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Non-commutative Complex Projective Spaces And The Standard Model*

Brian P. Dolan
Department of Mathematical Physics
National University of Ireland
Maynooth, Co. Kildare, Ireland
and
Dublin Institute for Advanced Studies
10, Burlington Rd., Dublin, Ireland
bdolan@thphys.may.ie

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Abstract

The standard model fermion spectrum, including a right handed neutrino, can be obtained as a zero-mode of the Dirac operator on a space which is the product of complex projective spaces of complex dimension two and three. The construction requires the introduction of topologically non-trivial background gauge fields. By borrowing from ideas in Connes' non-commutative geometry and making the complex spaces 'fuzzy' a matrix approximation to the fuzzy space allows for three generations to emerge. The generations are associated with three copies of space-time. Higgs' fields and Yukawa couplings can be accommodated in the usual way.

1 Introduction

Current descriptions of the force of gravity and the fundamental interactions of particle physics are set in the language of differential geometry and fibre bundles. A unified description of gravity and gauge theories has long been one of the main goals of modern theoretical physics and superstring theory is currently the most popular framework for this endeavour. Nevertheless it may be worthwhile

*Talk presented at the conference "Space-time and Fundamental Interactions: Quantum Aspects" in honour of A.P. Balachandran's 65th birthday, Vietri sul Mari, Salerno, Italy 26th-31st May, 2003 pursuing other avenues of investigation and a suggestion is made here, based on an observation of a curious connection between the standard model of particle physics and the Atiyah-Singer index theorem on complex projective spaces.

The motivation is to bring the geometrical descriptions of general relativity and Yang-Mills theory closer together. In a geometrical approach to fundamental interactions physical fields are tensors associated with the tangent space of an underlying four dimensional manifold, $\mathcal{M}$:

General Relativity	**Yang-Mills Theory**
Manifold: $\mathcal{M}$	Manifold: $\mathcal{M}$
Tangent Bundle: $T\mathcal{M}$	Vector Bundle: $\mathcal{E}(M)$
Lorentzian Metric: g	Connection: A
Spin Connection: ω	Curvature: F
Curvature: R	Local Gauge group:
Local tangent space rotations:	e.g. $U(n)$ or $SU(3) \times SU(2) \times U(1)$
$SO(3,1)$ (or $Sl(2,\mathbf{C})$)	

In Kaluza-Klein theories one brings these structures together by taking a compact co-set space, $\mathcal{C} \cong G/H$, of small radius and extending $\mathcal{M}$ to $\mathcal{M} \times \mathcal{C}$. The gauge group is identified with the isometry group G of $\mathcal{C}$. In the spirit of general relativity it is perhaps more natural to identify the holonomy group H of $\mathcal{C}$ with the Yang-Mills gauge group, since it is the holonomy group of $\mathcal{M}$ that takes the centre stage in gravity. If $\mathcal{C}$ is a complex manifold, with n complex dimensions and an hermitian metric, it has holonomy group $U(n)$ in general and so might furnish a $U(n)$ gauge theory.

If the Dirac operator on $\mathcal{M} \times \mathcal{C}$ decomposes as a direct sum

$$i\not{D} = i\not{D}_{\mathcal{M}} \otimes \mathbf{1} + \gamma_5 \otimes i\not{D}_{\mathcal{C}}, \tag{1}$$

where $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is the chirality operator on $\mathcal{M}$, then eigenspinors ζ of the internal Dirac operator,

$$i\not{D}_{\mathcal{C}}\zeta = \lambda\zeta, \tag{2}$$

will have eigenvalues of the order of the Ricci scalar on $\mathcal{C}$. In the spirit of Kaluza-Klein theory, if $\mathcal{C}$ has a very large curvature, only the zero eigenstates will contribute to the low energy spectrum seen in $\mathcal{M}$. It was shown by Witten [1] that the chiral fermion spectrum of the standard model in $\mathcal{M}$ cannot be obtained in this way purely from the metric and spin connection on $\mathcal{C}$ and this essentially killed the Kaluza-Klein programme in the early '80 s. Some extra ingredient is needed and we shall avoid Witten's theorem by introducing fundamental gauge fields.

