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Invariant Measures for

One-Dimensional Anderson Localisation

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Dedicated to Professor Leonid Pastur's 65-th birthday

Abstract. We compute the invariant measures for the Anderson model on two coupled chains. These measures live on a three-dimensional projective space, and we use a total set of functions on this space to characterise the measures. It turns out that there is a similar anomaly as first found by Kappus and Wegner for the single chain, but that, in addition, the measures take a different form on different regions of the spectrum.

1. Introduction: The Single Chain

The Hamiltonian is given by $H = H_0 + \eta V$, where

$$(H_0\psi)(n) = \psi(n+1) + \psi(n-1) \tag{1.1}$$

and

$$(V\psi)(n) = v_n\psi(n),$$

where the v_n are i.i.d. random variables. This is the well-known Anderson Hamiltonian. A great deal is known about it. In particular, it was proved by Goldsheid, Molchanov and Pastur [1] that the spectrum is entirely pure-point and all corresponding eigenfunctions are exponentially localised. Here, we will consider a different question, concerning the invariant measure for the corresponding eigenvalue equation. The eigenvalue equation is

$$\psi(n+1) + \psi(n-1) + \eta v_n\psi(n) = E\psi(n). \tag{1.2}$$

In terms of the variable $Z(n) = \psi(n)/\psi(n-1)$ this can be written as

$$Z(n+1) = E - \eta v_n - \frac{1}{Z(n)}. \tag{1.3}$$

The invariant measure $\nu_{\eta,E}$ for this transformation is defined by

$$\int f(x)\nu_{\eta,E}(dx) = \mathbb{E}\int f(E - \eta v - \frac{1}{x})\nu_{\eta,E}(dx). \tag{1.4}$$

It follows from Fürstenberg's theorem that this measure is unique for all $\eta > 0$. In 1971, Thouless [2] attempted to write down a perturbation series for $\nu_{\eta,E}$ about $\eta = 0$. However, Kappus and Wegner [3] discovered that this series is incorrect for the casse $E = 0$. They called this an *anomaly.* In fact, the limiting measure $\nu_{0,E}$ is dicontinuous at $E = 0$. The problem was further analysed by Derrida and Gardner [4]. They found that the perturbation series is also anomalous at the values $E = 2\cos\frac{p}{q}\pi$ for integer p and q. Bovier and Klein [5] then completed their investigation and derived the correct perturbation series in all cases. These series were subsequently shown to be asymptotic by Campanino and Klein [6] by means of a very sophisticated analysis.

Here we concentrate on the more restricted problem of showing the convergence of the measures as $\eta \to 0$:

$$\lim_{\eta \downarrow 0} \nu_{\eta,E} = \begin{cases} \dfrac{c}{x^2 - Ex + 1} dx & \text{if } E \neq 0; \\ \dfrac{c_0}{\sqrt{x^4+1}} dx & \text{if } E = 0. \end{cases} \tag{1.5}$$

We prove in a much simpler fashion than [6] that these limits hold in the sense of weak convergence of measures. Obviously, this is a much weaker result than that of [6]. (The convergence was not proved in [5].) We next generalise our approach to the case of two coupled chains. It turns out that this case is considerably more complicated. In particular, the limiting measure has a different appearance on different regions of the unperturbed spectrum.

2. Two Linked Chains

We consider two linked chains as in Figure 1. The corresponding Hamiltonian is again of the general form $H = H_0 + \eta V$ and H_0 and V are the natural generalisations of (1.1) and (1.2):

$$(H_0\psi)(n,s) = \psi(n+1,s) + \psi(n-1,s) + \psi(n,s\pm 1) \tag{2.1}$$

and

$$(V\psi)(n,s) = v_{n,s}\psi(n,s) \tag{2.2}$$

where $s = 1,2$ and the $v_{n,s}$ are i.i.d. random variables.

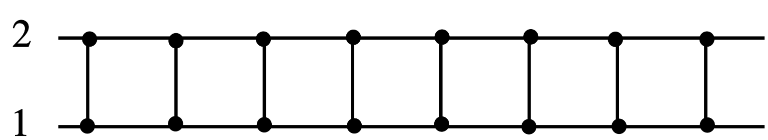

Figure 1. *Two linked chains.*

The unperturbed ($\eta = 0$) spectrum has two branches:

$$E(k) = 2\cos k \pm 1; \qquad k \in [-\pi, \pi]. \tag{2.3}$$

The dispersion relations (2.3) have been plotted in Figure 2.

