A Causal Approach to Massive Yang-Mills Theories

DIAS-STP-95-01

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Work supported by Schweizerischer Nationalfonds

Summary - We study quantized Yang-Mills theory with massive vector fields in the framework of causal perturbation theory. The most general form of the interaction which is invariant under operator gauge transformations is pointed out. The generator of these transformations generally fails to be nilpotent. This defect, however, is easily cured by including scalar fields in the gauge transformations. Due to gauge invariance these scalar gauge fields couple to the Yang-Mills fields with predicted strength. We also show that invariance under ghost charge conjugation fixes the form of the interaction completely. The coupling of the Yang-Mills fields and the scalar gauge fields to matter is investigated. It is proven that gauge invariance implies unitarity of the physical S-matrix. We always work in the Fock space of free quantum fields in which all expressions are mathematically well defined.

1. Introduction

Recently, massless Yang-Mills theory has been succesfully studied in the framework of causal perturbation theory [1-4]. The central object in this approach is the causal S-matrix

$$S[g] = 1 + \sum_{n=1}^{\infty} \int d^4x_1 \cdots d^4x_n T^{(n)}(x_1, \cdots, x_n) \tag{1.1}$$

$T^{(1)}$ specifies the theory. For massless Yang Mills theories it is given by

$$T^{(1)}(x) \stackrel{\text{def}}{=} -ief_{abc}\{\frac{1}{2} : A_{\mu a} A_{\nu b} F_c^{\mu\nu} : - : A_{\mu a} u_b \partial^\mu \tilde{u}_c :\}(x) \tag{1.2}$$

e is the coupling constant and f_{abc} are the structure constants of a non-abelian semi-simple compact gauge group G. A_a^μ are the *free* gauge fields, defined by

$$\partial \cdot \partial A_a^\mu(x) = 0, \quad [A_a^\mu(x), A_b^\nu(y)]_- = i\delta_{ab} g^{\mu\nu} D_0(x-y) \tag{1.3}$$

where D_0 is the Pauli - Jordan commutation function for $m = 0$. $F_a^{\mu\nu}$ are the *free* field strenghts:

$$F_a^{\mu\nu} \stackrel{\text{def}}{=} \partial^\mu A_a^\nu - \partial^\nu A_a^\mu \tag{1.4}$$

and u_a and $\tilde{u}_a$ are the *free* ghost fields:

$$\partial \cdot \partial u_a(x) = \partial \cdot \partial \tilde{u}_a(x) = 0, \quad \{u_a(x), \tilde{u}_b(y)\}_+ = -i\delta_{ab} D_0(x-y) \tag{1.5}$$

$$\{u_a(x), u_b(y)\}_+ = \{\tilde{u}_a(x), \tilde{u}_b(y)\}_+ = 0 \tag{1.6}$$

A detailed discussion of the algebraic properties of these ghost fields can be found in [5].

Differentiating (1.3) we get

$$[\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y)]_- = 0 \tag{1.7}$$

$$[\partial_\mu A_a^\mu(x), F_b^{\kappa\lambda}(y)]_- = 0 \tag{1.8}$$

Despite their simplicity, these equations have important consequences. For, let us consider the operator

$$Q \overset{\text{def}}{=} \int_{x_0=\text{const.}} d^3\vec{x}\, (\partial_\mu A_a^\mu(x))\, \overleftrightarrow{\partial_0}\, u_a(x) \tag{1.9}$$

Using the Leibnitz rule for graded algebras gives

$$Q^2 = \frac{1}{2}\{Q,Q\}_+ = \frac{1}{2} \int_{x_0=\text{const.}} d^3\vec{x} \int_{y_0=\text{const.}} d^3\vec{y} \Big\{ [\partial_\mu A_a^\mu(x), \partial_\nu A_b^\nu(y)]_- \, \overleftrightarrow{\partial_{x^0}}\, \overleftrightarrow{\partial_{y^0}}\, (u_a(x) u_b(y)) +$$
$$+ (\partial_\nu A_b^\nu(y) \partial_\mu A_a^\mu(x))\, \overleftrightarrow{\partial_{x^0}}\, \overleftrightarrow{\partial_{y^0}} \{u_a(x), u_b(y)\}_+ \Big\} = 0 \tag{1.10}$$

Thus eqs. (1.6, 1.7) make Q a differential operator in the sense of homological algebra. This allows for standard homological notions [6,7]: Let $\mathcal{F} = \{F\}$ be the field algebra consisting of the polynomials in the (smeared) gauge and ghost fields and their Wick powers. Consider the ghost charge operator [5]

$$Q_g \overset{\text{def}}{=} \int_{x_0=\text{const.}} d^3\vec{x} \tilde{u}_a(x)\, \overleftrightarrow{\partial_0}\, u_a(x) \tag{1.11}$$

and the corresponding derivation δ_g in $\mathcal{F}$

$$\delta_g F \overset{\text{def}}{=} [Q_g, F]_- \tag{1.12}$$

We say an operator F has ghost charge z if

$$\delta_g F = zF \tag{1.13}$$

Since Q_g has integer spectrum [5] we have $z \in \mathcal{Z}$. The operators F_z with ghost charge z form the subspace $\mathcal{F}_z$, and we obviously have

$$\mathcal{F} = \bigoplus_{z\in\mathcal{Z}} \mathcal{F}_z \tag{1.14}$$

which makes $\mathcal{F}$ a $\mathcal{Z}$-graded algebra. Consider the unitary operator [5]

$$E \overset{\text{def}}{=} (-1)^{Q_g}, \quad E^2 = 1 \tag{1.15}$$

It induces the canonical involution ω in $\mathcal{F}$ by

$$\omega F \overset{\text{def}}{=} EFE, \quad \omega^2 = 1 \tag{1.16}$$

We define the bosonic part F_b and the fermionic part F_f of an operator F by

$$F_{b(f)} \overset{\text{def}}{=} \frac{1}{2}(1 \underset{(-)}{+} \omega) F, \quad \Rightarrow \omega F_{b(f)} = \underset{(-)}{+} F_{b(f)} \tag{1.17}$$

and the graded bracket of two operators F and G by

$$[F,G] = [F_b + F_f, G_b + G_f] \overset{\text{def}}{=} [F_b, G_b]_- + [F_b, G_f]_- + [F_f, G_b]_- + \{F_f, G_f\}_+ \tag{1.18}$$

We also define on $\mathcal{F}$ the operator d_Q by

$$d_Q F \overset{\text{def}}{=} [Q, F] = QF - (\omega F) Q \tag{1.19}$$

This is a differential operator:

$$d_Q^2 = 0, \iff \{Q, [Q, F_b]_-\}_+ = [Q, \{Q, F_f\}_+]_- = 0 \tag{1.20}$$

and an antiderivation with respect to ω:

$$d_Q(FG) = (d_Q F)G + (\omega F)(d_Q G) \tag{1.21}$$

The commutator relation

$$[Q_g, Q] = -Q \tag{1.22}$$

implies

$$[\delta_g, d_Q]_- = -d_Q, \quad \Rightarrow d_Q \mathcal{F}_z \subseteq \mathcal{F}_{z-1} \tag{1.23}$$

i.e. d_Q is a homogeneous homomorphism of degree (-1) over $\mathcal{F}$. This implies in particular that it anticommutes with the canonical involution:

$$\{d_Q, \omega\}_+ = 0 \tag{1.24}$$

We conclude that the quadruplet $\{\mathcal{F}, \delta_g, \omega, d_Q\}$ fits well into the definition of a graded differential algebra [6].

Let us study the action of d_Q on $\mathcal{F}$ more explicitly. We find

$$d_Q A_a^\mu(x) = i\partial^\mu u_a(x) \tag{1.25}$$

$$d_Q u_a(x) = 0, \quad d_Q \tilde{u}_a(x) = -i\partial_\mu A_a^\mu(x) \tag{1.26}$$

Eqs. (1.7, 1.8) immediately give two gauge invariants:

$$d_Q \partial_\mu A_a^\mu(x) = d_Q F_a^{\mu\nu}(x) = 0 \tag{1.27}$$

The above actions of d_Q on F may be called *free* or asymptotic BRS variations since the (formally defined) full BRS variations of interacting fields [7] reduce to them in the absence of interaction. It is exactly these free varations we are intersted in when applying causal perturbation theory, since there we are looking for symmetries of the S-matrix which is defined in the Hilbert - Fock space H of free asymptotic fields. The algebra and homology of free BRS operators is well studied in [7,8]. The variations induced by d_Q are also called operator valued gauge transformations in [1] since they emerge from the usual asymptotic gauge variations in QED [12] by replacing the gauge function χ with the ghost operator u. We will often simply call this asymptotic BRS variations gauge variations and their invariants gauge invariants.

The interaction (1.2) is gauge invariant, i.e. we have

$$d_Q T^{(1)}(x) = \partial_\mu T^{(1)\mu}(x), \quad T^{(1)\mu}(x) \stackrel{\text{def}}{=} e f_{abc} : u_a\{A_{\nu a} F_c^{\mu\nu} + \frac{1}{2} u_b \partial^\mu \tilde{u}_c\} : (x) \tag{1.28}$$

The quintessence of causal perturbation theory is that all higher terms $T^{(n)}$, $n \geq 2$ in (1.1) are determined from $T^{(1)}$ by Poincaré invariance and causality [9-12]. This determination is unique up to some (finite!) normalization constants, which can be determined by the requirement of symmetries and (finitely many) normalization conditions. $T^{(n)}$ is given symbolically by

$$T^{(n)}(x_1, \cdots, x_n) = \Theta[T^{(1)}(x_1) \cdots T^{(n)}(x_n)] \tag{1.29}$$

where Θ means the time ordered product. This, however, *cannot* be constructed by multiplying with step functions, since this would lead to the well known UV-divergences[9,10]. Instead one has to use the method of distribution splitting, developed by Epstein and Glaser [11] and applied to QED, for example, by Scharf [12]. Using exactly this construction Duetsch et al. [1-4] have shown that the Yang-Mills theory specified by (1.2) is gauge invariant in all orders, i.e. the following equations hold true:

$$d_Q T^{(n)}(x_1, \cdots, x_n)] = \sum_{l=1}^{n} \partial_{\mu_l} T^{(n)\mu_l}_{\ \ l}(x_1, \cdots, x_n), \tag{1.30}$$

$$T^{(n)\mu_r}_{r}(x_1,\cdots,x_n)\stackrel{\rm def}{=}\Theta[T^{(1)}(x_1)\cdots T^{(1)\mu_r}(x_r)\cdots T^{(1)}(x_n)] \qquad (1.31)$$

Thus the gauge variation of the $T^{(n)}$ are total divergences. One would like to conclude from this the more convential form of gauge invariance:

$$\lim_{g\to 1} d_Q S[g] = 0 \qquad (1.32)$$

While this adabatic limit is well controlled in massive theories [13] the situation is far more difficult in massless theories, where it generally fails to exist in the S-matrix elements [12,14]. The strength of eq. (1.30) is to give a formulation of gauge invariance which is completely independent of the infrared problems encountered when passing to the adiabatic limit.

The importance of the gauge invariance (1.30) lies in the fact that it enables one to proof the unitary of the physical S-matrix $S_{\rm phys}$ defined in the physical subspace $H_{\rm phys}$ of the total Hilbert-Fock H. The latter one can be defined as the cohomology space of Q or, equivalently, as $\{{\rm Ker}Q\}\ominus_\perp\{{\rm Ran}Q\}$.

