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On Instanton Computations in N = 2 Supersymmetric Gauge Theory

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Abstract

Supersymmetry, of which D.V.Volkov was one of the earliest proponents, has had a number of spectacular successes. The most recent, which would have pleased him very much, is its use to obtain an exact expression for the low-energy effective Lagrangian in $N = 2$ supersymmetric gauge theory. The Seiberg-Witten Ansatz which provided this solution has been checked by direct computation in the 1 and 2 instanton approximations. In this note some puzzles presented by the instanton computations are pointed out and partially resolved.

1 Introduction

Recently it has been shown [1] by Seiberg and Witten (S-W) that the low energy (Cartan) sector of the $N = 2$ supersymmetric Yang-Mills theory is tractable in the sense that

(a) the effective Lagrangian for the Cartan fields can be expressed in terms of a single function $F(A)$ of the chiral scalar $N = 1$ superfield A and

(b) it is argued that $F(A)$ is a specific Fuchsian function, in fact is simply the ratio of the derivatives of two independent solutions of a specific hypergeometric equation.

The S-W result (a) is exact but (b) was obtained using an Ansatz based on electromagnetic duality. Accordingly, in order to check the validity of the Ansatz, a number of direct computations [2]-[6] have been made. As the perturbative part of the theory reduces to a one-loop contribution and the non-perturbative part of $F(A)$ is assumed to be due to instantons, these direct computations have concentrated on the instanton contributions. So far only the charge 1 and 2 instanton contributions have proved tractable but the results for these are in agreement with the S-W Ansatz for $F(A)$.

Although the instanton computations, especially the $N = 2$ ones, are technically impressive, they raise a number of puzzling questions of principle, as follows:

(a) Given that the computational results are claimed to be valid only for low orders in an expansion in gv, where g is the coupling constant and v is the vacuum value of the Higgs field, why are the results in *exact* agreement with the S-W Ansatz?

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(b) Given that the background configurations chosen for the computations are neither stationary with respect to the Action nor supersymmetrically invariant how can the computations based on them be exact?

(c) Given that the residual Lagrangian cannot then be supersymmetric, why should the bosonic and fermionic quantum fluctuations cancel, as is assumed?

The purpose of this note is to propose a resolution of the some of the above puzzles. The main point is that by using a specific realization of the supersymmetry algebra the background configurations can be made exactly supersymmetric.

2 Statement of the Problems

To present the problems just mentioned in a quantitative manner we recall that the $N = 2$ supersymmetric Yang-Mills Action takes the form

$$A_g = Tr \int \frac{1}{4}F^2 + \bar{\psi}\not{D}\psi + \frac{1}{4}(\phi^\dagger D^2\phi + \bar{\phi}D^2\phi^\dagger) + g[\bar{\psi},\psi]\phi - gD[\phi^\dagger,\phi] + \frac{1}{2}(D^2 + F^\dagger F) \quad (1)$$

where $F_{\mu\nu}$ is a gauge-field, ψ is a Dirac spinor, ϕ is a complex scalar field, the fields D and F are real and complex dummy-fields belonging the real-vector and chiral-scalar $N = 1$ sub-multiplets respectively, and all fields belong to the adjoint representation of a compact simple Lie group. The instanton computations are carried out in background in which the instanton number N is not zero and the scalar field ϕ satisfies the non-trivial boundary condition $\phi(x) \quad \to \quad v \neq 0$ as $x \to \infty$.

Because of the way in which the dummy fields occur in the Action, it is convenient to replace the dummy-fields F and D by a real triplet X_a defined by

$$F = X_1 + iX_2 \qquad D = X_3 + g[\phi^\dagger,\phi] \quad (2)$$

In that case the Action (1) becomes

$$A_g = Tr \int \frac{1}{4}F^2 + \bar{\psi}\not{D}\psi + \frac{1}{4}(\phi^\dagger D^2\phi + \phi D^2\phi^\dagger) + g[\bar{\psi},\psi]\phi - \frac{g}{2}[\phi^\dagger,\phi]^2 + \frac{1}{2}X_aX_a \quad (3)$$

The important point is that the dummy-fields X_a now decouple.