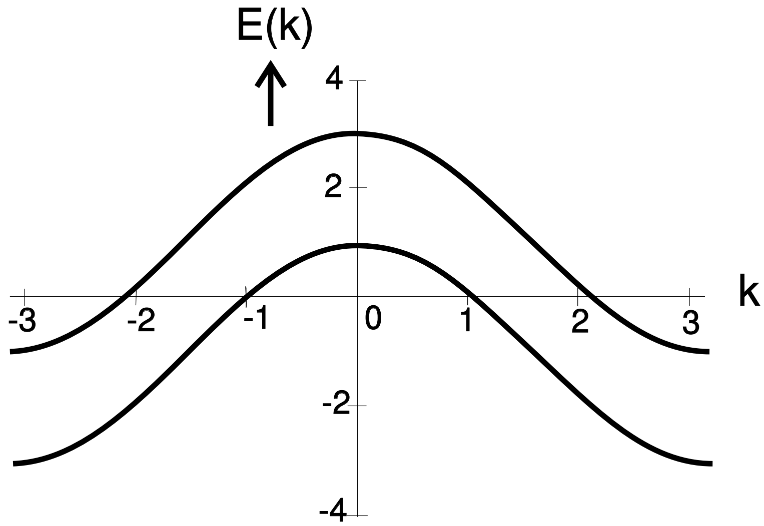

Figure 2. *The dispersion relation for two linked chains.*

We can write the Schrödinger equation for this case in transfer matrix form as follows:

$$\begin{pmatrix} \psi(n+1,1) \\ \psi(n+1,2) \\ \psi(n,1) \\ \psi(n,2) \end{pmatrix} = \begin{pmatrix} E-\eta v_{n,1} & -1 & -1 & 0 \\ -1 & E-\eta v_{n,2} & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi(n,1) \\ \psi(n,2) \\ \psi(n-1,1) \\ \psi(n-1,2) \end{pmatrix}. \quad (2.4)$$

This can be written more concisely as

$$\begin{pmatrix} \vec{\psi}(n+1) \\ \vec{\psi}(n) \end{pmatrix} = A_\eta \begin{pmatrix} \vec{\psi}(n) \\ \vec{\psi}(n-1) \end{pmatrix}, \quad (2.5)$$

with

$$A_\eta = \begin{pmatrix} C+\eta X & -I_2 \\ I_2 & 0 \end{pmatrix} \quad (2.6)$$

where

$$C = \begin{pmatrix} E & -1 \\ -1 & E \end{pmatrix} \text{ and } X = \begin{pmatrix} -v_{n,1} & 0 \\ 0 & -v_{n,2} \end{pmatrix}. \quad (2.7)$$

This formulation has the advantage that it generalises to an arbitrary number of lines.

Here we concentrate on the case of two lines, however. As in the case of a single line, the eigenvectors are defined up to a multiplicative constant, so only quotients of the components are relevant. These are points of the projective 3-sphere $\mathbb{RP}^3 = \mathbb{P}(\mathbb{R}^4)$. If we denote the quotient map of the linear map A_η on $\mathbb{RP}^3$ by $\tilde{A}_\eta$ then the equation for the invariant measure $\nu_{\eta,E}$ on $\mathbb{RP}^3$ reads:

$$\int_{\mathbb{RP}^3} f(x)\, \nu_{\eta,E}(dx) = \int_{\mathbb{RP}^3} \mathbb{E}\left[f(\tilde{A}_\eta x) \right] \nu_{\eta,E}(dx). \quad (2.8)$$

To clarify our method we first return to the case of a single chain.

