigma'_1 z + \ldots + \sigma'_{2k} z^{2k} = 0$ if z is one of the numbers $z_1, \ldots, z_k$ and, on dividing this equation by z^j, where j is one of the numbers $1, \ldots, k$, we obtain $\sigma'_0 \bar{z}^j + \ldots + \sigma'_{j-1} \bar{z} + \sigma'_j + \sigma'_{j+1} z + \ldots + \sigma'_{2k} z^{2k-j} = 0$. On replacing z by $\bar{z}$, subtracting and denoting $2(\sin p\alpha)i$ by $s_p(\alpha)$, we find that $\sigma'_0 s_{2k-j}(\alpha) + \sigma'_1 s_{2k-j-1}(\alpha) + \ldots + \sigma'_{j+1} s_1(\alpha) = \sigma'_{j-1} s_1(\alpha) + \ldots + \sigma'_0 s_j(\alpha)$, α being any one of the k angles $\alpha_1, \ldots, \alpha_k$. If, then, we denote simply by $s(\alpha)$ the $2k \times 1$ matrix whose elements are $(s_{2k-1}(\alpha), \ldots, s_1(\alpha), 0)$ the j^{th} and $(2k-j)^{th}$ elements of the $2k \times 1$ matrix $\Sigma_{2k} s(\alpha)$ are the same, Σ_{2k} being the $2k$-dimensional triangular matrix whose p^{th} row matrix is $(\sigma'_{1-p}, \sigma'_{2-p}, \ldots, \sigma'_{2k-p})$, it being understood that $\sigma'_q = 0$ if q is negative and that α is one of the k angles $\alpha_1, \dots, \alpha_k$. We have already seen that, if z is any one of the characteristic numbers of Z and ℓ is any non-negative integer, then

$$z^{\ell} = (h_{\ell}) \sum_{2k} \begin{pmatrix} z^{2k-1} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

where (h_{ℓ}) is the $1 \times 2k$ matrix $(h_{\ell-2k+1}, h_{\ell-2k+2}, \dots, h_{\ell})$; on replacing z by $\bar{z}$ and subtracting, we obtain

$$s_{\ell}(\alpha) = (h_{\ell}) \sum_{2k} s(\alpha)$$

Since the j^{th} and $(2k-j)^{th}$ elements of the $2k \times 1$ matrix $\sum_{2k} s(\alpha)$ are the same and since the $2k^{th}$ element of this matrix is zero this relation may be written more compactly as

$$s_{\ell}(\alpha) = (h_{\ell-k}, h_{\ell-k-1} + h_{\ell-k+1}, \dots, h_{\ell-2k+1} + h_{\ell-1}) \cdot \sum_k \begin{pmatrix} s_k(\alpha) \\ \cdot \\ \cdot \\ \cdot \\ s_1(\alpha) \end{pmatrix}$$

If ℓ is positive we may replace ℓ in this relation by $\ell+1$ and $\ell-1$ and obtain, on subtraction,

$$s_1(\alpha)\, c_{\ell}(\alpha) = (h'_{\ell-k+1}, h'_{\ell-k} + h'_{\ell-k+2}, \dots, h'_{\ell-2k+2} + h'_{\ell}) \sum_k \begin{pmatrix} s_k(\alpha) \\ \cdot \\ \cdot \\ \cdot \\ s_1(\alpha) \end{pmatrix}$$

where $c_{\ell}(\alpha) = 2 \cos \ell\alpha$, $h'_j = h_j - h_{j-2}$, and this relation remains valid when $\ell = 0$ since, then, the $1 \times 2k$ matrix on the right is $(0, 0, \dots, 0, 1)$ and $c_0(\alpha) = 1$. If $(\lambda) = (\lambda_1, \dots, \lambda_k)$ is any partition, involving not more than k non-zero parts, of any non-negative integer m, we set $\ell_1 = \lambda_1+k-1$, ..., $\ell_k = \lambda_k$, so that $\ell_1 > \ell_2 > \dots > \ell_k \geqslant 0$, and we denote by $[\lambda]$ the determinant of the k-dimensional matrix whose p^{th} row matrix is $(h'_{\ell_p-k+1}$,

$h'_{\ell_p-k} + h'_{\ell_p-k+2}$, ..., $h'_{\ell_p-2k+2} + h'_{\ell_p})$, $p = 1, \dots, k$,

and it follows that $C(\ell)$, which is the determinant of the k-dimensional matrix whose p^{th} row matrix is $(c_{\ell_p}(\alpha_1), \dots, c_{\ell_p}(\alpha_k))$, is the product of $[\lambda]$ by a number which is independent of (ℓ). When $(\lambda) = (0)$, $(\ell) = (k-1, \dots, 1, 0)$ and $[\lambda] = 1$, since it is the determinant of a triangular k-dimensional matrix all of whose diagonal elements are 1. Hence $C(\ell) = [\lambda]\, C(k-1, \dots, 0)$. Now h_j, being the character of an irreducible representation of the 2k-dimensional unitary group, is the character of a representation, in general reducible, of the 2k-dimensional rotation group, the latter group being a subgroup of the former, and so $[\lambda]$ is a generalised character of the 2k-dimensional rotation group. Since $[\lambda] = C(\ell) / C(k-1, \dots, 0)$, we know that the average of $|[\lambda]|^2$ over the 2k-dimensional rotation group is 1 when $\lambda_k = 0$ and 2 when $\lambda_k > 0$ and, hence, that $[\lambda]$ is, when $\lambda_k = 0$, either the character of an irreducible representation of this group or the negative of such a character. To settle this point we evaluate $[\lambda]$ at the identity element of the group, i.e., at $z_1 = z_2 = \dots = z_k = 1$. Since, when α is infinitesimal, $c_\ell(\alpha) = 2 - \ell^2\alpha^2$, if $\ell > 0$, while $c_0(\alpha) = 1$, the same argument as in the case of the unitary group shows that the limit as $\alpha_1, \dots, \alpha_k$ tend separately to zero, remaining different in the process, of $C(\ell) / C(k-1, \dots, 0)$

is, when $\lambda_k = 0$, the quotient of the difference product of the k integers $\ell_1^2, \ldots, \ell_{k-1}^2, 0$ by the difference product $2^{-(k-1)}(2k-2)!\,(2k-4)!\ldots 2!$ of the k integers $(k-1)^2, \ldots, 1, 0$ and that, when $\lambda_k > 0$, it is twice the quotient of the difference product of the k integers $\ell_1^2, \ldots, \ell_k^2$ by $2^{-(k-1)}(2k-2)!\ldots 2!$

Thus $C(\ell)\,/\,C(k-1, \ldots, 0)$ is, when $\lambda_k = 0$, the character of an irreducible continuous representation, which we denote either by $\Gamma_{[\lambda]}$ or by the symbol $[\lambda]$ for its character, of the $2k$-dimensional rotation group, and the various irreducible continuous representations of this group which we obtain in this way, by considering all partitions having not more than $k-1$ non-zero parts, of all non-negative integers m, are all different. On the other hand $[\lambda]$ is not, when $\lambda_k = \ell_k$ is positive, the character of an irreducible representation of the $2k$-dimensional rotation group but rather the sum or difference of two such characters. It is not hard to see that it is the sum of the characters $[\lambda]_+ = \mathrm{ch}\,\Gamma_{[\lambda]+}$ and $[\lambda]_- = \mathrm{ch}\,\Gamma_{[\lambda]-}$ of two irreducible representations $\Gamma_{[\lambda]+}$ and $\Gamma_{[\lambda]-}$ of the same dimension, the common dimension being, as we have seen above, $2^{k-1}\Delta(\ell^2)\,/\,(2k-2)!\ldots 2!$. Indeed, the character $\mathrm{ch}\,\Gamma$ of any representation Γ of the $2k$-dimensional rotation group is a symmetric function of the k angles $\alpha_1, \ldots, \alpha_k$ which is an even function of $k-1$ of these angles, say $\alpha_2, \ldots, \alpha_k$. Hence the product of $\mathrm{ch}\,\Gamma$ by $C(k-1, \ldots, 0)$ is an alternating function of the k angles $\alpha_1, \ldots, \alpha_k$ which is an even function of $\alpha_2, \ldots, \alpha_k$. $C(\ell)$ is the sum of $2^k k!$ terms of the type $\pm\exp(\pm\ell_{p_1}\alpha_1 + \ldots \pm \ell_{p_k}\alpha_k)i$ where $(p) = \uparrow\begin{pmatrix} p_1 & \ldots & p_k \\ 1 & \ldots & k \end{pmatrix}$

is any element of S_k and the sign in front of the exponential is + or - according as (p) is even or odd. If any one of these $2^k k!$ terms occurs in the product of ch $\Gamma_{[\lambda]+}$ by C (k-1, ..., 0) all the $2^{k-1} k!$ terms obtained from it by letting (p) run over S_k and maintaining the sign of the coefficient of α_1, the signs of the coefficients of $\alpha_2, \ldots, \alpha_k$ being freely varied, will also occur in this product. None of the remaining $2^{k-1} k!$ terms of C (ℓ) will appear in the product of ch $\Gamma_{[\lambda]+}$ by C (k-1, ..., 0) for, if one of these did, all of them would appear in this product which would, then, contain C (ℓ) and this cannot be since $\Gamma_{[\lambda]+}$ is irreducible. Thus C (ℓ) is the sum of two sets of $2^{k-1} k!$ terms, a typical term of the first set being $\pm \exp(\ell_{p_1}\alpha_1 \pm \pm \ldots \ell_{p_k}\alpha_k)$ and a typical term of the second set being $\pm \exp(-\ell_{p_1}\alpha_1 \pm \ldots \pm \ell_{p_k}\alpha_k)$. Since both $\exp(\ell_{p_1}\alpha_1 \pm \ldots \pm \ell_{p_k}\alpha_k)$ and $\exp(-\ell_{p_1}\alpha_1 \pm \ldots \pm \ell_{p_k}\alpha_k)$ have the same value, namely 1, when $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$ it follows that $[\lambda]$ is, when $\lambda_k > 0$, the sum of the characters of two irreducible representations $\Gamma_{[\lambda]+}$ and $\Gamma_{[\lambda]-}$ of the 2k-dimensional rotation group. The product of $[\lambda]_+ =$ = ch $\Gamma_{[\lambda]+}$ by C (k-1, ..., 1, 0) is the sum of the $2^{k-1} k!$ terms $\pm \exp(\ell_{p_1}\alpha_1 \pm \ldots \pm \ell_{p_k}\alpha_k)$ and the product of $[\lambda]_- =$ ch $\Gamma_{[\lambda]-}$ by C (k-1, ..., 1, 0) is the sum of the $2^{k-1} k!$ terms $\pm \exp(-\ell_{p_1}\alpha_1 \pm \pm \ell_{p_k}\alpha_k)$. Since the product $(\exp \ell_{p_1}\alpha_1 i + \exp - \ell_{p_1}\alpha_1 i) \ldots (\exp \ell_{p_k}\alpha_k i + \exp - \ell_{p_k}\alpha_k i)$ differs from the product $(\exp \ell_{p_1}\alpha_1 i - \exp - \ell_{p_1}\alpha_1 i) \ldots (\exp \ell_{p_k}\alpha_k i - \exp - \ell_{p_k}\alpha_k i)$ only in the signs prefixed to those terms in the product for which the number of $-$ signs prefixed to the ℓ's is odd, we have

$$S(\ell) = C(k-1, \ldots, 0)\left\{ \operatorname{ch} \Gamma_{[\lambda]+} - \operatorname{ch} \Gamma_{[\lambda]-} \right\}$$

where $S(\ell)$ is the determinant of the k-dimensional matrix whose p^{th} row matrix is $(s_{\ell_p}(\alpha_1), \ldots, s_{\ell_p}(\alpha_k))$, $p = 1, \ldots, k$, $s_j(\alpha) = (2 \sin j\alpha)\, i$ and, since

$$C(\ell) = C(k-1, \ldots, 0)\left\{ \operatorname{ch} \Gamma_{[\lambda]+} + \operatorname{ch} \Gamma_{[\lambda]-} \right\}$$

we have

$$\operatorname{ch} \Gamma_{[\lambda]+} = \tfrac{1}{2}\left\{ C(\ell) + S(\ell) \right\} / C(k-1, \ldots, 1, 0)$$

$$\operatorname{ch} \Gamma_{[\lambda]-} = \tfrac{1}{2}\left\{ C(\ell) - S(\ell) \right\} / C(k-1, \ldots, 1, 0)$$

These are the generalisations, for $k > 1$, of the formulas

$\exp m\alpha i = \cos m\alpha + (\sin m\alpha)\, i$; $\exp - m\alpha i = \cos m\alpha - (\sin m\alpha)\, i$ to which they reduce when $k = 1$.

The characters $[\lambda]$, where $\lambda_k = 0$, are even functions of all the k angles $\alpha_1, \ldots, \alpha_k$ while the characters $[\lambda]_+$ and $[\lambda]_-$, $\lambda_k > 0$, are even functions of only $k-1$ of these angles, a change of sign of α_1 interchanging $[\lambda]_+$ and $[\lambda]_-$. The irreducible representations $\Gamma_{[\lambda]+}$ and $\Gamma_{[\lambda]-}$ which we obtain, by considering all partitions with k non-zero parts of all positive integers m, are all different and they all differ from the irreducible representations $\Gamma_{[\lambda]}$, $\lambda_k = 0$.

The same argument as in the case of the n-dimensional unitary group shows that, if Γ is any continuous representation of the 2k-dimensional rotation group, then $\operatorname{ch} \Gamma$ is a linear combination, with positive integral coefficients, of terms $z_1^{m_1} \ldots z_k^{m_k}$ where the m's are integers. $\operatorname{ch} \Gamma$ is a symmetric function of $z_1, \ldots, z_k$ and it is also an even function of $z_2, \ldots, z_k$ and, hence, by the same argument as in the case of the unitary group, the product of $ch\,\Gamma$ by $C\,(k-1, \ldots, 1, 0)$ is a linear combination of the characters $[\lambda]$, $[\lambda]_+$, $[\lambda]_-$, the latter characters not appearing if $ch\,\Gamma$ is an even function of α_1. Thus the representations $\Gamma_{[\lambda]}$, $\Gamma^{\circ}_{[\lambda]+}$, $\Gamma_{[\lambda]-}$ exhaust the continuous irreducible representations of the 2k-dimensional rotation group.

2. The irreducible continuous representations of the (2k+1)-dimensional rotation group.

The characteristic numbers of a typical element Z of the (2k+1)-dimensional rotation group are $z_j = \exp \alpha_j i$, $\bar{z}_j = \exp -\alpha_j i$, $j = 1, \ldots, k$, and 1. We denote by $h_0, h_1, \ldots$ the complete symmetric functions of these 2k+1 characteristic numbers and by $h_0^*, h_1^*, \ldots$ the complete symmetric functions of the first 2k of them. Writing, as in the discussion of the 2k-dimensional rotation group, $h_j - h_{j-2} = h'_j$ we have

$$h'_0 + h'_1 t + \ldots = (1-t^2)(h_0 + h_1 t + \ldots) = (1+t)(h_0^* + h_1^* t + \ldots)$$

so that $h'_j = h_j^* + h_{j-1}^*$, $j = 0, 1, 2, \ldots$ It follows, then, from the relation

$$s_\ell(\alpha) = (h^*_{\ell-k},\ h^*_{\ell-k-1} + h^*_{\ell-k+1},\ \ldots,\ h^*_{\ell-2k+1} + h^*_{\ell-1}) \sum_k s(\alpha)$$

that

$$2(\cos\alpha/2)\, s_{\ell+\frac{1}{2}}(\alpha) = s_\ell(\alpha) + s_{\ell+1}(\alpha) = (h'_{\ell-k+1}, \ldots, h'_{\ell-2k+2} + h'_\ell) \sum_k s(\alpha)$$

and from this we deduce that $[\lambda]$, which is the determinant of the k-dimensional matrix whose p^{th} matrix row is $(h'_{\ell_p-k+1}, \ldots, h'_{\ell_p-2k+2} + h'_\ell)$, is the quotient of $S(\ell + 1/2) = S(\ell_1 + 1/2, \ldots, \ell_p + 1/2)$ by $S(k - 1/2, \ldots, 1/2)$ where $S(\ell + 1/2)$ is the determinant of the k-dimensional matrix whose p^{th} row matrix is $(s_{\ell_p + \frac{1}{2}}(\alpha_1), \ldots, s_{\ell_p + \frac{1}{2}}(\alpha_k))$, $s_j(\alpha)$ being $(2 \sin j\alpha)\, i$. From this it follows, as in the discussion of the 2k-dimensional rotation group, that $[\lambda]$ is the character of an irreducible representation $\Gamma_{[\lambda]}$ of the $(2k+1)$-dimensional rotation group and, since the character of any representation of this group is an even symmetric function of the k angles $\alpha_1, \ldots, \alpha_k$, the representations $\Gamma_{[\lambda]}$, which are all different, exhaust the continuous representations of the $(2k+1)$-dimensional rotation group. Since $s_p(\alpha) = 2p\alpha i\,(1 - p^2\alpha^2/3! + \ldots)$ the dimension of the irreducible representation $\Gamma_{[\lambda]}$ of the $(2k+1)$-dimensional rotation group is the quotient of $(\ell_1 + 1/2) \ldots (\ell_p + 1/2) \prod_{p<q} \{(\ell_p + 1/2)^2 - (\ell_q + 1/2)\}^2$

by $(k - 1/2) \ldots 1/2 \prod_{p<q} \{(k - p + 1/2)^2 - (k - q + 1/2)^2\}$.