First let us consider the puzzle (a) above. It is clear from the way that the coupling constant g occurs in the Action (1) that by rescaling the fields A_μ, ϕ, ψ and D by a factor g^{-1}, the coupling constant can be removed at the expense of an overall factor g^{-2}. Hence an expansion in g is not physically meaningful.

The Action (1.1) is also manifestly scale-invariant. Hence, since ϕ has scale-dimension -1, the boundary condition $|\phi| \to v \neq 0$ can be converted to $|\phi| \to 1$ without changing the Action. Thus for each instanton charge N the vacuum-value v can appear in the effective potential only in the factor $(\Lambda/v)^N$, where Λ is the renormalization parameter (see ()). It follows that an expansion in v-expansion within a given instanton sector is not physically meaningful.

It follows from these considerations that an expansion in powers of gv is not physically meaningful and thus, although the instanton results were obtained using such an expansion as a guide, they must actually be exact results. But this is a point that remains to be clarified.

3 Previous Background Configurations

We now turn to puzzles (b) and (c). The $N = 2$ supersymmetric transformations [7] are

$$[Q^i, \delta\phi_1] = i\psi^i \qquad \{Q^i, \delta\phi_2\} = \gamma_5\psi^i \qquad \{Q^i, \delta A_\mu\} = \gamma_\mu\psi^i \tag{4}$$

$$\{Q_i, \psi^j\} = \delta^i_j\Big(\frac{1}{2}\sigma.F + \not{D}\Phi + \gamma_5[\phi^\dagger, \phi]\Big) + (\tau \cdot X)^i_j \tag{5}$$

and

$$[Q^i, X_a] = (\tau_a)^i_j\Big(\not{D}\psi_j + g[\Phi, \psi_j]\Big) \qquad \Phi = \phi_1 - i\gamma_5\phi_2 \tag{6}$$

where the σ's and γ's are the usual Dirac matrices and the τ's are a set of Pauli matrices belonging to the $SU(2)_s$ group that connects chiral scalar and real vector $N = 1$ submultiplets.

The background configurations chosen in [2]-[6] are

$$F = F^* \qquad \not{D}\psi = 0 \qquad D^2\phi = g[\psi^\dagger, \psi] \tag{7}$$

Since the equations for ψ and ϕ in (7) are not the Euler-Lagrange field equations it is clear that the configurations are not stationary points of the Action. Furthermore, if we make a supersymmetric variation of (7) the first equation remains invariant but for the other two we obtain

$$[Q_i, \not{D}\psi_j] = \delta_{ij}\gamma_5 D[\phi^\dagger, \phi] + D(\tau.X)^i_j \tag{8}$$

and

$$[Q_i, D^2\phi - g[\bar{\psi}, \psi]] = g\Big([(\tau \cdot X + \gamma_5[\phi^\dagger, \phi]), \psi_i]\Big) \tag{9}$$

Thus the configurations (7) are manifestly not supersymmetric-invariant. Finally, for these background configurations we have from (3)

$$A \quad \rightarrow \quad \int d\Omega(\phi, D\phi) + \int d^4x(X.X - g^2[\phi^\dagger, \phi]^2) \tag{10}$$

where the 3-dimensional integral is over the surface Ω at space-time infinity. The surface term in (10) is the quantity that is actually computed in [2]-[6] and from (7) it would seem that there should be further classical contributions coming from the volume integral. Furthermore, since the background is not supersymmetric, one would expect that there would be still further contributions from the quantum fluctuations.

In [2]-[6] the dummy fields X_a are ignored and one possibility to get rid of the volume integral in (12) would be to choose the X_a so that the integrand vanishes. But this would lead to a further violation of supersymmetry, as can be seen either directly or by noting that it would correspond to a spontaneous breakdown of $SU(2)_s$ symmetry. Indeed so long as the dummy fields X_a are simple scalar fields there would appear to be no choice of their background values that would improve the situation. We turn now to the resolution of these puzzles.