3. The Case of a Single Chain

In this case the quotients $Z(n)$ can be seen as elements of the projective line $\mathbb{RP}^1 = \mathbb{P}(\mathbb{R}^2)$, which is homeomorphic to the circle S^1. We can introduce a parametrisation $t : \mathbb{RP}^1 \to [0, \pi)$ by

$$t(x_1 : x_2) = \cot^{-1}(x_2/x_1). \tag{3.1}$$

Now the matrix A_0 can be transformed to a rotation as follows. If we put $E = 2\cos\alpha$ then

$$A_\eta = \begin{pmatrix} 2\cos\alpha + \eta v & -1 \\ 1 & 0 \end{pmatrix} \tag{3.2}$$

and with

$$S = \begin{pmatrix} \sin\alpha & 0 \\ \cos\alpha & -1 \end{pmatrix} \tag{3.3}$$

we have

$$D_0 = S\, A_0\, S^{-1} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{3.4}$$

We now put

$$D_\eta = S\, A_\eta\, S^{-1} \tag{3.5}$$

and define the image measures $\tilde{\nu}_{\eta,E}$ and $\sigma_{\eta,E}$ by

$$\tilde{\nu}_{\eta,E} = \nu_{\eta,E} \circ \tilde{S}^{-1} \text{ and } \sigma_{\eta,E} = \tilde{\nu}_{\eta,E} \circ t^{-1}. \tag{3.6}$$

The invariance equation for $\sigma_{\eta,E}$ then reads

$$\int_0^\pi g(\theta)\, \sigma_{\eta,E}(d\theta) = \int_0^\pi (\mathcal{T}_\eta g)(\theta)\, \sigma_{\eta,E}(d\theta), \tag{3.7}$$

where the transformation $\mathcal{T}_\eta$ is given by

$$(\mathcal{T}_\eta g)(\theta) = \mathbb{E}\left[g\left(t \circ \tilde{D}_\eta \circ t^{-1}(\theta) \right) \right]. \tag{3.8}$$

The significance of this parametrisation is that $\mathcal{T}_0$ has a very simple form:

$$(\mathcal{T}_0 g)(\theta) = g\left(\theta - \alpha\, (\mathrm{mod}\ \pi)\right). \tag{3.9}$$

In the case of two chains, we shall seek an analogous parametrisation of $\mathbb{RP}^3$.

Clearly, if α is an irrational multiple of π then the ergodicity of the transformation $\mathcal{T}_0$ implies that the limiting measure $\sigma_{0,E} = \lim_{\eta\to 0}\sigma_{\eta,E}$ satisfies

$$\sigma_{0,E}(d\theta) = \frac{d\theta}{\pi}. \tag{3.10}$$

Now suppose that

$$\alpha = \frac{p}{q}\pi; \qquad p, q \in \mathbb{N}, (p,q) = 1. \tag{3.11}$$

Then $\mathcal{T}_0^q = \mathrm{id}$, the identity map. Following Bovier and Klein [5], we therefore iterate (3.7) q times to get

$$\int_0^\pi \left(\left(\mathcal{T}_0^q - \mathrm{id}\right) g\right)(\theta)\, \sigma_{0,E}(d\theta) = 0. \tag{3.12}$$

We then compute the limit

$$\lim_{\eta\downarrow 0} \eta^{-2} \left(\left(\mathcal{T}_\eta^q - \mathrm{id}\right) g\right)(\theta) \tag{3.13}$$

and show that it converges uniformly. Inserting this into (3.12) and taking $g(\theta) = e^{2in\theta}$ with $n \in \mathbb{Z}$ then yields equations for $\sigma_{0,E}$ which determine it uniquely.

To compute the limit (3.13) we write

$$\left(\mathcal{T}_\eta^q g\right)(\theta) = \mathbb{E}\left[g\left(t \circ \widetilde{S\, B_q\, S^{-1}} \circ t^{-1}(\theta)\right)\right], \tag{3.14}$$

where

$$B_q = \prod_{n=1}^{q} A_\eta^{(n)} \tag{3.15}$$

and the index (n) indicates that the random variables v_n in the different factors are i.i.d.