Since $\prod_{p<q} \{(k - p - 1/2)^2 - (k - q - 1/2)^2\}$, p fixed, $= (2k - 2p)!$ it follows that the dimension of $\Gamma_{[\lambda]}$ is the quotient of

$(2\ell_1 + 1) \ldots (2\ell_p + 1)\, \Delta(\ell) \prod_{p<q} (\ell_p + \ell_q + 1)$ by

$(2k-1)!\,(2k-3)! \ldots 1!$ For example, the dimension of the irreducible representation $[\lambda_1, \lambda_2]$ of the 5-dimensional rotation group is

$\frac{1}{6}(2\lambda_1 + 3)(2\lambda_2 + 1)(\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 + 2)$.

Lecture 14

1. The irreducible continuous representations of the n-dimensional orthogonal group.

We term an n-dimensional orthogonal matrix whose determinant is -1 an n-dimensional reflexion matrix and observe that every n-dimensional reflexion matrix is of the form $A Z$ where A is a fixed n-dimensional reflexion matrix and Z is a variable n-dimensional rotation matrix. We term the collection of matrices $A Z$ the reflexion part of the n-dimensional orthogonal group (the collection of matrices Z being the rotation part of this group). Any element $A Z$ of the reflexion part may be identified by the parameters which identify the element Z of the rotation part and we define the element of volume of the n-dimensional orthogonal group at az by equating it to the element of volume of the n-dimesnional rotation group at z (the element of volume of the n-dimensional orthogonal group at z being the same as the element of volume of the n-dimensional rotation group at z). This definition assures us that the element of volume of the n-dimensional orthogonal group is invariant under all left translations of the group, including the left translations which are induced by reflexion matrices as well as those which are induced by rotation matrices. It follows that the volume of the reflexion part of the n-dimensional orthogonal group is the same as the volume of the rotation part and so we refer to these two parts as halves of the group. The determinant representation of the n-dimensional orthogonal group assigns to Z the number 1 and to $A Z$ the number -1 and we denote this representation by Γ_{ε}, ch Γ_{ε} being ε where

$\mathcal{E}(Z) = 1$, $\mathcal{E}(A\,Z) = -1$. If Γ is any representation of the n-dimensional orthogonal group so also is $\Gamma_{\mathcal{E}}\Gamma$ and, since $\Gamma_{\mathcal{E}}^2$ is the identity representation, Γ bears to $\Gamma_{\mathcal{E}}\Gamma$ the same relation that $\Gamma_{\mathcal{E}}\Gamma$ bears to Γ ; we say that the two representations Γ and $\Gamma_{\mathcal{E}}\Gamma$ are associated, each being the associate of the other. A continuous representation Γ of the n-dimensional orthogonal group is self-associated when, and only when, ch Γ is zero over the reflexion half of the group.

The same argument as in the case of the n-dimensional rotation group shows that $[\lambda]$ is the character of a continuous representation $\Gamma_{[\lambda]}$ of the n-dimensional orthogonal group. The character of $\Gamma_{[\lambda]} + \Gamma_{\mathcal{E}}\Gamma_{[\lambda]}$ is zero over the reflexion half of the group and is $2[\lambda]$ over the rotation half of the group and so the integral of the squared modulus of the character over the group is four times the integral of $|[\lambda]|^2$ over the rotation half of the group. If $n = 2k+1$ is odd, or if $n = 2k$ is even and $\lambda_k = 0$, this is four times the volume of the rotation half of the group, i.e., twice the volume of the group while, if $n = 2k$ and $\lambda_k > 0$, it is four times the volume of the group. If, then $n = 2k+1$ or $n = 2k$ and $\lambda_k = 0$. $\Gamma_{[\lambda]} + \Gamma_{\mathcal{E}}\Gamma_{[\lambda]}$ contains precisely two irreducible representations, which must be $\Gamma_{[\lambda]}$ and $\Gamma_{\mathcal{E}}\Gamma_{[\lambda]}$, of the group so that $\Gamma_{[\lambda]}$ is irreducible and not self-associated. On the other hand, if $n = 2k$ and $\lambda_k > 0$, $\Gamma_{[\lambda]}$ is self-associated. Indeed, every reflexion matrix $A\,Z$ has the characteristic numbers ± 1 , since the 2×2 matrix $\begin{pmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{pmatrix}$, $c_1 = \cos\alpha_1$, $s_1 = \sin\alpha_1$, has the characteristic numbers ± 1 and a diagonal k-dimensional block representative of any class of reflexions of the 2k-dimensional orthogonal group has $\begin{pmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{pmatrix}$ as its first diagonal element. On denoting by

$\sigma_o^*, \ldots, \sigma_{2k-2}^*$ the elementary symmetric functions of the remaining 2k-2 characteristic numbers of A Z and writing $\sigma_j'^* = (-1)^j \sigma_j^*$, $j = 0, 1, \ldots,$ we have $(\sigma_o'^* + \sigma_1'^* t + \ldots + \sigma_{2k-2}'^* t^{2k-2})^{-1} = (1-t^2)(h_o + h_1 t + \ldots) = h_o' + h_1' t + \ldots,$ where $h_j' = h_j - h_{j-2}$. Hence $\sigma_o'^* h_j' + \sigma_1'^* h_{j-1}' + \ldots + \sigma_{2k-2}'^* h_{j-2k+2}' = 0, \; j = 1, 2, \ldots,$ and this yields, since $\sigma_p'^* = \sigma_{2k-2-p}'^*$, $h_{j-k+1}' \sigma_{k-1}'^* + (h_{j-k}' + h_{j-k+2}')\sigma_{k-2}'^* + \ldots + (h_{j-2k+2}' + h_j')\sigma_o'^* = 0, \; j = 1, 2, \ldots$. On assigning to j in turn the values $\ell_1, \ldots, \ell_k$, where $\ell_1 > \ell_2 > \ldots > \ell_k > 0$, we obtain k homogeneous relations connecting the k quantities $\sigma_{k-1}'^*, \ldots, \sigma_o'^*$ and it follows, since $\sigma_o'^* \neq 0$, that the $k \times k$ matrix of these relations does not possess a reciprocal. The determinant of this matrix is $[\lambda]$, where $\lambda_1 = \ell_1 - k + 1, \ldots, \lambda_k = \ell_k$, and so, if $\lambda_k > 0$, $[\lambda]$ is zero over the reflexion half of the 2k-dimensional rotation group. Hence $\Gamma_\varepsilon \Gamma_{[\lambda]} = \Gamma_{[\lambda]}$ so that $\Gamma_{[\lambda]} + \Gamma_\varepsilon \Gamma_{[\lambda]} = 2\Gamma_{[\lambda]}$ and it follows that $\Gamma_{[\lambda]}$ is irreducible; indeed the average of the squared modulus of the character of $\Gamma_{[\lambda]} + \Gamma_\varepsilon \Gamma_{[\lambda]}$ over the group is 4 so that the average of the squared modulus of the character of $\Gamma_{[\lambda]}$ over the group is 1. Thus, whether n is even or odd and whether λ_k is positive or zero, $\Gamma_{[\lambda]}$ is an irreducible representation of the n-dimensional orthogonal group.

Let, now, Γ be any continuous irreducible representation of the n-dimensional orthogonal group. If Γ is not self-associated the integral of $|\text{ch}\,\Gamma|^2$ over the rotation half of the group, being less than the volume of the orthogonal group, must be the volume of the rotation half of the group (since ch Γ is, over this rotation half, the character of a representation of the n-dimensional rotation group). Hence the representation of the n-dimensional rotation group which is induced by Γ is irreducible and, if $n = 2k$ is even, this irreducible representation must be a $\Gamma_{[\lambda]}$ for which $\lambda_k = 0$ (for ch Γ is an even function of all the k angles $\alpha_1, \ldots, \alpha_k$). It follows that Γ is either the irreducible representation $\Gamma_{[\lambda]}$ of the n-dimensional orthogonal group or is the product $\Gamma_\varepsilon \Gamma_{[\lambda]}$ of $\Gamma_{[\lambda]}$ by the determinant representation of the group; for, if not, the integral of $|[\lambda]|^2$ over the rotation half of the group $\pm$ the integral of $\overline{\mathrm{ch}\,\Gamma}\,[\lambda]$ over the reflexion half of the group would both be zero and this cannot be since the integral of $|[\lambda]|^2$ over the rotation half of the group is 1. If Γ is self-associated the representation of the n-dimensional rotation group which is induced by Γ is the sum of two irreducible representations of this rotation group (since the integral of $|\mathrm{ch}\,\Gamma|^2$ over the rotation half, being the same as the integral of $|\mathrm{ch}\,\Gamma|^2$ over the entire group, is twice the volume of the rotation group) and, since ch Γ is an even function of the k angles $\alpha_1, \ldots, \alpha_k$, $n = 2k$ must be even and the two irreducible representations of the 2k-dimensional rotation group are $\Gamma_{[\lambda]+}$ and $\Gamma_{[\lambda]-}$, $\lambda_k > 0$. Thus ch $\Gamma = [\lambda]$ over the rotation half of the group which implies, since $[\lambda]$ is zero over the reflexion half of the group, that Γ is the irreducible representation $[\lambda]$ of the 2k-dimensional orthogonal group. Hence the representations $\Gamma_{[\lambda]}$ exhaust the continuous irreducible representations of the n-dimensional orthogonal group; when $n = 2k+1$ is odd there are no continuous self-associated representations of the group and, when $n = 2k$ is even, the representations $\Gamma_{[\lambda]}$ for which $\lambda_k > 0$ are self-associated.

2. The analysis of the representations of the n-dimensional orthogonal and rotation groups which are induced by irreducible representations of the n-dimensional unitary group.

The character $[m]$ of $\Gamma_{[m]}$ is $h'_m = h_m - h_{m-2}$ and this may be written in the form $(1 - \xi^2)\, h_m$ where ξ is an operator which reduces the subscript of h_m by 2. Since $h_j = 0$ if j is negative it follows that $(1 + \xi^2 + \xi^4 + \dots + \xi^{2j})\, h'_m = h_m$ if $2j+2 > m$. We write this result in the form $h_m = (1 + \xi^2 + \dots)\, h'_m = h'_m + h'_{m-2} + \dots$ or, equivalently, $\{m\} = [m] + [m-2] + \dots$ Thus the symmetrized m^{th} power of the self-representation of the n-dimensional unitary group induces, since the character of this symmetrized m^{th} power is $\{m\}$, a representation of the n-dimensional orthogonal group which is the sum of the representations $\Gamma_{[m]}$, $\Gamma_{[m-2]}$, ... of this group and the same result holds for the n-dimensional rotation group. The representations $\Gamma_{[m]}$, $\Gamma_{[m-2]}$, ... of the n-dimensional orthogonal group are all irreducible and the representations $\Gamma_{[m]}$, $\Gamma_{[m-2]}$, ... of the n-dimensional rotation group are also irreducible save when $n = 2$ in which case $\Gamma_{[j]}$, whose dimension is 2 if $j > 0$, breaks up, if $j > 0$, into the sum of two 1-dimensional representations.

The character $\{\lambda_1, \lambda_2\}$ of $\Gamma_{(\lambda_1, \lambda_2)}$ may be written in the form $\begin{vmatrix} \xi_1 & 1 \\ \xi_2 & 1 \end{vmatrix} h_{\ell_1} h_{\ell_2}$ where ξ_1 and ξ_2 are operators which reduce by 1 the subscripts of the first and second factors, respectively, of the product $h_{\ell_1} h_{\ell_2}$; similarly the character of $\Gamma_{[\lambda_1, \lambda_2]}$ may be written in the form $\begin{vmatrix} \xi_1 & 1 + \xi_1^2 \\ \xi_2 & 1 + \xi_2^2 \end{vmatrix} h'_{\ell_1} h'_{\ell_2} =$

$$= (1-\xi_1^2)(1-\xi_2^2)(1-\xi_1\xi_2)\begin{vmatrix}\xi_1 & 1\\ \xi_2 & 1\end{vmatrix} h_{\ell_1} h_{\ell_2}$$

and so $[\lambda_1, \lambda_2] = (1-\xi_1^2)(1-\xi_2^2)(1-\xi_1\xi_2)\{\lambda_1, \lambda_2\}$ which implies that

$$\{\lambda_1, \lambda_2\} = (1+\xi_1^2+\dots)(1+\xi_2^2+\dots)(1+\xi_1\xi_2+\dots)[\lambda_1, \lambda_2]$$

In developing the product $(1+\xi_1^2+\dots)(1+\xi_2^2+\dots)(1+\xi_1\xi_2+\dots)$ we need keep only terms of degree $\leq m = \lambda_1+\lambda_2$. Thus

$$\{1^2\} = (1+\xi_1^2+\xi_2^2+\xi_1\xi_2)[1^2] = [1^2] + [-1, 1] + [1, -1] + [0]$$

Any disordered parenthesis $[\dots]$, i.e., one in which the terms are not in non-increasing order, may be rearranged according to the prescription $[\dots a\ b \dots] = -[\dots b{-}1\ a{+}1 \dots]$, this prescription being continued until the parenthesis is no longer disordered. For example $[-1, 1] = -[0]$. Furthermore, any parenthesis $[\dots]$, disordered or not, which ends with a negative term is to be discarded since the last row of the matrix whose determinant is $[\dots]$ consists, then, of zeros. Thus $\{1^2\} = [1^2]$, which tells us that the irreducible representation $\Gamma_{(1^2)}$, of dimension $n(n-1)/2$, of the n-dimensional unitary group induces, when $n > 3$, an irreducible representation of the n-dimensional orthogonal group and a representation of the n-dimensional rotation group which is irreducible when $n > 4$ but which reduces when $n = 4$, in which case it is of dimension 6, to the sum of two irreducible representations $\Gamma_{[1^2]+}$ and $\Gamma_{[1^2]-}$, each of dimension 3. When $n = 3$, the character τ_2 of $\{1^2\}$ is the product of $\overline{\sigma}_1$, the character of

$\overline{\Gamma}_{(1)}$, by det Z so that $\Gamma_{(1^2)} = (\det Z)\,\overline{\Gamma}_{(1)}$. Over the 3-dimensional orthogonal group $\overline{\Gamma}_{(1)} = \Gamma_{(1)}$, so that $\Gamma_{(1^2)} = \Gamma_{\Sigma}\Gamma_{(1)}$; over the 3-dimensional rotation group $\Gamma_{(1^2)} = \Gamma_{(1)}$

Exercise 1. Show that the adjoint representation of the n-dimensional orthogonal group, or of the n-dimensional rotation group, is the representation of the group in question which is induced by the representation $\Gamma_{(1^2)}$ of the n-dimensional unitary group so that this adjoint representation is irreducible save when $n = 4$ and the group in question is the rotation group.

Hint. If $M_{p,q}$ is any one of the $n(n-1)/2$ characteristic matrices of the n-dimensional rotation group and Z is any n-dimensional orthogonal matrix the coefficient of $M_{p,q}$ in $Z\,M_{p,q}\,Z^t$ is $(Z)^p_p\,(Z)^q_q - (Z)^p_q\,(Z)^q_p$ and so the character of the adjoint representation is the sum of the 2-rowed principal minors of Z , i.e., $\sigma_2 = \{1^2\}$.

Lecture 15

1. The parametrisation of the 2k-dimensional U-symplectic group.

The typical element Z of the 2k-dimensional U-symplectic group is of the form $\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$ where A and B are k-dimensional matrices which are such that A^tB is symmetric and $A^*A + B^*B = I_k$. Thus the 2k-dimensional U-symplectic matrices for which $B = 0$ are of the form $\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}$ where A is an arbitrary k-dimensional unitary matrix. If A is a plane k-dimensional unitary matrix of the form $U_{p,q}(\theta,\sigma)$, $p < q = 2, \dots, k$, where, for example,

$$U_{1,2}(\theta,\sigma) = \begin{pmatrix} c & -x+yi & \\ x+yi & c & 0 \\ & 0 & E_{k-2} \end{pmatrix}; \quad \begin{matrix} c = \cos\theta, & x = s\cos\sigma, \\ s = \sin\theta, & y = s\sin\sigma, \\ 0 \leq \theta \leq \pi/2, & -\pi < \sigma \leq \pi \end{matrix}$$

we denote the 2k-dimensional U-symplectic matrix

$\begin{pmatrix} U_{p,q}(\theta,\sigma) & 0 \\ 0 & \bar{U}_{p,q}(\theta,\sigma) \end{pmatrix}$ by $S_{p,q}(x,y)$ and we denote the plane 2k-dimensional U-symplectic matrix $U_{j,k+j}(\theta,\sigma)$, $j = 1, \dots, k$, by $S_j(x, y)$. Finally, we denote by $S'_{p,q}(x, y)$, $p < q = 2, \dots, k$, the 2k-dimensional U-symplectic matrix for which A is the k-dimensional diagonal matrix all of whose diagonal elements are 1 save the p^{th} and q^{th} which have the common value $c = (1 - x^2 - y^2)^{1/2}$ and for which B is the k-dimensional matrix all of whose elements are zero save those in the q^{th} row and p^{th} column and in the p^{th} row and q^{th} column which have the common value $x + yi$.