4 Proposed Supersymmetric Background

We first note that since the fields X_a decouple from all other fields in (4) we have a certain freedom in deciding how they should be realized. We shall assume that they are actually matrix-valued, in particular that they are of the form

$$X_a = (\tau_a)Y \tag{11}$$

where Y is a single scalar field. *The crucial point is that with the Ansatz (11) the supersymmetric transformations (4)-(6) still close.* In fact they reduce to

$$[Q^i, \delta\phi_1] = i\psi^i \qquad [Q^i, \delta\phi_2] = \gamma_5\psi^i \qquad [Q^i, \delta A_\mu] = \gamma_\mu\psi^i \tag{12}$$

$$\{Q_i, \psi^j\} = \delta^i_j\Big(\frac{1}{2}\sigma.F + \gamma.D\Phi + \gamma_5[\phi^\dagger, \phi] + Y\Big) \tag{13}$$

and

$$[Q^i, Y] = \gamma.D\psi_j + g[\Phi, \psi_j] \tag{14}$$

Thus, they consitute a realization of the original supersymmetric system (2). An interesting feature of this realization is that, since the three bosonic dummy fields X_a are *represented* by a single field Y the usual supersymmetric rule that the number of bosonic fermionic fields be equal is violated. Thus the realization is not a *faithful* one. But since the fields X_a decouple this has no physical consequences. In fact it is obvious that the Lagrangian (3) is invariant with respect to the supersymmetry transformations (12)-(14).

We now claim that a background configuration can be chosen so that

(a) it is invariant with respect to the supersymmetry (12)-(14) and

(b) The Action (3) reduces to the surface term.

To show this we choose as background configuration

$$F = F^* \qquad \not{D}\psi = 0 \qquad D^2\phi = g[\psi^\dagger, \psi] \quad \text{and} \quad Y = -g\gamma_5[\phi^\dagger, \phi] \tag{15}$$

These background conditions differ from the previous ones by the last equality. Note that setting Y equal to the ϕ-commutator is *not* the same as setting setting $D = g[\phi^\dagger, \phi]$ because the latter condition violates supersymmetry but (15) does not. In fact it is easy to verify that on the surface (15) we have

$$[Q_i, D^2\phi - g[\phi^\dagger, \phi]] = 0 \quad \text{and} \quad [Q_i, Y + g\gamma_5[\phi^\dagger, \phi]] = 0 \tag{16}$$

Furthermore, on this surface one sees by inspection that the four-dimensional integral in () vanishes identically so the Action reduces to the surface term $\int d\Omega(\phi, D\phi)$ as required. The surface term is the quantity computed in [2]-[6]. Since the background configuration is now supersymmetric it follows at once that the surface term must be supersymmetric. The supersymmetry of the surface term was verified in [3] for $N = 1$ and $N = 2$ using the explicit solutions of (15) but we see that in the present situation its supersymmetry is automatic and is valid for all N.

Thus we see puzzle (b) is solved by the observation that although the background was not supersymmetric with respect to the original supersymmetry (4)-(6) it is supersymmetric with repect to the realization (12)-(14).

5 Comparison of Instanton Computation and Ansatz

Finally to consider puzzle of the background fluctuations (c) we must consider the manner in which the S-W Ansatz is verified: According to S-W the effective low energy Action is

$$\int d^4x d^2\theta F(A) W_\alpha W_\alpha \tag{17}$$

where $A(x)$ and $W(x)$ are the chiral scalar and real vector $N = 1$ sub-superfields respectively, and $F(A)$ has the functional form

$$F(A) = \frac{i}{\pi} A^2 \ln(A^2) + \sum_N F_N(A) \quad \text{where} \quad F_N(A) = c_N \left(\frac{\Lambda}{A}\right)^{4N-2} \tag{18}$$