A lengthy but straightforward calculation now shows that

$$\left(B_q\right)_{i,j} = \left(B_{0,q}\right)_{i,j} + \eta^2 \left(B_{2,q}\right)_{i,j} + \mathcal{O}(\eta^3), \tag{3.16}$$

where $\mathbb{E}\left[(B_{2,q})_{i,j}\right] = 0$ and

$$B_{0,q} = (-1)^p \begin{pmatrix} 1-\eta X & \eta Y \\ -\eta Z & 1+\eta X \end{pmatrix}. \tag{3.17}$$

Here the random variables X, Y and Z are given by

$$X = \sum_{n=1}^{q} \tau(\alpha, n-1)\tau(\alpha, n) v_n, \tag{3.18}$$

$$Y = \sum_{n=1}^{q} \tau(\alpha, n)^2 v_n, \tag{3.19}$$

and

$$Z = \sum_{n=1}^{q} \tau(\alpha, n-1)^2 v_n, \tag{3.20}$$

where

$$\tau(\alpha, n) = \frac{\sin \alpha n}{\sin \alpha}. \tag{3.21}$$

The difference between $E = 0$ and $E \neq 0$ manifests itself in the expectations of X^2, etc.:

If $\alpha \neq \pi/2$,

$$\begin{aligned} \mathbb{E}(X^2) = \mathbb{E}(YZ) &= \frac{(3 - 2\sin^2\alpha)q}{8\sin^4\alpha}; \\ \mathbb{E}(Y^2) = \mathbb{E}(Z^2) &= \frac{3q}{8\sin^4\alpha}; \\ \mathbb{E}(XY) = \mathbb{E}(ZX) &= \frac{3q\cos\alpha}{8\sin^4\alpha}. \end{aligned} \tag{3.22}$$

If $\alpha = \pi/2$ then

$$\begin{aligned} &\mathbb{E}(X^2) = \mathbb{E}(XY) = \mathbb{E}(YZ) = \mathbb{E}(XZ) = 0 \\ &\mathbb{E}(Y^2) = \mathbb{E}(Z^2) = 1. \end{aligned} \tag{3.23}$$

We now insert for g the total set of functions $e^{2in\theta}$ with $n \in \mathbb{Z}$. Set $x = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$ and let

$$x' = S\,B_q\,S^{-1}x = \begin{pmatrix} \sin\theta' \\ \cos\theta' \end{pmatrix}. \tag{3.24}$$

Then

$$\tan\theta' = \tan\theta + \eta U + \eta^2 V + \mathcal{O}(\eta^3), \tag{3.25}$$

where U and V are linear combinations of X, Y and Z. Hence,

$$\begin{aligned} e^{2in\theta'} &= \left(\frac{1 + i\tan\theta'}{1 - i\tan\theta'}\right)^n \\ &= e^{2in\theta}\Big\{1 + 2i\eta n U\cos^2\theta \\ &\quad - 2in\theta^2\cos^4\theta\left[U^2(\tan\theta - in) - V\sec^2\theta\right] + \mathcal{O}(\eta^3)\Big\}. \end{aligned} \tag{3.25}$$

Taking expectations we get

$$\lim_{\eta \downarrow 0} \eta^{-2} \mathbb{E}\left[e^{2in\theta'} - e^{2in\theta}\right] = e^{2in\theta}\left(A_1 n + A_{11} n^2\right), \tag{3.26}$$

where, for $\alpha \neq \frac{\pi}{2}$,

$$A_1 = 0 \text{ and } A_{11} = -\frac{3q}{4\sin^2\alpha} \tag{3.27}$$

and for $\alpha = \frac{\pi}{2}$,

$$A_1 = -\frac{1}{2} i \sin 4\theta \text{ and } A_{11} = -\frac{1}{2}(3 + \cos 4\theta). \tag{3.28}$$

In general, we now have for all $n \in \mathbb{Z}$,

$$\int_0^\pi e^{2in\theta}\left(A_1(\theta) + nA_{11}(\theta)\right) \sigma_{0,E}(d\theta) = 0. \tag{3.29}$$

In case $\alpha \neq \frac{\pi}{2}$ this implies $\sigma_{0,E} = \frac{d\theta}{\pi}$.