The paper at hand aims at the construction of Yang-Mills theories with massive gauge (and ghost) fields. This is usally done via the Higgs mechanism which is known to give a renormalizable, gauge invariant, and unitary perturbation series. While not questioning the validity of this result we here want to develop a different approach.

To clearly enlighten this difference let us briefly summarize the logical steps used to derive the perturbation series for the Higgs model. There the starting point is a classical Yang-Mills theory defined by an action Σ which is invariant under the group of local classical gauge transformations $G_{\rm local}$. The gauge fields are coupled to scalar fields which interact among each other by a mexican hat potential. Then one considers the classical energy which is a functional on the configuration space of classical fields. Due to the peculiar form of the classical potential the classical ground state, i.e. the points in the classical configuration space which minimize the classical energy (often very misleadingly called the vacuum), is found to be degenerate. Then an arbitrary representative point from this state is chosen. This is called spontaneous symmetry breaking. After that new classical fields are defined as the original fields shifted by this reference point. Then a gauge fixing term is added to the action and also the corresponding Fadeev-Popov ghost term included. The total action is then shown to be BRS-invariant. Then the action is split into two parts: The free part being at most quadratically in the shifted fields and the interaction part being at least trilinear in these fields. It is only then that quantization comes into play: The quadratic part of the action defines the quantum kinematical setup, i.e. the shifted classical fields are quantized as free quantum fields in a Hilbert-Fock space H with unique (!) vacuum. These free quantized fields describe the in- and outgoing particles. The interacting part of the action describes the interaction of these particles and allows for the perturbative calculation of Green functions of the corresponding (only formally defined) interaction fields and, most important, determines the S-matrix in H. Again a physical subspace $H_{\rm phys}\subset H$, a formally defined set of physical interacting observables, and the physical S-matrix $S_{\rm phys}$ are defined via the homology of the BRS-transformation. It is then shown that these physical quantities depend neither on the representative point of the ground state of the classical energy chosen above nor on the the gauge fixing term. Eventually $S_{\rm phys}$ is shown to be unitary. In thhis step the BRS-invariance is again the key ingredient. The BRS-charge is often expressed in terms of the only formally defined interacting quantum fields. Due to its conservation it can, however, be expressed in terms of the free asymptotic fields as well. Since these are perfectly well defined the latter method is superior. Moreover, it is exactly this asymptotic BRS-invariance which is needed to proof unitarity of $S_{\rm phys}$.

This suggests our approach to massive Yang-Mills theories: We will not take any reference to the classical theory. So, neither the classical gauge group $G_{\rm local}$ nor the concept of spontaneous symmetry breaking will enter our reasoning. Instead, we immediately start with the quantum theory , defined by given asymptotic massive gauge and ghost fields and by the generator of the causal S-matrix, $T^{(1)}(x)$. This we demand to be invariant under asypmtotic BRS-transformations, and we give a classification of all all $T^{(1)}(x)$ with this property. Since we do not employ the notion of spontaneous symmetry breaking we have no reason to include scalar fields in our discussion from the very beginning. Instead, we derive the presence of these fields by a purely algebraic condition.

The paper is organized as follows: The next chapter deals once more with massless Yang-Mills theories showing that the interaction (1.2) admits gauge invariant generalizations and giving a complete list of them. Chapter 3 starts the investigation of massive Yang-Mills theories. We study theories with only gauge and ghost fields and construct their gauge invariant interactions. We will see, however, that the BRS-charge fails to be a differential operator in this case. Chapter 4 shows how to cure this defect: Scalar fields have to be included in the definition of the BRS-charge to restore its nilpotency. This changes the gauge transformations and we have to determine the gauge invariant interactions once again. It turns out that the scalar fields couple to the gauge fields with predicted strength. Chapter 5 shows how to incorporate matter fields into the theory. It is proven in chapter 6 that gauge invariance implies unitarity of S_{phys}. There the relation between anomalies and unitarity is clarified, too. The critical discussion can be found in the last chapter.

2. Gauge Invariant Interactions of Massless Yang-Mills Fields

Here we will generalize the interaction of massless Yang-Mills and ghost fields given by (1.1). So we have to classify their possible interactions $T := T^{(1)}(x)$. We will restrain the form of these interactions by requiring it to share the following structural properties with the interaction specified in (1.1):

1.) We demand the interaction to be normalizable. Then only normal products of three or four fields can appear in T. In the case of three fields one of them may be differentiated once.
2.) The interaction of the ghost and gauge fields shall be of Yang-Mills type, i.e. the coupling of the gauge and ghost fields, which are in the adjoint representation of the global Group G, shall be proportional to the structure constants f_{abc}.
3.) We constrain the interaction T to be invariant under the global group G. Thus the tensor of the stucure constants has to be contracted with three coloured fields. This excludes a posteriori the coupling of four fields in T. This is quite satisfactory. For, in the next order $T^{(2)}$, causality and gauge invariance will *create* these quartic couplings without further ado. The seagull graph in scalar QED and the four gluon coupling in Yang-Mills theory, for example, are generated this way.
4.) We require T to have vanishing ghost charge, i.e $\delta_{Q_g} T = 0$. This makes the two ghost fields u and $\tilde{u}$ always appear together and particularly implies that T is a bosonic operator.
5.) We will not give up invariance of T under the proper Poincaré group $P_+^\uparrow$, of course. Thus all Lorentz indices have to be contracted. This implies that one of the three fields coupled in T has to be differentiated, because otherwise the number of Lorentz indices would be odd.
6.) In order to have a pseudo-unitary S-matrix T should be anti-pseudo-hermitian.

There exist exactly four linear independent interaction terms T_i fulfilling these conditions:

$$T_1 \stackrel{\text{def}}{=} :-\frac{1}{2} ief_{abc} A_{\mu a} A_{\nu b} F_c^{\mu\nu}: \quad , \quad T_2 \stackrel{\text{def}}{=} :-ief_{abc} A_{\mu a} u_b \partial^\mu \tilde{u}_c: ,$$

$$T_3 \stackrel{\text{def}}{=} -ief_{abc} A_{\mu a} \partial^\mu u_b \tilde{u}_c: \quad , \quad T_4 \stackrel{\text{def}}{=} :ief_{abc} \partial_\mu A_a^\mu u_b \tilde{u}_c: \tag{2.1}$$

and any real linear combination of these terms fulfils these conditions, too. We therefore set

$$T = \sum_{i=1}^{4} \alpha_i T_i \tag{2.2}$$

with a priori arbitrary real constants α_i.

Now we demand, in addition, gauge invariance, i.e.

$$d_Q T = \partial_\mu T^\mu \tag{2.3}$$

Since T is different from the expression (1.2), $T^\mu := T^{(1)\mu}(x)$ will be different from (1.28), too. It shall, however, retain the following structural properties:

1'.) Normalizability: Only normal products of three or for fields may appear in T^μ. In the case of three fields one may be differentiated once.
2'.) T^μ shall be of Yang-Mills type, i.e. the coupling of the gauge and ghost fields shall be proportional to the structure constants f_{abc}.

3'.) G-invarinance: All colour indices in T^μ have to be contracted. This, again, excludes the appearence of normal products of four fields in T^μ.
4'.) T^μ must have ghost charge -1. This implies that either one u and no $\tilde{u}$ or two u and one $\tilde{u}$ are present in T^μ and that T^μ is fermionic.
5'.) $P_+^\uparrow$-covariance: All Lorentz indices except μ have to be contracted.
6'.) T^μ should be pseudo-hermitean.

These properties follow naturally from the corresponding properties 1.)-6.) of T and eq.(2.3). There exist exactly six linear independent terms fulfilling these conditions:

$$T_1^\mu \stackrel{\text{def}}{=} : ef_{abc} u_a A_{\nu b} F_c^{\mu\nu} : \quad , \quad T_2^\mu \stackrel{\text{def}}{=} : -ef_{abc} u_a A_b^\mu \partial_\nu A_c^\nu : ,$$

$$T_3^\mu \stackrel{\text{def}}{=} : -ef_{abc} u_a A_{\nu b} G_c^{\mu\nu} : \quad , \quad T_4^\mu \stackrel{\text{def}}{=} : \frac{1}{2} ef_{abc} u_a u_b \partial^\mu \tilde{u}_c : ,$$

$$T_5^\mu \stackrel{\text{def}}{=} =: ef_{abc} u_a \partial^\mu u_b \tilde{u}_c : \quad , \quad T_6^\mu \stackrel{\text{def}}{=} : ef_{abc} \partial^\nu u_a A_{\nu b} A_c^\mu : , \tag{2.4}$$

Here we have introduced $G_a^{\mu\nu} := \partial^\mu A_a^\nu + \partial^\nu A_a^\mu$. Any real linear combination of the six expressions above fulfils the requirements 1'-6', too. We therefore set

$$T^\mu = \sum_{j=1}^{6} \beta_j T_j^\mu \tag{2.5}$$

with a priori arbitrary real constants β_j.

To study (2.3) we need

$$d_Q T_1 =: ef_{abc} A_{\mu a} \partial_\nu u_b F_c^{\mu\nu} : \quad , \quad d_Q T_2 =: ef_{abc} \left(A_{\mu a} u_b \partial^\mu \partial_\nu A_c^\nu + \partial_\mu u_a u_b \partial^\mu \tilde{u}_c \right) : ,$$

$$d_Q T_3 =: ef_{abc} \left(A_{\mu a} \partial^\mu u_b \partial_\nu A_c^\nu + \partial_\mu u_a \partial^\mu u_b \tilde{u}_c \right) : \quad , \quad d_Q T_4 = 0 \tag{2.6}$$

$$\partial_\mu T_1^\mu =: ef_{abc} \left(A_{\mu a} \partial_\nu u_b F_c^{\mu\nu} + A_{\mu a} u_b \partial^\mu \partial_\nu A_c^\nu \right) : \quad , \quad \partial_\mu T_2^\mu =: ef_{abc} \left(A_a^\mu u_b \partial_\mu \partial_\nu A_c^\nu + A_a^\mu \partial_\mu u_b \partial_\nu A_c^\nu \right) : ,$$

$$\partial_\mu T_3^\mu =: ef_{abc} \left(A_{\mu a} \partial_\nu u_b G_c^{\mu\nu} + A_{\mu a} u_b \partial^\mu \partial_\nu A_c^\nu \right) : \quad , \quad \partial_\mu T_4^\mu =: ef_{abc} \partial_\mu u_a u_b \partial^\mu \tilde{u}_c : ,$$

$$\partial_\mu T_5^\mu =: ef_{abc} \left(\partial^\mu u_a \partial_\mu u_b \tilde{u}_c + \partial^\mu u_a u_b \partial_\mu \tilde{u}_c \right) : \quad , \quad \partial_\mu T_6^\mu =: ef_{abc} \left(-A_{\mu a} \partial^\mu u_b \partial_\nu A_c^\nu + A_{\mu a} \partial_\nu u_b \partial^\mu A_c^\nu \right) : \tag{2.7}$$

To derive these formulae we have used that the fields obey the wave equation. By inserting them in (2.3) we get a system of linear homogeneous equations for the coefficients α_i and β_j. Due to its homogeneity we can certainly choose freely the overall normalization of these coefficients, and we do that in setting $\alpha_1 = 1$. The solution of the equations turns out to have three additional free parameters which we call α, β, and γ. The general solution is:

$$T = T_1 + \left(\frac{1}{2} - \alpha \right) T_2 + \left(-\frac{1}{2} - \alpha \right) T_3 + (\alpha + \beta) T_4 \tag{2.8}$$

$$T^\mu = (1+\gamma) T_1^\mu + \left(-\frac{1}{2} - \alpha - 2\gamma \right) T_2^\mu + \gamma T_3^\mu + T_4^\mu + \left(-\frac{1}{2} - \alpha \right) T_5^\mu - 2\gamma T_6^\mu \tag{2.9}$$

Let us now study the structure of these expressions. We first remark that the special choice $\alpha = -\beta = -\frac{1}{2}$ leads us back to the original interaction (1.2). Setting, in addition, $\gamma = 0$ also reproduces (1.28). This choice is distinguished by its minimality: Firstly, there are only two terms in T and T^μ. Secondly, only four elemantary fields are used: A, F, u, and $\partial\tilde{u}$. The other four elementary fields G, $\partial \cdot A$, ∂u, and $\tilde{u}$ do not appear at all. This shortens lengthy higher order calculations and the very elaborate proof of gauge invariance in all orders [1-4] by a considerable amount.

