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Nonstandard Drinfeld-Sokolov reduction

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Abstract. Subject to some conditions, the input data for the Drinfeld-Sokolov construction of KdV type hierarchies is a quadruplet $(\mathcal{A}, \Lambda, d_1, d_0)$, where the d_i are **Z**-gradations of a loop algebra $\mathcal{A}$ and $\Lambda \in \mathcal{A}$ is a semisimple element of nonzero d_1-grade. A new sufficient condition on the quadruplet under which the construction works is proposed and examples are presented. The proposal relies on splitting the d_1-grade zero part of $\mathcal{A}$ into a vector space direct sum of two subalgebras. This permits one to interpret certain Gelfand-Dickey type systems associated with a nonstandard splitting of the algebra of pseudo-differential operators in the Drinfeld-Sokolov framework.

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1 Introduction

Developing the ideas of the pioneering papers [1, 2], recently a general Lie algebraic framework has been established in which to construct generalized KdV and modified KdV type integrable hierarchies [3, 4, 5]. This formalism contains many interesting systems as special cases [6, 7]. However, there exist some well-known systems which do not seem to fit in the approach as has been developed so far. For example, while the standard Gelfand-Dickey hierarchies [8]

$$\frac{\partial}{\partial t_m} L = \left[\left(L^{\frac{m}{n}} \right)_{\geq 0}, L \right] \quad \text{for} \quad L = \partial^n + \sum_{i=1}^{n} u_i \partial^{n-i} \tag{1.1}$$

have a well-known interpretation [1], which motivated the whole theory, their 'nonstandard' counterparts [9, 10, 11] defined by

$$\frac{\partial}{\partial t_m} L = \left[\left(L^{\frac{m}{n-1}} \right)_{\geq 1}, L \right] \quad \text{for} \quad L = \partial^{n-1} + \sum_{i=1}^{n-1} u_i \partial^{n-1-i} + \partial^{-1} u_n \tag{1.2}$$

so far resisted a similar interpretation.

In this paper we propose an extension of the above mentioned Lie algebraic framework of constructing integrable hierarchies. This will prove general enough to contain the nonstandard Gelfand-Dickey hierarchies as special cases.

The Drinfeld-Sokolov construction relies on the use of a classical r-matrix [12] given by the difference of two projectors, $\mathcal{R} = \frac{1}{2}(\mathcal{P}_\alpha - \mathcal{P}_\beta)$, associated with a splitting $\mathcal{A} = \alpha + \beta$ of an affine Lie algebra $\mathcal{A}$. So far it has been assumed [1, 2, 3, 4, 5] that the subalgebras $\alpha, \beta \subset \mathcal{A}$ have the form $\alpha = \mathcal{A}^{\geq 0}$ and $\beta = \mathcal{A}^{<0}$ in terms of a $\mathbf{Z}$-gradation $\mathcal{A} = \oplus_{n \in \mathbf{Z}} \mathcal{A}^n$ of $\mathcal{A}$. The essence of the proposal of this paper will be to use a more general r-matrix obtained by further splitting $\mathcal{A}^0$. That is we shall use

$$\mathcal{R} = \frac{1}{2}(\mathcal{P}_\alpha - \mathcal{P}_\beta) \quad \text{where} \quad \alpha = \mathcal{A}^{>0} + \alpha^0, \quad \beta = \mathcal{A}^{<0} + \beta^0 \tag{1.3}$$

in correspondence with a splitting $\mathcal{A}^0 = \alpha^0 + \beta^0$ subject to certain conditions. The standard construction will be recovered for $\beta^0 = \{0\}$.

The standard and nonstandard Gelfand-Dickey systems correspond to two splittings of the algebra of pseudo-differential operators that differ in the scalars being added to the subalgebras of purely differential or purely integral operators, which is reminiscent of the manipulation with the splittings in our construction. Since the nonstandard Gelfand-Dickey systems are recovered from it, we sometimes refer to our construction as the 'nonstandard Drinfeld-Sokolov construction'. However, it should be stressed that our construction is essentially just the standard one implemented under weakened conditions on the input data.

The layout of the paper is as follows. Section 2 is devoted to explaining the nonstandard Drinfeld-Sokolov construction. In subsection 2.1 modified KdV type systems are dealt with. The construction of KdV type systems is described in subsection 2.2. The nonstandard Gelfand-Dickey systems are derived as examples in section 3. Section 4 contains our conclusions. Throughout the paper, proofs are often omitted or kept short since they are similar to those in the standard case.

2 A general construction of integrable hierarchies

The standard construction [1, 3, 2, 4, 5] associates a modified KdV type system with a triplet $(\mathcal{A}, \Lambda, d_1)$ and a KdV type system with a quadruplet $(\mathcal{A}, \Lambda, d_1, d_0)$, where the d_i $(i = 0, 1)$ are $\mathbf{Z}$-gradations of an affine Lie algebra $\mathcal{A}$, $\Lambda \in \mathcal{A}$ is a semisimple element of d_1-grade $k > 0$, and some further conditions hold in the KdV case. Below we present a generalization of the standard construction based on a splitting of the d_1-grade zero subalgebra $\mathcal{A}^0$ into a direct sum of two subalgebras of a certain form, and weakened conditions on the gradations d_i.

2.1 Modified KdV type systems

Let $\mathcal{A}$ be an affine Lie algebra with vanishing center, that is a twisted loop algebra

$$\mathcal{A} = \ell(\mathcal{G}, \tau) \subset \mathcal{G} \otimes \mathbf{C}[\lambda, \lambda^{-1}] \tag{2.1}$$

attached to a finite dimensional complex simple Lie algebra $\mathcal{G}$ with an automorphism τ of finite order [13]. Let $\mathcal{A} = \oplus_{n\in\mathbf{Z}}\mathcal{A}^n$ denote a $\mathbf{Z}$-gradation of $\mathcal{A}$ given by the eigensubspaces of a derivation d_1 of $\mathcal{A}$ as $d_1(X) = nX$ for $X \in \mathcal{A}^n$. Consider a semisimple element $\Lambda \in \mathcal{A}^k$ for some $k > 0$, and two subalgebras α^0, β^0 of $\mathcal{A}^0$ in such a way that

$$\beta^0 \subset \mathrm{Ker}(\mathrm{ad}\,\Lambda) \quad \text{and} \quad \mathcal{A}^0 = \alpha^0 + \beta^0 \tag{2.2}$$

is a linear direct sum decomposition. The subsequent construction, which reduces to that in [1, 2, 3, 4, 5] for $\beta^0 = \{0\}$, defines a mod-KdV type system for any choice of the data $(\mathcal{A}, \Lambda, d_1; \alpha^0, \beta^0)$. By definition, the phase space of this system is the manifold Θ of first order differential operators given by

$$\Theta := \left\{ \mathcal{L} = \partial_x + \theta(x) + \Lambda \,|\, \theta(x) \in \mathcal{A}^{<k} \cap \mathcal{A}^{\geq 0} \right\}. \tag{2.3}$$

We use the notation $\mathcal{A}^{<k} = \oplus_{n<k}\mathcal{A}^n$ etc. Since $\mathcal{A}^{<k} \cap \mathcal{A}^{\geq 0}$ is a finite dimensional space, the field $\theta(x)$ encompasses a finite number of complex valued fields depending on the one-dimensional space variable x. We wish to exhibit a family of compatible evolution equations on Θ labelled by the graded basis elements of

$$\mathcal{C}(\Lambda) := (\mathrm{Cent}\,\mathrm{Ker}(\mathrm{ad}\,\Lambda))^{\geq 0}, \tag{2.4}$$

which is the positively graded part of the center of the Lie algebra $\mathrm{Ker}(\mathrm{ad}\,\Lambda)$. For this we shall use the well-known 'formal dressing procedure' based on the linear direct sum decomposition

$$\mathcal{A} = \mathrm{Ker}(\mathrm{ad}\,\Lambda) + \mathrm{Im}(\mathrm{ad}\,\Lambda), \qquad \mathrm{Ker}(\mathrm{ad}\,\Lambda) \cap \mathrm{Im}(\mathrm{ad}\,\Lambda) = \{0\}, \tag{2.5}$$

whose existence is guaranteed by the semisimplicity of Λ. We next recall the main points of this procedure in a slightly more general context than required in this subsection.

Lemma 1. *Let $j(x) \in \mathcal{A}^{<k}$ be an arbitrary formal series with smooth component functions. Consider the equation*

$$\mathcal{L} := (\partial_x + j(x) + \Lambda) \mapsto e^{\mathrm{ad}F}(\mathcal{L}) = \partial_x + h(x) + \Lambda, \tag{2.6}$$

where $F(x)$ and $h(x)$ are required to be formal series

$$F(x) \in \mathcal{A}^{<0}, \qquad h(x) \in (\mathrm{Ker}(\mathrm{ad}\,\Lambda))^{<k}. \tag{2.7}$$

Then (2.6) can be solved for $F(x)$ and in terms of a particular solution $F_0(x)$ the general solution is determined by

$$e^{\mathrm{ad}F} = e^{\mathrm{ad}K} e^{\mathrm{ad}F_0}, \tag{2.8}$$

where $K(x) \in (\mathrm{Ker}(\mathrm{ad}\,\Lambda))^{<0}$ is arbitrary. There is a unique solution $F(j(x)) \in (\mathrm{Im}(\mathrm{ad}\,\Lambda))^{<0}$, whose components are differential polynomials in the components of $j(x)$.