Any 2k-dimensional U-symplectic matrix Z may be factored as follows: We first determine $x_1, y_1, \dots, x_{2k-1}, y_{2k-1}$ so that all the elements in the last row-matrix of

$$Z' = Z\, S_{k-1,k}^{*}(x_1,y_1) \dots S_{1,k}^{*}(x_{k-1},y_{k-1})\, S_k^{*}(x_k,y_k)\, S'^{*}_{k-1,k}(x_{k+1},y_{k+1}) \dots S'^{*}_{1,k}(x_{2k-1},y_{2k-1})$$

are zero save the last so that the last row-matrix of Z' is of the form $(\exp -\beta_k i)\, e^*_{2k}$, the last column-matrix of Z' being $(\exp -\beta_k i) e_{2k}$. This implies that the element in the k^{th} row and k^{th} column of Z' is $\exp \beta_k i$ so that the k^{th} row-matrix and k^{th} column-matrix of Z' are $(\exp \beta_k i)\, e^*_k$ and $(\exp \beta_k i)\, e_k$, respectively. Thus, if we remove the k^{th} and $2k^{th}$ row-matrices and column-matrices from Z' we obtain a $(2k-2)$-dimensional U-symplectic matrix which we may factor in the same way, and so on. When we pass in this way from the 2k-dimensional U-symplectic group to the $(2k-2)$-dimensional symplectic group we lose $2(2k-1) + 1 = 4k - 1$ parameters and it follows, since $3 + 7 + \dots + (4k-1) = k(2k+1)$, that the 2k-dimensional U-symplectic group is a compact $k(2k+1)$-parameter group, each element of the group being of the form $D(\beta)\, S$ where $D(\beta)$ is the 2k-dimensional diagonal matrix whose diagonal elements are $\exp \beta_1 i, \dots, \exp \beta_k i, \exp -\beta_1 i, \dots, \exp -\beta_k i$ and S is a product of matrices of the types $S_{p,q}(x,y)$, $S_j(x,y)$, $S'_{p,q}(x,y)$. Taking as our parameters the β's and the x's and y's, the origin of the parametric space is the identity point and the $k(2k+1)$ characteristic matrices are of the following types:

1) k of the type $M_j = \begin{pmatrix} i H_j & 0 \\ 0 & -i H_j \end{pmatrix}$ where H_j, $j = 1, \dots, k$, is the k-dimensional diagonal Hermitian matrix all of whose diagonal elements are zero save the j^{th} which is 1 ; these characteristic matrices correspond to the parameters β.

2) $\frac{1}{2} k(k-1)$ of the type $(M_{p,q})_1 = \begin{pmatrix} i H_{p,q} & 0 \\ 0 & -i \bar{H}_{p,q} \end{pmatrix}$, where

$H_{p,q}$, $p < q = 2, \ldots, k$, is the k-dimensional Hermitian matrix all of whose elements are zero save those in the p^{th} column and q^{th} row and in the p^{th} row and q^{th} column, which are $-i$ and i, respectively, and $\frac{1}{2}k(k-1)$ of the type $(M_{p,q})_2 = \begin{pmatrix} i\,H'_{p,q} & 0 \\ 0 & -i\,H'_{p,q} \end{pmatrix}$, where $H'_{p,q}$, $p < q = 2, \ldots, k$, is the k-dimensional Hermitian matrix all of whose elements are zero save those in the p^{th} column and q^{th} row and in the p^{th} row and q^{th} column which are both 1. These correspond, respectively, to the parameters x, y which occur in $S_{p,q}(x,y)$. If we write $\frac{1}{2}\{(M_{p,q})_1 + i(M_{p,q})_2\} = (N_{p,q})_1$, $\frac{1}{2}\{(M_{p,q})_1 -$

$$- i(M_{p,q})_2\} = (N_{p,q})_2$$

all the elements of $(N_{p,q})_1$ are zero, save those in the p^{th} row and q^{th} column, and in the $(k+q)$-th row and $(k+p)$-th column, which are -1 and 1, respectively, and all the elements of $(N_{p,q})_2$ are zero save those in the q^{th} row and p^{th} column, and in the $(k+p)$-th row and $(k+q)$-th column, which are 1 and -1, respectively.

3) k of the type $M'_j = \begin{pmatrix} 0 & -H_j \\ H_j & 0 \end{pmatrix}$ and k of the type $M''_j = \begin{pmatrix} 0 & i\,H_j \\ i\,H_j & 0 \end{pmatrix}$, $j = 1, \ldots, k$, where H_j is the k-dimensional diagonal Hermitian matrix all of whose diagonal elements are zero save the j^{th} which is 1. These correspond, respectively, to the parameters x and y which occur in $S_j(x,y)$. If we write $\frac{1}{2}(M'_j + i\,M''_j) = N'_j$, $\frac{1}{2}(M'_j - i\,M''_j) = N''_j$ all the elements of N'_j are zero save the element in the j^{th} row and $(k+j)$-th column, which is -1, and all the elements of N''_j are zero save the element in the $(k+j)$-th row and j^{th} column, which is 1.

4) $\frac{1}{2}k(k-1)$ of the type $(M'_{p,q})_1 = \begin{pmatrix} 0 & -H'_{p,q} \\ H'_{p,q} & 0 \end{pmatrix}$ and $\frac{1}{2}k(k-1)$ of the type $(M'_{p,q})_2 = \begin{pmatrix} 0 & iH'_{p,q} \\ iH'_{p,q} & 0 \end{pmatrix}$, where $H'_{p,q}$, $p < q = 2, \ldots, k$, is the k-dimensional Hermitian matrix all of whose elements are zero save those in the p^{th} row and q^{th} column, and in the q^{th} row and p^{th} column which are both 1. These correspond, respectively, to the parameters x and y which occur in $S'_{p,q}(x,y)$. If we set $\frac{1}{2}\{(M'_{p,q})_1 + i(M'_{p,q})_2\} = (N'_{p,q})_1$, $\frac{1}{2}\{(M'_{p,q})_1 - i(M'_{p,q})_2\} = (N'_{p,q})_2$, all the elements of $(N'_{p,q})_1$ are zero save those in the p^{th} row and $(k+q)$-th column and in the q^{th} row and $(k+p)$-th column, which are both -1, and all the elements of $(N'_{p,q})_2$ are zero save those in the $(k+q)$-th row and q^{th} column and in the $(k+q)$-th row and p^{th} column, which are both 1.

2 <u>The class and in-class parameters of the 2k-dimensional U-symplectic group.</u>

If x and y are any two characteristic $n \times 1$ matrices of an n-dimensional unitary matrix Z, corresponding to the characteristic numbers λ and μ, respectively, of Z, so that $Zx = \lambda x$, $Zy = \mu y$ we have $y^*Z^* = \bar{\mu} y^*$ so that $y^*x = y^*Z^*Zx = \bar{\mu}\lambda y^*x$. Thus $y^*x = 0$ unless $\bar{\mu}\lambda = 1$ or, equivalently, since $\bar{\mu}\mu = 1$, unless $\mu = \lambda$. We express this result by the statement that any two characteristic $n \times 1$ matrices of an n-dimensional unitary matrix which correspond to different characteristic numbers are unitarily orthogonal. Since any linear combination of characteristic $n \times 1$ matrices which correspond to the same characteristic number is a characteristic $n \times 1$ matrix corresponding to this characteristic number, there correspond to any characteristic number of index q, i.e., which has corresponding to it q, and not more than q, linearly independent characteristic $n \times 1$ matrices, q characteristic $n \times 1$ matrices which are unitarily orthogonal and of unit magnitude (an $n \times 1$ matrix x being said to be of unit magnitude when $x^*x = 1$). Thus, since Z may be transformed to diagonal form, so that the characteristic $n \times 1$ matrices of Z contain amongst them n linearly independent ones, every n-dimensional unitary matrix Z possesses n characteristic $n \times 1$ matrices which are the column matrices of an n-dimensional unitary matrix, V, say, and this implies that $Z V = V D(z)$, where $D(z)$ is an n-dimensional diagonal matrix whose diagonal elements are the characteristic numbers $z_1, \ldots, z_n$ of Z. Thus $V^*ZV = D(z)$ so that the unitary n-dimensional matrix V transforms Z into diagonal form.

Let, now, $n = 2k$ be even and let Z be any element of the 2k-dimensional U-symplectic group. If x and y are any two characteristic $2k \times 1$ matrices of Z, corresponding to the characteristic numbers λ and μ, respectively, of Z, so that $Zx = \lambda x$, $Zy = \mu y$, we have $y^tZ^t = \mu y^t$ so that $y^tIx = y^tZ^tIZx = \mu\lambda y^tIx$ and $y^tIx = 0$ unless $\mu = 1/\lambda = \bar{\lambda}$. We express this result by the statement that any two characteristic $2k \times 1$ matrices of a 2k-dimensional U-symplectic matrix, which correspond to characteristic numbers which are not conjugate complex numbers, are symplectically orthogonal. If λ is a non-real characteristic number, of index q, of Z, let $x_1, \ldots, x_q$ be q unitarily orthogonal characteristic $2k \times 1$ matrices, of unit magnitude, of Z which correspond to λ. Then $Zx_j = \lambda x_j$, $j = 1, \ldots, q$, so that $\bar{Z}\bar{x}_j = \bar{\lambda}\bar{x}_j$ and, since $Z^tIZ = I$, $Z^*I\bar{Z} = I$ so that $Z\bar{I} = I\bar{Z}$ and $ZI\bar{x}_j = \bar{\lambda}I\bar{x}_j$. Thus $I\bar{x}_j$, $j = 1, \ldots, q$, is a characteristic ~~number~~ matrix of Z corresponding to the characteristic number $\bar{\lambda}$ of Z and, since I is unitary, the q

$2k \times 1$ matrices $I\bar{x}_j$, $j = 1, \ldots, q$, are of unit magnitude and unitarily orthogonal. Hence they are linearly independent so that the index of $\bar{\lambda}$ is at least as great as that of λ ; since λ is the conjugate complex of $\bar{\lambda}$, the index of λ is at least as great as that of $\bar{\lambda}$ and so the two characteristic numbers, λ and $\bar{\lambda}$, of Z have the same index. Since $I^t I = E_{2k}$, $(I\bar{x}_j)^t I x_p = x_j^* x_p$; which is zero unless $p = j$, in which case it is 1 ; thus, if $p \neq j$, the characteristic $2k \times 1$ matrices x_p and $I\bar{x}_j$ of Z are symplectically orthogonal.

If λ is a real characteristic number of Z, so that λ is either 1 or -1, its index is even. Indeed, if x_1 is any characteristic $2k \times 1$ matrix, of unit magnitude, of Z corresponding to λ so also is $I\bar{x}_1$, and x_1 and $I\bar{x}_1$ are unitarily orthogonal and, hence, linearly independent. Indeed, $(I\bar{x}_1)^* x_1 = -x_1^t I x_1 = 0$. Also $(I\bar{x}_1)^t I x_1 = x_1^* x_1 = 1$. If the index of λ is > 2, let x_2 be a characteristic $2k \times 1$ matrix, of unit magnitude, of Z corresponding to λ which is unitarily orthogonal to x_1 and $I\bar{x}_1$. Then $I\bar{x}_2$ is a characteristic $2k \times 1$ matrix, of unit magnitude, of Z corresponding to λ and $I\bar{x}_2$ is unitarily orthogonal to x_1, $I\bar{x}_1$ and x_2. Also x_2 is symplectically orthogonal to x_1 and $I\bar{x}_1$ since $x_2^t I x_1$ is the conjugate complex of $x_2^* I \bar{x}_1$ and $x_2^t I I \bar{x}_1$ is the conjugate complex of $-x_2^* x_1$ and this implies that $I\bar{x}_2$ is symplectically orthogonal to $I\bar{x}_1$ and x_1. Continuing in this way we see that the index $q = 2p$ of λ is even and that Z has, corresponding to λ, $2p$ unitarily orthogonal $2k \times 1$ characteristic matrices $x_1, \ldots, x_p$, $I\bar{x}_1, \ldots, I\bar{x}_p$ of unit magnitude which are such that any one of them is symplectically orthogonal to all but one of them (the only one to which x_j, $j = 1, \ldots, p$, is not symplectically orthogonal being $I\bar{x}_j$, which implies that the only one to which $I\bar{x}_j$ is not symplectically orthogonal is x_j). Furthermore $x_j^t\, I\, I\, \bar{x}_j = -1$, $j = 1, \ldots, p$. Thus we may arrange the $2k$ characteristic $2k \times 1$ matrices of Z in two sets of k each where, if $x_1, \ldots, x_k$ are the matrices of the first set, the matrices of the second set are $I\,\bar{x}_1, \ldots, I\,\bar{x}_k$; furthermore each of the $2k$ matrices $x_1, \ldots, x_k$, $I\,\bar{x}_1, \ldots, I\,\bar{x}_k$ is of unit magnitude and is unitarily orthogonal to all the other $2k-1$ matrices of the set and each is symplectically orthogonal to all but one of the set, x_j, $j = 1, \ldots, k$, failing to be symplectically orthogonal to $I\,\bar{x}_j$. In other words the matrix whose column $2k \times 1$ matrices are $x_1, \ldots, x_k$, $I\,\bar{x}_1, \ldots, I\,\bar{x}_k$ is a $2k$-dimensional U-symplectic matrix S_1^* and $Z\, S_1^* = S_1^*\, D(\alpha)$, where $D(\alpha)$ is the $2k$-dimensional diagonal matrix whose diagonal elements $\exp \alpha_1 i, \ldots, \exp \alpha_k i$, $\exp -\alpha_1 i, \ldots, \exp -\alpha_k i$ are the characteristic numbers of Z. Writing S_1 in the form $D\, S^*$, where D is a diagonal $2k$-dimensional U-symplectic matrix, we have $Z\, S\, D^* = S\, D^*\, D(\alpha)$ so that $Z\, S = S\, D^*\, D(\alpha)\, D = S\, D(\alpha)$ and $Z = S\, D(\alpha)\, S^*$. Taking as our parameters the k α's and the $2k^2$ parameters x, y, which occur in S, the α's are the class parameters and the $2k^2$ parameters x, y, are the in-class parameters of the $2k$-dimensional U-symplectic group.

Lecture 16

1. The element of volume of the 2k-dimensional U-symplectic group

We have seen that every element Z of the 2k-dimensional U-symplectic group may be written in the form $S\,D(\alpha)\,S^*$, where S is a function of the in-class parameters $x_1, y_1, \ldots, x_{k^2}, y_{k^2}$. From the relation $ZS = S\,D(\alpha)$ we derive, as in the case of the n-dimensional unitary group, the formula $S^*\,\delta Z\,S = D^{-1}(\alpha)\,\delta S\,D(\alpha) + \delta D(\alpha) - \delta S$. δS is a linear combination of the $2k^2$ characteristic matrices $(M_{p,q})_1$, $(M_{p,q})_2$, M'_j, M''_j, $(M'_{p,q})_1$, $(M'_{p,q})_2$ and, hence, of the matrices $(N_{p,q})_1$, $(N_{p,q})_2$, N'_j, N''_j, $(N'_{p,q})_1$ and $(N'_{p,q})_2$. Since $D^{-1}(\alpha)\,(N_{p,q})_1\,D(\alpha) = \frac{z_q}{z_p}(N_{p,q})_1$,

$D^{-1}(\alpha)\,(N_{p,q})_2\,D(\alpha) = \frac{z_p}{z_q}(N_{p,q})_2$, $D^{-1}(\alpha)\,N'_j\,D(\alpha) = \frac{1}{z_j^2}N'_j$,

$D^{-1}(\alpha)\,N''_j\,D(\alpha) = z_j^2\,N''_j$, $D^{-1}(\alpha)\,(N'_{p,q})_1\,D(\alpha) = \frac{1}{z_p z_q}(N'_{p,q})_1$,

$D^{-1}(\alpha)\,(N'_{p,q})_2\,D(\alpha) = z_p z_q\,(N'_{p,q})_2$ it follows, as in the case of the unitary group, that we may take $\left|\left\{\prod_{p<q}\left(\frac{z_q}{z_p}-1\right)\left(\frac{z_p}{z_q}-1\right)\left(1/(z_p z_q)-1\right)(z_p z_q-1)\right\}\left\{\prod_j\left(\frac{1}{z_j^2}-1\right)(z_j^2-1)\right\}\right|$ times $d(\alpha_1, \ldots, \alpha_k)$ as the class factor $dV(\alpha)$ of the element of volume of the 2k-dimensional U-symplectic group. Since $|z_p| = 1$ we may write $dV(\alpha)$ in the form

$$dV(\alpha) = \left|\left\{\prod_{p\leq q}(1 - z_p z_q)\right\}\Delta\right|^2 d(\alpha_1, \ldots, \alpha_k)\ ;\ \Delta = \prod_{p<q}(z_p - z_q)$$