Another preferred choice is $\alpha = \beta = \gamma = 0$. This gives

$$T = ief_{abc} : \left(-\frac{1}{2} A_{\mu a} A_b^\nu F_c^{\mu\nu} - \frac{1}{2} A_{\mu a} u_b \overleftrightarrow{\partial^\mu} \tilde{u}_c \right) : \tag{2.10}$$

$$T^\mu = ef_{abc} : u_a \left(A_{\nu b} F_c^{\mu\nu} + \frac{1}{2} A_b^\mu \partial_\nu A_c^\nu + \frac{1}{2} u_b \overleftrightarrow{\partial^\mu} \tilde{u}_c \right) : \tag{2.11}$$

This T has an additional symmetry: It is invariant under the ghost charge conjugation C_g [5]. This unitary operator reflects the gauge charge:

$$C_g Q_g C_g^{-1} = -Q_g \tag{2.12}$$

and acts on the ghost fields in the following way:

$$C_g u_a(x) C_g^{-1} = i\tilde{u}_a(x) \quad , \quad C_g \tilde{u}_a(x) C_g^{-1} = i u_a(x) \tag{2.13}$$

This implies indeed:

$$C_g T C_g^{-1} = T \tag{2.14}$$

and the choice $\alpha = \beta = 0$ is the only one making this equation hold true. This T is actually not only invariant under ghost charge conjugation; it is invariant under SU(1,1) - "rotations" in ghost space, too [5].

The three parameter freedom in T, T^μ has the following interpretation:

I.) The terms in T which are multiplied by α: T_α, are a pure divergence:

$$T_\alpha = -T_2 - T_3 + T_4 = \partial_\mu \left(ief_{abc} : A_a^\mu u_b \tilde{u}_c \right) : \stackrel{\text{def}}{=} \partial_\mu H^\mu \tag{2.15}$$

Since Q is x - independent, their gauge variation is a pure divergence, too:

$$\partial_Q T_\alpha = d_Q \partial_\mu H^\mu = \partial_\mu \left(d_Q H^\mu \right) = \partial_\mu T_\alpha^\mu \tag{2.16}$$

where

$$T_\alpha^\mu \stackrel{\text{def}}{=} -T_2^\mu - T_5^\mu \tag{2.17}$$

are exactly that terms in T^μ which are multiplied by α. It follows that the couple (T_α, T_α^μ) fulfils (2.3) separately.

II.) The term in T which is multiplied by β: T_β, is a pure gauge, i.e. a d_Q-boundary:

$$T_\beta = T_4 = d_Q L \quad , \quad L \stackrel{\text{def}}{=} -\frac{1}{2} ef_{abc} : u_a \tilde{u}_b \tilde{u}_c : \tag{2.18}$$

Since d_Q is a differential operator it is also a d_Q-cycle, i.e gauge invariant:

$$d_Q T_\beta = 0 \tag{2.19}$$

It can, therefore, freely be added to T without invalidating (2.3).

III.) The terms in T^μ which are multiplied by γ: T_γ^μ, are a conserved trilinear current:

$$T_\gamma^\mu = T_1^\mu - 2T_2^\mu + T_3^\mu - 2T_6^\mu = 2\partial_\nu \{ : ef_{abc} u_a A_b^\mu A_c^\nu : \} \stackrel{\text{def}}{=} K^\mu \quad , \quad \partial_\mu K^\mu = 0 \tag{2.20}$$

It follows that they can freely be added to T^μ without invalidating (2.3).

The discussion above allows to reformulate (2.8, 2.9) as

$$T = T_1 + \frac{1}{2}(T_2 - T_3) + \alpha\, \partial_\mu H^\mu + \beta\, d_Q L \ ,$$

$$T^\mu = T_1^\mu + T_4^\mu - \frac{1}{2}(T_2^\mu + T_5^\mu) + \alpha\, d_Q H^\mu + \gamma\, K^\mu \tag{2.21}$$

In the next chapter we will study how these structures change if the gauge fields and ghost fields are massive.

3. A Direct Route to Massive Yang-Mills Fields

To construct a theory of massive Yang-Mills fields we have to use free asymptotic massive gauge and ghost fields:

$$(\partial\cdot\partial+M^2)A_a^\mu(x)=(\partial\cdot\partial+M^2)u_a(x)=(\partial\cdot\partial+M^2)\tilde{u}_a(x)=0 \tag{3.1}$$

$$[A_a^\mu(x),A_b^\nu(y)]_-=i\delta_{ab}g^{\mu\nu}D_M(x-y) \tag{3.2}$$

$$\{u_a(x),\tilde{u}_b(y)\}_+=-i\delta_{ab}D_M(x-y),\ \ \{u_a(x),u_b(y)\}_+=\{\tilde{u}_a(x),\tilde{u}_b(y)\}_+=0 \tag{3.3}$$

where D_M, the Pauli-Jordan commutation function for mass $M>0$, appears. The free massive field strenghts $F_a^{\mu\nu}$ are defined as in (1.4). We have given all coloured fields the same mass, since we do not discuss breaking of the global group G here, while the ghost and the gauge fields have the same mass because they transform among each other under gauge transformations.

The nonvanishing of the mass M has simple but far reaching consequences: While (1.8) remains true,

$$[\partial_\mu A_a^\mu(x),F_b^{\kappa\lambda}(y)]_-=0 \tag{3.4}$$

(1.7) is altered to

$$[\partial_\mu A_a^\mu(x),\partial_\nu A_b^\nu(y)]_-=iM^2\delta_{ab}D_M(x-y) \tag{3.5}$$

Let us define the gauge charge Q by

$$Q\stackrel{\mathrm{def}}{=}\int\limits_{x_0=\mathrm{const.}}d^3\vec{x}\,(\partial_\mu A_a^\mu(x))\overleftrightarrow{\partial_0}\,u_a(x) \tag{3.6}$$

While this is the same *expression* as in the massless case, it is, of course, a different operator, because now the quantized fields in the integral are the massive ones. Its square is given by

$$Q^2=\frac{1}{2}\{Q,Q\}_+=\frac{1}{2}\int\limits_{x_0=\mathrm{const.}}d^3\vec{x}\int\limits_{y_0=\mathrm{const.}}d^3\vec{y}\cdot$$

$$\cdot\Big\{[\partial_\mu A_a^\mu(x),\partial_\nu A_b^\nu(y)]_-\overleftrightarrow{\partial_{x^0}}\,\overleftrightarrow{\partial_{y^0}}\,(u_a(x)u_b(y))+(\partial_\nu A_b^\nu(y)\partial_\mu A_a^\mu(x))\overleftrightarrow{\partial_{x^0}}\,\overleftrightarrow{\partial_{y^0}}\{u_a(x),u_b(y)\}_+\Big\} \tag{3.7}$$

and this is due to the nonvanishing commutator (3.5) unequal to zero, in contrast to the massless case (1.10). Instead it is given by

$$Q^2=iM^2Q_u\quad,\quad Q_u\stackrel{\mathrm{def}}{=}i\int\limits_{x_0=\mathrm{const.}}d^3\vec{x}u_a(x)\overleftrightarrow{\partial^0}\,u_a(x) \tag{3.8}$$

The charge Q_u has been discussed in the framework of the ghost charge algebra in [5]. So Q fails to be a differental operator and homological notions do not apply. This applies as well to the gauge variation d_Q, which is defined by (1.19) also in the massive case: (1.20) is changed to

$$d_Q^2=iM^2\delta_u \tag{3.9}$$

where δ_u is the derivation induced by Q_u. The algebraic Eqs. (1.11-1.19, 1.21-1.24) remain true in the massive theory, while the gauge variation of the basic fields changes from (1.25)-(1.27) to

$$d_QA_a^\mu(x)=i\partial^\mu u_a(x)\ ,\ d_QF_a^{\mu\nu}(x)=0\ ,\ d_Q\left(\partial_\mu A_a^\mu(x)\right)=iM^2u_a(x)\ , \tag{3.10}$$

$$d_Qu_a(x)=0\ ,\ d_Q\tilde{u}_a(x)=-i\partial_\mu A_a^\mu(x) \tag{3.11}$$

Now we look again for the general gauge invaraint interaction, i.e. any couple (T,T^μ) fulfilling the general conditions discussed in the preceding chapter and $d_QT=\partial_\mu T^\mu$. We can take over the expressions

(2.1) and (2.4) and the Ansätze (2.2) and (2.5). Since, however, the fields in these expressions are now massive (2.6) and (2.7) change to

$$d_Q T_1 =: ef_{abc} A_{\mu a} \partial_\nu u_b F_c^{\mu\nu} : \quad , \quad d_Q T_2 =: ef_{abc} \left(A_{\mu a} u_b \partial^\mu \partial_\nu A_c^\nu + \partial_\mu u_a u_b \partial^\mu \tilde{u}_c \right) : ,$$

$$d_Q T_3 =: ef_{abc} \left(A_{\mu a} \partial^\mu u_b \partial_\nu A_c^\nu + \partial_\mu u_a \partial^\mu u_b \tilde{u}_c \right) : \quad , \quad d_Q T_4 =: ef_{abc} M^2 u_a u_b \tilde{u}_c : \tag{3.12}$$

$$\partial_\mu T_1^\mu =: ef_{abc} \left(A_{\mu a} \partial_\nu u_b F_c^{\mu\nu} + A_{\mu a} u_b \partial^\mu \partial_\nu A_c^\nu \right) : \quad , \quad \partial_\mu T_2^\mu =: ef_{abc} \left(A_a^\mu u_b \partial_\mu \partial_\nu A_c^\nu + A_a^\mu \partial_\mu u_b \partial_\nu A_c^\nu \right) : ,$$

$$\partial_\mu T_3^\mu =: ef_{abc} \left(A_{\mu a} \partial_\nu u_b G_c^{\mu\nu} + A_{\mu a} u_b \partial^\mu \partial_\nu A_c^\nu \right) : \quad , \quad \partial_\mu T_4^\mu =: ef_{abc} \left(\partial_\mu u_a u_b \partial^\mu \tilde{u}_c - \frac{1}{2} M^2 u_a u_b \tilde{u}_c \right) : ,$$

$$\partial_\mu T_5^\mu =: ef_{abc} \left(\partial^\mu u_a \partial_\mu u_b \tilde{u}_c + \partial^\mu u_a u_b \partial_\mu \tilde{u}_c - M^2 u_a u_b \tilde{u}_c \right) : ,$$

$$\partial_\mu T_6^\mu =: ef_{abc} \left(-A_{\mu a} \partial^\mu u_b \partial_\nu A_c^\nu + A_{\mu a} \partial_\nu u_b \partial^\mu A_c^\nu \right) : \tag{3.13}$$

This leads to the result

$$T = T_1 + \frac{1}{2}(T_2 - T_3) + \alpha\, \partial_\mu H^\mu,$$

$$T^\mu = T_1^\mu + T_4^\mu - \frac{1}{2}\left(T_2^\mu + T_5^\mu \right) + \alpha\, d_Q H^\mu + \gamma\, K^\mu \tag{3.14}$$

where H^μ and K^μ are the expressions defined in (2.15) and (2.20).