We are thus led to an investigation of the zero modes of the Dirac operator on complex manifolds in the presence of background gauge fields but we first discuss Clifford algebras and complex vector spaces.

2 Clifford Algebras on Complex Vector Spaces

Let z^μ, $\mu = 1, \ldots, n$ be complex co-ordinates on a complex manifold with complex dimension n. Dirac fermions have 2^n components and Weyl fermions 2^{n-1} components. The γ-matrices can be chosen so that

$$\{\gamma^a, \gamma^b\} = \{\gamma^{\bar{a}}, \gamma^{\bar{b}}\} = 0, \qquad \{\gamma^a, \gamma^{\bar{b}}\} = 2\delta^{a\bar{b}} \tag{3}$$

with a, b indices labelling an orthonormal basis. This Clifford algebra is isomorphic to the algebra of of n-fermionic creation and annihilation operators [2]. Because of the fermionic nature of the creation and annihilation operators, $b^a = \frac{1}{\sqrt{2}}\gamma^a$ and $(b^a)^\dagger = \frac{1}{\sqrt{2}}\gamma^{\bar{a}}$, the Fock space is 2^n-dimensional. Denoting the vacuum state by $|\Omega>$ we can construct a basis for the Fock space

$$|\Omega>, \qquad |\Omega^{\bar{a}}>=\gamma^{\bar{a}}, \qquad |\Omega^{\bar{a}\bar{b}}>=\gamma^{\bar{a}\bar{b}} = -|\Omega^{\bar{b}\bar{a}}>, \qquad |\Omega^{\bar{a}\bar{b}\bar{c}}>=\gamma^{\bar{a}\bar{b}\bar{c}}, \quad \text{etc.,} \tag{4}$$

where $\gamma^{\bar{a}\bar{b}} = (1/2)[\gamma^{\bar{a}}, \gamma^{\bar{b}}]$, and higher rank products are similarly anti-symmetrised. A Dirac spinor can then be expanded as

$$\zeta = \phi|\Omega> + \phi_{\bar{a}}|\Omega^{\bar{a}}> + \phi_{\bar{a}\bar{b}}|\Omega^{\bar{a}\bar{b}}> + \cdots + \phi_{\bar{a}_1\cdots\bar{a}_n}|\Omega^{\bar{a}_1\cdots\bar{a}_n}>. \tag{5}$$

This can be decomposed into two Weyl spinors

$$\zeta_+ = \phi|\Omega> + \phi_{\bar{a}\bar{b}}|\Omega^{\bar{a}\bar{b}}> + \cdots \tag{6}$$

$$\zeta_- = \phi_{\bar{a}}|\Omega^{\bar{a}}> + \phi_{\bar{a}\bar{b}\bar{c}}|\Omega^{\bar{a}\bar{b}\bar{c}}> + \cdots. \tag{7}$$

We can thus read off how the spinor components $\phi_{\bar{a}_1\cdots\bar{a}_i}$ transform under the $SU(n)$ part of the holonomy:

$$\begin{array}{ccl}
\phi & \mathbf{1} & SU(n) \ \text{SINGLET} \\
\phi_{\bar{a}} & \overline{\mathbf{n}} & \text{ANTI-FUNDAMENTAL} \\
\vdots & & \\
\phi_{\bar{a}_1\cdots\bar{a}_{n-1}} & \mathbf{n} & \text{FUNDAMENTAL} \\
\phi_{\bar{a}_1\cdots\bar{a}_n} & \mathbf{1} & \text{SINGLET.}
\end{array} \tag{8}$$

The $U(1)$ charges are more subtle, since the creation and annihilation operators have $U(1)$ phases. Normalise the charge so that b^a has charge $+1$ — then the vacuum can have a charge, which will be denoted by k for the moment, and we shall fix later. The $U(1)$ charges of our Fock space basis are now

$$|\Omega>\sim k; \qquad |\Omega^{\bar{a}}>\sim k-1; \qquad \cdots \qquad |\Omega^{\bar{a}_1\cdots\bar{a}_n}>\sim k-n. \tag{9}$$