Thus the element of volume of the 2k-dimensional U-symplectic group is the product of the element of volume of the 2k-dimensional rotation group by $\left|\prod_p (1 - z_p^2)\right|^2 = 2^{2k} \sin^2\alpha_1 \ldots \sin^2\alpha_k$. Since $(2i)^k \sin\alpha_1 \ldots \sin\alpha_k\, C(k-1, \ldots, 0)$ is the determinant, $S(k, \ldots, 1)$, of the k-dimensional matrix whose p^{th} row matrix is $(s_{k-p+1}(\alpha_1), \ldots, s_{k-p+1}(\alpha_k))$ it follows that

$$dV(\alpha) = \{S(k, \ldots, 1)\}^2\, d(\alpha_1, \ldots, \alpha_k).$$

Since the angles $\alpha_1, \ldots, \alpha_k$ are longitude angles the volume of the group is, by the same argument as in the case of the n-dimensional unitary group, $(2\pi)^k k!$ so that the normalised element of volume is $(2\pi)^{-k} \frac{1}{k!} \{S(k, \ldots, 1)\}^2 d(\alpha_1, \ldots, \alpha_k)$. We have already seen that, if ℓ is any non-negative integer,

$$s_\ell(\alpha) = (h_{\ell-k}, h_{\ell-k-1} + h_{\ell-k+1}, \ldots, h_{\ell-2k+1} + h_{\ell-1}) \sum_k s(\alpha)$$

α being one of the k angles $\alpha_1, \ldots, \alpha_k$. If $(\lambda) = (\lambda_1, \ldots, \lambda_k)$ is any partition, having not more than k parts, of any non-negative integer m, we write $\ell'_p = \ell_p + 1 = \lambda_p + k - p + 1$, so that $\ell'_1 > \ell'_2 > \cdots > \ell'_p > 0$ and we denote by $\langle\lambda\rangle$ the determinant of the k-dimensional matrix whose k^{th} row matrix is

$$(h_{\ell'_p-k}, \ldots, h_{\ell'_p-2k+1} + h_{\ell'_p-1}) = (h_{\ell_p-k+1}, \ldots, h_{\ell_p-2k+2} + h_{\ell_p}).$$

Then $\langle\lambda\rangle$ is the quotient of $S(\ell') = S(\ell_1+1, \ldots, \ell_p+1)$, where $S(\ell')$ is the determinant of the k-dimensional matrix whose p^{th} row matrix is $(s_{\ell'_p}(\alpha_1), \ldots, s_{\ell'_p}(\alpha_k))$, by $S(k, \ldots, 1)$ and it follows, as in the case of the 2k-dimensional rotation group, that $\langle\lambda\rangle$ is the character of an irreducible representation $\Gamma_{\langle\lambda\rangle}$ of the 2k-dimensional U-symplectic group. The representations $\Gamma_{\langle\lambda\rangle}$ are all different and exhaust the continuous irreducible representations of this group. The dimension $d_{\langle\lambda\rangle}$ of $\Gamma_{\langle\lambda\rangle}$ is

$$d_{\langle\lambda\rangle} = \ell'_1 \cdots \ell'_k \prod_{p<q} (\ell'^2_p - \ell'^2_q) \div (2k-1)!\,(2k-3)! \cdots 3!$$

$$= (\ell_1+1) \cdots (\ell_k+1)\,\Delta(\ell) \prod_{p<q} (\ell_p+\ell_q+2) \div (2k-1)! \cdots 3!$$

2. The 2k-dimensional O-symplectic group.

The 2k-dimensional O-symplectic group is the collection of all real 2k-dimensional U-symplectic matrices, so that any element Z of the group is at once a 2k-dimensional rotation matrix and a 2k-dimensional symplectic matrix. It consists of the matrices $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ where A and B are real k-dimensional matrices which are such that A^tB is symmetric and $A^tA + B^tB = E_k$. It follows that $A + iB$ is a unitary k-dimensional matrix, and conversely, if $A + iB$ is a unitary k-dimensional matrix, A and B being real, A^tB is symmetric and $A^tA + B^tB = E_k$, so that $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ is a 2k-dimensional O-symplectic matrix. Since

$$2^{-1/2}\begin{pmatrix} E_k & iE_k \\ iE_k & E_k \end{pmatrix}\begin{pmatrix} A & -B \\ B & A \end{pmatrix} 2^{-1/2}\begin{pmatrix} E_k & -iE_k \\ -iE_k & E_k \end{pmatrix} = \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix},$$

the 2k-dimensional O-symplectic group may be presented as the subgroup of the 2k-dimensional U-symplectic group which consists of the 2k-dimensional U-symplectic matrices of the form $Z = \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}$, where

U is an arbitrary k-dimensional unitary matrix. If $Z \to Z'$ is any representation of this sub-group of the 2k-dimensional U-symplectic group, $U \to Z'$ is a representation of the k-dimensional unitary group, and, conversely, if $U \to Z'$ is any representation of the k-dimensional unitary group, then $Z \to Z'$ is a representation of the 2k-dimensional O-symplectic group. Thus, the 2k-dimensional O-symplectic group is a compact k^2-parameter group whose irreducible continuous representations are those of the k-dimensional unitary group, the characters of these irreducible representations being the functions $\{\lambda\}$ of the characteristic numbers $z_1, \ldots, z_k$ of $U = A + iB$ divided by non-negative integral powers of $\det U = \det (A + iB)$. On denoting by $H_0, H_1, \ldots$ the complete symmetric functions of $z_1, \ldots, z_k$, the complete symmetric functions of $z_1, \ldots, z_k, \bar{z}_1, \ldots, \bar{z}_k$ being denoted, as before, by $h_0, h_1, \ldots$, we have

$$h_0 + h_1 t + \ldots = (H_0 + H_1 t + \ldots)(\bar{H}_0 + \bar{H}_1 t + \ldots)$$

so that

$$h_1 = H_1 + \bar{H}_1, \quad h_2 = H_2 + H_1\bar{H}_1 + \bar{H}_2, \quad h_3 = H_3 + H_2\bar{H}_1 + H_1\bar{H}_2 + \bar{H}_3, \quad \ldots$$

Thus the self-representation of the 2k-dimensional unitary group induces a representation of the 2k-dimensional O-symplectic group which is the sum of the irreducible representation $\Gamma'_{(1)}$ of this group and its conjugate. Since $H_1\bar{H}_1$ is the character of the adjoint representation of the k-dimensional unitary group and since this adjoint representation is the sum of two irreducible representations, one of which is the identity representation, the symmetrized square of the 2k-dimensional unitary group induces a representation of the 2k-dimensional O-symplectic group which is the sum of four irreducible representations of this group, three of these being Γ_0 (the identity representation), $\Gamma_{(2)}$ and $\bar{\Gamma}_{(2)}$.

Lecture 17

The analysis of the product of irreducible continuous representations of the n-dimensional unitary group.

The character of the symmetrized m^{th} power $\Gamma_{(m)}$ of the self-representation of the n-dimensional unitary group is $\{m\} = h_m$ and so the character of $\Gamma_{(m)}\Gamma_{(m')}$ is $h_m h_{m'}$, where we may suppose, without lack of generality, since multiplication of representations is commutative, that $m \geqslant m'$. Now $\{m, m'\} = (\xi_1 - \xi_2) h_{m+1} h_{m'}$, where ξ_1 and ξ_2 are operators which decrease m and m', respectively, by 1. If δ_1 is an operator which increases m by 1, we may write this result in the form $\{m, m'\} = (1 - \delta_1 \xi_2) h_m h_{m'}$ and this implies, since $\xi_2^j \{m, m'\} = 0$ if $j > m'$, that

$$h_m h_{m'} = \{m, m'\} + \{m+1, m'-1\} + \dots + \{m + m'\}$$

so that

$$\Gamma_{(m)}\Gamma_{(m')} = \Gamma_{(m, m')} + \Gamma_{(m+1, m'-1)} + \dots + \Gamma_{(m+m')} .$$

This result furnishes the analysis of the product of any two irreducible continuous representations of the 2-dimensional unitary group; indeed, if $\lambda_2 > 0$, $\{\lambda_1, \lambda_2\} = (\det Z)^{\lambda_2}\{\lambda_1 - \lambda_2\}$ so that

$$\{\lambda_1, \lambda_2\}\{\lambda_1', \lambda_2'\} = (\det Z)^{\lambda_2 + \lambda_2'}\{\lambda_1 - \lambda_2\}\{\lambda_1' - \lambda_2'\} .$$

If (λ) and (λ') are partitions, involving not more than n non-zero parts, of any two positive integers m and m', respectively, we consider the expressions $\{\lambda'\}(\xi)\{\lambda\}$ and $\{\lambda'\}(\delta)\{\lambda\}$ where ξ_j and δ_j, $j = 1, \dots, n$, are operators which decrease and increase, respectively, λ_j by 1. Since $\{\lambda'\}(\xi) = A(\ell')(\xi) / \Delta(\xi)$ and $\{\lambda\} = \Delta(\xi)\, h_{\ell_1} \cdots h_{\ell_n}$, $\{\lambda'\}(\xi)\{\lambda\} = A(\ell')(\xi)\, h_{\ell_1} \cdots h_{\ell_n}$. $A(\ell')(\xi)$ is the determinant of the n-dimensional matrix whose p^{th} row-matrix, $p = 1, \dots, n$, is $(\xi_1^{\ell_1'}, \dots, \xi_1^{\ell_n'})$ and so $\{\lambda'\}(\xi)\{\lambda\}$, which we shall denote by $\{(\lambda) / (\lambda')\}$, is the determinant of the n-dimensional matrix whose p^{th} row-matrix is $(h_{\ell_p - \ell_1'}, h_{\ell_p - \ell_2'}, \dots, h_{\ell_p - \ell_n'})$. Since $\{\lambda\}$ is the determinant of the n-dimensional matrix whose p^{th} row-matrix is $(h_{\ell_p - n+1}, h_{\ell_p - n+2}, \dots, h_{\ell_p})$, the matrix whose determinant is $\{(\lambda) / (\lambda')\}$ is obtained from the matrix whose determinant is $\{\lambda\}$ by reducing the subscripts of the elements in the j^{th} column matrix of the latter by λ_j', $j = 1, \dots, n$. It follows that $\{(\lambda) / (\lambda')\}$ is zero if $\lambda_1' > \lambda_1$; it is also zero if $\lambda_1' \leq \lambda_1$ and $\lambda_2' > \lambda_2$ since, then, the first and second column-matrices of the matrix whose determinant is $\{(\lambda) / (\lambda')\}$ are both multiples of e_1; it is zero if $\lambda_1' \leq \lambda_1$, $\lambda_2' \leq \lambda_2$ and $\lambda_3' > \lambda_3$ for, then, the first, second and third column-matrices of the matrix whose determinant is $\{(\lambda) / (\lambda')\}$ are all linear combinations of e_1 and e_2. Continuing in this way, we see that $\{(\lambda) / (\lambda')\}$ is zero unless $\lambda_1' \leq \lambda_1, \dots, \lambda_n' \leq \lambda_n$. Thus $\{(\lambda) / (\lambda')\}$ is zero if $m' > m$ and, if $m' \leq m$, it is zero if the number of non-zero parts of (λ') is greater than the number of non-zero parts of (λ). If $m' = m$, $\{(\lambda) / (\lambda')\}$ is zero unless $(\lambda') = (\lambda)$, in which case it is $\{0\} = 1$. When (λ) involves $k < n$ non-zero parts we may write $((\lambda) / (\lambda'))$, when it is not zero, as the deter-

... minant of a k-dimensional, rather than an n-dimensional matrix.

$\{\lambda'\}(\delta)$ is the quotient of the determinant of the n-dimensional matrix whose p^{th} row-matrix is $(\delta_p^{\ell'_n}, \dots, \delta_p^{\ell'_1})$ by the determinant of the n-dimensional matrix whose p^{th} row-matrix is $(1, \delta_p, \dots, \delta_p^{n-1})$ and $\{\lambda\}$ is the result of operating on $h_{\lambda_1} h_{\lambda_2 - 1} \dots h_{\lambda_n - n+1}$ with the determinant of this latter matrix. Hence, $\{\lambda'\}(\delta)\{\lambda\}$ is the result of operating upon $h_{\lambda_1} \dots h_{\lambda_n - n+1}$ by the determinant of the n-dimensional matrix whose p^{th} row-matrix is $(\delta_p^{\ell'_n}, \dots, \delta_p^{\ell'_1})$. Since $\delta_p = \xi_p^{-1}$ this determinant is $(\delta_1 \dots \delta_n)^{\ell'_1}$ times the determinant of the n-dimensional matrix whose p^{th} row-matrix is $(\xi_p^{\ell'_1 - \ell'_n}, \dots, \xi_p^{\ell'_1 - \ell'_2}, 1)$ and so $\{\lambda'\}(\delta)\{\lambda\}$ is the result of operating on $h_{\lambda_1 + \ell'_1} \dots h_{\lambda_n - n+1+\ell'_1} = h_{\lambda'_1 + \ell_1} \dots h_{\lambda'_1 + \ell_n}$ with $A(\ell'_1 - \ell'_n, \dots, \ell'_1 - \ell'_2, 1)(\xi)$. Hence,

$$\{\lambda'\}(\delta)\{\lambda\} = \{(\lambda'_1 + \lambda_1, \dots, \lambda'_1 + \lambda_n) / (\lambda'_1 - \lambda'_n, \dots, \lambda'_1 - \lambda'_2)\}$$

In order to analyse $\Gamma_{(m')}\Gamma_{(\lambda)}$ where $(\lambda) = (\lambda_1, \dots, \lambda_k)$ is a partition, involving k non-zero parts, of m, we observe that

$$\{m', (\lambda)\} = (\tfrac{1}{\delta} - \xi_1) \dots (\tfrac{1}{\delta} - \xi_k)\, h_{m'+k}\{\lambda\} = (1 - \xi_1\delta) \dots (1 - \xi_k\delta)\, h_{m'}\{\lambda\}$$

where δ is an operator which increases m' by 1 and ξ_j, $j = 1, \dots, k$, is an operator which decreases λ_j by 1. Hence, since $\{m'\} = h_{m'}$,

$$\begin{aligned}\{m'\}\{\lambda\} &= [1 + h_1(\xi)\delta + h_2(\xi)\delta^2 + \dots + h_m(\xi)\delta^m]\{m', (\lambda)\} \\ &= \{m', (\lambda)\} + \{m'+1, \{(\lambda)/(1)\}\} + \dots + \{m'+m, \\ &\qquad \{(\lambda)/(m)\}\}.\end{aligned}$$

Here $\{(\lambda)/(1)\} = h_1(\xi)\{\lambda\} = \{\lambda_1-1, \lambda_2, \ldots, \lambda_k\} + \ldots +$

$+ \{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k-1\}$

$\{(\lambda)/(2)\} = h_2(\xi)\{\lambda\} = \{\lambda_1-2, \lambda_2, \ldots, \lambda_k\} + \ldots +$

$+ \{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k-2\} + \{\lambda_1-1, \lambda_2-1, \ldots \lambda_k\} +$

$+ \{\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1}-1, \lambda_k-1\}$

and so on. For example, in order to analyse $\Gamma_{(3)}\Gamma_{(41)}$ we first determine

$\{(41)/(1)\} = \{31\} + \{4\}$, $\{(41)/(2)\} = \{2\}\{1\} = \{3\} + \{21\}$,
$\{(41)/(3)\} = \{1\}\{1\} = \{2\} + \{1^2\}$,
$\{(41)/(4)\} = \{1\}$; then $\{3\}\{41\} = \{4^2\} + \{431\} + \{53\} +$
$+ \{521\} + \{62\} + \{61^2\} +$
$+ \{71\}$,

$\{341\}$ being zero.

If (λ') is a 2-part partition of m', we see, in the same way, that $\{\lambda'\}\{\lambda\}$ is the result of operating upon $\{(\lambda'), (\lambda)\}$ with the product $(1 + h_1(\xi)\,\delta_1 + \ldots)(1 + h_1(\xi)\,\delta_2 + \ldots)$ where δ_1 and δ_2 are operators which increase λ'_1 and λ'_2, respectively, by 1. The terms of degree j in δ_1 and δ_2 in this product are

$$h_j(\xi)\,\delta_1^j + h_{j-1}(\xi)\,\delta_1^{j-1}\delta_2 + \ldots + h_j(\xi)\,\delta_2^j$$

and, since $h_{j-1}h_1 = \{j\} + \{j-1, 1\}$, $h_{j-2}h_2 = \{j\} + \{j-1,1\} + \{j-2,2\}$ and so on, the coefficient of $\{j\}(\xi)$ in this expression is $\{j\}(\delta)$. The coefficient of $\{j-1, 1\}(\xi)$ is $\delta_1^{j-1}\delta_2 + \ldots + \delta_1\delta_2^{j-1} =$ $= \delta_1\delta_2\{j-2\}(\delta) = \{j-1, 1\}(\delta)$ and so on. Thus we obtain $\{\lambda'\}\{\lambda\}$ by operating upon $\{(\lambda'), (\lambda)\}$ with $\sum_{(\mu)}\{\mu\}(\delta)\{\mu\}(\xi)$, the summation being over all partitions (μ), involving not more than two non-zero parts, and whose first part is $\leq \lambda_1$, of all non-negative integers $\leq$ m. Thus a typical term of $\{\lambda'\}\{\lambda\}$ is found by prefixing $\{(\mu_1+\lambda_1', \mu_1+\lambda_2') / (\mu_1-\mu_2)$ to $\{(\lambda)/(\mu)\}$.

For example,

$$\{21\}\{21\} = \{2121\} + \{31+2^2, 2+1^2\} + \{41+32, 1\} + \{32, 1\} + \{42+3^2\}$$
$$= \{42\} + \{41^2\} + \{3^2\} + 2\{321\} + \{31^3\} + \{2^3\} + \{2^21^2\}$$

This result resolves the problem of analysing the product of all irreducible continuous representations of the 3-dimensional unitary group since, when $n = 3$, $\{\lambda_1, \lambda_2, \lambda_3\} = (\det Z)^{\lambda_3}\{\lambda_1-\lambda_3, \lambda_2-\lambda_3\}$.