The result has the same structure and interpretation as (2.21) in the massless theory, but for one difference: There is no free term of the form $\beta\, d_Q L$ here. The reason is clear: Since d_Q fails to be a differential operator such a term would not be gauge invariant. Since this term is absent, no analogon of the minimal choice exists in the massive theory. A C_g - symmetric interaction T, however, does exist, and is again uniquely given by setting $\alpha = 0$. For this case, the expressions T and T^μ are identical to those in the massless case. So one might say that this additional symmetry has stabilized the theory against perturbations by mass terms.

We have succeeded in constructing gauge invariant interactions for massive Yang-Mills fields. This theory, however, has no physical interpretation. For, we will see in chapter six that the not abundanable unitarity of the physical S-matrix is only guaranteed if both equations: $d_Q T = \partial_\mu T^\mu$ and $Q^2 = 0$ hold true, but the latter one fails here. These problems have also been studied in the canonical framework [7].

We would like to point out that the theory defined above may none the less be quite useful: It is a gauge invariant infrared regulator for the massless theory, and a properly done comparsion of the two can give important results.

Knowing now exactly where the straightforward approach to massive quantized Yang - Mills fields fails we will not find it to difficult to cure this problem. This is done in the next chapter.

4. The Algebraic Introduction of Scalar Gauge Fields

From (3.7) we learn that the the reason why Q fails to be a differential operator is simply the nonvanishing commutator (3.5). Consider now $\dim G$ hermitean free quantized Klein-Gordon fields $h_a(x)$ which are, like the gauge fields A_a^μ and the ghosts fields u_a and $\tilde{u}_a$, in the adjoint representation of G and which have the same mass as these fields. They obey

$$(\partial \cdot \partial + M^2) h_a(x) = 0 \ , \ [h_a(x), h_b(y)]_= - i\delta_{ab} D_M(x-y) \tag{4.1}$$

Their commutuator has the opposite sign to (3.5). It follows that the fields $\partial_\mu A_a^\mu(x) + M h_a(x)$ have vanishing commutators with themselves:

$$[\partial_\mu A_a^\mu(x) + M h_a(x), \partial_\nu A_b^\nu(y) + M h_b(y)]_- = 0 \tag{4.2}$$

This suggests the following new definition for Q:

$$Q \stackrel{\text{def}}{=} \int\limits_{x_0=\text{const.}} d^3\vec{x}\, \left(\partial_\mu A_a^\mu(x) + M h_a(x) \right) \overleftrightarrow{\partial_0}\, u_a(x) \tag{4.3}$$

For, this implies

$$Q^2 = \frac{1}{2} \int\limits_{x_0=\text{const.}} d^3\vec{x} \int\limits_{y_0=\text{const.}} d^3\vec{y} \Big\{ [\partial_\mu A_a^\mu(x) + Mh_a(x), \partial_\nu A_b^\nu(y) + Mh_b(y)]_- \overleftrightarrow{\partial}_{x^0} \overleftrightarrow{\partial}_{y^0} (u_a(x) u_b(y)) +$$

$$+ \big(\partial_\nu A_b^\nu(y) + Mh_b(y)\big)\big(\partial_\mu A_a^\mu(x) + Mh_a(x)\big) \overleftrightarrow{\partial}_{x^0} \overleftrightarrow{\partial}_{y^0} \{u_a(x), u_b(y)\}_+ \Big\} = 0 \tag{4.4}$$

So we have managed to recover this important algebraic property which, together with gauge invariance, guarantees unitarity of S_{phys}! Note that all eqs. (1.11-1.20) hold true anew.

The gauge variations of the elementary fields are given by

$$d_Q A_a^\mu(x) = i\partial^\mu u_a(x) \ , \ d_Q h_a(x) = iMu_a(x) \ , \ d_Q u_a(x) = 0 \ , \ d_Q \tilde{u}_a(x) = -i\big(\partial_\mu A_a^\mu(x) + Mh_a(x)\big) \tag{4.5}$$

Consequently,

$$d_Q \partial_\mu A_a^\mu(x) = -iM^2 u_a(x) \ , \ d_Q F_a^{\mu\nu}(x) = d_Q\big(\partial_\mu A_a^\mu(x) + Mh_a(x)\big) = 0 \tag{4.6}$$

We see that the scalar fields $h_a(x)$ are effected by the gauge variation d_Q and that they appear in the gauge variations of other fields. Hence it is appropriate to call them scalar gauge fields. We will see in chapter six that these fields are unphysical, i.e. their projections onto H_{phys} vanish.

We now have to determine the possible gauge invariant interactions (T, T^μ) once more. We will not dispense with the structural conditions discussed in chapter 2. These conditions, however, can now be fulfiled by more expressions T_i and T_i^μ than in the preceding chapters, since the presence of the scalar fields allows for the constructions of new terms. We give the following complete list:

$$T_1 \stackrel{\text{def}}{=} : -\frac{1}{2} ie f_{abc} A_{\mu a} A_{\nu b} F_c^{\mu\nu} : \ , \ T_2 \stackrel{\text{def}}{=} : -ie f_{abc} A_{\mu a} u_b \partial^\mu \tilde{u}_c : \ , \ T_3 \stackrel{\text{def}}{=} -ie f_{abc} A_{\mu a} \partial^\mu u_b \tilde{u}_c : \ ,$$

$$T_4 \stackrel{\text{def}}{=} : ie f_{abc} \partial_\mu A_a^\mu u_b \tilde{u}_c : \ , \ T_5 \stackrel{\text{def}}{=} : ie f_{abc} M h_a u_b \tilde{u}_c : \ , \ T_6 \stackrel{\text{def}}{=} : \frac{1}{2} ie f_{abc} A_{\mu a} h_b \overleftrightarrow{\partial^\mu} h_c : \tag{4.7}$$

$$T_1^\mu \stackrel{\text{def}}{=} : e f_{abc} u_a A_{\nu b} F_c^{\mu\nu} : \ , \ T_2^\mu \stackrel{\text{def}}{=} : -e f_{abc} u_a A_b^\mu \partial_\nu A_c^\nu : \ , \ T_3^\mu \stackrel{\text{def}}{=} : -e f_{abc} u_a A_{\nu b} G_c^{\mu\nu} : \ ,$$

$$T_4^\mu \stackrel{\text{def}}{=} : \frac{1}{2} e f_{abc} u_a u_b \partial^\mu \tilde{u}_c : \ , \ T_5^\mu \stackrel{\text{def}}{=} =: e f_{abc} u_a \partial^\mu u_b \tilde{u}_c : \ , \ T_6^\mu \stackrel{\text{def}}{=} : e f_{abc} \partial^\nu u_a A_{\nu b} A_c^\mu : \ ,$$

$$T_7^\mu \stackrel{\text{def}}{=} : -e f_{abc} M u_a A_b^\mu h_c : \ , \ T_8^\mu \stackrel{\text{def}}{=} -\frac{1}{2} e f_{abc} u_a h_b \overleftrightarrow{\partial^\mu} h_c \tag{4.8}$$

Using

$$d_Q T_1 =: e f_{abc} A_{\mu a} \partial_\nu u_b F_c^{\mu\nu} : \ , \ d_Q T_2 =: e f_{abc} \left(A_{\mu a} u_b \partial^\mu \big[\partial_\nu A_c^\nu + M h_c\big] + \partial_\mu u_a u_b \partial^\mu \tilde{u}_c \right) :$$

$$d_Q T_3 =: e f_{abc} \left(A_{\mu a} \partial^\mu u_b \big[\partial_\nu A_c^\nu + M h_c\big] + \partial_\mu u_a \partial^\mu u_b \tilde{u}_c \right) : \ , \ d_Q T_4 =: e f_{abc} \left(-M \partial_\mu A_a^\mu u_b h_c + M^2 u_a u_b \tilde{u}_c \right) : \ ,$$

$$d_Q T_5 =: e f_{abc} \left(M \partial_\mu A_a^\mu u_b h_c - M^2 u_a u_b \tilde{u}_c \right) : \ ,$$

$$d_Q T_6 =: e f_{abc} \left(-\partial_\mu u_a h_b \partial^\mu h_c - M A_{\mu a} u_b \partial^\mu h_c - M A_{\mu a} h_b \partial^\mu u_c \right) : \tag{4.9}$$

$$\partial_\mu T_1^\mu =: e f_{abc} \left(A_{\mu a} \partial_\nu u_b F_c^{\mu\nu} + A_{\mu a} u_b \partial^\mu \partial_\nu A_c^\nu \right) : \ , \ \partial_\mu T_2^\mu =: e f_{abc} \left(A_a^\mu u_b \partial_\mu \partial_\nu A_c^\nu + A_a^\mu \partial_\mu u_b \partial_\nu A_c^\nu \right) : \ ,$$

$$\partial_\mu T_3^\mu =: e f_{abc} \left(A_{\mu a} \partial_\nu u_b G_c^{\mu\nu} + A_{\mu a} u_b \partial^\mu \partial_\nu A_c^\nu \right) : \ , \ \partial_\mu T_4^\mu =: e f_{abc} \left(\partial_\mu u_a u_b \partial^\mu \tilde{u}_c - \frac{1}{2} M^2 u_a u_b \tilde{u}_c \right) : \ ,$$

$$\partial_\mu T_5^\mu =: e f_{abc} \left(\partial^\mu u_a \partial_\mu u_b \tilde{u}_c + \partial^\mu u_a u_b \partial_\mu \tilde{u}_c - M^2 u_a u_b \tilde{u}_c \right) : \ ,$$

$$\partial_\mu T_6^\mu =: e f_{abc} \left(-A_{\mu a} \partial^\mu u_b \partial_\nu A_c^\nu + A_{\mu a} \partial_\nu u_b \partial^\mu A_c^\nu \right) : \ ,$$

$$\partial_\mu T_7^\mu =: e f_{abc} M \left(\partial_\mu A_a^\mu u_b h_c + A_a^\mu \partial_\mu u_b h_c + A_a^\mu u_b \partial_\mu h_c \right) : \ ,$$

$$\partial_\mu T_8^\mu =: e f_{abc} h_a \partial_\mu u_b \partial^\mu h_c \tag{4.10}$$

we find the general gauge invariant interaction as follows:

$$T = T_1 + (\frac{1}{2} - \alpha)T_2 - (\frac{1}{2} + \alpha)T_3 + (\alpha + \beta)T_4 + \beta\, T_5 + \frac{1}{2}T_6 =$$

$$= T_1 + \frac{1}{2}(T_2 - T_3 + T_6) + \alpha\, \partial_\mu H^\mu + \beta\, d_Q L \tag{4.11}$$

$$T^\mu = (1+\gamma)T_1^\mu - (\frac{1}{2} + \alpha + 2\gamma)T_2^\mu + \gamma\, T_3^\mu + T_4^\mu - (\frac{1}{2} + \alpha)T_5^\mu - 2\gamma\, T_6^\mu - \alpha\, T_7^\mu + \frac{1}{2}T_8^\mu =$$

$$= T_1^\mu + T_4^\mu + \frac{1}{2}(T_8^\mu - T_2^\mu - T_5^\mu) + \alpha\, d_Q H^\mu + \gamma\, K^\mu \tag{4.12}$$

where

$$H^\mu \stackrel{\text{def}}{=} :ie f_{abc} A_a^\mu u_b \tilde{u}_c: \;,\; L \stackrel{\text{def}}{=} :-\frac{1}{2} e f_{abc} u_a u_b \tilde{u}_c: \;,\; K^\mu \stackrel{\text{def}}{=} 2\partial_\nu \{: e f_{abc} u_a A_b^\mu A_c^\nu :\} \tag{4.13}$$