If $\alpha = \frac{\pi}{2}$ ($E = 0$), integration by parts yields, assuming that $\sigma_{0,E}(d\theta) = \rho_0(\theta_0(\theta)d\theta$ (this assumption can be justified),

$$\begin{aligned}\int_0^\pi e^{2in\theta} A_1(\theta)\rho_0(\theta)d\theta &= -2in \int_0^\pi e^{2in\theta} d\theta \int_0^\theta A_1(\theta')\rho_0(\theta')d\theta' \\ &= -n \int_0^\pi e^{2in\theta} A_{11}(\theta)\rho_0(\theta)d\theta.\end{aligned} \tag{3.30}$$

Since n is arbitrary, this implies, by the fact that the functions $e^{2in\theta}$ are total,

$$A_{11}(\theta)\rho_0(\theta) = 2i \int_0^\theta A_1(\theta')\rho_0(\theta')d\theta' + K, \tag{3.31}$$

for an arbitrary constant K. This equation is easily solved to yield

$$\rho_0(\theta) = \frac{C}{\sqrt{3 + \cos 4\theta}}, \tag{3.32}$$

which, of course, is the same as the right-hand side of the second equation of (1.5).

4. The Case of Two Chains

We first introduce a parametrisation analogous to (3.1). There are many possibilities, but the following seems to be appropriate in this case: $t : \mathbb{RP}^3 \to \Omega = \Omega_{(0,\frac{\pi}{2})} \cup \Omega_0 \cup \Omega_{\pi/2}$, where

$$\begin{aligned}\Omega_{(0,\frac{\pi}{2})} &= [0, 2\pi] \times [0, \pi) \times (0, \frac{\pi}{2}), \\ \Omega_0 &= [0, \pi) \times \{0\}, \\ \Omega_{\pi/2} &= [0, \pi) \times \{\frac{\pi}{2}\}.\end{aligned} \tag{4.1}$$

The map t is essentially given by

$$\begin{cases} x_1 = & \sin\theta_1 \, \sin\theta_3; \\ x_2 = & \cos\theta_1 \, \sin\theta_3; \\ x_3 = & \sin\theta_2 \, \cos\theta_3; \\ x_4 = & \cos\theta_2 \, \cos\theta_3. \end{cases} \tag{4.2}$$

(The subscript of Ω refers to the value of θ_3; in the extremal cases, the projective space reduces to a circle.)

The analogue of the total set of functions $e^{2in\theta}$ is now a family of three subclasses of functions:

$$\begin{aligned} &\left\{e^{i(n_1\theta_1+n_2\theta_2)} \sin 2n_3\theta_3 \cdot 1_{\Omega_{(0,\pi/2)}} \middle| \, n_1, n_2 \in \mathbb{Z}, n_3 \in \mathbb{N}, n_1+n_2 \text{ even}\right\} \\ &\cup \left\{e^{2in_1\theta_1} \sin\theta_3 \cdot 1_{\Omega_{(0,\pi/2)}} + e^{2in_1\theta_1} 1_{\Omega_{\pi/2}} \middle| \, n_1 \in \mathbb{Z}\right\} \\ &\cup \left\{e^{2in_2\theta_2} \cos\theta_3 \cdot 1_{\Omega_{(0,\pi/2)}} + e^{2in_2\theta_2} 1_{\Omega_0} \middle| \, n_2 \in \mathbb{Z}\right\}. \end{aligned} \tag{4.3}$$

We have to consider different parts of the spectrum separately. We start by considering the case $E \in [-1, 1]$, which will have to be subdivided further at a later stage. In this case we can set

$$E = 2\cos\alpha - 1 = 2\cos\beta + 1. \tag{4.4}$$

The matrix

$$S = \begin{pmatrix} 1 & -1 & -\cos\alpha & \cos\alpha \\ 0 & 0 & \sin\alpha & -\sin\alpha \\ -\cos\beta & -\cos\beta & 1 & 1 \\ -\sin\beta & -\sin\beta & 0 & 0 \end{pmatrix} \tag{4.5}$$

then transforms A_0 (given by (2.6)) into two rotations:

$$S\, A_0\, S^{-1} = \begin{pmatrix} R_\alpha & 0 \\ 0 & R_\beta \end{pmatrix} \quad \text{with} \quad R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{4.6}$$