As an example consider the case of a 4-dimensional space with $n = 2$. Without any complex structure the holonomy group is $SO(4) \approx SU(2) \times SU(2)$ but this can be restricted to $SU(2) \times U(1)$ when a complex structure and a compatible hermitian metric are introduced. From the decomposition under $SU(2) \to U(1)$

$$\mathbf{2} \to \mathbf{1}_1 + \mathbf{1}_{-1} \tag{10}$$

we see the following structure:

$$\begin{array}{ccccc} \underline{SO(4) \approx SU(2) \times SU(2)} & \longrightarrow & \underline{SU(2) \times U(1)} & \longrightarrow & \underline{SU(2) \times U(1)} \\ (\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2}) & \longrightarrow & \mathbf{2}_0 + (\mathbf{1}_1 + \mathbf{1}_{-1}) & \longrightarrow & \mathbf{2}_{-1} + (\mathbf{1}_0 + \mathbf{1}_{-2}) \\ \text{DIRAC} & & |\Omega^{\bar{a}}> +(|\Omega> + |\Omega^{\bar{a}\bar{b}}>) & & \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \nu_R \quad e_R \\ & & (k=1) & & (k=0) \end{array} \tag{11}$$

It is natural to take $k = 1$, as indicated in the middle column. But if we shift k to zero we see that the states in the last column have the quantum numbers of the electro-weak sector of the standard model. I first became aware of this assignment of quantum numbers to the components of a spinor for $n = 2$ from Balachandran [3].

Now consider a 6-dimensional space with $n = 3$. Without any complex structure the holonomy group is $SO(6) \approx SU(4)$ but this can be restricted to $SU(3) \times U(1)$ when a complex structure and a compatible hermitian metric are introduced:

$$\begin{array}{ccc} \underline{SO(6) \approx SU(4)} & \longrightarrow & \underline{SU(3) \times U(1)} \\ \mathbf{4} + \bar{\mathbf{4}} & \longrightarrow & (\mathbf{3}_{-1/2} + \mathbf{1}_{3/2}) + (\bar{\mathbf{3}}_{1/2} + \mathbf{1}_{-3/2}) \\ \text{DIRAC} & & (|\Omega^{\bar{a}\bar{b}}> \ |\Omega>) \quad (|\Omega^{\bar{a}}> |\Omega^{\bar{a}\bar{b}\bar{c}}>) \\ & & (k = 3/2) \end{array} \tag{12}$$

In this instance we see that $k = 3/2$ gives the correct $U(1)$ charges for the decomposition $\mathbf{4} \to \mathbf{3}_{-1/2} + \mathbf{1}_{3/2}$ (the normalisation of the $U(1)$ charge is at our disposal). If k is shifted to zero a single Weyl fermion reduces to

$$\mathbf{4} \to \mathbf{3}_{-2} + \mathbf{1}_0. \tag{13}$$

If we now take the tensor product of a Dirac fermion for $n = 2$ with $k = 0$, dividing the $U(1)$ charges by two to give $\mathbf{2}_{-1/2} + (\mathbf{1}_0 + \mathbf{1}_{-1})$, and and Weyl fermion for $n = 3$, dividing the $U(1)$ charges by -3 to give $\mathbf{3}_{2/3} + \mathbf{1}_0$, we find

$$\begin{aligned} (\mathbf{3}_{2/3} \;\; + \;\; \mathbf{1}_0)^+ \otimes ((\mathbf{2}_{-1/2})^- + (\mathbf{1}_0 + \mathbf{1}_{-1}))^+ = \\ (\mathbf{3},\mathbf{2})^-_{1/6} + (\mathbf{3},\mathbf{1})^+_{2/3} + (\mathbf{3},\mathbf{1})^+_{-1/3} + (\mathbf{1},\mathbf{2})^-_{-1/2} + (\mathbf{1},\mathbf{1})^+_0 + (\mathbf{1},\mathbf{1})^+_{-1} \end{aligned} \tag{14}$$

with the superscript $\pm$ denoting the chirality. The fermion spectrum of the standard model emerges, including a right-handed neutrino. This structure can be summarised by putting the 16 fermion states into a 4×4 matrix,