Thanks to the lemma, which goes back to Drinfeld-Sokolov [1] (see also[3, 4]), for any constant $b \in \mathcal{C}(\Lambda)$ and any function $j(x) \in \mathcal{A}^{<k}$ one can define

$$B_b(j) := e^{-\mathrm{ad}F(j)}(b). \tag{2.9}$$

The components of $B_b(j)$ are uniquely determined differential polynomials in the components of j and one has $[B_b(j), (\partial_x + j + \Lambda)] = 0$ as a result of $[b, (\partial_x + h + \Lambda)] = 0$. For later purpose, note also that the formula

$$\mathcal{H}_b(j) := \int dx\, h_b(j(x)) \qquad \text{with} \quad h_b(j(x)) := \langle b, h(j(x)) \rangle \tag{2.10}$$

yields a well-defined functional of $j(x) \in \mathcal{A}^{<k}$ if we assume that the integral of a total derivative is zero. Here $\langle\ ,\ \rangle$ is a nondegenerate, invariant, symmetric bilinear form on $\mathcal{A}$. (Such a bilinear form exists and is unique up to a constant; the density $h_b(j(x))$ is well-defined only up to a total derivative in general.) According to a standard calculation [1], the functional derivative of $\mathcal{H}_b(j)$ defined using this bilinear form can be taken to be

$$\frac{\delta \mathcal{H}_b}{\delta j} = B_b(j) \qquad \text{for} \quad j(x) \in \mathcal{A}^{<k}, \quad b \in \mathcal{C}(\Lambda). \tag{2.11}$$

Since the conditions of Lemma 1 hold on Θ in (2.3), we can apply the dressing procedure to construct an integrable hierarchy on this manifold. For this we now introduce the splitting,

$$\mathcal{A} = \alpha + \beta \quad \text{with} \quad \alpha = \mathcal{A}^{>0} + \alpha^0, \quad \beta = \mathcal{A}^{<0} + \beta^0 \tag{2.12}$$

using (2.2), and the corresponding classical r-matrix of $\mathcal{A}$,

$$\mathcal{R} = \frac{1}{2}(\mathcal{P}_\alpha - \mathcal{P}_\beta), \tag{2.13}$$

where $\mathcal{P}_\alpha$, $\mathcal{P}_\beta$ project $\mathcal{A}$ onto the respective subalgebras α, β. By definition, the evolution equation associated with $b \in \mathcal{C}(\Lambda)$ is given by the following vector field $\frac{\partial}{\partial t_b}$ on Θ:

$$\frac{\partial}{\partial t_b}\theta := [\mathcal{R}(B_b(\theta)), \mathcal{L}] = [\mathcal{P}_\alpha(B_b(\theta)), \mathcal{L}] = -\,[\mathcal{P}_\beta(B_b(\theta)), \mathcal{L}] \quad \text{at} \quad \mathcal{L} = (\partial_x + \theta + \Lambda) \in \Theta. \tag{2.14}$$

The second and third equalities hold since $[B_b(\theta), \mathcal{L}] = 0$. It is easy to see from these equalities that $\frac{\partial}{\partial t_b}\theta$ is a differential polynomial in the components of $\theta(x)$ with values in $\mathcal{A}^{<k} \cap \mathcal{A}^{\geq 0}$, which guarantees that (2.14) makes sense as a vector field on Θ. Using that $[B_a(\theta), B_b(\theta)] = 0$ for

all $a, b \in \mathcal{C}(\Lambda)$ and that as a consequence of (2.14) $\frac{\partial}{\partial t_a} B_b(\theta) = [\mathcal{R}(B_a(\theta)), B_b(\theta)]$, together with the modified classical Yang-Baxter equation [12] for $\mathcal{R}$, it can be shown that the vector fields associated with different elements of $\mathcal{C}(\Lambda)$ commute:

$$\left[\frac{\partial}{\partial t_a}, \frac{\partial}{\partial t_b}\right] = 0 \qquad \forall\, a, b \in \mathcal{C}(\Lambda). \tag{2.15}$$

The functions $h_b(\theta)$ are of course the densities of corresponding conserved currents.

Having presented the algebraic construction of the hierarchy of evolution equations on Θ, now we describe the hamiltonian formulation of these equations. For this it will be convenient to first introduce a Poisson bracket on the local functionals on the space

$$\mathcal{M} := \{\mathcal{L} = \partial_x + j(x) + \Lambda \,|\, j(x) \in \mathcal{A}^{<k}\} \tag{2.16}$$

which contains Θ. A local functional $f : \mathcal{M} \to \mathbf{C}$ is given by $f(j) = \int dx\, p(x, j(x), \ldots, j^{(n)}(x))$, where p is a differential polynomial in the components of j whose coefficients are smooth functions of x. We let $\frac{\delta f}{\delta j(x)} \in \mathcal{A}$ denote the functional derivative of f, and introduce the r-bracket

$$[X, Y]_{\mathcal{R}} = [\mathcal{R}X, Y] + [X, \mathcal{R}Y] \qquad X, Y \in \mathcal{A}. \tag{2.17}$$

Imposing, for example, periodic boundary condition on $j(x)$, the following formula defines a Poisson bracket on the local functionals:

$$\{f, g\}_{\mathcal{R}}(j) := \int dx \left\langle j + \Lambda, \left[\frac{\delta f}{\delta j}, \frac{\delta g}{\delta j}\right]_{\mathcal{R}} \right\rangle - \left\langle \mathcal{R}\frac{\delta f}{\delta j}, \partial_x \frac{\delta g}{\delta j} \right\rangle - \left\langle \frac{\delta f}{\delta j}, \partial_x \mathcal{R}\frac{\delta g}{\delta j} \right\rangle. \tag{2.18}$$

The hamiltonian vector field, δ_f, corresponding to f is given by

$$\delta_f j = [\mathcal{R}\frac{\delta f}{\delta j}, \mathcal{L}] + \mathcal{R}^t[\frac{\delta f}{\delta j}, \mathcal{L}] \qquad \text{at} \quad \mathcal{L} = (\partial_x + j + \Lambda) \in \mathcal{M}. \tag{2.19}$$

Using the projectors associated with the decomposition $\mathcal{A} = \alpha^\perp + \beta^\perp$, where $\alpha^\perp$, $\beta^\perp$ denote the annihilators of α, β in $\mathcal{A}$, we have $\mathcal{R}^t = \frac{1}{2}(\mathcal{P}_{\beta^\perp} - \mathcal{P}_{\alpha^\perp})$. More explicitly, the hamiltonian vector field reads as

$$\delta_f j = \mathcal{P}_{\alpha^\perp}[j + \Lambda, \mathcal{P}_\beta \frac{\delta f}{\delta j}] - \mathcal{P}_{\beta^\perp}[j + \Lambda, \mathcal{P}_\alpha \frac{\delta f}{\delta j}] + \mathcal{P}_{\alpha^\perp}\mathcal{P}_\beta \partial_x \frac{\delta f}{\delta j} - \mathcal{P}_{\beta^\perp}\mathcal{P}_\alpha \partial_x \frac{\delta f}{\delta j}. \tag{2.20}$$

We now have two important statements to make. The first is that $\Theta \subset \mathcal{M}$ is a Poisson submanifold, and therefore one can trivially restrict the Poisson bracket to Θ. The second is that the flow in (2.14) is hamiltonian with respect to the Poisson bracket $\{\ ,\ \}_{\mathcal{R}}$ on Θ and the Hamiltonian $\mathcal{H}_b(\theta)$ obtained by restricting (2.10) to Θ. The first statement requires one to check that if $j(x) \in \mathcal{A}^{<k} \cap \mathcal{A}^{\geq 0}$ then $\delta_f j(x)$ lies in the same space, which is readily done with the aid of (2.20). The second statement follows by combining (2.11) with (2.19).

Remark 1. One can naturally extend the definition of the commuting vector fields $\frac{\partial}{\partial t_b}$ to the whole manifold $\mathcal{M}$ in (2.16) on which Lemma 1 applies. The flows of the resulting hierarchy on $\mathcal{M}$ can be written in hamiltonian form using $\{\ ,\ \}_{\mathcal{R}}$ in (2.18) and the Hamiltonian $\mathcal{H}_b(j)$ in (2.10). This system on $\mathcal{M}$ is conceptually useful to consider since both the mod-KdV type and (as we shall see) the KdV type systems are reductions of it. In general, it is an interesting problem to find all consistent subsystems of the hierarchy on $\mathcal{M}$ that involve finitely many independent fields.