We notice that the term T_6 enters in T with a fixed coefficient. This term describes the interaction of the scalar gauge fields h_a with the Yang-Mills fields A_a^μ. Thus the strength of this interaction is fixed by the condition of gauge invariance. The free terms multiplied by α, β, and γ have the same interpretation as in the massless case: $\partial_\mu H^\mu$ is a pure divergence, $d_Q L$ is pure gauge, and K^μ is a conserved current. The condition of C_g-invariance on T uniquely fixes $\alpha = \beta = 0$. Since the current K^μ is not related to the gauge structure of the theory, $\gamma = 0$ is certainly a sensible choice, too. In this case the interaction is given explicitly by

$$T = ie f_{abc} : \left\{ -\frac{1}{2} A_{\mu a} A_{\nu b} F_c^{\mu\nu} - \frac{1}{2} A_{\mu a} u_b \overleftrightarrow{\partial^\mu}\, \tilde{u}_c + \frac{1}{4} A_{\mu a} h_b \overleftrightarrow{\partial^\mu}\, h_c \right\} \;,$$

$$T^\mu = e f_{abc} : u_a \left\{ A_{\nu b} F_c^{\mu\nu} + \frac{1}{2} u_b \overleftrightarrow{\partial^\mu}\, \tilde{u}_c + \frac{1}{2} A_b^\mu \partial_\nu A_c^\nu - \frac{1}{4} h_b \partial^\mu h_c \right\} : \tag{4.14}$$

We have now succeeded in constructing gauge invariant interactions of Yang-Mills fields, ghost fields, and scalar gauge fields. In the next chapter we study how these fields couple to matter fields.

5. The Coupling of Matter Fields

All fields we have studied so far, the Yang-Mills fields A_a^μ, the scalars h_a, and the ghosts u_a and $\tilde{u}_a$ may be called gauge fields, since they transformform among each other under the gauge variation d_Q. Now we will study additional fields, which we call matter fields. We will study scalar matter fields: ϕ_i (adjoint fields: ϕ_i^+) and Dirac matter fields: ψ_i (Dirac-adjoint fields: $\overline{\psi}_i = \psi_i^+ \gamma_5$). i is an internal index numbering different fields, while the bispinor indices α for the Dirac fields are always supressed. Let us assume that the fields ϕ_i, ψ_i transform under a certain irreducible representation$R$ of G in which the hermitean generators are given by R_a^{ij}. The matter fields will form bilinear currents in the following way:

1.) The scalar fields constitute the currents

$$S_a^\mu =: \phi_i^+ R_a^{ij}\, i \overleftrightarrow{\partial^\mu}\, \phi_j : \tag{5.1}$$

If we demand strictly G-invariance, all fields ϕ_i should have the same mass m_S:

$$(\partial \cdot \partial + m_S^2)\phi_i = 0 \tag{5.2}$$

In this case the currents are conserved:

$$\partial_\mu S_a^\mu = 0 \tag{5.3}$$

Let us, however allow for G-breaking in the mass sector of the scalar matter fields, i.e. replace (5.2) by

$$\left(\partial \cdot \partial\, \delta^{ij} + (m_S^2)^{ij}\right) \phi_j = 0 \tag{5.4}$$

where (m_S^2) is the positive mass-square matrix of the scalar fields in R. Then (5.3) does not hold. Instead we find

$$\partial_\mu S_a^\mu =: \phi_i^+ i \left[(m_S^2), R_a\right]_-^{ij} \phi_j : \tag{5.5}$$

2.) The Dirac fields form two kinds of currents. The vector currents are defined by

$$V_a^\mu =: \overline{\psi}_i R_a^{ij} \gamma^\mu \psi_j : \tag{5.6}$$

while the axial currents are given by

$$X_a^\mu =: \overline{\psi}_i R_a^{ij} \gamma^\mu \gamma_5 \psi_j : \tag{5.7}$$

Strict G-invariance would require that all Dirac fields in R have the same mass m_D:

$$(i\gamma^\mu \partial_\mu - m_D)\psi_i = 0 \tag{5.8}$$

In this case the vector current would be conserved and the divergence of the axial current would be the pseudo-scalar

$$\partial_\mu V_a^\mu = 0 \ , \ \partial_\mu X_a^\mu = 2m_D : \overline{\psi}_i R_a^{ij} i\gamma_5 \psi_j : \tag{5.9}$$

Let us, however allow for G-breaking in the mass sector of the Dirac matter fields, i.e. replace (5.8) by

$$\left(i\gamma^\mu \partial_\mu \delta^{ij} - (m_D)^{ij}\right) \psi_j = 0 \tag{5.10}$$

where (m_D) is the hermitean mass matrix of the Dirac fields in R. Then (5.9) does not hold. Instead we find

$$\partial_\mu V_a^\mu =: \overline{\psi}_i i \left[(m_D), R_a\right]_-^{ij} \psi_j : \ , \ \partial_\mu X_a^\mu =: \overline{\psi}_i \left\{(m_D), R_a\right\}_+^{ij} i\gamma_5 \psi_j : \tag{5.11}$$

The interaction T, T^μ constructed in the last chapter contains only gauge fields. So let us call it T_{gauge}, T^μ_{gauge} from now on. We now add to it T_{matter}, T^μ_{matter} to desribe the interaction of the matter fields with the ghost fields:

$$T = T_{\text{gauge}} + T_{\text{matter}} \ , \ T^\mu = T^\mu_{\text{gauge}} + T^\mu_{\text{matter}} \tag{5.12}$$

T_{matter} is constructed by coupling the above currents to the gauge fields: Let us introduce the total current

$$J_a^\mu \stackrel{\text{def}}{=} i(e_S S_a^\mu + e_V V_a^\mu + e_X X_a^\mu) \tag{5.13}$$

and set

$$T_{\text{matter}} = iJ_a^\mu A_{\mu a} + \cdots \tag{5.14}$$

The dots indicate that we will soon add other terms to this expression. For, let us study gauge invariance. Since Q (4.3) is entirely composed of gauge fields it commutes with the matter fields and their currents:

$$d_Q \phi_i = d_Q \psi_i = d_Q J_a^\mu = 0 \tag{5.15}$$

This leads to

$$d_Q T_{\text{matter}} = -J_a^\mu \partial_\mu u_a + d_Q(\cdots) = \partial_\mu \left\{-J_a^\mu u_a\right\} + \left(\partial_\mu J_a^\mu\right) u_a + d_Q(\cdots) \tag{5.16}$$

The first term on the right hand side of this equation is already a divergence. The second term vanishes iff the currents J_a^μ are conserved but is not a divergence otherwise. So it has to be compensated by the third term, i.e. we get the condition

$$d_Q(\cdots) = -\left(\partial_\mu J_a^\mu\right) u_a \tag{5.17}$$

which is easily solved by

$$\cdots = iM^{-1} \left(\partial_\mu J_a^\mu\right) h_a \tag{5.18}$$

Such we have found the following gauge invariant interaction of the matter fields with the gauge fields:

$$T_{\text{matter}} = i\left\{J_a^\mu A_{\mu a} + M^{-1}\partial_\mu J_a^\mu h_a\right\} \ , \ T^\mu_{\text{matter}} = -J_a^\mu u_a \ , \ d_Q T_{\text{matter}} = \partial_\mu T^\mu_{\text{matter}} \tag{5.19}$$

Let us interpret this result. In the case of conserved currents the matter fields couple only to the Yang-Mills fields and this interaction has the same form as the coupling of matter fields to massless Yang-Mills fields [15]. More interesting is the case of nonconserved currents: There the matter fields couple to the scalar gauge fields, too. We conclude that the scalar gauge fields are a very important part of the whole theory: They allow a consistent treatment of massive Yang-Mills fields and of nonconserved currents at the same time. We also notice that the coupling of the nonconseved currents to the scalar gauge fields is proportional to the inverse mass of the gauge fields. That is only possible if this mass does not vanish, which is in agreement with experiment: The conserved strong vector currents couple to massless (though confined) gluons while the nonconserved weak axial currents couple to the massive weak bosons!

Let us remark that the currents S_a^μ, V_a^μ, and X_a^μ couple with different coupling constants e_S, e_V, and e_X to the Yang-Mills fields, as described in (5.13, 5.14). We also have always used the same irreducible reprentation R for the matter fields. This is, of course, not necessary: The scalar fields will generally occur in other representations than the Dirac fields and the vector currents will generally be made out of fermions in representations different from those to which the fermions constituting the axial currents belong. Moreover, the representations R need not to be irreducible. Instead, they can be the direct sum of several irreducible parts. The above formulae remain true if one defines the total current (5.13) as the sum of all currents in the various representations R. Then each representation R can have its own coupling constant e_R. We remark, however, that this is a typical first order phenomenon. The condition of gauge invariance in second order will certainly give restrictions on the various coupling constants which seem to arbitrary at the moment. This also happens in massless Yang-Mills theory, where the gauge invariance of certain second order tree graphs implies the equality of the Yang-Mills self-coupling constant with the one in the coupling of the Yang-Mills fields to matter[15].

We now have carefully studied the possible interactions of *quantized* Yang-Mills fields, ghost fields, scalar gauge fields, and scalar and Dirac matter fields. Though only working in first order T^1 we have discovered very interesting structures. To complete the theory, we would have to study gauge invariance in all orders, eq.(1.30). Before we take on this Herculean task we like to know what we get if we succeed. It is the unitarity of S_{phys}. This is shown in the next chapter.

6. Gauge Invariance and Unitarity of the Physical S-Matrix

Unitarity of the physical S-matrix in the case of massless Yang-Mills fields was proven in [4]. Here we treat the massive case, i.e. the interaction constructed in the two preceding chapters.