$$\zeta = \begin{pmatrix} u_L^r & u_L^g & u_L^b & \nu_L \\ u_d^r & d_L^g & d_L^b & e_L \\ u_R^r & u_R^g & u_R^b & \nu_R \\ d_R^r & d_R^g & d_R^b & e_R \end{pmatrix}, \tag{15}$$

and the action of $SU(3) \times SU(2) \times U(1)$ on this matrix is represented by

$$\zeta \to g_2 \zeta g_3^* \tag{16}$$

where $g_2 \in U(2)$ and $g_3 \in U(3)$ — the $U(1)$ action is just the difference of the $U(1)$'s in $U(3)$ and $U(2)$, if the $k = 0$ charge assignments in (11) and (13) are multiplied by $\frac{1}{2}$ and $\frac{1}{3}$ respectively and then subtracted as implied by (16).

3 Global Spinor Fields

The considerations of the last section were purely algebraic and it is a much more involved question to decide whether these structures can be defined globally on a given complex space. Indeed it is well known that the complex projective space CP^2 ($n = 2$) does not admit a globally well defined spin structure [4]. Fortunately even when a complex manifold manifold does not admit a spin structure it always admits a spin^c structure, obtained by introducing a topologically non-trivial $U(1)$ background gauge field. An essential tool in determining the zero modes of the Dirac operator in a background field is the Atiyah-Singer index theorem and an analysis of the index on CP^n was given in Ref. [5] (the particular case of a $U(1)$ field on CP^2 was first analysed, in physicists language, in Ref. [6]).

Here we quote the results in Ref. [5] for the index of the Dirac operator for a fermion on CP^n in a background $U(n)$ gauge field obtained by identifying the gauge connection with the spin connection, $A \approx \omega$. For a fermion which is a $SU(n)$ singlet and has $U(1)$ charge $Y_{(n)} = q$ the index is

$$\nu_q = \frac{(q+1)\cdots(q+n)}{n!}, \tag{17}$$

where the $U(1)$ charge is normalised so that the fundamental unit of charge is an integer and $q = 0$ is the spin^c structure.[1]

A fermion in the fundamental representation, $\mathbf{n}$, of $SU(n)$, with $U(1)$ charge $Y_{(n)} = q + \frac{1}{n}$, has index

$$\nu_{q,\mathbf{n}} = \frac{(q+1)\cdots(q+n-1)(q+n+1)}{(n-1)!} \tag{18}$$

[1]Since the Euler characteristic of CP^n is $\chi_n = n + 1$ the first Chern class of the spin connection is $n + 1$, by the Gauss-Bonnet theorem. Thus a 'natural' unit of charge might be $1/\chi_n = 1/(n+1)$ — in these units a fermion of unit charge would couple to the gauge connection with the same weight as to the spin connection. Relative to this 'natural' unit the charges q in the text have been scaled by χ_n so as to make the fundamental unit of charge equal to unity rather than $1/\chi_n$.

(the $U(1)$ charge is $q+\frac{1}{n}$ because a $U(n)$ instanton on CP^n has first Chern class one, so the $U(1)$ generator is $\frac{1}{n}\mathbf{1}$ where $\mathbf{1}$ is the unit $n \times n$ matrix, Ref. [5]).

Using these formulae it can be shown that the algebraic structure described in section 2 is, in fact, global on $CP^2 \times CP^3$ [5]. We construct a Dirac spinor on CP^2 and positive chirality Weyl spinor on CP^3 by taking the following combinations:

$$\bullet \quad CP^2 \quad - \left\{ \begin{array}{llll} SU(2) & \text{SINGLET WITH} & q=0, & \nu_0 = 1, & Y_{(2)} = 0 \\ SU(2) & \text{SINGLET WITH} & q=-3, & \nu_{-3} = 1, & Y_{(2)} = -3 \\ SU(2) & \text{DOUBLET WITH} & q=-2, & \nu_{-2,\mathbf{2}} = -1, & Y_{(2)} = -\frac{3}{2} \end{array} \right.$$

$$\bullet \quad CP^3 \quad - \left\{ \begin{array}{llll} SU(3) & \text{SINGLET WITH} & q=0, & \nu_0 = 1, & Y_{(3)} = 0 \\ SU(3) & \text{TRIPLET WITH} & q=-3, & \nu_{-3,\mathbf{3}} = 1, & Y_{(3)} = -\frac{8}{3}. \end{array} \right. \tag{19}$$