We may proceed, in the same way, with the general problem of analysing $\{\lambda'\}\{\lambda\}$, for the n-dimensional unitary group, when the number k' of non-zero parts in (λ') is greater than 2, it being assumed, without loss of generality, that $k' \leq k$, the number of non-zero parts in (λ). We obtain $\{\lambda'\}\{\lambda\}$ by operating upon $\{(\lambda'),(\lambda)\}$ with $\sum_{(\mu)} \{\mu\}(\delta)\mu(\xi)$ the summation being over all partitions (μ), involving not more than k' non-zero parts, and whose first part $\leq \lambda_1$, of all non-negative integers $\leq$ m. We may obtain a master formula by first applying this method to $\{0, \ldots, 0\}\{\lambda\}$ and we then obtain $\{\lambda'\}\{\lambda\}$ adding $\lambda_1', \ldots, \lambda_{k'}'$ to the first k parts of each term in the master formula. For example, since, when $k' = 2$, $\{1\}(\delta) = \delta_1 + \delta_2$, $\{2\}(\delta) = \delta_1^2 + \delta_1\delta_2 + \delta_2^2$, $\{1^2\}\delta = \delta_1\delta_2$, $\{21\}(\delta) = \delta_1\delta_2\{1\}(\delta) = \delta_1^2\delta_2 + \delta_1\delta_2^2$ the master formula for $\{\lambda_1', \lambda_2'\}\{2,1\}$ is $\{0,0\}\{21\} = \{0021\} + \{102\} + \{101^2\} + \{012\} + \{01^3\} + \{201\} + 2\{1^3\} + \{021\} + \{21\} + \{12\}$

so that

$$\{\lambda'_1, \lambda'_2\}\{2,1\} = \{\lambda'_1, \lambda'_2, 21\} + \{\lambda'_1+1, \lambda'_2, 2\} + \{\lambda'_1+1, \lambda'_2, 1^2\} +$$
$$+ \{\lambda'_1, \lambda'_2+1, 2\} + \{\lambda'_1, \lambda'_2+1, 1^2\} + \{\lambda'_1+2, \lambda'_2, 1\} +$$
$$+ 2\{\lambda'_1+1, \lambda'_2+1, 1\} + \{\lambda'_1, \lambda'_2+2, 1\} +$$
$$+ \{\lambda'_1+2, \lambda'_2+1\} + \{\lambda'_1+1, \lambda'_2+2\} .$$

For example,

$$\{43\}\{21\} = \{4321\} + \{532\} + \{531^2\} + \{4^2 2\} + \{4^2 1^2\} + \{631\} +$$
$$+ \quad 2\{541\} + \{64\} + \{5^2\},$$ the term $\{451\}$ vanishing.

Lecture 18

1. The analysis of the product of irreducible continuous representations of the orthogonal and symplectic groups.

The character $[\lambda]$ of the continuous irreducible representation $\Gamma_{[\lambda]}$ of the n-dimensional orthogonal group is

$$[\lambda] = \prod_{p \leq q}^{k} (1 - \xi_p \xi_q)\{\lambda\} = \prod_{p \leq q}^{k} (1 - \xi_p \xi_q) \Delta(\xi) h_{\ell_1} \cdots h_{\ell_k}$$

$(\lambda) = (\lambda_1, \ldots, \lambda_k)$ being a partition, involving not more than k non-zero parts, of any non-negative integer m, where $n = 2k$ if it even and $= 2k+1$ if it is odd, and $\ell_1, \ldots, \ell_k$ being defined by the formulas $\ell_1 = \lambda_1 + k - 1, \ldots, \ell_k = \lambda_k$. If (λ) involves more than k, but not more than $2k$, non-zero parts we introduce the function

$$[\lambda] = \prod_{p \leq q}^{2k} (1 - \xi_p \xi_q)\{\lambda\} = \prod_{p \leq q}^{2k} (1 - \xi_p \xi_q) \Delta(\xi) h_{\ell_1} \cdots h_{\ell_{2k}}$$

where $\ell_1 = \lambda_1 + 2k - 1, \ldots, \ell_{2k} = \lambda_{2k}$, and observe that although $[\lambda]$ is not now the character of an irreducible representation of the n-dimensional orthogonal group it is a generalised character of this group. The character $[\lambda][\lambda']$ of the product $\Gamma_{[\lambda]}\Gamma_{[\lambda']}$ of the two continuous irreducible representations $\Gamma_{[\lambda]}$ and $\Gamma_{[\lambda']}$ of the n-dimensional orthogonal group is found by operating on $\{\lambda\}\{\lambda'\}$

with $\left(\prod_{p \leq q}^{k}{}_{1} (1 - \xi_p \xi_q)\right)\left(\prod_{p' \leq_{k+1} q'}^{2k} (1 - \xi_{p'} \xi_{q'})\right)$

and this is the same as operating on $[\{\lambda\}\{\lambda'\}]$ with

$\prod_{\substack{1 \leq p \leq k \\ k+1 \leq p' \leq 2k}} (1 - \xi_p \xi_{p'})^{-1}$ where, if $\{\lambda\}\{\lambda'\} = \sum_{(\alpha)} c_{(\alpha)} \{\alpha\}$,

(α) being a partition of m+m', we understand by $[\{\lambda\}\{\lambda'\}]$ the expression $\sum_\alpha c_\alpha [\alpha]$. Now $(1-\xi_1\xi_1')^{-1}(1-\xi_1\xi_2')^{-1}\ldots(1-\xi_k\xi_k')^{-1}$ $= \sum_{(\mu)} \mu(\xi)\mu(\xi')$, the sum being over all partitions (μ), involving not more than k non-zero parts, of all non-negative integers and so

$$[\lambda][\lambda'] = \left[\sum_{(\mu)} \{(\lambda)/(\mu)\}\{\lambda'/\mu'\}\right] .$$

For example, the product of the two irreducible representations $\Gamma_{[m]}$ and $\Gamma_{[m']}$ of the 3-dimensional orthogonal group is furnished by the formula

$$[m][m'] = [\{m\}\{m'\} + \{m-1\}\{m'-1\} + \ldots + \{m-m'\}], \quad m \geqslant m'$$

Here $[\{m\}\{m'\}] = [m+m'] + [m+m'-1,1] + \ldots + [m,m']$ involves terms $[\lambda]$ where (λ) has more than k=1 part and it is necessary to determine the corresponding generalised character of the group. The rule which enables us to determine these is known as the modification rule for the 3-dimensional orthogonal group. This rule is that $[\lambda_1, \lambda_2]$ is zero unless $\lambda_2 = 1$ in which case it is $\varepsilon[\lambda_1]$ where ε is 1 over the rotation half, and -1 over the reflexion half, of the group.

Thus $[m][m'] = [m+m'] + \varepsilon[m+m'-1] + [m+m'-2] + \ldots$

This is known as the Clebsch-Gordan formula for the product of irreducible representations of the 3-dimensional orthogonal group.

The argument which furnished $[\lambda][\lambda']$ is applicable to the 2k-dimensional U-symplectic group; thus

$$\langle\lambda\rangle\langle\lambda'\rangle = \left\langle\sum_{(\mu)} \{(\lambda)/(\mu)\}\{(\lambda')/(\mu)\}\right\rangle$$

In particular, $\langle m\rangle\langle m'\rangle = \langle\{m\}\{m'\} + \{m-1\}\{m'-1\} + \ldots + \{m-m'\}\rangle, \quad m \geqslant m'$

The modification rule for the 2-dimensional U-symplectic group, i.e., the unimodular 2-dimensional unitary group, is different from that for the 3-dimensional orthogonal group; it is that all $\langle \lambda_1, \lambda_2 \rangle$ for which $\lambda_2 > 0$ are zero. Thus the Clebsch-Gordan formula for the unimodular 2-dimensional unitary group is

$$\langle m \rangle \langle m' \rangle = \langle m+m' \rangle + \langle m+m'-2 \rangle + \dots + \langle m-m' \rangle \,, \quad m \geq m' \,.$$

2. <u>The modification rule for the 2k-dimensional rotation group</u>.

If $(z) = (z_1, \dots, z_{2k})$ are the characteristic numbers of a typical element Z of the 2k-dimensional rotation group and $(t) = t_1, \dots, t_{2k}$ is any set of $2k$ indeterminates we denote $(1 - z_1 t_j) \dots (1 - z_{2k} t_j)$, $j = 1, \dots, 2k$, by $f(t_j)$ so that $f(t_j) = 1 + \sigma'_1 t_j + \dots + \sigma'_{2k} t_j^{2k} = (1 + t_j^{2k}) + \sigma'_1 (t_j + t_j^{2k-1}) + \dots + \sigma'_k t_j^k$ and $(f(t_j))^{-1} = 1 + h_1 t_j + \dots$. Then $\prod_{j=1}^{2k} (f(t_j))^{-1} = \sum_{(\lambda)} \{\lambda\}(z)\{\lambda\}(t)$, the summation being over all partitions (λ), involving not more than $2k$ non-zero parts, of all non-negative integers. On writing the product $\prod_{j=1}^{2k} (f(t_j))^{-1}$ in the form $\sum_{(j)} h_{j_1} \dots h_{j_{2k}} t_1^{j_1} \dots t_{2k}^{j_{2k}}$ we see that the result of applying the operator $\xi_1^{q_1} \dots \xi_{2k}^{q_{2k}}$ to this product is $t_1^{q_1} \dots t_{2k}^{q_{2k}} \sum_{(j')} h_{j'_1} \dots h_{j'_{2k}} t_1^{j'_1} \dots t_{2k}^{j'_{2k}}$, where $j'_p = j_p - q_p$, $p = 1, \dots, 2k$. Thus the effect of operating on $\sum_{(\lambda)} \{\lambda\}(z)\{\lambda\}(t)$ with $\xi_1^{q_1} \dots \xi_{2k}^{q_{2k}}$ is to multiply it by $t_1^{q_1} \dots t_{2k}^{q_{2k}}$. Since $[\lambda](z)$ is the result of operating on $\{\lambda\}(z)$ with the operator $\prod_{1}^{p \leq q} {}^{2k} (1 - \xi_p \xi_q)$ it follows that

$$\sum_{(\lambda)} [\lambda](z)\{\lambda\}(t) = \prod_{p \leq q}^{2k} (1-t_p t_q) \sum_{(\lambda)} \{\lambda\}(z)\{\lambda\}(t)$$

$$= \prod_{p=1}^{2k} (1-t_p^{\,2}) \prod_{\substack{p<q \\ 1}}^{2k} (1-t_p t_q) \prod_{j=1}^{2k} (f(t_j))^{-1}$$

If we set $t_{k+1} = t_{k+2} = \dots = t_{2k} = 0$ in this relation $\{\lambda\}(t)$ is zero if (λ) contains more than k non-zero parts and we obtain

$$\sum_{(\lambda')} [\lambda'](z)\{\lambda'\}(t') = \prod_{p=1}^{k} (1-t_p'^{\,2}) \prod_{\substack{p<q \\ 1}}^{k} (1-t_p' t_q') \prod_{j=1}^{k} (f(t_j'))^{-1}$$

where $(t') = (t_1, \dots, t_k)$ is any set of k, rather than $2k$, indeterminates and (λ') is any partition, involving not more than k, rather than $2k$, non-zero parts, of any non-negative integer. Now the product of $\prod_{\substack{p<q \\ 1}}^{2k} (1-t_p t_q)$ by $\Delta(t)$ is the determinant of the $2k$-dimensional matrix whose p^{th} row-matrix is $(t_p^{2k-1}, t_p^{2k-2}+t_p^{2k}, \dots, 1+t_p^{4k-2})$, $p = 1, \dots, 2k$, and we may change the last k columns of this $2k$-dimensional matrix, without affecting its determinant, as follows. Since $f(t_p) = (1+t_p^{2k}) + \sigma_1'(t_p+t_p^{k-1}) + \dots + \sigma_k' t_p^k$ we may replace $1 + t_p^{4k-2}$ by $f(t_p)(1+t_p^{2k-2})$, $t_p+t_p^{4k-3}$ by $f(t_p)(t_p+t_p^{2k-3})$ and so on to $t_p^{k-1} + t_p^{3k-1}$ which may be replaced by $f(t_p)\,t_p^{k-1}$. We expand the determinant of the resulting $2k$-dimensional matrix as a sum of products of k-rowed determinants formed from its first k columns and its last k columns. If $((\tau), (\tau')) = (\tau_1, \dots, \tau_k, \tau_1', \dots, \tau_k')$ is an even permutation of $(t) = (t_1, \dots, t_{2k})$ one such product is $f(\tau_1') \dots f(\tau_k')\, D(\tau_1, \dots, \tau_k)\, D'(\tau_1', \dots, \tau_k')$ where $D(\tau_1, \dots, \tau_k)$ is the determinant of the k-dimensional matrix whose p^{th} row-matrix is $(\tau_p^{2k-1}, \tau_p^{2k-2}+\tau_p^{2k}, \dots, \tau_p^{k} + \tau_p^{3k-2})$ and $D'(\tau_1', \dots, \tau_k')$ is

the determinant of the k-dimensional matrix whose p^{th} row-matrix is

$((\tau'_p)^{k-1}, (\tau'_p)^{k-2} + (\tau'_p)^k, \ldots, 1 + (\tau'_p)^{2k-2})$. Thus

$\sum_{(\lambda)} [\lambda](z)\{\lambda\}(t)\,\Delta(t)$ is $(1/k!)^2$ the sum over all such even

permutations $((\tau), (\tau'))$ of (t) of

$$\prod_{p=1}^{k}(1-\tau_p^2)\prod_{j=1}^{k}(f(\tau_j))^{-1}\,D(\tau_1, \ldots, \tau_k)\prod_{p=1}^{k}(1-\tau_p'^2)\,D'(\tau'_1, \ldots \tau'_k).$$

The factor $(\prod_{p=1}^{k}(1-\tau_p^2))\,D(\tau_1, \ldots, \tau_k)\prod_{j=1}^{k}(f(\tau_j))^{-1} =$

$$= \{\sum_{(\lambda')}[\lambda'](z)\{\lambda'\}(\tau)\}(\tau_1 \cdots \tau_k)^k\,\Delta(\tau) =$$

$= \sum_{(\lambda')}[\lambda'](z)\,A(\ell_1, \ldots, \ell_k)(\tau)$ where $\ell_1 = \lambda'_1+2k-1, \ldots, \ell_k = \lambda'_k+k$

and the factor $\prod_{p=1}^{k}(1-\tau_p'^2)\,D'(\tau'_1, \ldots, \tau'_k) = \{\prod_{1}^{p \leq q \leq k}(1-\tau'_p\tau'_q)\}.\Delta(\tau')$

$= [1 - \{2\}(\tau') + (\{2\}\otimes\{1^2\})(\tau') - \ldots]\,\Delta(\tau')$. If, then,
$\{\mu\}$ is any term of $\{2\}\otimes\{1^m\}$, for which the partition (μ)

of $2m$ does not involve more than k non-zero parts, the product
$\prod_{p=1}^{k}(1-\tau_p'^2)\,D'(\tau'_1, \ldots \tau'_k)$ is the sum over all non-negative integers m
and all such partitions (μ) of $2m$, of $(-1)^m\{\mu\}(\tau')\Delta(\tau') =$
$= (-1)^m A(\ell_{k+1}, \ldots, \ell_{2k})(\tau')$ where $\ell_{k+1} = \mu_1 + k - 1, \ldots, \ell_{2k} = \mu_{2k}$.