2.2 KdV type systems

We below describe a construction that yields systems that we call 'systems of KdV type'. The construction requires that data of the form $(\mathcal{A}, \Lambda, d_1, d_0; \alpha^0, \beta^0)$ be given, where $\mathcal{A}$, Λ, d_1 satisfy the previous conditions and d_0 is another $\mathbf{Z}$-gradation of $\mathcal{A}$. There are further conditions on the data that we specify next.

We first of all assume that the two $\mathbf{Z}$-gradations of $\mathcal{A}$ are compatible, which means that $[d_0, d_1] = 0$ and we have a bi-gradation of $\mathcal{A}$:

$$\mathcal{A} = \oplus_{m,n\in\mathbf{Z}} \mathcal{A}^n_m, \qquad \mathcal{A}^n_m := \{\, X \in \mathcal{A} \,|\, d_1(X) = nX, \;\; d_0(X) = mX \,\}, \tag{2.21}$$

where superscripts/subscripts denote d_1/d_0-grades. We need the nondegeneracy condition

$$\mathrm{Ker}(\mathrm{ad}\,\Lambda) \cap \mathcal{A}^{<0}_0 = \{0\}, \tag{2.22}$$

which is a nontrivial condition if $\mathcal{A}^{<0}_0 \neq \{0\}$. We finally suppose that

$$\mathcal{A}^{>0} \subset \mathcal{A}_{\geq 0}, \qquad \mathcal{A}^{<0} \subset \mathcal{A}_{\leq 0}, \tag{2.23}$$

and require a splitting of $\mathcal{A}^0$ into a vector space direct sum of subalgebras of the form

$$\mathcal{A}^0 = \alpha^0 + \beta^0, \qquad \alpha^0 = \mathcal{A}^0_{\geq 0}, \qquad \beta^0 \subset \mathrm{Ker}(\mathrm{ad}\,\Lambda). \tag{2.24}$$

The conditions on the two gradations used in [4] (see also [3]) are stronger than (2.23) in that they include the additional condition $\mathcal{A}^0 \subset \mathcal{A}_0$. This extra condition guarantees the existence of a splitting of the form (2.24), given by $\alpha^0 = \mathcal{A}^0$, $\beta^0 = \{0\}$. In general, the existence of a subalgebra $\beta^0 \subset \mathrm{Ker}(\mathrm{ad}\,\Lambda)$ which is complementary to $\mathcal{A}^0_{\geq 0}$ in $\mathcal{A}^0$ is a nontrivial question. As is easy to see, a necessary condition for the existence of such a subalgebra is that

$$\mathcal{A}^0_{>0} \cap (\mathrm{Ker}(\mathrm{ad}\,\Lambda))^{\perp} = \{0\}, \tag{2.25}$$

which in particular implies that

$$\dim\left(\mathcal{A}^0_{<0}\right) \leq \dim\left(\mathcal{A}^0 \cap \mathrm{Ker}(\mathrm{ad}\,\Lambda)\right). \tag{2.26}$$

In the derivation of the system (1.2) presented in subsection 3.2, we have

$$\beta^0 = \mathcal{A}^0 \cap \mathrm{Ker}(\mathrm{ad}\,\Lambda). \tag{2.27}$$

Our construction of KdV type systems will proceed quite similarly to the standard one, except that we shall use the r-matrix $\mathcal{R}$ given by (2.12), (2.13) together with (2.24) to define the commuting vector fields. In the case when the condition $\mathcal{A}^0 \subset \mathcal{A}_0$ is satisfied the construction reduces to the standard one.

By definition, the phase space of the KdV type system is the factor space $Q/\mathcal{N}$, where

$$Q := \left\{\mathcal{L} = \partial_x + q(x) + \Lambda \,|\, q(x) \in \mathcal{A}^{<k} \cap \mathcal{A}_{\geq 0}\right\} \tag{2.28}$$

and $\mathcal{N}$ is the group of 'gauge transformations' e^{γ} acting on Q according to

$$e^{\gamma} : \mathcal{L} \mapsto e^{\mathrm{ad}\gamma}(\mathcal{L}) = e^{\gamma}\mathcal{L}e^{-\gamma}, \qquad \mathcal{L} \in Q, \quad \gamma(x) \in \mathcal{A}^{<0}_0. \tag{2.29}$$

Let us present a model of $Q/\mathcal{N}$. Take a d_1-graded vector space V appearing in a direct sum decomposition

$$\mathcal{A}^{<k} \cap \mathcal{A}_{\geq 0} = \left[\Lambda, \mathcal{A}_0^{<0}\right] + V. \tag{2.30}$$

Define $Q_V \subset Q$ by

$$Q_V := \{ \mathcal{L} = \partial_x + q_V(x) + \Lambda \,|\, q_V(x) \in V \}. \tag{2.31}$$

Due to the nondegeneracy condition (2.22) and the grading assumptions, the action of $\mathcal{N}$ on Q is a free action and the following result holds.

Lemma 2. *The submanifold $Q_V \subset Q$ is a global cross section of the gauge orbits defining a one-to-one model of $Q/\mathcal{N}$. When regarded as functions on Q, the components of $q_V(x) = q_V(q(x))$ are differential polynomials, which thus provide a free generating set of the ring of gauge invariant differential polynomials on Q.*

Since this lemma also goes back to [1] (see also [14]), Q_V is referred to as a *DS gauge*. To construct a hierarchy on $Q/\mathcal{N}$, we first exhibit commuting vector fields on Q by means of the dressing procedure (recall Remark 1). That is for any $b \in \mathcal{C}(\Lambda)$ in (2.4), we define commuting vector fields $\frac{\partial}{\partial t_b}$ on Q, similarly to (2.14), by

$$\frac{\partial}{\partial t_b} q := [\mathcal{R}(B_b(q)), \mathcal{L}] = [\mathcal{P}_\alpha(B_b(q)), \mathcal{L}] = -\left[\mathcal{P}_\beta(B_b(q)), \mathcal{L}\right] \quad \text{at} \quad \mathcal{L} = (\partial_x + q + \Lambda) \in Q. \tag{2.32}$$

Here $B_b(q)$ is obtained from (2.9) and the splitting (2.24) is used to define $\mathcal{R}$ by (2.12), (2.13). The conditions in (2.23), (2.24) ensure that (2.32) gives a consistent evolution equation on Q.

The evolution equation defined by (2.32) has a gauge invariant meaning. Algebraically speaking, this means that $\frac{\partial}{\partial t_b}$ induces a derivation of the ring of $\mathcal{N}$-invariant differential polynomials in q. The corresponding geometric statement is that the vector field $\frac{\partial}{\partial t_b}$ on Q can be consistently projected on $Q/\mathcal{N}$. The projectability of $\frac{\partial}{\partial t_b}$ follows from the uniqueness property of the formal dressing procedure stated by equation (2.8). This leads to the equality $B_b(e^\gamma \mathcal{L} e^{-\gamma})) = e^\gamma B_b(\mathcal{L}) e^{-\gamma}$, where $\gamma(x) \in \mathcal{A}_0^{<0}$ parametrizes a gauge transformation and we put $B_b(\mathcal{L}) := B_b(q)$. Using this, it is in fact straightforward to show the projectability of $\frac{\partial}{\partial t_b}$.

If we use Q_V as the model of $Q/\mathcal{N}$ and let $\pi : Q \to Q_V$ denote the natural mapping, then the projected vector field $\pi_*\left(\frac{\partial}{\partial t_b}\right)$ on Q_V is described by an equation of the form

$$\pi_* \left(\frac{\partial}{\partial t_b}\right) q_V = [\mathcal{R}(B_b(q_V)) + \eta_b(q_V), \partial_x + q_V + \Lambda], \tag{2.33}$$

where $\eta_b(q_V(x)) \in \mathcal{A}_0^{<0}$ is a uniquely determined differential polynomial in $q_V(x)$. These vector fields generate the commuting flows of the KdV type hierarchy on $Q_V = Q/\mathcal{N}$.