Let us begin with discussing the Krein structure [5,8,16,17] in the Hilbert-Fock space of the gauge fields. The massive Yang-Mills fields are quantized as

$$A_a^\mu(x) = \sum_{\lambda=0}^{3} \int dk \, \left\{ \epsilon_\lambda^\mu(k) a_{\lambda,a}(k) e^{-ikx} + \epsilon_\lambda^\mu(k) a_{\lambda,a}^K(k) e^{ikx} \right\} \tag{6.1}$$

k is always on the mass shell $\mathcal{M}$:

$$k \stackrel{\text{def}}{=} (k_0, \vec{k}) \ , \ k_0 \stackrel{\text{def}}{=} +[(\vec{k})^2 + M^2]^{\frac{1}{2}} \ , \ dk \stackrel{\text{def}}{=} \frac{d^3\vec{k}}{2k_0(2\pi)^3} \ , \ \delta(k-k') \stackrel{\text{def}}{=} 2k_0(2\pi)^3 \delta^{(3)}\left(\vec{k} - \vec{k}'\right) \tag{6.2}$$

ϵ_λ^μ are four polarisation vectors satisfying

$$\epsilon_0^\mu(k) \stackrel{\text{def}}{=} \frac{k^\mu}{M} \ , \ g_{\mu\nu}\epsilon_\lambda^\mu(k)\epsilon_\kappa^\nu(k) = g_{\kappa\lambda} \ ,$$

$$\sum_{\lambda=1}^{3} \epsilon_\lambda^\mu(k)\epsilon_\lambda^\nu(k) = -\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}\right] \ , \ \sum_{\lambda=0}^{3} g_{\lambda\lambda}\epsilon_\lambda^\mu(k)\epsilon_\lambda^\nu(k) = g^{\mu\nu} \ , \ \overline{\epsilon_\lambda^\mu(k)} = \epsilon_\lambda^\mu(k) \tag{6.3}$$

$a_{\lambda,a}(k)$ are $4(\dim G)$ standard (distributional) bosonic annihilation operators [12,17,18,19] acting in the Hilbert- Fock space

$$H_A = \bigoplus_{n=0}^{\infty} \left\{ \bigvee^{n} \left\{ \bigoplus_{\lambda=0}^{3} \bigoplus_{a=1}^{\dim G} \left[L^2(\mathcal{M}, dk)\right]_{\lambda,a} \right\} \right\} \tag{6.4}$$

which is equipped with the standard positive scalar product $(\underline{a}, \underline{b})_A$. The operator O^+ denotes the adjoint of O with respect to this scalar product. The Fock space operators fulfil

$$[a_{\lambda,a}(k), a^+_{\kappa,b}(k')]_- = \delta_{\lambda\kappa}\delta_{ab}\delta(k-k') \tag{6.5}$$

The number operators for a given polarization λ are defined by

$$N_\lambda \stackrel{\text{def}}{=} \int dk\, a^+_{(\lambda),a}(k) a_{(\lambda),a}(k) \tag{6.7}$$

The Krein operator J_A in H_A [5,8,16,17] is defined by

$$J_A \stackrel{\text{def}}{=} (-1)^{N_0} \tag{6.8}$$

It defines a pseudo-conjugation O^K [4,5,12,16,17] of an operator O by

$$O^K \stackrel{\text{def}}{=} J_A O^+ J_A \tag{6.9}$$

Sometimes the form $< \underline{a}, \underline{b} >_A := (\underline{a}, J_A \underline{b})_A$ is called an indefinite scalar product. We will not follow this terminology here. The word orthogonal (hermitean, unitary) will always mean orthogonal (hermitean, unitary) with respect to the positive inner product. Otherwise we say pseudo-orthogonal (pseudo-hermitean, pseudo-unitary).

The gauge invariant *physical* Yang-Mills fields A_{phys} have only the three transversal polarisations:

$$(A_{\text{phys}})^\mu_a(x) = \sum_{\lambda=1}^{3} \int dk\, \left\{ \epsilon^\mu_\lambda(k) a_{\lambda,a}(k) e^{ikx} + \epsilon^\mu_\lambda(k) a^+_{\lambda,a}(k) e^{-ikx} \right\} \ , \ d_Q A_{\text{phys}} = 0 \tag{6.10}$$

and the commutator

$$[(A_{\text{phys}})^\mu_a(x), (A_{\text{phys}})^\nu_b(y)]_- = -\left[g^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{M^2} \right] \delta_{ab}(-i) D_m(x-y) \tag{6.11}$$

The *unphysical* Yang Mills A_{unphys} fields are given by

$$(A_{\text{unphys}})^\mu_a(x) \stackrel{\text{def}}{=} A^\mu_a(x) - (A_{\text{phys}})^\mu_a(x) = \frac{-1}{M^2} \partial^\mu \partial_\nu A^\nu_a(x) \tag{6.12}$$

The following conjugation properties are easily checked:

$$A = A^K \ , \ A_{\text{phys}} = A^K_{\text{phys}} = A^+_{\text{phys}} \ , \ A_{\text{unphys}} = A^K_{\text{unphys}} = -A^+_{\text{unphys}} \tag{6.13}$$

We also note

$$F^{\mu\nu}_a \stackrel{\text{def}}{=} \partial^\mu A^\nu_a - \partial^\nu A^\mu_a = \partial^\mu (A_{\text{phys}})^\nu_a - \partial^\nu (A_{\text{phys}})^\mu_a = (F^{\mu\nu}_a)^K = (F^{\mu\nu}_a)^+ \tag{6.14}$$

$$\partial_\mu A^\mu_a = \partial_\mu (A_{\text{unphys}})^\mu_a = (\partial_\mu A^\mu_a)^K = -(\partial_\mu A^\mu_a)^+ \tag{6.15}$$

The representation of the proper Poincaré group $P^\uparrow_+$ in H_A is defined by

$$U_A(a,\Lambda) A^\mu_a(x) U_A(a,\Lambda)^{-1} = \Lambda^\mu{}_\nu A^\nu_a(\Lambda x + a) \ , \ U_A(a,\Lambda)\Omega_A = \Omega_A \tag{6.16}$$

where Ω_A is the vacuum in H_A. It is pseudo-unitary:

$$U_A(a,\Lambda) U_A(a,\Lambda)^K = 1 \tag{6.17}$$

and, since it commutes with J_A:

$$J_A U_A(a,\Lambda) J_A = U_A(a,\Lambda) \tag{6.18}$$

unitary as well:

$$U_A(a,\Lambda)U_A(a,\Lambda)^+ = 1 \tag{6.19}$$

The last two equations fail in the massless case where $\epsilon_0^\mu(k)$ cannot be chosen covariantly.

Next we come to the hermitean scalar gauge fields which are quantized in the usual way:

$$h_a(x) = \int dk\ \left\{b_a(k)e^{-ikx} + b_a^+(k)e^{ikx}\right\} = h_a^+(x) \tag{6.20}$$

as operators in the Hilbert-Fock space

$$H_h = \bigoplus_{n=0}^{\infty}\left\{\bigvee^{n}\left\{\overset{\dim G}{\underset{a=1}{\oplus}}\left[L^2(\mathcal{M},dk)\right]_a\right\}\right\} \tag{6.21}$$

with standard positive scalar product $(\underline{a},\underline{b})_h$. We have

$$[b_a(k), b_b^+(k')]_- = \delta_{ab}\delta(k-k') \tag{6.22}$$

We do not introduce an additional Krein structure in H_h. This is equivalent to saying that $J_h = 1$ and that the two conjugations agree over H_h: $O^K = O^+$. The same is true for the two forms over H_h: $<\underline{a},\underline{b}>_h = (\underline{a},\underline{b})_h$. The representation of $P_+^\uparrow$ in H_h: $U_h(a,\Lambda)$ is unitary. The total number of scalar gauge particles is given by

$$N_h = \int dk\, b_a^+(k)b_a(k) \tag{6.23}$$

Now we consider the ghost fields. They have been extensively studied in [5]. So we summarize only the most important formulae here. The ghost fields

$$u_a(x) = \int dk\ \left\{c_{1,a}(k)e^{-ikx} + c_{-1,a}^+(k)e^{ikx}\right\}\ ,\ \tilde{u}_a(x) = \int dk\ \left\{-c_{-1,a}(k)e^{-ikx} + c_{1,a}^+(k)e^{ikx}\right\} \tag{6.24}$$

are defined in the Hilbert-Fock space

$$H_g = \bigoplus_{n=0}^{\infty}\left\{\bigwedge_n\left\{\underset{i=\pm1}{\oplus}\ \overset{\dim G}{\underset{a=1}{\oplus}}\left[L^2(\mathcal{M},dk)\right]_{i,a}\right\}\right\} \tag{6.25}$$

with positive inner product $(\underline{a},\underline{b})_g$. The index i distinguishes ghost from antighost particles. We have

$$\{c_{i,a}(k), c_{j,b}^+(k')\}_+ = \delta_{ij}\delta_{ab}\delta(k-k') \tag{6.26}$$

The Krein operator in H_g is defined by

$$J_g = i^{N_g - \Gamma} \tag{6.27}$$

Here N_g denotes the total number of ghost and antighost particles:

$$N_g = \int dk\, c_{i,a}^+(k)c_{i,a}(k) \tag{6.28}$$

while Γ is defined by

$$\Gamma = \int dk\ \left\{c_1^+(k)c_{-1}(k) + c_{-1}^+(k)c_1(k)\right\} \tag{6.29}$$

Again one considers the indefinite form $<\underline{a},\underline{b}>_g := (\underline{a}, J_g\underline{b})$ and defines $O^K = J_g O^+ J_g$. The representation of $P_+^\uparrow$ in H_g: $U_g(a,\Lambda)$ is unitary and, since it commutes with J_g, pseudo-unitary as well [5].

The scalar and Dirac matter fields are quantized in their own Hilbert-Fock space H_{matter} in the usual way. The scalar product in this space is again positive and the Krein structure J_{matter} is the unit operator. The representation $U_{\text{matter}}(a, \Lambda)$ is unitary.

The Hilbert space H of the total system is the tensor-product of the spaces above:

$$H = H_A \otimes H_h \otimes H_g \otimes H_{\text{matter}} \tag{6.30}$$

The Krein operator and the representation of $P_+^\uparrow$ factorize accordingly:

$$J = J_A \otimes J_h \otimes J_g \otimes J_{\text{matter}} \tag{6.31}$$

$$U(a, \Lambda) = U_A(a, \Lambda) \otimes U_h(a, \Lambda) \otimes U_g(a, \Lambda) \otimes U_{\text{matter}}(a, \Lambda) \tag{6.32}$$

This U is unitary and pseudo-unitary, since it commutes with J. The positive scalar product in H is denoted by $(\underline{a}, \underline{b})$ and $||\underline{a}|| := (\underline{a}, \underline{a})$, $< \underline{a}, \underline{b} >:= (\underline{a}, J\underline{b})$, $O^K := JO^+J$.

Our next task is to study more closly the gauge charge Q (4.3). It is expressed in momentum space as

$$Q = M \int dk \, \left\{c_{-1,a}^+(k) \left[a_{0,a}(k) + ib_a(k)\right] - \left[a_{0,a}^+(k) + ib_a^+(k)\right] c_{1,a}(k)\right\} \tag{6.33}$$

Its adjoint Q^+ is given by:

$$Q^+ = M \int dk \, \left\{c_{1,a}^+(k) \left[-a_{0,a}(k) + ib_a(k)\right] + \left[a_{0,a}^+(k) - ib_a^+(k)\right] c_{-1,a}(k)\right\} \tag{6.34}$$

Q and Q^+ are both pseudo-hermitean $P_+^\uparrow$ invarinat differential operators:

$$Q^2 = \left(Q^+\right)^2 = 0 \; , \; Q = Q^K \; , \; \left(Q^+\right)^K = Q^+ \; , \; U(a, \Lambda) Q^{(+)} U(a, \Lambda)^{-1} = Q^{(+)} \tag{6.35}$$

We now follow Razumov and Rybkin [8] who showed that the physical Hilbert space of a gauge theory with quadratic BRS charge Q can be defined as

$$H_{\text{phys}} \stackrel{\text{def}}{=} \text{kernel} \, \{Q, Q^+\}_+ \tag{6.36}$$

Razumov showed the equivalence of this definition with the more conventional one using equivalent classes in semidefinite metric spaces [7]. Razumovs definition is advantageous because it realizes H_{phys} as a concrete subspace of the Hilbert space H which has a clear particle interpretation. To work this out we only have to calculate the above anticommutator. We find:

$$\{Q, Q^+\}_+ = 2[N_0 + N_h + N_g] \stackrel{\text{def}}{=} 2N \tag{6.37}$$

i.e. N is the number of longitudinal Yang-Mills fields plus the number of scalar gauge fields plus the number of ghost and antighost particles. Thus all these particles are unphysical. The only physical particles are the transverse quanta of the Yang-Mills fields and the matter particles.