On summing over all even permutations $((\tau), (\tau'))$ of t the product
$A(\ell_1, \ldots \ell_k)(\tau)\,A(\ell_{k+1}, \ldots, \ell_{2k})(\tau')$ and dividing by $(k!)^2$ we
obtain $A(\ell_1, \ldots, \ell_{2k})(t)$ and so

$$\sum_{(\lambda)}[\lambda](z)\{\lambda\}(t)\,\Delta(t) = \sum_{m,(\lambda'),(\mu)}(-1)^m[\lambda'](z)\,A(\ell_1 \cdots \ell_{2k})(t)$$

or, equivalently,

$$\sum_{(\lambda)} [\lambda](z)\{\lambda\}(t) = \sum_{m,(\lambda'),(\mu)} (-1)^m [\lambda'](z)\{(\lambda'),(\mu)\}(t) .$$

Since the functions $\{\lambda\}(t)$ are linearly independent it follows that $[\lambda](z)$ is zero unless (λ) is of the form $((\lambda'), (\mu))$ where (λ') involves not more than k non-zero parts and $\{\mu\}$ is a term of $\{2\} \otimes \{1^m\}$, m=0, 1, 2, ... and that $[(\lambda'), (\mu)](z) = (-1)^m[\lambda'](z)$. This is the modification rule for the 2k-dimensional rotation group. For example, for the 4-dimensional rotation group, $[2^3] = -[2^2]$, $[31^2] = -[302] = [3]$ and so on. It is easy to see that $[\lambda_1, \ldots, \lambda_{2k-1}, 0] = [\lambda_1, \lambda_1-\lambda_{2k-1}, \ldots, \lambda_1-\lambda_2, 0]$ for the 2k-dimensional rotation group. Indeed $\{\lambda_1, \ldots, \lambda_{2k-1}, 0\}\Delta(z)$ is the determinant of the 2k-dimensional matrix whose p^{th} row-matrix is $(z_1^{\ell_p}, \ldots, z_{2k}^{\ell_p})$ and, since $z_1z_2\ldots z_{2k} = 1$, the determinant of this matrix is unaffected if we multiply its j^{th} column matrix by $z_j^{-\ell_1}$ and its $(k+j)^{th}$ column matrix by $(z_{k+j})^{-\ell_1}$, j = 1, ..., k. The determinant remains unaffected if we then invert the order of the rows of the matrix (which would multiply it by $(-1)^{k(2k-1)}$) and interchange the j^{th} and $(k+j)^{th}$ columns, j = 1, ..., k, (which would multiply it by $(-1)^k$). The resulting matrix is the 2k-dimensional matrix whose determinant is $A(\ell_1, \ell_1-\ell_{2k-1}, \ldots, \ell_1-\ell_2, 0)$, proving that $\{\lambda_1, \ldots, \lambda_{2k-1}, 0\} = \{\lambda_1, \lambda_1-\lambda_{2k-1}, \ldots, \lambda_1-\lambda_2, 0\}$. Since $[\lambda]$ is the result of applying the operator $\prod_{p \leq q}^{2k} (1-\xi_p\xi_q)$ to $\{\lambda\}$ it follows that $[\lambda_1, \ldots, \lambda_{2k-1}, 0] = [\lambda_1, \lambda_1-\lambda_{2k-1}, \ldots, \lambda_1-\lambda_2, 0]$. If λ_{2k} is not zero we use the relation $\{\lambda_1, \ldots, \lambda_{2k}\} =$

$$= (\det Z)^{\lambda_{2k}}\{\lambda_1-\lambda_{2k}, \ldots, \lambda_{2k-1}-\lambda_{2k}, 0\} = \{\lambda_1-\lambda_{2k}, \ldots,$$

$\lambda_{2k-1}-\lambda_{2k}, 0\}$ to show that $[\lambda_1, \ldots, \lambda_{2k}] =$

$$= [\lambda_1 - \lambda_{2k}, \ldots, \lambda_{2k-1} - \lambda_{2k}, 0] = [\lambda_1 - \lambda_{2k}, \lambda_1 - \lambda_{2k-1}, \ldots \lambda_1 - \lambda_{2k-1}, 0].$$

3. The modification rule for the 2k-dimensional U-symplectic group.

The only change in the argument which furnished the modification rule for the 2k-dimensional rotation group which is necessary when we pass to the 2k-dimensional U-symplectic group is that we must replace the expression $\prod_{\substack{p \leq q \\ 1}}^{k} (1 - \zeta'_p \zeta'_q)' = 1 - \{2\}(\zeta') + (\{2\} \otimes \{1^2\})(\zeta') - \ldots$ by the expression $\prod_{\substack{p < q \\ 1}}^{k} (1 - \zeta'_p \zeta'_q) = 1 - \{1^2\}(\zeta') + (\{1^2\} \otimes \{1^2\})(\zeta') - \ldots$. Thus the modification rule for the 2k-dimensional U-symplectic group is as follows: $\langle\lambda\rangle$ is zero unless (λ) is of the form $((\lambda'), (\mu))$ where (λ') involves not more than k non-zero parts and $\{\mu\}$ is a term of $\{1^2\} \otimes \{1^m\}$, $m = 0, 1, 2, \ldots$, and, when this is the case, $\langle(\lambda'), (\mu)\rangle = (-1)^m \langle\lambda'\rangle$. For example, when $k = 1$, $\langle\lambda_1, \lambda_2\rangle = 0$ if $\lambda_2 > 0$.

4. The modification rule for the (2k+1)-dimensional rotation group.

In discussing the (2k+1)-dimensional rotation group we separate the characteristic number $z_{2k+1} = 1$ from the remaining 2k characteristic numbers $(z) = (z_1, \ldots, z_{2k})$ of a typical element Z of the group. The argument proceeds as in the case of the 2k-dimensional rotation group, the only difference being that $f(t) = (1-z_1t) \ldots (1-z_{2k}t)(1-t)$ is replaced by $f(t) / (1-t)$; this has the consequence that the product $\prod_{p \leqq q}^{k} (1-\zeta'_p \zeta'_q)$ is replaced by $\prod_{j=1}^{k} (1+\zeta'_j) \prod_{p<q}^{k} (1-\zeta'_p \zeta'_q) =$

$$= \{1+\{1\}(\zeta')+\{1^2\}(\zeta')+\ldots\}\{1-\{1^2\}(\zeta')+(\{1^2\}\otimes\{1^2\})(\zeta')-\ldots\}$$

$$= 1+\{1\}(\zeta')-(\{1\}\{1^2\}-\{1^3\})(\zeta')+(\{1^4\}-\{1^2\}\{1^2\}+ (\{1^2\}\otimes\{1^2\}))(\zeta')-$$

$$= 1+\{1\}(\zeta')-\{21\}(\zeta')-\{2^2\}(\zeta')+\{31^2\}(\zeta')+\{321\}(\zeta')-\{3^22\}(\zeta')-\{3^3\}(\zeta')+\ldots$$

The parentheses $\{\mu\}$ in this expression are those for which (μ) is a self-associated partition and the sign prefixed to each such $\{\mu\}$ is determined by writing (μ) in the form $\begin{pmatrix} a_1 & a_2 & \ldots & a_s \\ a_1 & a_2 & \ldots & a_s \end{pmatrix}$ which is described at the beginning of the next lecture; for example $(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $(21) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $(2^2) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$; $(31^2) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $(321) = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$, (3^22) as $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$, $(3^3) = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ and so on. Then the sign prefixed to $\{\mu\}(\zeta')$ is $(-1)^{a_1+\ldots+a_s}$. Thus the modification rule for the $(2k+1)$-dimensional rotation group is as follows: $[\lambda]$ is zero unless (λ) is of the form $\left((\lambda'), \begin{pmatrix} a_1 & \ldots & a_s \\ a_1 & \ldots & a_s \end{pmatrix} \right)$, where (λ') involves not more than k non-zero parts and $\left[(\lambda'), \begin{pmatrix} a_1 & \ldots & a_s \\ a_1 & \ldots & a_s \end{pmatrix} \right] =$

$= (-1)^{a_1+\ldots+a_s} [\lambda']$. For example, for the 3-dimensional rotation group $[\lambda_1, \lambda_2] = 0$ if $\lambda_2 > 1$ while $[\lambda\ 1] = [\lambda]$.

5. The modification rule for the 2k-dimensional orthogonal group.

In order to determine the value of $[\lambda]$, where (λ) involves more than k non-zero parts, over the reflexion half of the 2k-dimensional orthogonal group we first observe that, since every 2-dimensional reflexion matrix $\begin{pmatrix} c & s \\ s & -c \end{pmatrix}$ has the characteristic numbers 1 and -1, every 2k-dimensional reflexion matrix Z has the characteristic numbers $z_{2k-1} = 1$, $z_{2k} = -1$. Separating these out from the remaining $2k-2$ characteristic numbers $(z) = (z_1, \ldots, z_{2k-2})$ of Z we proceed as in the case of the 2k-dimensional rotation group, $f(t) = (1 - z_1 t) \ldots (1 - z_{2k-2} t)(1 - t)(1 + t)$ being replaced by $(1 - z_1 t) \ldots (1 - z_{2k-2} t) = f(t) / (1 - t^2)$. This has the consequence that the product $\prod_{p \leq q}^{k} (1 - \zeta'_p \zeta'_q)$ is replaced by $\prod_{p < q}^{k+1} (1 - \zeta'_p \zeta'_q) =$

$= 1 - \{1^2\}(\zeta') + (\{1^2\} \otimes \{1^2\})(\zeta') - \ldots$ and we expand the determinant of the 2k-dimensional matrix which we encounter as a sum of products of $(k-1)$-rowed determinants formed from its first $k-1$ columns by $(k+1)$-rowed determinants formed from its last $k+1$ columns. It follows that $[\lambda_1, \ldots, \lambda_{2k}]$ is zero over the reflexion-half of the 2k-dimensional orthogonal group unless (λ) is of the form $(\lambda'_1, \ldots, \lambda'_{k-1}, (\mu))$ where $\{\mu\}$ is a term of $\{1^2\} \otimes \{1^m\}$. We shall see in the next lecture that every term of $\{1^2\} \otimes \{1^m\}$ is of the form $\left\{\begin{pmatrix} a_1 & \ldots & a_s \\ a_1+1 & \ldots & a_s+1 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} a_1 & \ldots & a_s \\ a_1+1 & \ldots & a_s+1 \end{pmatrix}\right\} =$

$= (-1)^s \left\{0, \begin{pmatrix} a_1+1 & \ldots & a_s+1 \\ a_1 & \ldots & a_s \end{pmatrix}\right\}$. For example $\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\} = \{21^2\}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = (31)$ so that $\left\{0, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} = \{031\} = -\{21^2\}$.

$\left\{\begin{pmatrix} a_1+1 & \dots & a_s+1 \\ a_1 & \dots & a_s \end{pmatrix}\right\}$ is a term of $\{2\} \otimes \{1^m\}$ and so $[\lambda]$ is zero over the reflexion-half of the 2k-dimensional rotation group unless $\{\lambda\}$ can be written in the form $\{\lambda'_1, \dots, \lambda'_{k-1}, 0, (\mu)\}$ where $\{\mu\}$ is a term of $\{2\} \otimes \{1^m\}$ and when this is the case, so that $(\mu) = \begin{pmatrix} a_1+1 & \dots & a_s+1 \\ a_1 & \dots & a_s \end{pmatrix}$, $[\lambda] = (-1)^{m+s}[\lambda'_1, \dots \lambda'_{k-1}]$. On combining this result with that already obtained for the rotation-half of the group we obtain the following modification rule for the 2k-dimensional orthogonal group:

If (λ) contains more than k non-zero parts, $[\lambda]$ is zero over the group save when (λ) is of the form $(\lambda'_1, \dots, \lambda'_{k-1}, 0, (\mu))$ where $\{\mu\}$ is a term of $\{2\} \otimes \{1^m\}$, $m = 1, 2, \dots$, and, when this is the case $[\lambda] = (-1)^m \varepsilon^s[\lambda'_1, \dots, \lambda'_{k-1}]$, where ε is 1 over the rotation-half, and -1 over the reflexion-half, of the group.

6. The modification rule for the (2k+1)-dimensional orthogonal group.

The only difference between the discussion of the reflexion-half of the (2k+1)-dimensional orthogonal group and that of the rotation-half is that the characteristic number which we separate from the remaining 2k characteristic numbers $(z) = (z_1, \dots, z_{2k})$ is -1 rather than 1. This has the consequence that instead of dealing with $1 + \{1\}(\zeta') + \{1^2\}(\zeta') + \dots$ times $1 - \{1^2\}(\zeta') + (\{1^2\} \otimes \{1^2\})(\zeta') + \dots$ we have to deal with $\{1 - \{1\}(\zeta') + \{1^2\}(\zeta') - \dots\}\{1 - \{1^2\}(\zeta') + (\{1^2\} \otimes \{1^2\})(\zeta') - \dots\} = 1 - \{1\}(\zeta') + \{21^2\}(\zeta') - \dots$. This furnishes the following modification rule for the (2k+1)-dimensional orthogonal group: If (λ) involves more than k non-zero parts, $[\lambda]$

is zero over the group unless (λ) is of the form $\left((\lambda'),\ \begin{pmatrix} a_1 \ldots a_s \\ a_1 \ldots a_s \end{pmatrix}\right)$,

where (λ') involves not more than k non-zero parts and

$$\left[(\lambda'),\ \begin{pmatrix} a_1 & \ldots & a_s \\ a_1 & \ldots & a_s \end{pmatrix}\right] = (-1)^{a_1+\ldots+a_s}\,\mathcal{E}^s[\lambda']$$

where $\mathcal{E}$ is 1 over the rotation-half, and -1 over the reflexion-half, of the group.

Lecture 19

1. Associated irreducible representations of the n-dimensional unitary group.

We may associate with any partition $(\lambda) = (\lambda_1, \ldots, \lambda_k)$ of any positive integer m a dot diagram which has λ_1 dots in the first row, λ_2 dots in the second row and so on to the k^{th} row which has λ_k dots. We denote by (λ^*) the partition of m whose dot diagram is obtained by interchanging the rows and columns of the dot diagram of (λ) and we term (λ) and (λ^*) associated partitions of m. It is clear that $\lambda_1^* = k$ and that $\lambda_1 = k^*$. The irreducible representations $\Gamma_{(\lambda)}$ and $\Gamma_{(\lambda^*)}$ of the n-dimensional unitary group are said to be associated and if $(\lambda^*) = (\lambda)$, in which case the partition (λ) of m is termed self-associated, the irreducible representation $\Gamma_{(\lambda)}$ is said to be self-associated. The associate (m^*) of (m) is (1^m) and we have seen that the fact that the $(m+1)$-dimensional matrix whose p^{th} row-matrix is $(h_{1-p}, h_{2-p}, \ldots, h_{m+1-p})$ is the reciprocal of the $(m+1)$-dimensional matrix whose p^{th} row-matrix is $(\sigma'_{1-p}, \ldots, \sigma'_{m+1-p})$ assures us that $\{m\} = \{1^m\}(\sigma) = \{m^*\}(\sigma)$, where by $\{1^m\}(\sigma)$ we mean the result of applying the operator $\Delta(\xi)$ to $\sigma_m\sigma_{m-1}\ldots\sigma_1$ rather than to $h_m h_{m-1}\ldots h_1$. The same argument shows that $\{\lambda\}$, which is the result of applying the operator $\Delta(\xi) = (\xi_1 - \xi_2)\ldots(\xi_{k-1} - \xi_k)$ to $h_{\ell_1}\ldots h_{\ell_k}$, where $\ell_1 = \lambda_1 + k - 1, \ldots, \ell_k = \lambda_k$, is equal to $\{\lambda^*\}(\sigma)$, which is the result of applying the operator $\Delta^*(\xi) =$

$= (\xi_1 - \xi_2) \cdots (\xi_{k^*-1} - \xi_{k^*})$ to $\sigma_{\ell_1^*} \cdots \sigma_{\ell_{k^*}^*}$, where

$\ell_1^* = \lambda_1^* + k_1^* - 1, \ldots, \ell_{k^*}^* = \lambda_{k^*}^*$.

The elementary symmetric functions $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the n characteristic numbers $(z) = (z_1, \ldots, z_n)$ of a typical element Z of the n-dimensional unitary group may be expressed in terms of the power sums $(s) = (s_1, \ldots, s_n)$, where $s_j = z_1^j + \cdots + z_n^j$, $j = 1, \ldots, n$, of these characteristic numbers as follows. Since $1 - \sigma_1 t + \cdots = \prod_{j=1}^{n}(1 - z_j t)$, $\log(1 - \sigma_1 t + \cdots) = \sum_{j=1}^{n} \log(1 - z_j t)$ $= -(s_1 t + \frac{s_2}{2} t^2 + \cdots)$ so that $1 - \sigma_1 t + \cdots = \exp(-s_1 t) \times \exp(-\frac{s_2}{2} t^2) \cdots$ which implies that $(-1)^j \sigma_j =$

$$\sum_{(\alpha)} (-1)^{\alpha_1 + \alpha_2 + \cdots + \alpha_j} \frac{1}{\alpha_1!} s_1^{\alpha_1} \frac{1}{\alpha_2!} \left(\frac{s_2}{2}\right)^{\alpha_2} \cdots \frac{1}{\alpha_j!} \left(\frac{s_j}{n}\right)^{\alpha_j}$$

where the summation on the right is over all sets $(\alpha) = (\alpha_1, \ldots \alpha_j)$ of j non-negative integers which are such that $\alpha_1 + 2\alpha_2 + \cdots + j\alpha_j = j$. Thus

$$\sigma_j = \sum_{(\alpha)} (-1)^{\alpha_2 + \alpha_4 + \cdots} \frac{1}{\alpha_1!} s_1^{\alpha_1} \frac{1}{\alpha_2!} \left(\frac{s_2}{2}\right)^{\alpha_2} \cdots \frac{1}{\alpha_n!} \left(\frac{s_n}{n}\right)^{\alpha_n}$$

$$= \sum_{(\alpha)} \frac{1}{\alpha_1!} s_1^{\alpha_1} \frac{1}{\alpha_2!} \left(-\frac{s_2}{2}\right)^{\alpha_2} \cdots \frac{1}{\alpha_n!} \left(\frac{(-1)^{n-1} s_n}{n}\right)^{\alpha_n}$$

Since $1 + h_1 t + \cdots = \prod_{j=1}^{n} (1 - z_j t)^{-1}$, $\log(1 + h_1 t + \cdots) =$ $= s_1 t + \frac{s_2}{2} t^2 + \cdots$ and

$$h_j = \sum_{(\alpha)} \frac{1}{\alpha_1!} s_1^{\alpha_1} \frac{1}{\alpha_2!} \left(\frac{s_2}{2}\right)^{\alpha_2} \cdots \frac{1}{\alpha_n!} \left(\frac{s_n}{n}\right)^{\alpha_n} .$$