To deal with the hamiltonian formalism, notice that Q in (2.28) is a Poisson submanifold of $\mathcal{M}$ in (2.16) with respect to the Poisson bracket $\{\ ,\ \}_\mathcal{R}$ in (2.18). This is easy to verify by combining (2.20) with (2.23), (2.24). It then follows immediately that the derivative of a local functional f on Q with respect to the vector field $\frac{\partial}{\partial t_b}$ on Q is given by

$$\frac{\partial}{\partial t_b} f = \{f, \mathcal{H}_b\}_\mathcal{R}, \tag{2.34}$$

where $\mathcal{H}_b$ is obtained from (2.10). By the projectability of $\frac{\partial}{\partial t_b}$, we know that the right hand side of (2.34) must be gauge invariant if f is gauge invariant. Since $\mathcal{H}_b$ is a gauge invariant local functional on Q, by the uniqueness property (2.8), we are naturally lead to suspect that the Poisson bracket $\{f, g\}_{\mathcal{R}}$ of any two gauge invariant local functionals on Q is again gauge invariant. Indeed, under an infinitesimal gauge transformation $\delta_\gamma \mathcal{L} = [\gamma, \mathcal{L}]$ with $\gamma(x) \in \mathcal{A}_0^{<0}$, one finds for any two local functionals f, g on Q that

$$\delta_\gamma \{f, g\}_{\mathcal{R}} = \int dx \left\langle \frac{\delta g}{\delta q}, [\gamma_f, \mathcal{L}] \right\rangle - \left\langle \frac{\delta f}{\delta q}, [\gamma_g, \mathcal{L}] \right\rangle \tag{2.35}$$

with

$$\gamma_f = [\gamma, \mathcal{R}\frac{\delta f}{\delta q}] - \mathcal{R}[\gamma, \frac{\delta f}{\delta q}], \qquad \gamma_g = [\gamma, \mathcal{R}\frac{\delta g}{\delta q}] - \mathcal{R}[\gamma, \frac{\delta g}{\delta q}]. \tag{2.36}$$

Inspection shows that $\gamma_f(x), \gamma_g(x) \in \mathcal{A}_0^{<0}$. Hence $\delta_\gamma\{f, g\}_{\mathcal{R}}$ vanishes if f and g are gauge invariant, proving that the Poisson bracket of gauge invariant local functionals is gauge invariant.

To summarize the outcome of the above, a Poisson bracket is defined on $Q/\mathcal{N}$ by identifying the local functionals on $Q/\mathcal{N}$ with the gauge invariant local functionals on Q and determining the Poisson bracket on these functionals by (2.18). The KdV type hierarchy on $Q/\mathcal{N}$ is generated by the commuting Hamiltonians $\mathcal{H}_b$ with respect to this induced Poisson bracket.

In the standard case, for which $\mathcal{A}^0 \subset \mathcal{A}_0$ in addition to (2.23), one can show that our induced Poisson bracket on Q coincides with the 'second' Poisson bracket given in [5]. In this case one also has an alternative interpretation of the 'second' Poisson bracket on $Q/\mathcal{N}$ based on the r-matrix associated with the splitting $\mathcal{A} = \mathcal{A}_{\geq 0} + \mathcal{A}_{<0}$ (see e.g. [15]). However, in the case of a splitting in (2.24) for which $\beta^0 \neq \{0\}$ this second r-matrix does not even lead to consistent flows on Q, and hence we do not have such an alternative interpretation.

Given the data $(\mathcal{A}, \Lambda, d_1, d_0; \alpha^0, \beta^0)$ with which we associated a KdV type system, we also have the modified KdV type system corresponding to the data $(\mathcal{A}, \Lambda, d_1; \alpha^0, \beta^0)$. This modified KdV system can be restricted to the subspace of its phase space Θ in (2.3) given by

$$\Xi := \Theta \cap Q = \left\{ \mathcal{L} = \partial_x + \xi(x) + \Lambda \,|\, \xi(x) \in \mathcal{A}^{<k} \cap \mathcal{A}^{\geq 0} \cap \mathcal{A}_{\geq 0} \right\}. \tag{2.37}$$

In fact, $\Xi \subset \Theta \subset Q \subset \mathcal{M}$ is in this case a chain of Poisson submanifolds with respect to $\{\ ,\ \}_{\mathcal{R}}$ in (2.18). One can further check that

$$\dim\left(\mathcal{A}^{<k} \cap \mathcal{A}^{\geq 0} \cap \mathcal{A}_{\geq 0}\right) = \dim\left(\mathcal{A}^{<k} \cap \mathcal{A}_{\geq 0}\right) - \dim\left(\mathcal{A}_0^{<0}\right), \tag{2.38}$$

which means that the number of the components of $\xi(x)$ coincides with the number of the independent KdV fields. The map

$$\mu : \Xi \to Q/\mathcal{N} \tag{2.39}$$

induced by the natural projection $Q \to Q/\mathcal{N}$ is a generalization of the well-known Miura map. This maps converts the modified KdV type flows on Ξ to the KdV type flows on $Q/\mathcal{N}$. In the hamiltonian setting, μ is a Poisson map with respect to the (linear) Poisson bracket $\{\ ,\ \}_{\mathcal{R}}$ on Ξ and the (nonlinear) Poisson bracket on $Q/\mathcal{N}$ obtained as reduction of the Poisson bracket $\{\ ,\ \}_{\mathcal{R}}$ on Q.

Remark 2. One can naturally extend the action of $\mathcal{N}$ in (2.29) to the manifold $\mathcal{M}$ in (2.16). Although (like the action on Q) this action of $\mathcal{N}$ on $\mathcal{M}$ does not leave the Poisson bracket $\{\ ,\ \}_{\mathcal{R}}$ in (2.18) invariant, it is an 'admissible' action in the sense that the $\mathcal{N}$-invariant functions close under the Poisson bracket. This property permits to consider hamiltonian symmetry reduction of the hierarchy on $\mathcal{M}$, mentioned in Remark 1, with respect to $\mathcal{N}$. Then $Q/\mathcal{N} \subset \mathcal{M}/\mathcal{N}$ can be interpreted as a Poisson submanifold and the KdV type hierarchy as a subsystem of the reduced hierarchy on $\mathcal{M}/\mathcal{N}$.

3 Application to nonstandard Gelfand-Dickey hierarchies

In fact, our original motivation for this work was to understand whether the nonstandard Gelfand-Dickey hierarchy [9, 10, 11] defined by (1.2) can be interpreted in the Drinfeld-Sokolov framework. We here explain that it can indeed be obtained as an example of the general construction presented in the previous section. It turns out that one needs to use a nontrivial splitting with $\beta^0 \neq \{0\}$ in (2.24) to derive this system. For completeness, we also give the interpretation of the other nonstandard Gelfand-Dickey hierarchy [10, 11] defined by

$$\frac{\partial}{\partial t_m} L = [(L^{\frac{m}{n+1}})_{>0}, L] \qquad \text{for} \qquad L = \left(\partial^n + \sum_{i=1}^{n} u_i \partial^{n-i}\right)\partial. \tag{3.1}$$

In this case the input data satisfy the standard conditions, for which $\mathcal{A}^0 \subset \mathcal{A}_0$.

In the next subsection, we collect known results that will be used in the derivation of the nonstandard Gelfand-Dickey systems presented in subsections 3.2 and 3.3.

3.1 Generalized KP systems

We now recall some basic facts about the standard and nonstandard KP systems from which the Gelfand-Dickey systems arise by reduction to Poisson submanifolds. For a detailed exposition, see e.g. [8, 11] and references therein. Note that in this paper only the quadratic Poisson brackets will be considered, even though these systems are known to possess linear Poisson brackets as well.

We denote by $\mathcal{D}$ the associative algebra of scalar pseudo-differential operators (PDOs) of the form

$$L = \sum_{i=0}^{\infty} u_i \partial^{n-i} \qquad \forall n \in \mathbf{Z}. \tag{3.2}$$

The transposition $L \to L^t = \sum_{i=0}^{\infty} (-\partial)^{n-i} u_i$ is an anti-involution of $\mathcal{D}$. The projectors $P_{\geq 0}$, $P_{<0}$, P_0, $P_{>0}$ and $P_{\leq 0}$ on the corresponding associative subalgebras of $\mathcal{D}$ are given by

$$P_{\geq 0}(L) = \sum_{i=0}^{n} u_i \partial^{n-i}, \qquad P_{<0}(L) = \sum_{i=1}^{\infty} u_{n+i} \partial^{-i}$$
$$P_0(L) = u_n, \qquad P_{>0} = P_{\geq 0} - P_0, \qquad P_{\leq 0} = P_{<0} + P_0. \tag{3.3}$$

The Adler trace [16] on $\mathcal{D}$ is given by $\mathrm{Tr}(L) = \int dx\, \mathrm{res}(L)$ with $\mathrm{res}(L) = u_{n+1}$. We shall use the linear functionals on $\mathcal{D}$ defined by $l_X(L) = \mathrm{Tr}(LX)$, where X is some constant PDO. For a fixed positive integer n, we shall often consider the affine subspace $\mathcal{D}_n$ of $\mathcal{D}$,

$$\mathcal{D}_n = \{ L = \partial^n + \sum_{i=1}^{\infty} u_i \partial^{n-i} \}. \tag{3.4}$$

