DUALITY IN LIOUVILLE THEORY AS A REDUCED SYMMETRY

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Abstract

The origin of the rather mysterious duality symmetry found in quantum Liouville theory is investigated by considering the Liouville theory as the reduction of a WZW-like theory in which the form of the potential for the Cartan field is not fixed *a priori*. It is shown that in the classical theory conformal invariance places no condition on the form of the potential, but the conformal invariance of the classical reduction requires that it be an exponential. In contrast, the quantum theory requires that, even before reduction, the potential be a sum of two exponentials. The duality of these two exponentials is the fore-runner of the Liouville duality. An interpretation for the reflection symmetry found in quantum Liouville theory is also obtained along similar lines.

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A single real scalar field governed by the so-called Liouville Action [1] plays a ubiquitous role in the study of two-dimensional conformal and integrable field theories, and is of considerable importance in the context of quantum gravity and string theory [2]. Amongst its many unusual properties is a particularly intriguing one, namely, the invariance of the quantum theory under a duality symmetry which is absent in the classical theory. This duality symmetry is responsible for the doubling of a one-dimensional lattice of poles (in the parameter space) of the coefficient of the three-point function of vertex operators, as explained in detail in [3]. It was also shown there that the three-point function, with the correct pole structure, may be derived by incorporating the duality into the definition of the path integral for the Liouville theory on a sphere.

As is well-known, the classical Liouville theory may be obtained by imposing a set of linear first class constraints on the currents of an SL(2, R) Wess-Zumino-Witten (WZW) model [4]. In the present paper, we investigate the origin of the Liouville duality by considering the Liouville theory as the reduction of a WZW-like theory in which the form of the potential for the Cartan field is not fixed *a priori*. It will be shown that the conformal invariance of the classical reduction forces this theory to be the SL(2, R) WZW model, but an essential difference between the classical and quantum reductions lies at the heart of the duality symmetry of the quantum Liouville theory.

The WZW-like theory we start with is defined by the following Action:

$$S = k \int d^2x \sqrt{g}\ [\frac{1}{2} g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) + g^{\mu\nu}(\partial_\mu a)(\partial_\nu c)V(\phi)] \tag{1}$$

In the above equation, ϕ, a and c are real scalar fields and $V(\phi)$ is an arbitrary function of ϕ. Obviously, (1) describes a conformally invariant model at the classical level. Upon the addition of a topological term $-ik\epsilon^{\mu\nu}(\partial_\mu a)(\partial_\nu c)V(\phi)$, the Action becomes

$$S = k \int d^2z\ [\frac{1}{2}(\bar{\partial}\phi)(\partial\phi) + (\bar{\partial}a)(\partial c)V(\phi)] \tag{2}$$

where we have used complex coordinates $z = \frac{x_0 + ix_1}{2}$, $\bar{z} = \frac{x_0 - ix_1}{2}$ in terms of which $\partial = \partial_0 - i\partial_1$, $\bar{\partial} = \partial_0 + i\partial_1$ and $d^2z = 2idzd\bar{z}$. Note that choosing the topological term with the opposite sign would have resulted in $\partial \leftrightarrow \bar{\partial}$ in (2). The classical energy-momentum tensor, $T^{\alpha\beta} \equiv -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\alpha\beta}}$, is given by

$$\begin{aligned} T^{\alpha\beta} = & \frac{k}{2} g^{\alpha\beta} g^{\mu\nu} [\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi) + (\partial_\mu a)(\partial_\nu c)V(\phi)] \\ & - k g^{\mu\alpha} g^{\beta\nu} [\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi) + (\partial_\mu a)(\partial_\nu c)V(\phi)] \end{aligned} \tag{3}$$

As usual, it is traceless, symmetric, and conserved, and does not receive any contribution from the topological term as the latter is independent of the metric.

Note that for $V(\phi) = e^{-\phi}$, Eq. (2) simplifies to the SL(2, R) WZW Action, with a and c locally parametrising the nilpotent subgroups, and ϕ the abelian subgroup, respectively, in a Gauss decomposition of the group-valued WZW field. In the following, however, we shall allow $V(\phi)$ to be arbitrary to begin with, and examine the restrictions imposed on its form, by requiring the Action (2) to be amenable to a conformally invariant reduction. It will be shown that for this requirement to be satisfied, $V(\phi)$ is governed by a second order functional differential equation whose two solutions are dual to each other. In the classical limit only one of the two solutions of this equation has a non-trivial (ϕ-dependent) value which corresponds to the standard SL(2, R) WZW case.