From this it follows, on denoting $(s_1, -s_2, \ldots (-1)^{n-1}s_n)$ by $s*$, that $h_j(s) = \sigma_j(s*)$ and this implies that $\{\lambda\}_{(s)} = \{\lambda^*\}_{(s*)}$, where $\{\lambda\}_{(s)} = \{\lambda\}(z)$ and $\{\lambda^*\}_{(s*)}$ is the result of replacing (s) by (s*) in $\{\lambda^*\}_{(s)}$. More generally, the same argument shows that $\{(\lambda)/(\mu)\}_{(s)} = \{(\lambda^*)/(\mu^*)\}_{(s*)}$ so that the partitions which appear in the development of $\{(\lambda^*)/(\mu^*)\}$ are the associates of those which appear in the development of $\{(\lambda)/(\mu)\}$. The function $\{\lambda\}_{(s)}$ is of the form $\sum_{(\alpha)} c_{(\alpha)} s_1^{\alpha_1} \ldots s_m^{\alpha_m}$ where the summation is over all sets $(\alpha) = (\alpha_1, \ldots, \alpha_m)$ of m non-negative integers which are such that $\alpha_1 + 2\alpha_2 + \ldots + m\alpha_m = m$, (λ) being a partition of m, and we denote this function by S_1. The result of replacing (s) by (s*) in $\{\lambda\}_{(s)}$, i.e., $\{\lambda\}_{(s*)}$, is $\sum_{(\alpha)} (-1)^{\alpha_2+\alpha_4+\ldots} c_{(\alpha)} s_1^{\alpha_1} \ldots s_m^{\alpha_m}$ and we denote this function by S_1^*. The values of S_1 and S_1^* at the element Z^j of the n-dimensional unitary group, where j is any positive integer, are obtained by replacing s_1 by s_j, s_2 by s_{2j}, and so on, and we denote these functions by S_j and $(S*)_j$, respectively, so that

$$S_j = \sum_{(\alpha)} c_{(\alpha)} s_j^{\alpha_1} \ldots s_{mj}^{\alpha_m} \; ; \; (S*)_j = \sum_{(\alpha)} (-1)^{\alpha_2+\alpha_4+\ldots} c_{(\alpha)} s_j^{\alpha_1} \ldots s_{mj}^{\alpha_m}.$$

The function $\{\lambda\}_{(s)} \otimes \{\lambda'\}_{(s)}$ is, by definition, $\sum_{(\beta)} c'_{(\beta)} S_1^{\beta_1} \ldots S_{m'}^{\beta_{m'}}$, $\{\lambda'\}_{(s)}$ being $\sum_{(\beta)} c'_{(\beta)} s_1^{\beta_1} \ldots s_{m'}^{\beta_{m'}}$, where the summation is over all sets $(\beta) = (\beta_1, \ldots, \beta_{m'})$ of m' non-negative integers which are such that $\beta_1 + 2\beta_2 + \ldots + m'\beta_{m'} = m'$, (λ') being a partition of m'. If j is odd, $(S*)_j = \sum_{(\alpha)} (-1)^{\alpha_2+\alpha_4+\ldots} c_{(\alpha)} \times$

$$s_j^{\alpha_1} \ldots s_{mj}^{\alpha_m} =$$

$= (S_j)^*$ while, if j is even, $(S_j)^* = \sum_{(\alpha)} (-1)^{\alpha_1+\alpha_2+\cdots}$ x $c_{(\alpha)} s_j^{\alpha_1} \cdots s_{mj}^{\alpha_m}$ and, since $m = \alpha_1 + 2\alpha_2 + \cdots$, this is $(-1)^m \sum_{(\alpha)} (-1)^{\alpha_2+\alpha_4+\cdots} c_{(\alpha)} s_j^{\alpha_1} \cdots s_{mj}^{\alpha_m} = (-1)^m (S^*)_j$. If, then, m is even, $(S^*)_j = (S_j)^*$ for all values of j so that $\{\lambda\}_{(s^*)} \otimes \{\lambda'\}_{(s)}$ is obtained from $\{\lambda\}_{(s)} \otimes \{\lambda'\}_{(s)}$ by replacing (s) by (s^*); in other words, $\{\lambda^*\} \otimes \{\lambda'\}$ is the associate of $\{\lambda\} \otimes \{\lambda'\}$. On the other hand, if m is odd, $(S^*)_j = (S_j)^*$ if j is odd, while $(S^*)_j = -(S_j)^*$ if j is even. Thus

$$\{\lambda^*\} \otimes \{\lambda'\} = \sum_{(\beta)} (-1)^{\beta_2+\beta_4+\cdots} c'_{(\beta)} ((S_1)^*)^{\beta_1} \cdots ((S_{m'})^*)^{\beta_{m'}}$$

which is the associate of $\{\lambda\} \otimes \{\lambda'^*\}$. We have, then, the following rule: If m is even, $\{\lambda^*\} \otimes \{\lambda'\}$ is the associate of $\{\lambda\} \otimes \{\lambda'\}$ and, if m is odd, $\{\lambda^*\} \otimes \{\lambda'\}$ is the associate of $\{\lambda\} \otimes \{\lambda'^*\}$. For example, $\{2\} \otimes \{m'\}$ is the associate of $\{1^2\} \otimes \{m'\}$ while $\{3\} \otimes \{m'\}$ is the associate of $\{1^3\} \otimes \{1^{m'}\}$.

2. The analysis of the representations $\{1^2\} \otimes \{1^m\}$ and $\{1^2\} \otimes \{m\}$ of the n-dimensional unitary group.

The characteristic numbers of a typical element Z' of the anti-symmetrized square $\Gamma_{(1^2)}$ of the self-representation of the n-dimensional unitary group are $z_1 z_2, \ldots, z_{n-1} z_n$, where $z_1, \ldots, z_n$ are the characteristic numbers of a typical element Z of this group, and the elementary symmetric functions of these characteristic numbers are the characters of the various representations $\{1^2\} \otimes \{1^m\}$, $m = 0, 1, 2, \ldots$, of the n-dimensional unitary group. On separating z_1 from the remaining $n-1$ characteristic numbers $z_2, \ldots, z_n$ of Z we see that $1 - \{1^2\}t + (\{1^2\} \otimes \{1^2\})t^2 - \ldots$ is the product of $1 - \{1\}'z_1 t + \{1^2\}'z_1^2 t^2 - \ldots$ and $1 - \{1^2\}'t + (\{1^2\} \otimes \{1^2\})'t^2 - \ldots$, where the prime attached to any $\{\ldots\}$ indicates that it is a function of $(z') = (z_2, \ldots, z_n)$ rather than of $(z) = (z_1, z_2, \ldots, z_n)$. Thus $\{1^2\} = \{1\}'z_1 + \{1^2\}'$; $\{1^2\} \otimes \{1^2\} = \{1^2\}'z_1^2 + \{1\}'\{1^2\}'z_1 + (\{1^2\} \otimes \{1^2\})'$ and so on. Hence the first term of $\{1^2\} \otimes \{1^2\}$ is $\{21^2\}$ and, generally, the first term of $\{1^2\} \otimes \{1^m\}$ is $\{m1^m\}$. Since $\{(21^2)/(1)\} = \{1\}\{1^2\}$ there are no other terms in $\{1^2\} \otimes \{1^2\}$ which is, therefore, $\{21^2\}$. Since $\{(31^3)/(2)\} = \{1\}\{1^3\} = \{21^2\} + \{1^4\}$ and since $\{1^2\} \otimes \{1^3\} = \{1^3\}'z_1^3 + \{1^2\}'\{1^2\}'z_1^2 + \{1\}'(\{1^2\} \otimes \{1^2\})'z_1 + (\{1^2\} \otimes \{1^3\})' = \{1^3\}'z_1^3 + (\{2^2\}' + \{21^2\}' + \{1^4\}')z_1^2 + (\{31^2\}' + \{2^21\}' + (21^3)')z_1 + (\{1^2\} \otimes \{1^3\})'$ we have $\{1^2\} \otimes \{1^3\} = \{31^3\} + \{2^3\}$. Similarly $\{1^2\} \otimes \{1^4\} = \{41^4\} + \{32^21\}$, $\{1^2\} \otimes \{1^5\} = \{51^5\} + \{42^21^2\} + \{3^22^2\}$. The partitions which appear in $\{1^2\} \otimes \{1^m\}$ may be conveniently described as follows. In the dot diagram of any partition let s be the number of rows for which $\lambda_j \geqslant j$ and write $a_j = \lambda_j - j$, $j = 1, \ldots, s$, so that $a_1 > a_2 > \ldots > a_s \geqslant 0$. The number s for (λ^*) is the same as the number s for (λ) and we write $b_j = \lambda_j^* - j$, $j = 1, \ldots, s$; then (λ) is characterised by the $2s$ numbers

a_j, b_j and we write $(\lambda) = \begin{pmatrix} a_1 & \dots & a_s \\ b_1 & \dots & b_s \end{pmatrix}$. This being understood the partitions of 2m which appear in $\{1^2\} \otimes \{1^m\}$ are those for which $b_j = a_j + 1$, $j = 1, \dots, s$. Since $2m = \sum_{j=1}^{s}(a_j+b_j+1) = 2\sum_{j=1}^{s}(a_j+1)$, $\sum_{j=1}^{s} a_j = m-s$. For example, $\{1^2\} \otimes \{1^6\} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 5 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \{61^6\} + \{52^21^3\} + \{3^4\}$.

Since $\{2\} \otimes \{1^m\}$ is the associate of $\{1^2\} \otimes \{1^m\}$ and since $(\lambda^*) = \begin{pmatrix} b_1 & \dots & b_s \\ a_1 & \dots & a_s \end{pmatrix}$ if $(\lambda) = \begin{pmatrix} a_1 & \dots & a_s \\ b_1 & \dots & b_s \end{pmatrix}$ the partitions which appear in $\{2\} \otimes \{1^m\}$ are of the form $\begin{pmatrix} b_1 & \dots & b_s \\ a_1 & \dots & a_s \end{pmatrix}$, where $b_j = a_j + 1$, $j = 1, \dots, s$. For example, $\{2\} \otimes \{1^2\} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \{31\}$; $\{2\} \otimes \{1^3\} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \{41^2\} + \{3^2\}$;

$\{2\} \otimes \{1^4\} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} = \{51^3\} + \{431\}$

$\{2\} \otimes \{1^5\} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \{61^4\} + \{4321\} + \{4^22\}$

In the same way we see that $1 + \{1^2\}t + (\{1^2\} \otimes \{2\})t^2 + \dots$ is the product of $1 + \{1\}'z_1t + \{2\}'z_1^2t^2 + \dots$ and $\{1\}+\{1^2\}'t + (\{1^2\} \otimes \{2\})'t^2 + \dots$ so that $\{1^2\} \otimes \{2\}$, for example, $= \{2\}'z_1^2 + \{1\}'\{1^2\}'z_1 + (\{1^2\} \otimes \{2\})'$. Hence $\{1^2\} \otimes \{2\}$ starts off with $\{2^2\}$ and, since $\{(2^2) / (1)\} = h_1h_2 - h_3 = \{21\}$, the remaining term of $\{1^2\} \otimes \{2\}$ is $\{1^4\}$ so that $\{1^2\} \otimes \{2\} = \{2^2\} + \{1^4\}$. Similarly, $\{1^2\} \otimes \{3\} = \{3^2\} + \{2^21^2\} + \{1^6\}$, $\{1^2\} \otimes \{4\} = \{4^2\} + \{3^21^2\} + \{2^4\} + \{2^21^4\} + \{1^6\}$. In general, the partitions of 2m which appear in $\{1^2\} \otimes \{m\}$ are obtained by taking all the partitions of m and repeating each part of each such partition. For example, since the partitions of 5 are (5), (41), (32), (31^2), $(2^2 1)$, (21^3), (1^5), $\{1^2\} \otimes \{5\} = \{5^2\} + \{4^2 1^2\} + \{3^2 2^2\} + \{3^2 1^4\} + \{2^4 1^2\} + \{2^2 1^6\} + \{1^{10}\}$. It follows that the partitions of 2m which appear in $\{2\} \otimes \{m\}$ are obtained by taking all the partitions of m and doubling each part of each such partition. For example,

$$\{2\} \otimes \{2\} = \{4\} + \{2^2\} \ ; \ \{2\} \otimes \{3\} = \{6\} + \{42\} + \{2^3\}$$
$$\{2\} \otimes \{4\} = \{8\} + \{62\} + \{4^2\} + \{42^2\} + \{2^4\}$$
$$\{2\} \otimes \{5\} = \{10\} + \{82\} + \{64\} + \{62^2\} + \{4^2 2\} + \{42^3\} + \{2^5\}$$

and so on.

3. The relation between the characters $\langle \lambda \rangle$ of irreducible representations of the 2k-dimensional U-symplectic group and the characters $[\lambda]$ of irreducible representations of the 2k-dimensional orthogonal group.

The characters $\langle \lambda \rangle$ and $[\lambda]$ of irreducible representations of the 2k-dimensional symplectic and orthogonal groups, respectively, are furnished by the formulas

$$\langle \lambda \rangle = \prod_{\substack{p<q \\ 1}}^{k} (1 - \xi_p \xi_q) \Delta(\xi)\, h_{\ell_1} \cdots h_{\ell_k} = \prod_{\substack{p<q \\ 1}}^{k} (1 - \xi_p \xi_q) \{\lambda\}$$

$$[\lambda] = \prod_{\substack{p \leq q \\ 1}}^{k} (1 - \xi_p \xi_q) \Delta(\xi)\, h_{\ell_1} \cdots h_{\ell_k} = \prod_{\substack{p \leq q \\ 1}}^{k} (1 - \xi_p \xi_q) \{\lambda\}$$

and, since $\prod_{\substack{p<q\\1}}^{k}(1-\xi_p\xi_q) = 1 - \{1^2\}(\xi) + (\{1^2\} \otimes \{1^2\})(\xi) - \ldots$

$$= 1 - \{1^2\}(\xi) + \{21^2\}(\xi) - \ldots$$

$$\prod_{\substack{p\leq q\\1}}^{k}(1-\xi_p\xi_q) = 1 - \{2\}\xi + (\{2\} \otimes \{1^2\})\xi - \ldots$$

$$= 1 - \{2\}(\xi) + \{31\}(\xi) - \ldots$$

we have $\langle\lambda\rangle = \{\lambda\} - \{(\lambda)/(1^2)\} + \{(\lambda)/(21^2)\} - \ldots$

$$[\lambda] = \{\lambda\} - \{(\lambda)/(2)\} + \{(\lambda)/(31)\} - \ldots$$

Thus $\langle\lambda\rangle^*$, the associate of $\langle\lambda\rangle$, $= \{\lambda^*\} - \{(\lambda)^*/(2)\} + \{(\lambda)^*/(31)\} -$

which is $[\lambda^*]$. In other words, $[\lambda]$ is the associate of $\langle\lambda^*\rangle$ and $\langle\lambda\rangle$ is the associate of $[\lambda^*]$ where, by the associate of $\langle\lambda^*\rangle$ we mean $\sum_{(\mu)} c_{(\mu)}\{\mu^*\}$ if $\langle\lambda^*\rangle = \sum_{(\mu)} c_{\mu}\{\mu\}$ and by the associate of $[\lambda^*]$ we mean $\sum_{(\mu)} c_{\mu}\{\mu^*\}$ if $[\lambda^*] = \sum_{(\mu)} c_{(\mu)}\{\mu\}$.

The relations

$$\langle\lambda\rangle = \prod_{\substack{p<q\\1}}^{k}(1-\xi_p\xi_q)\{\lambda\}; \quad [\lambda] = \prod_{\substack{p\leq q\\1}}^{k}(1-\xi_p\xi_q)\{\lambda\}$$

yield

$$\{\lambda\} = \{1 + \{1^2\}(\xi) + (\{1^2\} \otimes \{2\})(\xi) + \ldots\}\langle\lambda\rangle$$
$$= \langle\lambda\rangle + \langle\{(\lambda)/(1^2)\}\rangle + \langle\{(\lambda)/(2^2)+(1^4)\}\rangle + \ldots$$

$$\{\lambda\} = \{1 + \{2\}(\xi) + (\{2\} \otimes \{2\})(\xi) + \ldots\}[\lambda]$$
$$= [\lambda] + [\{(\lambda)/(2)\}] + [\{(\lambda)/(4)+(2^2)\}] + \ldots$$

where by $[\{(\lambda)/(1^2)\}]$, for example, we understand the result of replacing each of the parentheses $\{\ldots\}$ which occur in $\{(\lambda)/(1^2)\}$ by the corresponding parenthesis $[\ldots]$. These relations furnish the analyses of the representations of the 2k-dimensional U-symplectic and orthogonal groups, respectively, which are induced by the irreducible representations $\Gamma_{(\lambda)}$ of the 2k-dimensional unitary group, where (λ) is a partition, involving not more than k non-zero parts, of any non-negative integer m. The second of the two formulas also furnishes the analysis of the representation of the (2k+1)-dimensional orthogonal group which is induced by the irreducible representation $\Gamma_{(\lambda)}$ of the (2k+1)-dimensional unitary group.