3.1.1. The standard KP hierarchy. The KP hierarchy is defined with the aid of the splitting of $\mathcal{D}$ into differential and purely pseudo-differential operators, which yields the antisymmetric r-matrix $R = \frac{1}{2}(P_{\geq 0} - P_{<0})$. In association with R, there exists a one parameter family of *local* quadratic Poisson brackets on $\mathcal{D}$

$$\{l_X, l_Y\}^{\nu}_{\mathrm{GD}}(L) = \mathrm{Tr}\,(LXR(LY) - XLR(YL)) + \nu \int dx (D^{-1}\mathrm{res}[L,X])\mathrm{res}[L,Y], \tag{3.5}$$

where $\partial_x(D^{-1}f) = f$. Notice that the constant ambiguity in the definition of $(D^{-1}f)$ drops out from formula (3.5), which defines a local Poisson bracket since $\mathrm{res}[L,X]$ is a total derivative. The 'second' Adler-Gelfand-Dickey bracket [8] corresponds to $\nu = 0$. The possibility to add the term proportional with ν on the right hand side of (3.5) was apparently first noticed in [17]. For any $n > 0$, $\mathcal{D}_n \subset \mathcal{D}$ is a Poisson submanifold with respect to the family of brackets in (3.5). For any complex number c, the local (differential polynomial) map F_c given on $\mathcal{D}_n$ by

$$F_c(L) = e^{c\Phi} L e^{-c\Phi}, \qquad \Phi = D^{-1} u_1, \tag{3.6}$$

which is invertible except for $c = \frac{1}{n}$, is a Poisson map according to

$$\{l_X \circ F_c, l_Y \circ F_c\}^{\nu}_{\mathrm{GD}} = \{l_X, l_Y\}^{\nu_c}_{\mathrm{GD}} \circ F_c, \qquad \nu_c = \nu + c(2 - nc)(1 - n\nu). \tag{3.7}$$

Note that $\nu = \frac{1}{n}$ is a fixed point for this transformation. The operators $L \in \mathcal{D}_n$ satisfying the condition $L = P_{\geq 0}(L)$ form a Poisson submanifold with respect to $\{\ ,\ \}^{\nu}_{\mathrm{GD}}$ for any ν, whereas those of the form

$$L = \partial^n + \sum_{i=2}^{n} u_i \partial^{n-i} \tag{3.8}$$

form a Poisson submanifold for the value $\nu = \frac{1}{n}$ only.

On $\mathcal{D}_n$, the commuting KP flows[1]

$$\frac{\partial}{\partial t_m} L = [R(L^{\frac{m}{n}}), L] = [(L^{\frac{m}{n}})_{\geq 0}, L] \qquad m = 1, 2, \ldots \tag{3.9}$$

are generated by the Hamiltonians $H_m(L) = \frac{n}{m}\mathrm{Tr}(L^{\frac{m}{n}})$ with respect to any of the brackets in (3.5). Restriction of the KP flows (3.9) to (3.8) gives the standard n^{th} KdV hierarchy.

Let us now write the operator $L = P_{\geq 0}(L) \in \mathcal{D}_n$ in the factorized form

$$L = \partial^n + \sum_{i=1}^{n} u_i \partial^{n-i} = (\partial + \xi_n)(\partial + \xi_{n-1}) \cdots (\partial + \xi_1) \tag{3.10}$$

which yields the Miura transformation, each field u_i being expressed as a differential polynomial in the ξ_i. Then the generalization in [18] of the Kupershmidt-Wilson theorem [19] asserts that the quadratic bracket $\{f, g\}^{\nu}_{\mathrm{GD}}(L)$ is equal to the bracket

$$\int dx \sum_{i,l=1}^{n} \left(\frac{\delta f}{\delta \xi_i}\right)_x (\delta_{il} - \nu) \left(\frac{\delta g}{\delta \xi_l}\right) \tag{3.11}$$

[1] Originally, only the hierarchy with the value $n = 1$ was called the KP hierarchy.

when the u_i and the ξ_i are related through the Miura transformation. Notice that this bracket is invariant under any permutation of the ξ_i and under a global change of sign of them.

3.1.2. The nonstandard KP hierarchy. The relevance of 'nonstandard' splittings of $\mathcal{D}$ to soliton equations was apparently first noticed in [20], see also [9, 21]. The definition of the nonstandard KP hierarchy of our interest is based on the splitting of $\mathcal{D}$ into purely differential operators and pseudo-differential operators containing the constant term. This splitting gives rise to the non-antisymmetric r-matrix

$$\hat{R} = \frac{1}{2}(P_{>0} - P_{\leq 0}). \tag{3.12}$$

In correspondence with this r-matrix, two quadratic *local* Poisson brackets have been defined on $\mathcal{D}$ in [10, 11]:

$$\begin{aligned}\{l_X, l_Y\}^A_{\mathrm{O}}(L) &= \mathrm{Tr}(LX\hat{R}(LY) - XL\hat{R}(YL)) \\ &+ \mathrm{Tr}([L,Y]_0 XL + [L,Y](LX)_0 + (D^{-1}\mathrm{res}[L,Y])[X,L]),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\{l_X, l_Y\}^B_{\mathrm{O}}(L) &= \mathrm{Tr}(LX\hat{R}(LY) - XL\hat{R}(YL)) \\ &+ \mathrm{Tr}([L,Y]_0 LX + [L,Y](XL)_0 - (D^{-1}\mathrm{res}[L,Y])[X,L]).\end{aligned} \tag{3.14}$$

For any fixed $n > 0$, $\mathcal{D}_n \subset \mathcal{D}$ is a Poisson subspace with respect to both of the above Poisson brackets. The two brackets admit different Poisson subspaces with a finite number of fields. In the case of bracket (3.13), the operators L^A of the form

$$L^A = \partial^{n-1} + \sum_{i=1}^{n-1} u_i \partial^{n-1-i} + \partial^{-1} u_n \tag{3.15}$$

form a Poisson submanifold for any $n \geq 1$. In the case of bracket (3.14), the operators

$$L^B = (\partial^n + \sum_{i=1}^{n} u_i \partial^{n-i})\partial \tag{3.16}$$

form a Poisson submanifold for any $n \geq 1$.

On the space $\mathcal{D}_n$, the nonstandard KP hierarchy[2] is given by the commuting flows

$$\frac{\partial}{\partial t_m} L = [\hat{R}(L^{\frac{m}{n}}), L] = [(L^{\frac{m}{n}})_{>0}, L] \qquad m = 1, 2, \dots . \tag{3.17}$$

These flows are generated by the Hamiltonians $H_m = \frac{n}{m}\mathrm{Tr}(L^{\frac{m}{n}})$ with respect to any of the brackets (3.13), (3.14). In contrast to the standard case, the coefficient u_1 of the subleading term of L is now dynamical, but $H_0 := \int dx\, u_1$ is still constant with respect to the flows. Restriction of these flows to the sets of operators $\{L^A\}$ in (3.15) and $\{L^B\}$ in (3.16) yields the nonstandard Gelfand-Dickey hierarchies in (1.2) and in (3.1), respectively. In the case (A) one obtains nontrivial flows for $n \geq 2$ only.

There exists a Poisson equivalence between the brackets $\{\ ,\ \}^{A,B}_{\mathrm{O}}$ and $\{\ ,\ \}^{\nu}_{\mathrm{GD}}$ with $\nu = \pm 1$:

$$\{l_X \circ p_A, l_Y \circ p_A\}^{+1}_{\mathrm{GD}} = \{l_X, l_Y\}^A_{\mathrm{O}} \circ p_A \tag{3.18}$$

$$\{l_X \circ p_B, l_Y \circ p_B\}^{-1}_{\mathrm{GD}} = \{l_X, l_Y\}^B_{\mathrm{O}} \circ p_B \tag{3.19}$$

[2]The term 'modified' KP hierarchy is also used in the literature, especially in the $n = 1$ case [22, 23].

where p_A, p_B are two invertible maps on $\mathcal{D}$ defined by $p_A(L) = \partial^{-1}L$ and $p_B(L) = L\partial$. This result, given in [24, 18] (see also [25]), allows to derive the properties of the brackets $\{\ ,\ \}_{\rm O}^{A,B}$ from familiar properties of the bracket $\{\ ,\ \}_{\rm GD}^{\nu}$. In particular, it allows for a straightforward derivation [18, 26] of generalized Kupershmidt-Wilson theorems for these brackets. For this purpose, one writes the operators $L^A \in p_A(P_{\geq 0}(\mathcal{D}_n))$ in (3.15) and $L^B \in p_B(P_{\geq 0}(\mathcal{D}_n))$ in (3.16) in a multiplicative form as

$$L^A = \partial^{-1}(\partial + \xi_n)(\partial + \xi_{n-1})\cdots(\partial + \xi_1) \tag{3.20}$$

$$L^B = (\partial + \xi_n)(\partial + \xi_{n-1})\cdots(\partial + \xi_1)\partial. \tag{3.21}$$

Then the Kupershmidt-Wilson theorem for $\{\ ,\ \}_{\rm GD}^{\nu}$ with $\nu = \pm 1$ and the relations (3.18), (3.19) imply that if the u_i are expressed through the ξ_i by the Miura transformations (3.20), (3.21), then the brackets $\{\ ,\ \}_{\rm O}^{A,B}$ on $\{L^{A,B}\}$ are equal to

$$\int dx \sum_{i,l=1}^{n} \left(\frac{\delta f}{\delta \xi_i}\right)_x (\delta_{il} - \nu^{A,B}) \left(\frac{\delta g}{\delta \xi_l}\right) \tag{3.22}$$

with $\nu^A = -\nu^B = 1$, respectively. These Miura transformations will be quite useful for us.