We now define D_η as in (3.5) and consider the image measure $\sigma_{\eta,E}$ which now lives on Ω. The analogue of (3.7) reads

$$\int_\Omega g(\theta_1, \theta_2, \theta_3)\, \sigma_{\eta,E}(d\theta_1, d\theta_2, d\theta_3) = \int_\Omega (\mathcal{T}_\eta g)\, (\theta_1, \theta_2, \theta_3)\, \sigma_{\eta,E}(d\theta_1, d\theta_2, d\theta_3), \tag{4.7}$$

where

$$\left(\mathcal{T}_\eta^q g\right)(\omega) = \mathbb{E}\left[g\left(t \circ S\,\widetilde{B_q}\,S^{-1} \circ t^{-1}(\omega)\right)\right], \tag{4.8}$$

and $\omega = (\theta_1, \theta_2, \theta_3)$. However, in this case, even if α and β are both irrational multiples of π, it is non-trivial to determine $\sigma_{\eta,E}$. In all cases, assuming that the disorder is diagonal as in (2.2), i.e. $v_n = \mathrm{diag}(v_n^{(1)}, v_n^{(2)})$, we can write an analogue of (3.16): (we include factors S and S^{-1} for convenience)

$$S\,B(m)\,S^{-1} = B_0(m) + \eta B_1(m) + \eta^2 B_2(m)(v_1, \ldots, v_m) + \mathcal{O}(\eta^3) \tag{4.9}$$

for general m, where $\mathbb{E}(B_2(m)) = 0$,

$$B_0(m) = \begin{pmatrix} R_{m\alpha} & 0 \\ 0 & R_{m\beta} \end{pmatrix}, \tag{4.10}$$

and

$$B_1(m) = \sum_{n=1}^{m} v_n^- C_n(m) + \sum_{n=1}^{m} v_n^+ D_n(m), \tag{4.11}$$

with $v_n^\pm = \frac{1}{2}(v_n^{(1)} \pm v_n^{(2)})$ and block matrices $C_n(m)$ and $D_n(m)$ given by

$$C_n(m) = -2\begin{pmatrix} 0 & \frac{1}{\sin\alpha}\left(R_{\frac{\pi}{2}-n\beta-(m-n+1)\alpha} - R_{\frac{\pi}{2}-n\beta+(m-n+1)\alpha}\sigma_z\right) \\ \frac{1}{\sin\beta}\left(R_{\frac{\pi}{2}-(n-1)\alpha-(m-n)\beta} - R_{\frac{\pi}{2}-(n-1)\alpha+(m-n)\beta}\sigma_z\right) & 0 \end{pmatrix}, \tag{4.12}$$

where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and

$$D_n(m) = 2\begin{pmatrix} \frac{1}{\sin\alpha}\left(R_{\frac{\pi}{2}-m\alpha} - R_{\frac{\pi}{2}-(2n-2-m)\alpha}\sigma_z\right) & 0 \\ 0 & \frac{1}{\sin\beta}\left(R_{\frac{\pi}{2}-m\beta} - R_{\frac{\pi}{2}-(2n-m)\beta}\sigma_z\right) \end{pmatrix} \tag{4.13}$$

An explicit calculation, which will be published in full elsewhere, yields the analogue of (3.25),

$$\tan\theta_1' = \tan(\theta_1 - m\alpha) + \eta U_1 + \eta^2 V_1 + \mathcal{O}(\eta^3), \tag{4.14}$$

$$\tan\theta_2' = \tan(\theta_2 - m\beta) + \eta U_2 + \eta^2 V_2 + \mathcal{O}(\eta^3), \tag{4.15}$$

and

$$\tan\theta_3' = \tan\theta_3 + \eta U_3 + \eta^2 V_3 + \mathcal{O}(\eta^3). \tag{4.16}$$

The expressions for U_i and V_i are long and complicated. Eventually, we have to take expectations as in (3.26) and these simplify to some extent. The general result is:

$$\begin{aligned}\mathbb{E}\left[e^{i(n_1\theta_1'+n_2\theta_2'+n_3\theta_3')}\right] &= e^{i(n_1\theta_1+n_2\theta_2+n_3\theta_3)}e^{-im(n_1\alpha+n_2\beta)} \\ &\quad\times\big\{1+\eta^2[A_1n_1+A_2n_2+A_3n_3+A_{11}n_1^2+A_{22}n_2^2+A_{33}n_3^2 \\ &\qquad + A_{12}n_1n_2+A_{23}n_2n_3+A_{31}n_3n_1]\big\}+\mathcal{O}(\eta^3),\end{aligned} \tag{4.17}$$

where $A_k = \mathbb{E}(B_k)$ and $A_{kl} = \mathbb{E}(B_{kl})$ and the latter are defined by

$$B_k = i(V_k + V_k\tan^2(\theta - m\delta_k) - U_k^2\tan(\theta_k - m\delta_k))\cos^4(\theta_k - m\delta_k), \tag{4.18}$$

$$B_{kk} = -\frac{1}{2}U_k^2\cos^4(\theta_k - m\delta_k), \tag{4.19}$$

and for $k \neq l$),

$$B_{kl} = -U_kU_l\cos^2(\theta_k - m\delta_k)\cos^2(\theta_l - m\delta_l). \tag{4.20}$$