4. <u>The analysis of the representations $\{m\} \otimes \{2\}$ and $\{m\} \otimes \{1^2\}$ of the n-dimensional unitary group.</u>

If we change the sign of t in the relation

$$\{(1-z_1t)\ldots(1-z_nt)\}^{-1} = 1 + \{1\}t + \{2\}t^2 + \ldots$$

we obtain

$$\{(1+z_1t)\ldots(1+z_nt)\}^{-1} = 1 - \{1\}t + \{2\}t^2 - \ldots$$

so that

$$\{(1-z_1^2t^2)\ldots(1-z_n^2t^2)\}^{-1} = 1 + (2\{2\} - \{1\}^2)t^2 + \\ + (2\{4\} - 2\{3\}\{1\} + \{2\}^2)t^4 +$$

and this implies that the value of $\{m\}$ at z^2 is

$$2\{2m\} - 2\{2m-1\}\{1\} + \ldots + (-1)^{m-1}2\{m+1\}\{m-1\} + (-1)^m\{m\}^2 .$$

If we denote $\{m\}$ by S_1 the value of $\{m\}$ at Z^2 is S_2 and, since $\{m\} \otimes \{2\} = \frac{1}{2}(S_1{}^2 + S_2)$ we have $\{m\} \otimes \{2\} =$ $\{2m\} - \{2m-1\}\{1\} + \dots$, the last term being $\{m+1\}\{m-1\}$, if m is odd, and $\{m\}^2$ if m is even. Since $\{2m-2\}\{2\} -$ $- \{2m-1\}\{1\} = \{2m-2, 2\}$ and so on, we have

$$\{m\} \otimes \{2\} = \{2m\} + \{2m-2, 2\} + \dots$$

the last term being $\{m+1, m-1\}$, if m is odd, and $\{m, m\}$, if m is even.

Similarly, $\{m\} \otimes \{1^2\} = \frac{1}{2}(S_1{}^2 - S_2) = -\{2m\} + \{2m-1, 1\} - \dots$

$$= \{2m-1, 1\} + \{2m-3, 3\} + \dots$$

the last term being $\{m, m\}$, if m is odd, and $\{m+1, m-1\}$ if m is even.

Lecture 20

1. Hermite's Law of Reciprocity.

If $Z \to Z'$ is the irreducible representation $\Gamma_{(m)}$ of the 2-dimensional unimodular unitary group the characteristic numbers of Z' are $z_1^m, z_1^{m-1}z_2, \ldots, z_2^m$ where z_1 and $z_2 = 1/z_1$ are the characteristic numbers of Z. Since $h_m = z_1^m + z_2 h_{m-1}$ the product of $\{(1 - z_1^m t) \ldots (1 - z_2^m t)\}^{-1} = 1 + \{m\}t +$
$+ (\{m\} \otimes \{2\})t^2 + \ldots$ by $1 - z_1^m t$ is $1 + \{m-1\} z_2 t +$
$+ (\{m-1\} \otimes \{2\}) z_2^2 t^2 + \ldots$ so that

$$(\{m-1\} \times \{j\})\, z_2^j = (\{m\} \otimes \{j\}) - (\{m\} \otimes \{j-1\})\, z_1^m, \qquad j = 1, 2, \ldots$$

Upon interchanging z_1 and z_2, subtracting, and dividing by $z_1 - z_2$ we obtain

$$(\{m-1\} \otimes \{j\})\{j-1\} = (\{m\} \otimes \{j-1\})\{m-1\}$$

Setting $j=m$, we obtain $\{m-1\} \otimes \{m\} = \{m\} \otimes \{m-1\}$. Setting $j = m+1$ we obtain $(\{m-1\} \otimes \{m+1\})\{m\} = (\{m\} \otimes \{m\})\{m-1\} =$
$= (\{m+1\} \otimes \{m-1\})\{m\}$ so that $\{m-1\} \otimes \{m+1\} = \{m+1\} \otimes \{m-1\}$. Continuing in this way we see that, if m and m' are any two positive integers, $\{m\} \otimes \{m'\} = \{m'\} \otimes \{m\}$. This is known as Hermite's Law of Reciprocity for the 2-dimensional unimodular unitary group.

Since, for the n-dimensional unitary group,

$$\{m\} \otimes \{2\} = \{2m\} + \{2m-2, 2\} + \ldots$$
$$\{m\} \otimes \{1^2\} = \{2m-1, 1\} + \{2m-3, 3\} + \ldots$$

and since, for the 2-dimensional unimodular unitary group,

$\{\lambda_1, \lambda_2\} = \{\lambda_1 - \lambda_2\}$ we have, for this latter group,

$\{m\} \otimes \{2\} = \{2m\} + \{2m-4\} + \ldots = \{2\} \otimes \{m\}$

$\{m\} \otimes \{1^2\} = \{2m-2\} + \{2m-6\} + \ldots$

2. The analysis of $\{m\} \otimes \{\lambda\}$ for the 2-dimensional unimodular unitary group.

The function $\{m\} \otimes \{1^j\}$; $j = 1, \ldots, m+1$, are the elementary symmetric functions $\sum_1, \ldots, \sum_{m+1}$ of the $m+1$ characteristic numbers z_1^m, $z_1^{m-1}z_2, \ldots, z_2^m$ of a typical matrix Z' of the representation $\Gamma_{(m)}$ of the 2-dimensional unimodular unitary group and, since the reciprocal of each of these $m+1$ characteristic numbers is one of the set of $m+1$ characteristic numbers, $\sum_j = \sum_{m+1-j}$, $j = 1, \ldots, m+1$. The function $\{m\} \otimes \{\lambda^*\}$ is the determinant of the k-dimensional matrix whose p^{th} row-matrix is $(\sum_{\ell_p-k+1}, \ldots, \sum_{\ell_p})$, k being the number of non-zero parts in (λ) and ℓ_p being λ_p+k-p , $p = 1, \ldots, k$. On denoting by λ'_{k-p+1} the complement $m+1-\lambda_p$ of λ_p in $m+1$, the k-dimensional matrix obtained by inverting the order of the rows and columns of the matrix whose determinant is $\{m\} \otimes \{\lambda^*\}$ has $\sum_{\lambda_k} = \sum_{\lambda'_1}$, $\sum_{\lambda_{k-1}} = \sum_{\lambda'_2}$, and so on, as its diagonal elements and its determinant is $\{m\} \otimes \{\lambda'^*\}$. For example, when $(\lambda^*) = (j^m)$, $(\lambda) = (m^j)$, $(\lambda') = (1^j)$, $(\lambda'^*) = j$ and we have the result

$$\{m\} \otimes \{j^m\} = \{m\} \otimes \{j\}$$

When $m = 2$, we may take $k \leq 2$ since, for the $m+1 = 3$-dimensional unitary group, $\{\lambda_1, \lambda_2, \lambda_3\} = \{\lambda_1 - \lambda_3, \lambda_2 - \lambda_3\}$. Writing $(\lambda^*) = (p,q)$ we have $(\lambda) = (2^{q}1^{p-q})$, $(\lambda') = (2^{p-q}1^{q})$, $(\lambda'^*) = (p, p-q)$ so that

$$\{2\} \otimes \{p,q\} = \{2\} \otimes \{p, p-q\}$$

Similarly $\{3\} \otimes \{p, q, r\} = \{3\} \otimes \{p, p-r, p-q\}$

$$\{4\} \otimes \{p, q, r, s\} = \{4\} \otimes \{p, p-s, p-r, p-q\}$$

and, generally,

$$\{m\} \otimes \{\lambda_1, \ldots, \lambda_m\} = \{m\} \otimes \{\lambda_1, \lambda_1 - \lambda_m, \ldots, \lambda_1 - \lambda_2\}.$$

It is easy to develop recurrence formulas which enable us to analyse $\{m\} \otimes \{p\}$. Separating z_1^{m} from the remaining m characteristic numbers $z_1^{m-1}z_2, \ldots, z_2^{m}$ of a typical matrix Z' of $\Gamma_{(m)}$ we see that $1 + \{m-1\}z_2 t + (\{m-1\} \otimes \{2\})z_2^2 t^2 + \ldots$ is the product of $1 + \{m\}t + (\{m\} \otimes \{2\})t^2 + \ldots$ and $1 - z_1^{m}t$ and so

$$(\{m-1\} \otimes \{p\})z_2^{p} = \{m\} \otimes \{p\} - (\{m\} \otimes \{p-1\})z_1^{m}, \quad p=1,2,\ldots$$

or, equivalently,

$$\{m\} \otimes \{p\} = (\{m-1\} \otimes \{p\})z_2^{p} + (\{m\} \otimes \{p-1\})z_1^{m}.$$

Upon interchanging z_1 and z_2 and decreasing p by 1 we obtain

$$\{m\} \otimes \{p-1\} = (\{m-1\} \otimes \{p-1\})z_1^{p-1} + (\{m\} \otimes \{p-2\})z_2^{m}, \quad p = 2,3, \ldots$$

so that

$$\{m\} \otimes \{p\} = (\{m-1\} \otimes \{p\})z_2^{p} + (\{m-1\} \otimes \{p-1\})z_1^{m+p-1} + \{m\} \otimes \{p-2\}$$

since $z_1 z_2 = 1$. Upon interchanging z_1 and z_2, and m and p, in this equation we obtain, making use of Hermite's Law of Reciprocity, the relation

$$\{m\} \otimes \{p\} = (\{m\} \otimes \{p-1\})z_1^{m} + (\{m-1\} \otimes \{p-1\})z_2^{m+p-1} + \\ + \{m-2\} \times \{p\}$$

which yields

$$(\{m-1\} \otimes \{p\})z_2^{p} = (\{m-1\} \otimes \{p-1\})z_2^{m+p-1} + \{m-2\} \otimes \{p\} .$$

Hence

$$\{m\} \otimes \{p\} = \{m\} \otimes \{p-2\} + (\{m-1\} \otimes \{p-1\})(z_1^{m+p-1} + z_2^{m+p-1}) \\ + (\{m-2\} \otimes \{p\})$$

and, since $\{m+p-1\} = z_1^{m+p-1} + z_1 z_2 \{m+p-3\} + z_2^{m+p-1} =$

$$= z_1^{m+p-1} + \{m+p-3\} + z_2^{m+p-1} ,$$

we obtain the recurrence relation

$$\{m\} \otimes \{p\} = \{m\} \otimes \{p-2\} + (\{m-1\} \otimes \{p-1\})(\{m+p-1\} - \\ - (m+p-3)) + (\{m-2\} \otimes \{p\}) .$$

For example

$$\{3\} \otimes \{3\} = 2\{3\} + (\{4\} + \{0\})(\{5\} - \{3\}) = \\ = \{9\} + \{5\} + \{3\}$$

$\{4\}\{5\}$, for example, being $\{9\} + \{81\} + \{72\} + \{63\} + \{54\}$ $= \{9\} + \{7\} + \{5\} + \{3\} + \{1\}$ so that $\{4\}(\{5\} - \{3\}) = \{9\}$. Similarly,

$$\{3\} \otimes \{4\} = \{12\} + \{8\} + \{6\} + \{4\} + \{0\}$$

$$\{4\} \times \{4\} = \{16\} + \{12\} + \{10\} + 2\{8\} + \{0\}.$$

In order to obtain a formula for $\{m\} \otimes \{p-1, 1\}$ we proceed as follows. Upon multiplying the relation

$$\{m\} \otimes \{p\} = (\{m-1\} \otimes \{p\})z_2^{p} + (\{m\} \otimes \{p-1\})z_1^{m}$$

by z_1 we obtain, since $z_1 z_2 = 1$,

$$(\{m\} \otimes \{p\})z_1 = (\{m-1\} \otimes \{p\})z_2^{p-1} + (\{m\} \otimes \{p-1\})z_1^{m+1}$$

Upon interchanging z_1 and z_2 , subtracting and dividing by z_1-z_2 , we find that

$$\{m\} \otimes \{p\} = (\{m\} \otimes \{p-1\})\{m\} - (\{m-1\} \otimes \{p\})\{p-2\}$$

Now the relation $\{p-1\}\{1\} = \{p\} + \{p-1, 1\}$ tells us, since $\{m\} = \{m\} \otimes \{1\}$, that

$$(\{m\} \otimes \{p-1\})\{m\} = (\{m\} \otimes \{p\}) + \{m\} \otimes \{p-1,1\}$$

and so

$$\begin{aligned}\{m\} \otimes \{p-1,1\} &= (\{m\} \otimes \{p-1\})\{m\} - (\{m\} \otimes \{p\}) \\ &= (\{m-1\} \otimes \{p\})\{p-2\}\end{aligned}$$

For example, $\{2\} \times \{21\} = \{3\}\{1\} = \{4\} + \{2\}$.

Since $1 - \{m\}t + (\{m\} \otimes \{1^2\})t^2 - \dots$ is the product of $1 - \{m-1\}z_2 t + (\{m-1\} \otimes \{1^2\})z_2^2 t^2 - \dots$ by $1 - z_1^m t$ we have

$$\{m\} \otimes \{1^p\} = (\{m-1\} \otimes \{1^p\})z_2^p + (\{m-1\} \otimes \{1^{p-1}\})z_1^{m-p+1}$$

so that

$$(\{m\} \otimes \{1^p\})z_1 = (\{m-1\} \otimes \{1^p\})z_2^{p-1} + (\{m-1\} \otimes \{1^{p-1}\}) \times z_1^{m-p+2} .$$

Upon interchanging z_1 and z_2 , subtracting and dividing by z_1-z_2 , we obtain

$$\begin{aligned}\{m\} \otimes \{1^p\} = (\{m-1\} \otimes \{1^{p-1}\})\{m-p+1\} - \\ - (\{m-1\} \otimes \{1^p\})\{p-2\} .\end{aligned}$$

This relation enables us to prove, by induction, that

$$\{m\} \otimes \{1^p\} = \{m-p+1\} \otimes \{p\} , \quad p = 1, 2, \dots$$

Indeed, this relation is true for every positive integer p if $m = 1$ since $\{1^2\} = \{0\}$ and since both sides of the relation are zero if $p > 2$. Assuming that the relation is true, for every positive integer p, for any given positive integer $m-1$ we show, as follows, that it is true for m. We have $\{m\} \otimes \{1^p\} =$

$= (\{m-p+1\} \otimes \{p-1\})\{m-p+1\} - (\{m-p\} \otimes \{p\})\{p-2\} =$

$= (\{m-p+1\} \otimes \{p\}) + (\{m-p+1\} \otimes \{p-1, 1\}) -$

$- (\{m-p\} \otimes \{p\})\{p-2\}$. On replacing m by $m-p+1$ in the relation $\{m\} \otimes \{p-1, 1\} = (\{m-1\} \otimes \{p\})\{p-2\}$ we see that $\{m-p+1\} \otimes \{p-1, 1\} = (\{m-p\} \otimes \{p\})\{p-2\}$ so that

$\{m\} \otimes \{1^p\} = \{m-p+1\} \otimes \{p\}$.

Thus, in particular, $\{m\} \otimes \{1^m\} = \{m\}$, $\{m\} \otimes \{1^{m+1}\} = \{0\}$, and $\{m\} \otimes \{1^p\} = 0$ if $p > m+1$.

Since $(\{m\} \otimes \{1^p\})\{m\} = (\{m\} \otimes \{21^{p-1}\}) +$

$+ (\{m\} \otimes \{1^{p+1}\})$ we have

$\{m\} \otimes \{21^{p-1}\} = (\{m-p+1\} \otimes \{p\})\{m\} - (\{m-p\} \otimes \{p+1\})$

and, in particular,

$\{m\} \otimes \{21^{m-1}\} = \{m\}\{m\} - \{0\} = \{2m\} + \{2m-2\} + \dots + \{2\}$

When $m = 2$ we may suppose, in evaluating $\{2\} \otimes \{\lambda\}$ that $(\lambda) = (\lambda_1, \lambda_2)$ does not involve more than two non-zero parts. Since $\{2\} \otimes \{\lambda_1, \lambda_2\} = \{2\} \otimes \{\lambda_1, \lambda_1 - \lambda_2\}$ we may limit our attention to the cases where $2\lambda_2 \leq \lambda_1$. We readily derive the following formulas

$\{2\} \otimes \{2p+1, 2q\} = (\{4p-2q+2\} + \{4p-2q-2\} + \dots + \{2q+2\})\{2q\}$

$$\{2\} \otimes \{2p+1,\ 2q+1\} = (\{4p-2q+1\} + \{4p-2q-3\} + \\ + \dots + \{2q+1\})\{2q+1\} - \\ - \{4q\} - \{4q-4\} - \dots - \{0\}$$

$$\{2\} \otimes \{2p,\ 2q+1\} = (\{4p-2q-1\} + \{4p-2q-5\} + \dots + \\ + \{2q+3\})\{2q+1\}$$

$$\{2\} \otimes \{2p,2q\} = (\{4p-2q\} + \{4p-2q-4\} + \dots + \{2q\})\{2q\} - \\ - \{4q-2\} - \{4q-6\} - \dots - \{2\}\ .$$

For example $\{2\} \otimes \{52\} = (\{8\} + \{4\})\{2\} = \{10\} + \{8\} + + 2\{6\} + \{4\} + \{2\}$

$$\{2\} \otimes \{51\} = (\{9\} + \{5\} + \{1\})\{1\} - \{0\} = \\ = \{10\} + \{8\} + \{6\} + \{4\} + \{2\}$$

$$\{2\} \otimes \{63\} = (\{9\} + \{5\})\{3\} = \{12\} + \{10\} + \\ + 2\{8\} + 2\{6\} + \{4\} + \{2\}$$

$$\{2\} \otimes \{62\} = (\{10\} + \{6\} + \{2\})\{2\} - \{2\} = \\ = \{12\} + \{10\} + 2\{8\} + \{6\} + 2\{4\} + \\ + \{0\}\ .$$