Although they intertwine the Poisson brackets, neither p_A or p_B converts the standard Gelfand-Dickey hierarchy into the nonstandard one since the commuting Hamiltonians are not intertwined by these maps.

Remark 3. It is well-known [12] that the quadratic Adler-Gelfand-Dickey bracket is a version of the Sklyanin bracket and its Jacobi identity depends on the antisymmetry of the r-matrix $R = \frac{1}{2}(P_{\geq 0} - P_{<0})$. General results on quadratic Poisson brackets on Lie groups associated with non-antisymmetric r-matrices have been obtained in [27], and equivalent results are found in [28] in the context of associative algebras, see also [29, 30] which deal with special cases. The brackets $\{\ ,\ \}_{\rm GD}^{\nu}$ as well as the brackets $\{\ ,\ \}_{\rm O}^{A,B}$ are identified in [25] as special cases of the '(a, b, c, d)-scheme' of [27], with nonlocal operators a, b, c, d. For the brackets $\{\ ,\ \}_{\rm O}^{A,B}$ an equivalent identification in terms of the notation of [28] is contained in [11].

3.2 Nonstandard Gelfand-Dickey system of type A

Now we show that the nonstandard Gelfand-Dickey system in (1.2) can be recovered within the generalized Drinfeld-Sokolov formalism developed in section 2 with an appropriate choice of the sextuplet $(\mathcal{A}, \Lambda, d_1, d_0; \alpha^0, \beta^0)$. Our demonstration below will be purely deductive; the right choice of data was originally found by a long explicit inspection of the linear problem for L in (1.2), which involved some guesswork too.

We consider the algebra $\mathcal{A} = sl(n) \otimes \mathbf{C}[\lambda, \lambda^{-1}]$. We denote by $e_{i,j}$ the $n \times n$ matrix with 1 at the intersection of line i and column j and 0 everywhere else, and introduce the two compatible gradations of $\mathcal{A}$

$$d_1 = (n-1)\lambda\partial_\lambda + \mathrm{ad}\,\mathcal{K}_A, \quad \mathcal{K}_A = \textstyle\sum_{k=1}^{n} \frac{n+1-2k}{2}\, e_{k,k} \tag{3.23}$$

$$d_0 = \lambda\partial_\lambda. \tag{3.24}$$

Note that d_0 is the homogeneous gradation like in the derivation of the standard system (1.1) [1], but d_1 is not the principal gradation. These gradations satisfy the conditions in (2.23), but do not satisfy the condition $\mathcal{A}^0 \subset \mathcal{A}_0$. Then for $n \geq 3$ we choose[3]

$$\Lambda := \sum_{k=1}^{n-1} e_{k,k+1} + \lambda(e_{n-1,1} + e_{n,2}), \tag{3.25}$$

which has d_1-grade one. One can check that Λ is a regular semisimple element, that is to say $\mathcal{A} = \mathrm{Ker}(\mathrm{ad}\,\Lambda) \oplus \mathrm{Im}(\mathrm{ad}\,\Lambda)$, and $\mathrm{Ker}(\mathrm{ad}\,\Lambda) \subset \mathcal{A}$ is an abelian subalgebra. The vector space $\mathrm{Ker}(\mathrm{ad}\,\Lambda)$ is generated by the homogeneous elements $\Lambda_m \in \mathcal{A}^m$ given by

$$\Lambda_{l(n-1)+r} = (2\lambda)^l(\Lambda^r - \delta_{r,n-1}\frac{2(n-1)\lambda}{n}I_n) \qquad \text{with} \qquad 1 \leq r \leq n-1, \quad l \in \mathbf{Z}. \tag{3.26}$$

The splitting of $\mathcal{A}^0$ of the form in (2.24) is in this case defined by

$$\beta^0 := (\mathrm{Ker}(\mathrm{ad}\,\Lambda))^0 = \mathrm{span}\{\Lambda_0\}. \tag{3.27}$$

3.2.1. The modified KdV type system. We shall first identify the mod-KdV type system associated with the above sextuplet as the modified nonstandard Gelfand-Dickey system of type A based on the factorized Lax operator in (3.20). An element of the mod-KdV phase space Ξ in (2.37) can now be parametrized by

$$\xi(x) = \sum_{k=1}^{n}(\xi_k - \sigma)e_{k,k} + \lambda(\xi_1 + \xi_n)e_{n,1}, \qquad \sigma = \frac{1}{n}\sum_{k=1}^{n}\xi_k. \tag{3.28}$$

The explicit evaluation of the Poisson bracket (2.18) on Ξ yields exactly the bracket in (3.22) that corresponds to the Miura map for the bracket $\{\ ,\ \}_0^A$. The identification of the conserved quantities can be performed through the linear problem and the elimination procedure following, e.g., the lines of [7]. If $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)^t$ then one gets from the linear problem $\mathcal{L}\Psi = 0$ the eigenvalue equation

$$(\partial - \sigma)^{-1}(\partial + \xi_n - \sigma)(\partial + \xi_{n-1} - \sigma)\cdots(\partial + \xi_1 - \sigma)\psi_1 = 2(-1)^{n-1}\lambda\psi_1, \tag{3.29}$$

which is equivalent to

$$L\tilde{\psi}_1 = \partial^{-1}(\partial + \xi_n)(\partial + \xi_{n-1})\cdots(\partial + \xi_1)\tilde{\psi}_1 = 2(-1)^{n-1}\lambda\tilde{\psi}_1 \quad \text{with} \quad \tilde{\psi}_1 := e^{-(D^{-1}\sigma)}\psi_1. \tag{3.30}$$

According to subsection 2.1, the commuting Hamiltonians of the mod-KdV type system can be chosen as $\mathcal{H}_k = \int dx\, h_k(x)$ where $e^{\mathrm{ad}F}(\mathcal{L}) = \partial_x + \Lambda + \sum_{m=0}^{\infty} h_m(x)\Lambda_{-m}$ is defined by (2.6). Then $\tilde{\psi}_1$ may then be computed in two different ways, from $\mathcal{L}\Psi = 0$ and from (3.30), along the lines of [7]. By comparison of the two results, one obtains

$$\begin{aligned} \mathcal{H}_0 &= \tfrac{1}{n-1}\int dx(\textstyle\sum_{k=1}^{n}\xi_k) \\ \mathcal{H}_m &= -\tfrac{a_m}{m}(-1)^m\mathrm{Tr}(L^{\frac{m}{n-1}}) \quad \text{for} \quad m \geq 1, \end{aligned} \tag{3.31}$$

with $a_m = n$ if m is zero modulo $(n-1)$, and $a_m = 1$ in the other cases. Except for normalization, the Hamiltonians $\mathcal{H}_m$ for $m \geq 1$ are thus identical with the H_m that generate the flows in (1.2)

[3]The $n = 2$ case is special and will be treated separately.

and $\mathcal{H}_0$ gives the conserved charge H_0 mentioned after (3.17). This proves the equivalence of the two modified systems.

3.2.2. The KdV type system. We shall now identify the generalized KdV type system that results from the construction of subsection 2.2 using the above sextuplet as the nonstandard Gelfand-Dickey system in (1.2). The nondegeneracy condition (2.22) holds in our case since Λ is a regular element. We can parametrize the phase space $Q/\mathcal{N}$ of the KdV system by the DS gauge slice Q_V in (2.31) whose general element is written as

$$q_V(x) = 2\lambda v_1 e_{n,1} + \sum_{k=1}^{n-1}(-1)^{n-k} v_{n+1-k} e_{n,k}. \tag{3.32}$$

From the linear problem $\mathcal{L}\Psi = 0$ for $\mathcal{L} \in Q$ (2.28), we obtain a gauge invariant eigenvalue equation on ψ_1, which in the DS gauge becomes

$$(\partial - v_1)^{-1}(\partial^n + \sum_{k=2}^{n} v_k \partial^{n-k})\psi_1 = 2(-1)^{n-1}\lambda\psi_1, \tag{3.33}$$

or equivalently

$$L\tilde{\psi}_1 = (\partial^{n-1} + \sum_{k=1}^{n-1} u_k \partial^{n-1-k} + \partial^{-1} u_n)\tilde{\psi}_1 = 2(-1)^{n-1}\lambda\tilde{\psi}_1, \qquad \tilde{\psi}_1 = e^{-(D^{-1}v_1)}\psi_1. \tag{3.34}$$

The u_k have differential polynomial expressions in terms of the v_k. These relations are invertible and the v_k are also differential polynomials of the u_k. Hence the points of Q_V may be parametrized by the functions u_k. The correspondence between the two parametrization of L in (3.30) and in (3.34) provides a model for the Miura map $\mu : \Xi \to Q/\mathcal{N} = Q_V$ in (2.39), which is a Poisson map when Ξ is equipped with the linear bracket (2.18) and $Q/\mathcal{N} = Q_V$ is equipped with the nonlinear bracket obtained as the reduction of the linear bracket on Q. Since this map is given by just the factorization of L, using the identification between the linear bracket on Ξ with the bracket in (3.22) that appears in the generalized Kupershmidt-Wilson theorem for the bracket $\{\ ,\ \}_{\mathrm{O}}^{A}$ (3.13), we conclude that the nonlinear bracket on $Q/\mathcal{N} = Q_V$ coincides with the bracket $\{\ ,\ \}_{\mathrm{O}}^{A}$ on the set of the Lax operators $L = L^A$ in (3.15). The identification between the respective sets of commuting Hamiltonians has already been established in (3.31).