We shall first present the analysis of conformal invariance in the classical case. The expressions for the momenta π conjugate to the fields $\phi,\ a,\ c$ follow from the Action (2) and are given by

$$\pi_\phi = k\dot{\phi}, \quad \pi_a = kV(\phi)\partial c, \quad \text{and} \quad \pi_c = kV(\phi)\bar{\partial} a \tag{4}$$

The non-vanishing Poisson brackets, evaluated at equal time, are given by

$$\{\phi(x,t), \pi_\phi(y,t)\} = \{a(x,t), \pi_a(y,t)\} = \{c(x,t), \pi_c(y,t)\} = \delta(x-y) \tag{5}$$

The Virasoro generator T is given by

$$T = \int dz\ \epsilon(z)[-\frac{k}{2}(\partial\phi)^2 - k\pi_a\pi_c V^{-1} + 2i\pi_a(\partial_1 a)](z) \tag{6}$$

The other Virasoro generator $\bar{T}$, may be obtained by making the exchanges $a \leftrightarrow c,\ \partial \leftrightarrow \bar{\partial},\ \epsilon \to \bar{\epsilon}$. As usual, the Poisson bracket of the two Virasoros vanishes, and it suffices to focus our attention on one Virasoro.

The conformal transformation of a field $\mathcal{O}$, generated by T_ϵ, is given by $\delta_\epsilon \mathcal{O} \equiv \{T_\epsilon, \mathcal{O}\}$. Hence the fields have the following conformal variations:

$$\delta_\epsilon a = \epsilon \partial a, \quad \delta_\epsilon c = \epsilon \partial c, \quad \delta_\epsilon \phi = \epsilon \partial \phi, \quad \delta_\epsilon V(\phi) = \epsilon \partial V(\phi) \tag{7}$$

$$\delta_\epsilon \pi_a = \epsilon \partial \pi_a + (\partial \epsilon)\pi_a, \quad \delta_\epsilon \pi_c = 0 \tag{8}$$

The momentum π_ϕ does not transform like a conformal primary field:

$$\delta_\epsilon \pi_\phi = \epsilon \partial \pi_\phi + (\partial \epsilon)\frac{k}{2}\partial \phi$$

This is expected because $\pi_\phi = (k\dot{\phi})$ is a mixture of ∂ and $\bar{\partial}$ derivatives of a scalar field. As can be seen from the following equation, however, $\partial\phi$ transforms like a conformal primary field:

$$\delta_\epsilon \partial \phi = \epsilon \partial \partial \phi + (\partial \epsilon) \partial \phi \tag{9}$$

It is also easy to check that

$$\delta_\epsilon T = \epsilon \partial T + 2(\partial \epsilon) T \tag{10}$$

It follows from the above equations that classically $a,\ c,\ \phi,\ \pi_c$ have conformal weights zero; $\pi_a,\ \partial\phi$ have weights one, and T has a weight two – as expected. The weights with respect to the other Virasoro are given by the same numbers if the exchanges $a \leftrightarrow c$, $\epsilon \leftrightarrow \bar{\epsilon}$, and $\partial \leftrightarrow \bar{\partial}$ are done.

At this juncture it is worth noting that the Virasoro (6) is unique if we insist on keeping $V(\phi)$ arbitrary. However, if we are interested in a special class of theories for which V is restricted to be of the form

$$V = e^{\lambda \phi} \tag{11}$$

the Virasoro is no longer unique.* This is because it is not necessary that π_a and π_c should have conformal weights (1,0) and (0,1) respectively, for the Action (2) to be conformally invariant; it is only necessary that the combination $\pi_a\pi_c V^{-1}(\phi)$ has a weight (1,1). This freedom is expressible in terms of improvement terms which, when added to the standard Virasoro (6), redistribute the weights of the various fields in a way which preserves the weight of the above combination of the fields. It is straightforward to see that the improvement term t, for the Virasoro T, is given by

$$t_\epsilon = -\alpha \int dz\ (\partial\epsilon)[k\partial\phi - \lambda a\pi_a](z) \tag{12}$$

where α is an arbitrary parameter. Clearly, with respect to the full Virasoro $T+t$, π_a behaves like a primary field of weight $(1+\lambda\alpha, 0)$. It is also easy to see that the field $e^{-\lambda\phi}$ transforms as follows under the action of the full Virasoro $T+t$:

$$\delta_\epsilon e^{-\lambda\phi} = -\lambda\alpha(\partial\epsilon)e^{-\lambda\phi} + \epsilon\partial e^{-\lambda\phi} \tag{13}$$