Defining the hypercharge as

$$Y := \frac{1}{3}Y_{(2)} - \frac{1}{4}Y_{(3)} \tag{20}$$

the tensor product of these zero-modes is

$$\begin{aligned} (\mathbf{3}_{2/3} \quad + \quad \mathbf{1}_0)^+ \otimes \left((\mathbf{2}_{-1/2})^- + (\mathbf{1}_0 + \mathbf{1}_{-1})\right)^+ = \\ (\mathbf{3},\mathbf{2})^-_{1/6} + (\mathbf{3},\mathbf{1})^+_{2/3} + (\mathbf{3},\mathbf{1})^+_{-1/3} + (\mathbf{1},\mathbf{2})^-_{-1/2} + (\mathbf{1},\mathbf{1})^+_0 + (\mathbf{1},\mathbf{2})^+_{-1}, \end{aligned} \tag{21}$$

which is precisely that of (14).

The structure here was obtained by identifying the spin connection with the gauge connection, $A \approx \omega$, but the existence of zero-modes does not require this, the connections can be varied independently and the zero-modes are guaranteed to persist by the index theorem — provided the connections are kept within their topological classes. Unlike standard Kaluza-Klein theory the structure here does not rely on any special isometry symmetry being present.

4 Harmonic Expansion of zero-modes

The zero-modes of the Dirac operator on CP^n are closely related to the representation theory of $SU(n+1)$. Consider, for example, the index for fermions on CP^2 which are $SU(2)$ singlets with $U(1)$ charge q. Equation (17), with $n=2$, gives

$$\nu_q = \frac{(q+1)(q+2)}{2}. \tag{22}$$

For $q = 0,1,2,3,\ldots$ this gives $\nu_q = 1,3,6,10,\ldots$ and it is no coincidence that these are the dimensions of irreducible representations of $SU(3)$ — more specifically the symmetric irreducible representations. The same is true for CP^3. The representations required in section 3 all had index with $|\nu| = 1$ and so correspond to singlets of $SU(3)$ on CP^2 and singlets of $SU(4)$ on CP^3. This has the

the immediate consequence that there exists a metric and a gauge connection in the relevant topological sector for which these zero-modes are *constant* spinors. Although the existence of zero-modes, being a topological statement, does not require any specific metric on CP^n, one can choose to work with the $SU(n+1)$ symmetric metric (the Fubini-Study metric [7]). There is then a linear combination of spinor components for which the gauge connection exactly cancels the spin connection in the Dirac equation [8]

$$i\not{D}_C\zeta = i\gamma^a e^\mu{}_a \left(\partial_\mu + \omega_\mu + i(\tfrac{p}{\chi_n})A_\mu\right)\zeta, \tag{23}$$

so a solution is to take $\zeta = const$ (p here is the $U(1)$ charge coupling to the $U(1)$ gauge connection — there is a contribution to q from the spin connection too [5]).

The fact that the zero-modes have such a simple structure suggests introducing a new ingredient to the construction presented here. The full space $\mathcal{M} \times \mathcal{C}$ discussed in the introduction is a fourteen dimensional space and so one cannot hope that the standard model would be renormalisable in this space, but we can expect a renormalisable theory if the internal space $\mathcal{C}$ had only a finite number of degrees of freedom rather than the infinite number of a continuum manifold. Borrowing from ideas of Connes [9] this suggests that one might replace the complex projective spaces by finite matrix approximations — fuzzy CP^n's [10]. We are thus led to a picture of the standard model involving non-commutative geometry similar in spirit, but different in detail, to the Connes-Lott model [11]. Fuzzy CP^2 has finite matrix approximations of dimension 1×1, 3×3, $6\times 6,\ldots, d_L\times d_L$, where $d_L = (L+1)(L+2)/2$ are the dimensions of the symmetric representations of $SU(3)$, and fuzzy CP^3 similarly requires matrix algebras whose size is dictated by the symmetric representations of $SU(4)$ [10]. If we replace the continuum $CP^2\times CP^3$ with its fuzzy version, $CP^2_F \times CP^3_F$, then chiral spinors become matrices