The spectrum of the number operator N are the natural numbers and 0:

$$N = \sum_{n=0}^{\infty} n P_n \tag{6.38}$$

where P_n is the orthogonal projector on the subspace with n unphysical particles. (6.36) means that the orthogonal projector on H_{phys} is given by P_0:

$$H_{\text{phys}} = P_0 H \tag{6.39}$$

The operator N can be inverted on the orthogonal complement of its kernel:

$$N^{\sim 1} \stackrel{\text{def}}{=} 0P_0 + \sum_{n=1}^{\infty} n^{-1} P_n \ , \ \ N^{\sim 1}N = NN^{\sim 1} = (1 - P_{\text{phys}}) \tag{6.40}$$

Since Q and Q^+ are $P_+^\uparrow$ invariant, so are N, $N^{\sim 1}$, and P_n:

$$U(a,\Lambda)\left\{N; N^{\sim 1}; P_n\right\} U(a,\Lambda)^{-1} = \left\{N; N^{\sim 1}; P_n\right\} \tag{6.41}$$

We also note that N, P_n, and $N^{\sim 1}$ commute with Q and Q^+:

$$[Q^{(+)}, N]_- = [Q^{(+)}, P_n]_- = [Q^{(+)}, N^{\sim 1}]_- = 0 \tag{6.42}$$

We now follow again [8] and introduce the following subspaces of H:

$$H_K \stackrel{\text{def}}{=} \text{kernel}\, Q \ , \ \ H_{K^+} \stackrel{\text{def}}{=} \text{kernel}\, Q^+ \ , \ \ H_R \stackrel{\text{def}}{=} \text{range}\, Q \ , \ \ H_{R^+} \stackrel{\text{def}}{=} \text{range}\, Q^+ \tag{6.43}$$

Let us study the relations between these spaces and H_{phys}. Let $\underline{a}_0 \in H_{\text{phys}}$. By

$$0 = (\underline{a}_0, \{Q, Q^+\}_+ \underline{a}_0) = ||Q^+ \underline{a}_0||^2 + ||Q\underline{a}_0||^2 \tag{6.44}$$

we find

$$H_{\text{phys}} = H_K \cap H_{K^+} \tag{6.45}$$

Since $Q^2 = (Q^+)^2 = 0$ we have

$$H_R \subseteq H_K \ , \ \ H_{R^+} \subseteq H_{K^+} \tag{6.46}$$

Let $\underline{a}_K \in H_K$, $\underline{b}_{R^+} \in H_{R^+}$. Since

$$(\underline{a}_K, \underline{b}_{R^+}) = (\underline{a}_K, Q^+ \underline{b}) = (Q\underline{a}_K, \underline{b}) = 0 \tag{6.47}$$

H_K and H_{R^+} are orthogonal to each other, and replacing Q by Q^+ shows that the same holds true for H_R and H_{K^+}:

$$H_K \perp H_{R^+} \ , \ \ H_R \perp H_{K^+} \tag{6.48}$$

Combining this with (6.46) gives

$$H_R \perp H_{R^+} \tag{6.49}$$

while (6.45) now implies

$$H_{\text{phys}} \perp H_R \ , \ \ H_{\text{phys}} \perp H_{R^+} \tag{6.50}$$

We conclude that the three spaces H_{phys}, H_R, and H_{R^+} are all mutually orthogonal. Let now $\underline{a} \in H$. Then we can write

$$\underline{a} = P_0\underline{a} + (1 - P_0)\underline{a} = P_0\underline{a} + NN^{\sim 1}\underline{a} = P_0\underline{a} + \frac{1}{2}QQ^+N^{\sim 1}\underline{a} + \frac{1}{2}Q^+QN^{\sim 1}\underline{a} \stackrel{\text{def}}{=} \underline{a}_0 + \underline{a}_R + \underline{a}_{R^+} \tag{6.51}$$

where $\underline{a}_0 \in H_{\text{phys}}$, $\underline{a}_R \in H_R$, and $\underline{a}_{R^+} \in H_{R^+}$. This shows that the Hilbert space H is the direct orthogonal sum of the three spaces H_{phys}, H_R, and H_{R^+}:

$$H = H_{\text{phys}} \oplus_\perp H_R \oplus_\perp H_{R^+} \tag{6.52}$$

Since the first two of this spaces are subspaces of H_K and the third is orthogonal to it we can also write:

$$H = H_K \oplus_\perp H_{R^+} \ , \ \ H_K = H_{\text{phys}} \oplus_\perp H_R \tag{6.53}$$

i.e the physical Hilbert space is also given by

$$H_{\text{phys}} = H_K \ominus_\perp H_R \tag{6.54}$$

Moreover, since H_{phys} and H_{R+} are subspaces of H_{K+} and this space is orthogonal to H_R one can also write

$$H = H_{K+} \oplus_\perp H_R \; , \; H_{K+} = H_{\text{phys}} \oplus_\perp H_{R+} \tag{6.55}$$

i.e. we get one more characterization of H_{phys} as

$$H_{\text{phys}} = H_{K+} \ominus_\perp H_{R+} \tag{6.56}$$

The orthogonal decompositions above were already given in [8], and, in the specific context of massless Yang-Mills theories, in [4]. We denote the orthogonal projections on $\{H_{\text{phys}}; H_K; H_R; H_{K+} H_{R+}\}$ by $\{P_0; P_K; P_R; P_{K+}; P_{R+}\}$ and the vectors in these spaces by $\{\underline{a}_0; \underline{a}_K; \underline{a}_R; \underline{a}_{K+}; \underline{a}_{R+}\}$. From the preceding equations we find:

$$P_0 P_R = P_0 P_{R+} = P_R P_{R+} = 0 \; , \; P_0 + P_R + P_{R+} = 1 \; , \; P_0 = P_0^+ \; , \; P_R = P_R^+ \; , \; P_{R+} = P_{R+}^+ \tag{6.57}$$

$$P_R = \frac{1}{2} Q Q^+ N^{\sim 1} \; , \; P_{R+} = \frac{1}{2} Q^+ Q N^{\sim 1} \tag{6.58}$$

Let us now study the strucure of some important operators with respect to the orthogonal decomposition (6.52). The operators Q and Q^+ map the complements of their kernels onto their range. This gives:

$$Q = P_R Q P_{R+} \; , \; Q^+ = P_{R+} Q^+ P_R \tag{6.59}$$

Then (6.37) implies that the decomposition of N and $N^{\sim 1}$ are given by

$$N^{(\sim 1)} = P_R N^{(\sim 1)} P_R + P_{R+} N^{(\sim 1)} P_{R+} \tag{6.60}$$

Let now I be a gauge invariant operator, i.e. $d_Q I = 0$. Then H_K and H_R are stable under the action of I, that is

$$I P_K = P_K I P_K \; , \; I P_R = P_R I P_R \tag{6.61}$$

Thus we get

$$I = P_0 I P_0 + P_0 I P_{R+} + P_R I P_0 + P_R I P_R + P_R I P_{R+} + P_{R+} I P_{R+} \tag{6.62}$$

Next we use the pseudo-hermiticity of Q and Q^K to get information about the Krein operator J. Let $\underline{a}_K \in H_K$, $\underline{b}_R = Q\underline{c} \in H_R$. Then we have

$$(\underline{a}_K, J\underline{b}_R) =< \underline{a}_K, Q\underline{c} >=< Q\underline{a}_K, \underline{c} >= 0 \tag{6.63}$$

This means:

$$P_K J P_R = 0 \tag{6.64}$$

Taking the adjoint gives:

$$P_R J P_K = 0 \tag{6.65}$$

Using Q^+ instead of Q in the argument above gives

$$P_{K+} J P_{R+} = P_{R+} J P_{K+} = 0 \tag{6.66}$$

A direct inspection of J in (6.31) gives the additional information that J agrees on H_{phys} with the unit operator:

$$P_0 J P_0 = P_0 \tag{6.67}$$

The last four equations are summarized in

$$J = P_0 J P_0 + P_R J P_{R+} + P_{R+} J P_R \tag{6.68}$$

The second of eqs. (6.53) means that H_K can be interpreted as a linear fiber bundle: H_{phys} is the base space and the fibers are the elements of H_R. Eqs. (6.64,6.65) show that the fibers are pseudo-orthogonal to any vector in H_K. Moreover, writing $\underline{a}_K = \underline{a}_0 + \underline{a}_R$ according to the orthogonal decomposition (6.53) gives

$$< \underline{a}_K, \underline{b}_K >= (\underline{a}_0, \underline{b}_0) \tag{6.69}$$

This shows that that the form $<,>$ agrees on H_{phys} with the positive form $(,)$, that it is positive semidefinite in H_K, and that its kernel as a quadratic form in this space are the fibers:

$$H_R = \text{kernel } <,>_K \tag{6.70}$$

where $<,>_K$ means the restriction of the form $<,>$ to H_K. So we get another expression forH_{phys}:

$$H_{\text{phys}} = H_K \ominus_\perp \text{kernel } <,>_K \tag{6.72}$$

The form $<,>_K$ is constant along the fibers in both arguments separately:

$$< \underline{a}_K + \underline{a}_R, \underline{b}_K + \underline{b}_R >_K =< \underline{a}_K, \underline{b}_K >_K \tag{6.73}$$

and the same holds true for the matrix elements of any gauge invariant operator I with respect to this form:

$$< \underline{a}_K + \underline{a}_R, I\,(\underline{b}_K + \underline{b}_R) >_K =< \underline{a}_K, I\underline{b}_K >_K \tag{6.73}$$

This allows to choose any linear cross section H_S in H_K, i.e. any subspace of H which is a (pseudo-orthogonal but generally not orthogonal) complement of H_R (in H_K) as a realization of the physical Hilbert space. The scalar product in H_S is the restriction of the form $<,>$ to this space, and there it is positive definite. All this spaces are unitarily equivalent, and the matrix elements of gauge invariant operators do not depend on the section chosen. So one might also consider the equivalence class of all this spaces and that is what is usually done in the literature [7]. We prefer to use H_{phys} as a concrete realization of the physical Hilbert space since it is the only section which is orthogonal to the fibers and which allows for a simple interpretation of the quanta of the elementary fields as physical or unphysical particles. The projections onto the sections along the fibers are also called gauge transformations. For H_{phys}, and only for it, they agree with the orthogonal projection.