N denoting the contribution from the N-th instanton sector. The form of $F(A)$ is determined on general grounds. What the S-W Ansatz determines is the actual values of the numerical coefficients c_N. Because of the integration over Grassman variables the effective Lagrangian (17) describes only truncated n-point functions of the form

$$< \Phi^n > \;=\; < \phi^{n-4}\psi^4 > \qquad < \phi^{n-3}\psi^2 F_{\mu\nu} > \quad \text{and} \quad < \phi^{n-2} F^2_{\mu\nu} > \tag{19}$$

Expanding $A(x)$ in the form $A(x) = v + \hat{A}(x)$ one sees that these are simply

$$< \Phi^n >= \frac{\partial^n}{(\partial v)^n} F(v) \qquad \text{or} \qquad < \Phi^n >_N = \frac{\partial^n}{(\partial v)^n} \frac{c_N}{v^{4N-2}} \qquad n \geq 2 \tag{20}$$

The verification of the S-W Ansatz consists of comparing the truncated n-point functions (20) with those obtained by direct instanton computations i.e. the truncated n-point functions obtained from

$$< \Phi^n >= \int D(\Phi) \Phi^n e^{-S_N(\Phi)} \qquad n \geq 2 \tag{21}$$

The trick used to compute (21) is to restrict oneself to the background configurations (15) and those configurations that can be obtained from them by supersymmetric transformations. The advantage of this is that the volume integral in the Action vanishes, leaving only the surface integral. Furthermore, from (15) one sees that the configurations $\Phi(\alpha)$ and the value of the surface term, $\sigma(\alpha)$ say, are determined by the parameters α which describe the instantons and their zero-modes and were classified by Atiyah, Hitchin, Drinfeld Manin and Nahm (AHDMN). Hence, if one assumes that the quantum fluctuations do not contribute on account of supersymmetric cancellation, equation (21) reduces to the ordinary integral

$$< \Phi^n >_N = \int d\mu_N(\alpha) \Phi(\alpha)^n e^{-\sigma(\alpha)} \qquad n \geq 2 \tag{22}$$

where $d\mu_N(\alpha)$ is the measure in the space of the instanton parameters. The measure $d\mu_N(\alpha)$ is known explicitly only for the $N = 1$ and $N = 2$ sectors and this is why the S-W Ansatz has been checked only for these two sectors. From the properties of the background fields and their supersymmetrc vaiations it is possible to show that (22) can be written in the form

$$< \Phi^n >_N = \frac{\partial^n}{(\partial \lambda)^n} \int d\mu_N(\alpha) e^{-\lambda\sigma(\alpha)} \quad \text{at} \quad \lambda = 0 \qquad n \geq 2 \tag{23}$$

It is the quantity (23) that has been computed and compared successfully with (20) in references [2]-[6]. An interesting consequence of the above results is that, although they are valid only for $n \geq 2$, they imply that

$$F_N(v) = \int d\mu_N(\alpha)\epsilon^{-\sigma(\alpha)} \tag{24}$$

This is an interesting result from a number of points of view. Mathematically, it shows that (for $v = 1$) the coefficients c_N in the asymptotic expansion Fuchsian function $F(v)$ are equal to the volume of the instanton parameter-space compactified with the exponential factor shown. Physically, it shows that in each instanton sector c_N is a kind of partition function for the background fields. Finally equation (24) encapsulates all the information concerning the n-point functions.

We turn now to the question of the quantum fluctuations. Now that the background fields are truly supersymmetric, the fact that the n-point functions are restricted to these fields and their supersymmetric variations, would seem to justify the assumption the bosonic and fermionic contributions, thus resolving puzzle (c) above. But a number of important details remain to be clarified. First, it should be demonstrated explicitly that the bosonic and fermionic quantum fluctuations do indeed cancel. Second, the role of the supersymmetric zero-modes i.e. the instanton modes which are *not* lifted by the exponential term in (22)-(24), needs to be clarified. Finally, in view of the simplicity and importance of equation (24) there ought to be a more direct way of obtaining it. These are problems to which we hope to consider in the future.

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