3.2.3. The case $n = 2$. This case is slightly different from the generic one because of the form of Λ, now we choose $\Lambda = e_{1,2} + \lambda^2 e_{2,1}$. The element Λ is semisimple and regular with $\mathrm{Ker}(\mathrm{ad}\,\Lambda) = \mathrm{span}\{\lambda^k\Lambda \mid k \in \mathbf{Z}\}$. Apart from this, the sextuplet $(\mathcal{A}, \Lambda, d_1, d_0; \alpha^0, \beta^0)$ does not differ from the generic case. We briefly describe the identification of the associated KdV type system with the nonstandard Gelfand-Dickey system in (1.2) for $n = 2$. A convenient Drinfeld-Sokolov gauge is defined by

$$q_V(x) = (-\lambda u_1 - \frac{1}{4}(4u_2 + 2u_{1x} - u_1^2))e_{2,1}. \tag{3.35}$$

From the linear problem $\mathcal{L}\Psi = 0$, we obtain the eigenvalue equation on ψ_1

$$(\partial^2 + u_2 + \frac{u_{1x}}{2} - \frac{u_1^2}{4} + \lambda u_1)\psi_1 = \lambda^2\psi_1, \tag{3.36}$$

or equivalently

$$L\tilde{\psi}_1 = (\partial + u_1 + \partial^{-1}u_2)\tilde{\psi}_1 = 2\lambda\tilde{\psi}_1, \qquad \tilde{\psi}_1 = \exp(-\lambda x - \frac{1}{2}D^{-1}u_1)\psi_1. \tag{3.37}$$

Thanks to the low number of fields, the Poisson brackets of the u's may easily be computed by reduction of (2.18), and coincide with those obtained from the bracket in (3.13). The identification of the commuting Hamiltonians is obtained as in the other cases.

Remark 4. Another example for which the technique developed in section 2 applies is the following. The algebra $\mathcal{A}$ in the sextuplet $(\mathcal{A}, \Lambda, d_1, d_0; \alpha^0, \beta^0)$ is now $\mathcal{A} = sl(n+1) \otimes \mathbf{C}[\lambda, \lambda^{-1}]$, but we keep the same matrix Λ of equation (3.25), and the same gradations (3.23,3.24) as before. Clearly, Λ still is a semisimple element. A basis for $(\mathrm{Ker}(\mathrm{ad}\,\Lambda))^0$ is now given by the two elements Λ_0 in (3.26) and $\Omega = e_{n+1,n+1} - \frac{1}{n}\sum_{k=1}^{n} e_{k,k}$, and we take

$$\beta^0 := \mathrm{span}\{\Lambda_0\}. \tag{3.38}$$

A Drinfeld-Sokolov gauge may be parametrized by

$$q_V(x) = 2\lambda(v_1 - \frac{\vartheta}{n})e_{n,1} + \sum_{k=1}^{n-1}(-1)^{n-k}v_{n+1-k}e_{n,k} + \vartheta\Omega + (-1)^n\varphi e_{n,n+1} + \chi e_{n+1,1}. \tag{3.39}$$

Then the standard elimination procedure leads from the linear problem $\mathcal{L}\Psi = 0$ to the eigenvalue equation on ψ_1

$$(\partial - v_1)^{-1}((\partial - \frac{\vartheta}{n})^n + \sum_{k=2}^{n} v_k(\partial - \frac{\vartheta}{n})^{n-k} + \varphi(\partial - \vartheta)^{-1}\chi)\psi_1 = 2(-1)^{n-1}\lambda\psi_1, \tag{3.40}$$

or equivalently, with $\tilde{\psi}_1 = e^{-(D^{-1}v_1)}\psi_1$,

$$L\tilde{\psi}_1 = (\partial^{n-1} + \sum_{k=1}^{n-1} u_k\partial^{n-1-k} + \partial^{-1}u_n + \partial^{-1}\varphi(\partial + w)^{-1}\chi)\tilde{\psi}_1 = 2(-1)^{n-1}\lambda\tilde{\psi}_1. \tag{3.41}$$

The fields u_k and w have invertible differential polynomial expressions in terms of the v_k and ϑ. It is not hard to check that the nonstandard KP equations (3.17) define consistent flows for the operator L in (3.41). Moreover, the field w does not evolve under these flows, and may be set to zero. Then, if one introduces a field Φ which is a primitive of φ, $\Phi = (D^{-1}\varphi)$, the Lax operator L may be brought to the form

$$L = \partial^{n-1} + \sum_{k=1}^{n-1} u_k\partial^{n-1-k} + \partial^{-1}(u_n - \Phi\chi) + \Phi\partial^{-1}\chi, \tag{3.42}$$

which is one of the Poisson subspaces of the bracket (3.13) given in [11]. The restricted Poisson bracket is nonlocal in this parametrization [11]. One should note that the set of Lax operators L in (3.41) also defines a Poisson subspace of the bracket (3.13), which may be extended to a *local* bracket on the set of fields $\{u_k, w, \varphi, \chi\}$. Finally, it is clear that using the natural embedding of $sl(n)$ into $sl(n+m)$ for any $m \geq 1$ while keeping the same element Λ, one would reach a Lax operator containing m-component vector versions of φ and χ. More generally, constrained nonstandard matrix KP systems having Lax operators similar in form to L in (3.41) can also be derived by a slight modification of this example.

Remark 5. Restricting to the case $n = 2l+1$, we notice that the element Λ in (3.25) may be conjugated to

$$\hat{\Lambda} = \sum_{i=1}^{l} e_{i,i+1} - \sum_{i=l+1}^{2l} e_{i,i+1} - \lambda e_{2l,1} + \lambda e_{2l+1,2}, \tag{3.43}$$

which is thus semisimple and has d_1-grade one. We then consider the involution ζ of the algebra $\mathcal{A}$ defined on some element X by

$$X \mapsto \zeta(X) = -\eta X^t \eta, \qquad \eta = \sum_{i=1}^{2l+1} e_{i,2l+2-i}. \tag{3.44}$$

The elements in $\mathcal{A}$ invariant under this involution form the loop algebra of $so(2l+1)$ and $\hat{\Lambda}$ is an invariant element. The gradation d_1 in (3.23) and the homogeneous gradation d_0 both commute with the involution ζ. Moreover, the r-matrix $\mathcal{R}$ in (2.13) with α_0 as in (2.24) and β_0 as in (3.27) also commutes with ζ. Therefore, using an invariant element $b \in \mathcal{C}(\Lambda)$, one may restrict the flow (2.32) to those elements q which are invariant under ζ. The map $\mathcal{P}_{\beta_0}$ is identically zero on the invariant subalgebra. As a consequence, the restricted nonstandard hierarchy is just the standard Drinfeld-Sokolov hierarchy [1] based on the algebra $so(2l+1) \otimes \mathbf{C}[\lambda, \lambda^{-1}]$, whose Lax operator satisfies $\partial^{-1} L^t \partial = L$. If $n = 2l$, then a reduction of (1.2) to Lax operators satisfying $\partial^{-1} L^t \partial = -L$ is possible [9], but in this case we do not have an interpretation in the Drinfeld-Sokolov approach at present.

3.3 Nonstandard Gelfand-Dickey system of type B

The nonstandard Gelfand-Dickey hierarchy of the type in (3.1) can be recovered within the usual Drinfeld-Sokolov formalism [4, 5] with an appropriate choice of the quadruplet $(\mathcal{A}, \Lambda, d_1, d_0)$.

We consider $\mathcal{A} = sl(n) \otimes \mathbf{C}[\lambda, \lambda^{-1}]$ endowed with the two compatible gradations

$$d_1 = n\lambda\partial_\lambda + \mathrm{ad}\,\mathcal{K}_B, \qquad \mathcal{K}_B = \textstyle\sum_{k=1}^{n} \frac{n+1-2k}{2} e_{k,k} \tag{3.45}$$

$$d_0 = 2\lambda\partial_\lambda + \mathrm{ad}\,\mathcal{K}, \qquad \mathcal{K} = \tfrac{1-n}{n} e_{n,n} + \tfrac{1}{n} \textstyle\sum_{k=1}^{n-1} e_{k,k}. \tag{3.46}$$

Here d_1 is the principal gradation, but d_0 is not the homogeneous gradation. The assumptions in (2.23) as well as the condition $\mathcal{A}^0 \subset \mathcal{A}_0$ are satisfied. We choose for Λ the standard regular semisimple element of d_1-grade one:

$$\Lambda = \sum_{k=1}^{n-1} e_{k,k+1} + \lambda e_{n,1}. \tag{3.47}$$

The abelian algebra $\mathrm{Ker}(\mathrm{ad}\,\Lambda)$ is generated by the matrices Λ^m for m not a multiple of n.