We now consider again operators A, B, $C\cdots$ over H. We define the orthogonal projection of A on H_{phys}: A_0 by

$$A_0 \stackrel{\text{def}}{=} P_0 A P_0 \tag{6.74}$$

A_0 is still an operator from H to$H$. It is zero on the orthogonal complement of H_{phys}: $H^\perp_{\text{phys}}$. Since this zero is certainly not very interesting we define A_{phys} to be the restriction of A_0 to H_{phys}:

$$A_{\text{phys}} \stackrel{\text{def}}{=} (A_0)_{\downarrow H_{\text{phys}}} \tag{6.75}$$

The map $A \longrightarrow A_{\text{phys}}$ is certainly linear:

$$(\alpha A)_{\text{phys}} = \alpha A_{\text{phys}}\ ,\ (A+B)_{\text{phys}} = A_{\text{phys}} + B_{\text{phys}} \tag{6.76}$$

More interesting is the projection of the product of two operators. We calculate:

$$P_0 A B P_0 = P_0 A (P_0 + P_R + P_{R+}) B P_0 = P_0 A P_0 P_0 B P_0 + P_0 A \frac{1}{2} Q Q^+ N^{\sim 1} B P_0 + P_0 A \frac{1}{2} N^{\sim 1} Q^+ Q B P_0 \tag{6.77}$$

Let us concentrate on the second summand: X. Since $P_0 Q=0$ we can replace AQ by $\{A,Q\}_\pm$. We take the anticommutator if A is fermionic and the commutator if it is bosonic. This gives

$$X = P_0 \frac{1}{2}\{A,Q\}_\pm Q^+ N^{\sim 1} B P_0 \tag{6.78}$$

Now we use $P_0 Q^+ = 0$ to replace that by

$$X = P_0 \frac{1}{2}\left\{\{A,Q\}_\pm, Q^+\right\}_\mp N^{\sim 1} B P_0 \tag{6.79}$$

And finally we use $P_0 N^{\sim 1} = 0$ to write

$$X = P_0 \frac{1}{2}\left[\left\{\{A,Q\}_\pm, Q^+\right\}_\mp, N^{\sim 1}\right]_- B P_0 \tag{6.80}$$

There are always two commutators and one anticommutator in this expression. Thus it can be uniquely written as

$$X = P_0 (\mathcal{T} A) B P_0 \tag{6.81}$$

where the *triple variation* $\mathcal{T}$ is defined by

$$\mathcal{T} \stackrel{\text{def}}{=} \frac{1}{2} \delta_{(N^{\sim 1})} d_{(Q^+)} d_Q \tag{6.82}$$

Here $\delta_{(N^{\sim 1})}$ is the derivation induced by $N^{\sim 1}$, and d_Q and $d_{(Q^+)}$ are the antiderivations induced by Q and Q^+, repectively (see chapter 1). Note that the triple variation of gauge invariant operators vanishes. In the same way the third summand in (6.77): Y is written as

$$Y = P_0 A (\mathcal{T} B) P_0 \tag{6.83}$$

We thus have found the important *projection formula*:

$$A_{\text{phys}} B_{\text{phys}} = (AB)_{\text{phys}} - \{(\mathcal{T} A) B + A (\mathcal{T} B)\}_{\text{phys}} \tag{6.84}$$

This implies

Theorem I: The product of the physical projections of two operators with vanishing triple variation, especially of two gauge invariant operators, is identical to the physical projection of their product. The physical projection of a group (of an algebra) of operators with vanishing triple variation, especially of gauge invariant operators, is a representation of this group (algebra).

Next we consider the physical projection of the pseudo-adjoint A^K of an operator A. So we have to study

$$P_0 A^K P_0 = P_0 J A^+ J P_0 \tag{6.85}$$

Now we use that (6.68) implies

$$P_0 J = J P_0 = P_0 J P_0 = P_0 \tag{6.86}$$

to conclude:

$$P_0 A^K P_0 = P_0 A^+ P_0 = (P_0 A P_0)^+ \tag{6.87}$$

which means

$$\left(A^K\right)_{\text{phys}} = (A_{\text{phys}})^+ \tag{6.88}$$

Thus we have found

Theorem II: The physical projection of the pseudo-adjoint operator is identical to the adjoint of the physical projection of this operator. The physical projection of a pseudo-hermitean operator is hermitean.

Combining the two theorems above gives:

Theorem III: The physical projection of a pseudo-unitary operator with vanishing triple variation, especially of a pseudo-unitary gauge invariant operator, is an unitary operator.

Now we are well equipped to tackle unitarity of the physical S-matrix. The interaction $T^{(1)}(x)$ constructed in the preceding chapters is anti-pseudo-hermitean:

$$\left(T^{(1)}(x)\right)^K = -T^{(1)}(x) \tag{6.89}$$

This guarantees the pseudo-unitarity of $S[g]$ [11,12,15]:

$$S[g]S^K[g] = 1 \tag{6.90}$$

This is [11] equivalent to

$$\sum_{I\oplus J=N} T(I)T^K(J) = 0, \;\; \forall N \neq \emptyset \tag{6.91}$$

Here $T(I)$ means $T^{(r)}(x_{i_1}, \cdots, x_{i_r})$, $T^K(J)$ means $\left(T^{(s)}(x_{j_1}, \cdots, x_{j_s})\right)^K$, where $r+s = n$, and the sum Σ runs over all direct decompositions of the set $N = \{1, \cdots, n\}$ into two subsets $I = \{i_1, \cdots, i_r\}$ and $J = \{j_1, \cdots, j_s\}$. Let us now *assume* that gauge invariance holds true in all orders, i.e. we have

$$d_Q T^{(n)} = \sum_{k=1}^{4n} \partial^k T_k^{(n)} \tag{6.92}$$

Taking the pseudo-conjugate of this equation gives:

$$d_Q \left(T^{(n)}\right) = -\sum_{k=1}^{4n} \partial^k \left(T_k^{(n)}\right)^K \tag{6.93}$$

Then (6.84,6.88) and (6.90-6.92) imply

$$\sum_{I\oplus J=X} T_{\text{phys}}(I)\,(T_{\text{phys}}(J))^+ = \sum_{k=1}^{4n} \partial^k W_k^{(n)}(X)\ ,$$

$$W_k^{(n)}(X) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{I\oplus J=X} \left\{-\theta_I(k)\left(\delta_{(N^{-1})} d_{(Q+)} T_k(I)\right) T^K(J) + \theta_J(k) T(I)\left(\delta_{(N^{-1})} d_{(Q+)} T_k^K(J)\right)\right\}_{\text{phys}} \tag{6.94}$$

where

$$\theta_I(k) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } k \in I \\ 0, & \text{if } k \notin I \end{cases} \tag{6.95}$$

This is the exact *perturbative, pre-adiabatic* expression for the unitarity of the physical S-matrix. *If* the adiabtic limit:

$$S_{\text{phys}} = \lim_{g\to 1} \left(S[g]\right)_{\text{phys}} \tag{6.96}$$

exists and has the same analytic properties as in the saclar theory discussed in [13], and *if* the boundary terms $\int \partial^k W_k^{(n)}$ vanish, (6.94) will imply

$$S_{\text{phys}} S^K_{\text{phys}} = 1 \tag{6.97}$$

We note, however, that this adiabatic limit may have additional subtleties if the heavy gauge particles are coupled to light matter, since then the gauge fields become unstable, i.e are not really asymptotic fields.

Let us end this long chapter with a short discussion of $P_+^\uparrow$-invariance. The remark after (6.32) and eq. (6.41) immediately imply that $U(a, \Lambda)_{\text{phys}}$ is an unitary representation of $P_+^\uparrow$, while the $P_+^\uparrow$-invariance of $S[g]$:

$$U(a,\Lambda)S[g]U(a,\Lambda)^{-1} = S[g_{a,\Lambda}]\ ,\ g_{a,\Lambda}(x) \stackrel{\text{def}}{=} g\left(\Lambda^{-1}(x-a)\right) \tag{6.98}$$

and (6.41) lead direct to

$$U(a,\Lambda)S[g]_{\text{phys}}U(a,\Lambda)^{-1} = S[g_{a,\Lambda}]_{\text{phys}} \tag{6.99}$$

which is the $P_+^\uparrow$-invariance of the physical S-matrix.

The situation is different in the massless case where J is not $P_+^\uparrow$-invariant. However, the three theorems above and the projection formula (6.84) hold true in this case, too. Since Q is $P_+^\uparrow$-invariant also in the massless case, the theorems show that $U(a,\Lambda)_{\text{phys}}$ is indeed an unitary representation of $P_+^\uparrow$, while the $P_+^\uparrow$-invariance of $S[g]$ and (6.84,6.88) implie that (6.99) is violated only by boundary terms. The latter *should* vanish in physical quantities like cross sections, for example.

7. Discussion

We first have shown that the interaction of massless Yang-Mills fields studied in [1-4] admits generalizations. Then we have constructed gauge invariant first order interactions between massive Yang-Mills fields, scalar gauge fields, and matter fields. We have shown how invariance under ghost charge conjugation fixes the interaction uniquely. The scalar gauge fields had to be introduced for a pure algebraic reason, i.e. to have $Q^2{=}0$. Moreover, we have proven that gauge invariance implies unitarity of the physical S-matrix and that the latter is Poincaré invariant. However, gauge invariance were only shown to hold true in the first order. This is certainly not enough. Before we can claim to have a completely consistent theory we have to proof gauge invariance in all orders, i.e. the absence of anomalies. We intend to tackle this labourious though important task in future publications.

The analysis of unitarity has shown that *all* scalar gauge fields are unphysical. This means that our (pure) gauge theory is not a Higgs-model. One could, of course, introduce a physical scalar particle in the matter sector of the theory. But there seems to be no logical reason for doing so (unless one finds out that this would be the only way to avoid anomalies) since, in contrast to the Higgs model, the matter sector and the gauge sector are independent structures in our theory. Our theory has structural similarity to Stueckelberg type gauge theories. There [20] all scalar gauge fields are unphysical, too. We remark, however, that our theory is not identical to the Stueckelberg models discussed in [20], it differs from them in the detailed structure of the propagators and couplings. Stueckelberg type theories are most often discussed under the aspect of classical gauge symmetry and gauge independence while BRS-invariance is only a derived concept. We have followed a completely different route and have made (free) BRS-invariance a first principle. This seems appropriate since it is exactly this invariance which implies unitarity. All Stueckelberg models studied in [20] where either nonrenormalizable or nonunitary. This was shown by direct calculation, but the reason why remained a mystery. Our model is renormalizable by construction, and unitarity follows from gauge invariance. So, *should* it fail to be consistent, we know at least why: There have to be anomalies. One could then study these anomalies to learn how to avoid them.

To proof unitarity of the physical S-matrix we have extended Razumovś and Rybkinś investigation of quadratic BRS systems [8] by giving the important projection formula (6.84). This formula shows that the triple variation $\mathcal{T}$ is more inportant for the algebraic structure of the theory than the gauge variation is. Indeed, it follows from our calculations in the last chapter that a theory which is not gauge-invariant but is invariant under the triple variation $\mathcal{T}$ would still have an unitary physical S-matrix! Thus one might call anomalies of gauge invariance which do not violate triple invariance *weak anomalies* and anomalies which do violate it *strong anomalies*. Well, it could happen that all *known anomalies* are strong ones, but this is far from obvious. A careful algeraic analysis of anomalies with respect to the triple variation is certainly worth doing, and we plan to investigate this interesting structure in the future.

Another thing which one has to do before our theory can make contact with "physics" is the extension to the nonsimple group $G = \mathrm{U}(2)$ and an analysis of global G-breaking which allows for different masses of the various gauge fields.

Let us summarize: A direct algebraic analysis of massive *quantized* Yang-Mills theories done in the framework of causal perturbation theory and free of any classical notions as spontaneuos symmetry breaking has naturally led to the construction of a Higgs free model of massive gauge fields and has revealed new interesting algebraic structures. Further investigations remain to be done.

Acknowledgments - I would like to thank: Ivo Schorn for providing me with an excellent TEXnical enviroment and for his constant and patient advise how to use it. Dr. Michael Duetsch for carefully proofreading the manuscript. The Swiss National Science Foundation for financial support. And the Dublin Institute for Advanced Studies and the University of Zuerich for warm hospitality.

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