Thus the combination $\pi_a\pi_c V^{-1}$ has weight one with respect to the full Virasoro $T+t$. Note that the ratio of the coefficients of the two terms in (12) is fixed by the parameter λ in the potential. It is straightforward to see that this ratio is also equal to the corresponding ratio in the improvement terms $\bar{t}$ for the other Virasoro $\bar{T}$. Further, the requirement that the Poisson bracket of the two Virasoros is zero implies that the improvement terms t and $\bar{t}$ have the same overall coefficient α relative to T and $\bar{T}$. It may also be noted that for the theory to remain conformally invariant, it is not possible to have only λ to be zero. If λ is zero, then α has to be necessarily zero.

It is worth mentioning that there is a good reason for considering the improved Virasoro $T+t$, and the concomitant freedom in redistributing the weights of the various fields in the theory, namely, that it is necessary for the conformal reduction of the SL(2,

* Note that by scaling the fields ϕ, a, and c, appropriately, (2) can be seen to represent the SL(2, R) WZW model for this choice of V.

R) WZW theory to the Liouville theory. The constraints that accomplish this reduction are

$$\pi_a = m_a \quad \text{and} \quad \pi_c = m_c \tag{14}$$

where m_a and m_c are constants. Note that unless the improvement terms are present with $\lambda\alpha = -1$, it is not possible to impose the above constraints in a conformally invariant manner since π_a and π_c have non-zero conformal dimensions. Thus one is forced to use the full Virasoro $T+t$ with $\lambda\alpha = -1$. Indeed, the improvement term with $\lambda\alpha = -1$ automatically provides the usual improvement terms of the Liouville theory upon reduction. Having thus motivated the need for the improvement terms, it may be noted that the argument can be reversed: Given the full Virasoro $T+t$ and the Action (2), conformal invariance requires V to be of the form (11).

The situation in the quantum theory is remarkably different from that in the classical theory. The entire analysis leading up to Eq. (10) can be carried over to the quantum theory in a relatively straightforward way by replacing Poisson brackets with commutators and the appropriate factors of $i\hbar$, as long as care is exercised in taking the ordering of the operators. The commutator of π_a with the Virasoro can be calculated without ado as the Virasoro is linear in a, the conjugate field. As expected, π_a remains a conformal primary field of weight (1, 0). The commutator of V^{-1} with the Virasoro is trickier because T is quadratic in π_ϕ. A short calculation shows that

$$\delta_\epsilon V^{-1} \equiv [T_\epsilon,\ V^{-1}]_- = (\partial\epsilon)\Big(\frac{-i\hbar^2}{2k}\frac{\delta^2 V^{-1}}{\delta\phi^2}\Big) + i\hbar\epsilon(\partial V^{-1}) \tag{15}$$

where we have used $-\partial\delta(z-z') = i[\delta(z-z')]^2$ [5]. Thus, at the quantum mechanical level, V^{-1} ceases to behave like a conformal scalar for general V, with respect to T_ϵ. However, as in the classical case, if one adds improvement terms, it is sufficient to require that the combination $\pi_a\pi_c V^{-1}(\phi)$ has a weight (1,1). It is easy to check then that with respect to the full Virasoro $T+t$, π_a behaves like a primary field of weight $(1+\lambda\alpha, 0)$ even at the quantum level. The conformal variation of V^{-1} with respect to the full quantum Virasoro is

$$\delta_\epsilon V^{-1} \equiv [T_\epsilon + t_\epsilon,\ V^{-1}]_- = (\partial\epsilon)\Big(\frac{-i\hbar^2}{2k}\frac{\delta^2 V^{-1}}{\delta\phi^2} + i\hbar\alpha\frac{\delta V^{-1}}{\delta\phi}\Big) + i\hbar\epsilon(\partial V^{-1}) \tag{16}$$