Here $\delta_1 = \alpha$, $\delta_2 = \beta$ and $\delta_3 = 0$. The explicit expressions for A_k and A_{kl} have to be worked out separately in the individual cases.

Here we consider only the case where α/π and β/π are irrational.

Ergodicity then implies that $\sigma_{0,E}$ must still be Lebesgue measure in the two variables θ_1 and θ_2, i.e. $\sigma_{0,E}(d\theta_1, d\theta_2, d\theta_3) = d\theta_1 d\theta_2\,\tilde{\sigma}_{0,E}(d\theta_3)$. To determine $\tilde{\sigma}_{0,E}$ we can take $m = 1$ in the above formulas and integrate over the variables θ_1 and θ_2. Using (4.17) with $n_1 = n_2 = 0$ we can then compute

$$\int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \mathbb{E}(e^{in_3\theta_3'}) = e^{in_3\theta_3}\big\{1+\eta^2[C_3n_3 + C_{33}n_3^2]\big\} + \mathcal{O}(\eta^3), \tag{4.21}$$

where $C_3 = \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \mathbb{E}(A_3)$ and $C_{33} = \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \mathbb{E}(A_{33})$. In terms of $\phi = 2\theta_3$ they are given by

$$C_3 = i\frac{\pi^2}{8\sin\phi}\left(\frac{(1-\cos\phi)(5\cos\phi - \cos^2\phi + 2)}{\sin^2\alpha} + \frac{(1+\cos\phi)(5\cos\phi + \cos^2\phi - 2)}{\sin^2\beta}\right) \tag{4.22}$$

and

$$C_{33} = -\frac{\pi^2}{16}\left(\frac{3+4\cos\phi+\cos^2\phi}{\sin^2\beta} + \frac{3-4\cos\phi+\cos^2\phi}{\sin^2\alpha}\right). \tag{4.23}$$

Inserting the identity

$$\lim_{\eta\downarrow 0}\eta^{-2}\int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \mathbb{E}\left(e^{in_3\theta_3'} - e^{in_3\theta_3}\right) = n_3 e^{in_3\theta_3}(C_3 + C_{33}n_3) \tag{4.24}$$

into the analogue of (3.12) we obtain the following equation for $\tilde{\sigma}_{0,E}$:

$$\int_0^{\pi/2} \big\{2n_3 C_{33}\cos(2n_3\theta_3) + iC_3\sin(2n_3\theta_3)\big\}\rho(\theta_3)d\theta_3 = 0, \tag{4.25}$$

assuming that $\tilde{\sigma}_{0,E}$ has a density $\rho(\theta_3)$. Integrating by parts, this leads to the differential equation

$$iC_3\rho(\phi) - 2\frac{d}{d\phi}(C_{33}\rho(\phi)) = 0. \tag{4.26}$$

This can be solved exactly. It is helpful to write

$$\rho(\phi) = S(\cos\phi)\sin\phi. \tag{4.27}$$

The solution takes different forms depending on the value of E. Consider the case $E > 0$ and let $E_0 = (\sqrt{13}-2)/\sqrt{3}$. If $0 < E < E_0$ then

$$S(t) = \frac{C}{(a-t)^2 + b^2}\exp\left(\frac{a}{b}\tan^{-1}\frac{b}{a-t}\right), \tag{4.28}$$

where $a = \dfrac{4E}{3-E^2}$ and $b = \dfrac{\sqrt{3E^4 - 34E^2 + 27}}{3-E^2}$. For $E = E_0$,

$$S(t) = \frac{C}{(a-t)^2}\exp\left(\frac{a}{a-t}\right) \tag{4.29}$$

and if $E > E_0$,

$$S(t) = \frac{C}{(a-t)^2 - c^2}\left(\frac{a+c-t}{a-c-t}\right)^{\frac{a}{2c}}, \tag{4.30}$$

where $c = \dfrac{\sqrt{34E^2 - 3E^4 - 27}}{3-E^2}$. Figure 3 shows a graph of $\rho(\theta_3)$ in the various cases.