$$\zeta(z) \qquad \to \qquad \hat{\zeta} \in Mat_{d_L} \otimes \mathbf{C}^{2(n-1)}, \tag{24}$$

where Mat_{d_L} is the algebra of $d_L \times d_L$ matrices, z is a point in CP^n and $\mathbf{C}^{2(n-1)}$ is chiral spin space. In fact, for the constant spinors that we require for our zero-modes, we only need the trivial representations, $d_L = 1$, for both CP^2_F and CP^3_F.

5 Generations and Yukawa Couplings

The construction described in the previous section only provides one generation of the standard model, because the index of the Dirac operator is ± 1. An obvious question is whether or not it is possible to obtain 3 generations in some way. Since the inclusion of generations requires an $SU(3)$ symmetry it is natural to ask whether or not the $SU(3)$ isometry group of $CP^2 \cong SU(3)/U(2)$ might be able to provide the extra generations, so let us focus on CP^2_F (fuzzy CP^2 was analysed in detail in Ref. [3]). Since the $SU(3)$ generation symmetry is broken in the real world, let us assume that it is broken in our model too. For example we could deform the Fubini-Study metric so that it no longer has $SU(3)$ as its group of Killing vectors. This will change the spin connection ω_μ in equation (23), while leaving the gauge connection A_μ unchanged. We no longer have exact cancellation between the gauge and spin connections, there are still zero-modes but they must become non-trivial functions of position, $\partial_\mu \zeta \neq 0$. An immediate consequence of this is that singlets alone are no longer sufficient for a harmonic expansion of $\zeta(z)$, higher dimensional representation of $SU(3)$ must be included. The simplest possibility is that d_L in (24) is three and $\hat{\zeta}$ is then a 3×3 matrix, with $SU(3)$ representation content $\bar{\mathbf{3}} \times \mathbf{3} = \mathbf{1} + \mathbf{8}$, whose entries depend on the parameters describing the metric deformation. The Dirac operator (1) on $\mathcal{M} \times \mathcal{C}$ acts on spinors $\Psi(x) = \psi(x) \otimes \hat{\zeta}$ where $\psi(x)$ is spinor on $\mathcal{M}$, $(x \in \mathcal{M})$, and we are led to consider

$$\overline{\Psi} i \not{D} \Psi(x) = \left(\overline{\psi(x)} \otimes \hat{\zeta}^\dagger\right)\left(i\not{D}_{\mathcal{M}} \otimes \mathbf{1}\right)\left(\psi(x) \otimes \hat{\zeta}\right) = \left(\overline{\psi(x)} i \not{D}_{\mathcal{M}} \psi(x)\right) \otimes \hat{\zeta}^\dagger \hat{\zeta}. \tag{25}$$

Now $\hat{\zeta}^\dagger \hat{\zeta}$ is a 3×3 hermitian matrix and so can be diagonalised by an $SU(3)$ transformation and the three eigenstates would look like three generations in $\mathcal{M}$.

One can introduce Yukawa couplings in the usual way: $\hat{\zeta}$ represents the set of zero-modes which we shall denote by

$$\hat{Q}_L := \begin{pmatrix} \hat{U}_L \\ \hat{D}_L \end{pmatrix}_{1/6} \qquad \begin{matrix} (\hat{U}_R)_{2/3} \\ (\hat{D}_R)_{-1/3} \end{matrix} \qquad \hat{L}_L := \begin{pmatrix} \hat{N}_L \\ \hat{E}_L \end{pmatrix}_{-1/2} \qquad \begin{matrix} (\hat{E}_R)_{-1} \\ (\hat{N}_R)_0, \end{matrix} \tag{26}$$

where each of these states is a 3×3 matrix (the subscript denotes the hypercharge). Introduce Higgs scalars

$$\Phi = \begin{pmatrix} \varphi_+ \\ \varphi_0 \end{pmatrix}_{1/2} \qquad \Phi_C = (i\sigma_2)\Phi^* = \begin{pmatrix} (\varphi_0)^* \\ -\varphi_- \end{pmatrix}_{-1/2}, \tag{27}$$