The necessary condition for V^{-1} to have the correct conformal properties, can be read off from the above equation to be

$$\frac{-\hbar}{2k\alpha}\frac{\delta^2 V^{-1}}{\delta\phi^2} + \frac{\delta V^{-1}}{\delta\phi} = -\lambda V^{-1} \tag{17}$$

Note that the above equation is a quadratic equation and has two solutions while its classical counterpart obtained by setting $\hbar = 0$ is a first order equation which leads to $V = e^{\lambda\phi}$. Thus the quantum theory is slightly less stringent about the form of V, than the classical theory, with respect to the demands imposed by conformal invariance. The general solution for V^{-1} takes the form

$$V^{-1} = e^{\omega_+\phi} + \mu e^{\omega_-\phi} \tag{18}$$

where

$$\omega_\pm = \frac{\alpha k}{\hbar} \mp \frac{\alpha k}{\hbar}\sqrt{1 + \frac{2\hbar\lambda}{\alpha k}} \tag{19}$$

and μ is a constant. Note that $\omega_+\omega_- = -\frac{2k\lambda\alpha}{\hbar}$ — showing that the two solutions are dual to each other. Also note that in the classical limit, $\hbar \to 0$, $\omega_+ \to -\lambda$ and $\omega_- \to \infty$. In this limit the duality disappears and only (11) survives.

In passing we mention that the above results can be readily applied to work out the transformation properties of the vertex operators $U(\phi) = e^{2\gamma\phi}$ of the field ϕ. It follows from Eq. (16) that

$$\delta_\epsilon(e^{2\gamma}) \equiv [T_\epsilon + t_\epsilon,\ e^{2\gamma}]_- = (\partial\epsilon)(i\hbar)\Big(\frac{2\gamma}{k}(k\alpha - \hbar\gamma)\Big)e^{2\alpha} + i\hbar\epsilon(\partial e^{2\gamma}) \tag{20}$$

Thus the conformal weight of the vertex operator is given by

$$\Delta(e^{2\gamma}) = \frac{2\gamma}{k}(k\alpha - \hbar\gamma) \tag{21}$$

which is manifestly symmetric under the exchange $\gamma \rightarrow k\alpha - \gamma$ in units of $\hbar = 1$. This is the reflection symmetry of the conformal weight of a vertex operator in the quantum theory.

Substituting the solution for V from Eqs. (18) and (19) in Eq. (2), we have

$$S = -\int d^2z \; [\frac{1}{2}(\bar{\partial}\phi)(\partial\phi) + (\bar{\partial}a)(\partial c)(e^{\omega_+\phi} + \mu e^{\omega_-\phi})^{-1}] \tag{22}$$

What we have shown is that the above highly non-linear Action is conformally invariant with respect to the full Virasoro $T+t$ in the quantum theory, and reduces to the standard SL(2, R) WZW case in the classical limit. It easily follows that the constraints (14) reduce (22) to

$$S = -\int d^2z \; [\frac{1}{2}(\bar{\partial}\phi)(\partial\phi) + m_a m_c(e^{\omega_+\phi} + \mu e^{\omega_-\phi})] \tag{23}$$

This is precisely the theory proposed in [3] to explain the duality symmetry of quantum Liouville theory and the pole structure of its three-point function.

To summarise the results of this paper, we have shown how the duality symmetry that appears rather mysteriously in quantum Liouville theory may be interpreted as a reduced symmetry. Thus the origin of this symmetry can be traced to the fact that the conditions imposed by conformal invariance on the structure of the potential in the Liouville theory are different in the classical and quantum reductions, the latter being less stringent than the former. This is because the $\hbar$-dependent operator ordering effects in quantum theory produce a second order functional differential equation for the potential, whose solutions are dual to each other, while the absence of such effects in the classical theory produces a first order functional differential equation whose unique solution does not allow for the possibility of a duality symmetry. A natural interpretation for the reflection symmetry in quantum Liouville theory was also obtained along these lines. It would be interesting to see if the results of this paper can be generalised to discuss the duality and reflection invariance of Toda theories as reduced symmetries.

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