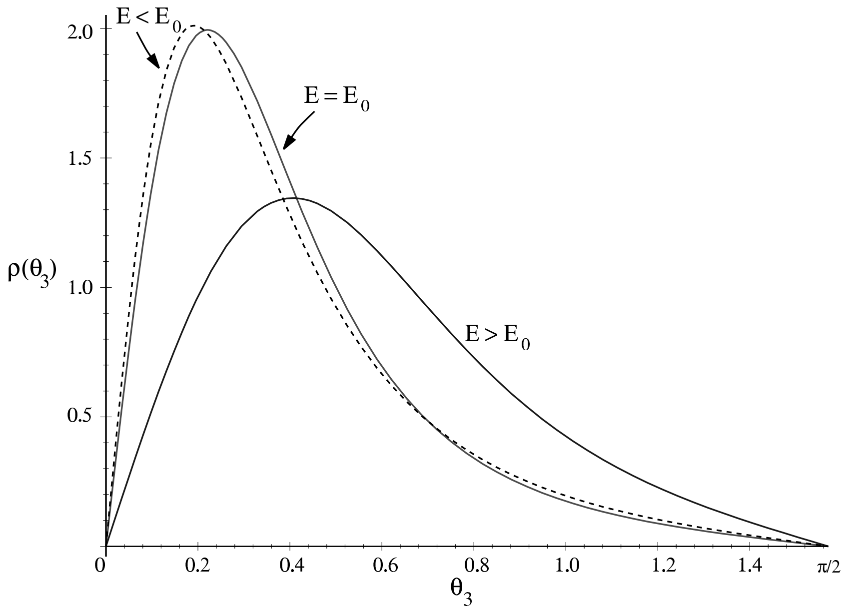

Figure 3. *The density of the measure for* $|E| \in (0,1)$.

In the case (4.4) holds with α/π rational and β/π irrational $\sigma_{0,E}$ is only Lebesgue in θ_2 and we can only integrate the expectations w.r.t. this variable. We have to use (4.17) with $n_2 = 0$. The resulting differential equation is much more complicated, but it turns out that the solution is in fact the same as in the previous case.

Other cases are even more complicated. Here, we restrict ourselves to a few remarks. Details will be published elsewhere. In particular the case $E = 0$ is the most complicated. It corresponds to $\alpha = \frac{\pi}{3}$ and $\beta = \frac{2\pi}{3}$ and we have to take $m = 6$ in (4.9) - (4.20). We can show that the density of $\sigma_{0,E}$ only depends on ϕ and $\psi = 2\theta_1 + 2\theta_2 + \frac{\pi}{3}$. The resulting differential equation for the density is very complicated and we have not been able to solve it exactly.

A final remark about the case $|E| \in (1,3)$. In this case we cannot write (4.4). Instead, we put

$$E = 2\cos\alpha - 1 = 1 - 2\cosh\gamma. \tag{4.31}$$

assuming $-3 < E < -1$. (The case $E > 0$ is similar.) All the formulas (4.5), (4.6), (4.10), (4.12) and (4.13) need to be modified. For example, S is now given by

$$S = \begin{pmatrix} 1 & -1 & -\cos\alpha & \cos\alpha \\ 0 & 0 & \sin\alpha & -\sin\alpha \\ e^{-\gamma} & e^{-\gamma} & 1 & 1 \\ e^{\gamma} & e^{\gamma} & 1 & 1 \end{pmatrix} \tag{4.32}$$

and

$$S\,A_0\,S^{-1} = \begin{pmatrix} R_\alpha & 0 \\ 0 & -\tilde{R}_\gamma \end{pmatrix} \quad \text{with} \quad \tilde{R}_\gamma = \begin{pmatrix} e^{-\gamma} & 0 \\ 0 & e^{\gamma} \end{pmatrix}. \tag{4.33}$$

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