which transform under $SU(2) \times U(1)$ as $\mathbf{2}_{1/2}$ and $\mathbf{2}_{-1/2}$ respectively and are constant on CP^2_F. $SU(3)$ singlets can now be constructed in the usual manner:

$$\hat{D}_R^\dagger \mathcal{D}\left(\Phi^\dagger \hat{Q}_L\right) + \hat{U}_R^\dagger \mathcal{U}\left(\Phi_C^\dagger \hat{Q}_L\right) + \hat{E}_R^\dagger \mathcal{L}\left(\Phi^\dagger \hat{L}_L\right), \tag{28}$$

where $\mathcal{D}, \mathcal{U}$ and $\mathcal{L}$ are arbitrary complex 3×3 matrices of Yukawa couplings. The usual argument to derive the CKM matrix goes through without modification — $U(3)$ rotations act on the left of the fermion fields to diagonalise $\mathcal{U}$ and $\mathcal{L}$ and the CKM matrix is in $\mathcal{D}$.

The point of this discussion is to show that all the usual arguments used to derive the CKM matrix work with this fuzzy construction because they only require acting on the fermion generations from the left by $U(3)$, and this does not interfere with the fact that we have diagonalised $\hat{\zeta}^\dagger \hat{\zeta}$ by acting on $\hat{\zeta}$ from the right. All the standard arguments for the neutrino sector go through as well

— one can introduce three Dirac and three Majorana neutrino masses as well as twelve complex couplings [12]. Of course the individual eigenstates here will not be zero-modes of the Dirac operator, in general, and more work is necessary to determine whether or not this suggestion for the origin of the generations is viable.

6 Conclusions

It has been argued that one generation of the standard model can be obtained as a zero-mode of the Dirac operator on $CP^2 \times CP^3$, by introducing fundamental $U(2) \times U(3)$ gauge fields and identifying the spin connection with the gauge connection. The discussion circumvents Witten's no-go theorem for chiral fermions in Kaluza-Klein theories precisely by introducing fundamental gauge fields — indeed fundamental gauge are essential as soon as one considers CP^2 because CP^2 does not admit spinors without them. Three generations can be obtained by considering the $SU(3)$ isometry group of CP^2 to be related to generation symmetry. Introducing concepts from non-commuting geometry and making the complex projective spaces fuzzy allows one to represent the zero-modes as finite matrices and distorting the metric on CP^2 away from the $SU(3)$ symmetric metric can lead to 3×3 matrices whose eigenstates we identify with the three generations.

The picture presented here borrows from many ideas that are in the air at the moment, but it modifies them slightly and puts them together in a rather different way than usual. Kaluza-Klein theory uses the isometry group as the gauge group, but here it is the holonomy group. In Connes' non-commutative version of the standard model two copies of space-time are introduced to accommodate the two-component Higgs field while here three copies of space-time are being introduced to accommodate three generations. Also the three copies used here are being directly related to a fuzzy space and I am not aware of any such interpretation in the literature of the Connes-Lott model — though the two copies used there do look very like a fuzzy sphere and may have such an interpretation (associating the generations with different copies of space-time was suggested in Ref. [13]). Another difference between the construction presented here and the Connes-Lott model is that the gauge symmetries here are not automorphisms of the matrix algebra.

Many questions remain to be investigated in this approach. The holonomy group of $CP^2 \times CP^3$ is $U(2) \times U(3)$, which has two $U(1)$ factors, but only one linear combination, dictated by (20), has been used. This seems natural in view of the structure in equations (15) and (16), but it raises the question of the significance, if any, of the orthogonal combination of $U(1)$ generators, which remains open. The Higgs' fields and Yukawa couplings were introduced by hand here but it would clearly be preferable to have a geometrical interpretation — it would be satisfying if the Yukawa couplings could be incorporated into the Dirac operator on the fuzzy spaces as in the Connes-Lott model. The rôle of the $SU(4)$ isometry group of CP^3 has not been discussed here either — it is tempting the think that it may be related to the 4×4 structure of equation (15) in some, way but this remains to be investigated.

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