3.3.1. The modified KdV type system. We wish to identify the mod-KdV type system defined by the above quadruplet with the system belonging to the factorized Lax operator, of order n, of the form $L = L^B$ in (3.16). For this we now parametrize the phase space Ξ in (2.37) by

$$\xi(x) = \sum_{k=1}^{n-1} (\xi_k - \sigma) e_{k,k} - \sigma e_{n,n}, \qquad \sigma = \frac{1}{n} \sum_{k=1}^{n-1} \xi_k. \tag{3.48}$$

The explicit evaluation of the Poisson bracket (2.18) on Ξ yields exactly the bracket given in (3.22). The linear problem $\mathcal{L}\Psi = 0$, where $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)^t$, leads to the eigenvalue equation

$$L\tilde{\psi}_n = (\partial + \xi_{n-1})(\partial + \xi_{n-2}) \cdots (\partial + \xi_1)\partial\tilde{\psi}_n = (-1)^n \lambda \tilde{\psi}_n \tag{3.49}$$

with $\tilde{\psi}_n = e^{-(D^{-1}\sigma)}\psi_n$. The result of the dressing procedure in (2.6) applied to $\mathcal{L} \in \Xi$ may be parametrized as $e^{\mathrm{ad}F}(\mathcal{L}) = \partial_x + \Lambda + \sum_{0<m\neq pn} h_m(x)\Lambda^{-m}$. Then one finds

$$\mathcal{H}_m = \int dx\, h_m(x) = -\frac{1}{m}(-1)^n \mathrm{Tr}(L^{\frac{m}{n}}), \tag{3.50}$$

whereby the identification of the respective modified systems is complete.

3.3.2. The KdV type system. In order to identify the KdV type system associated with the above quadruplet as the corresponding nonstandard Gelfand-Dickey system, we now parametrize the elements of a convenient DS gauge Q_V in (2.31) as

$$q_V(x) = \sum_{k=1}^{n-1} \frac{v_1}{n-1} e_{k,k} - v_1 e_{n,n} - \sum_{k=2}^{n-1} (-1)^{k+1} v_k e_{n-1,n-k}. \tag{3.51}$$

From the linear problem $\mathcal{L}\Psi = 0$, we obtain the eigenvalue equation on ψ_n

$$((\partial + \frac{v_1}{n-1})^{n-1} + \sum_{k=2}^{n-1} v_k (\partial + \frac{v_1}{n-1})^{n-1-k})(\partial + v_1)\psi_n = (-1)^n \lambda \psi_n \tag{3.52}$$

or equivalently

$$L\tilde{\psi}_n = (\partial^{n-1} + \sum_{k=1}^{n-1} u_k \partial^{n-1-k})\partial\tilde{\psi}_n = (-1)^n \lambda \tilde{\psi}_n, \qquad \tilde{\psi}_n = e^{(D^{-1}v_1)}\psi_n. \tag{3.53}$$

By the same argument as in paragraph 3.2.2, we conclude that the nonlinear bracket on $Q/\mathcal{N} = Q_V$, parametrized by the functions u_k, is the same as the bracket obtained from (3.14). This establishes the desired identification.

4 Discussion

The Gelfand-Dickey system (1.1) and its variants in (1.2), (3.1) are reductions of the KP hierarchy and of its nonstandard counterpart. They represent three distinct generalized KdV hierarchies whose flows are hamiltonian with respect to the same nonlinear Poisson bracket structure given by the $\mathcal{W}_n \oplus U(1)$ classical extended conformal algebra. Drinfeld and Sokolov [1] showed how to view the Gelfand-Dickey hierarchy in an affine Lie algebraic setting. Their derivation used the grade one element of the principal Heisenberg subalgebra of the loop algebra $sl(n) \otimes \mathbf{C}[\lambda, \lambda^{-1}]$ in a hamiltonian reduction procedure in which the interplay between the homogeneous and the principal gradations played an important role. In this paper, we found that the nonstandard Gelfand-Dickey hierarchy of type B (3.1) admits a similar derivation in which the homogeneous gradation is replaced by another gradation. In this way, the system in (3.1) is interpreted as a special case of the systems obtained by the generalized Drinfeld-Sokolov reduction procedure defined in [4], which associates a KdV type system with a quadruplet $(\mathcal{A}, \Lambda, d_1, d_0)$, where the gradations d_i of the loop algebra $\mathcal{A}$ satisfy that every element with positive d_0-grade is positive in the d_1-gradation too.

More interestingly, we found that the Gelfand-Dickey system of type A (1.2) cannot be obtained in the framework of [4]. In fact, we gave a matrix Lax formulation for the hierarchy (1.2) by using two gradations d_0 and d_1 which satisfy weaker conditions than those above. In particular, there may exist elements of the loop algebra $\mathcal{A}$ with positive d_0 grade and zero d_1 grade, provided they are not orthogonal to the kernel of the adjoint action of $\Lambda \in \mathcal{A}$. The general Lie algebraic setting which we used could be applied to other cases as well. A series of such applications was mentioned in Remark 4.

The kernel of the regular semisimple element Λ in (3.25) is a maximal abelian subalgebra of the loop algebra $\mathcal{A} = sl(n) \otimes \mathbf{C}[\lambda, \lambda^{-1}]$, which would become a Heisenberg algebra after introducing the central charge. The inequivalent Heisenberg subalgebras are classified by the conjugacy classes of the Weyl group of $sl(n)$ [31], which are in one-to-one correspondence with the partitions of n. The Heisenberg subalgebra to which Λ belongs corresponds in the Kac-Peterson classification to the conjugacy class in the Weyl group associated with the partition $(n-1, 1)$. Indeed, after a suitable rescaling of the loop parameter, Λ may be shown to be equivalent to

$$\tilde{\Lambda} = \sum_{i=1}^{n-2} e_{i,i+1} + \tilde{\lambda} e_{n-1,1} \tag{4.1}$$

which is a generator of the principal Heisenberg algebra of the subalgebra $sl(n-1) \otimes \mathbf{C}[\tilde{\lambda}, \tilde{\lambda}^{-1}]$. Beside the constrained version (1.2) of the nonstandard KP hierarchy derived in section 3.2, there exists also a constrained version of the standard KP hierarchy associated with the same Heisenberg subalgebra of $\mathcal{A}$. This constrained KP hierarchy [32] has a scalar Lax operator of the form

$$\tilde{L} = \partial^{n-1} + \tilde{u}_1 \partial^{n-2} + \cdots + \tilde{u}_{n-1} + \tilde{\varphi}(\partial + \tilde{w})^{-1} \tilde{\chi} \quad \text{with} \quad \tilde{w} = -\tilde{u}_1, \tag{4.2}$$

whose derivation in the Drinfeld-Sokolov framework is described in [7]. The derivation uses the element $\tilde{\Lambda} \in \mathcal{A}$ and the homogeneous gradation together with a gradation $\tilde{d}_1$ for which the condition $\mathcal{A}^0 \subset \mathcal{A}_0$ is satisfied. More precisely, in [7] the $gl(n)$ case is considered for which $\tilde{w}$ in (4.2) is independent of $\tilde{u}_1$; neither $\tilde{u}_1$ nor $\tilde{w}$ has nontrivial dynamics. By setting $\tilde{w} = 0$ and conjugating by $\tilde{\varphi}^{-1}$, which is a singular map at the zeros of $\tilde{\varphi}$, one may convert the flows and the Poisson brackets of the system based on $\tilde{L}$ in (4.2) into those of the system in (1.2) (see [10]). Note also that the map $p_A : \tilde{L} \to \partial^{-1}\tilde{L}$ connects the second Poison bracket (but not the flows) of the system in (4.2) to that of a system of the type mentioned in Remark 4. It should be possible to interpret these connections between the standard and nonstandard constrained KP hierarchies as consequences of their closely related affine Lie algebraic origin.

Finally, we wish to stress that in our opinion the most interesting problem arising from this paper is to find new input data whereby the general construction of section 2 might give rise to new integrable hierarchies. As another problem which remains to be settled, let us also remark that although the nonstandard Gelfand-Dickey hierarchies of type A are known from the scalar Lax formalism to be bi-hamiltonian [10], we have not yet been able to identify the first Poisson bracket in the matrix Lax formalism.

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References

[1] V.G. Drinfeld and V.V. Sokolov, Sov. Math. Dokl. **23** (1981) 457; J. Sov. Math. **30** (1985) 1975.

[2] G.W. Wilson, Ergod. Th. and Dynam. Sys. **1** (1981) 361.

[3] I.R. McIntosh, PhD Thesis, Dept. Math., Imperial College, London, 1988.

[4] M.F. de Groot, T.J. Hollowood and J.L. Miramontes, Commun. Math. Phys. **145** (1992) 57.

[5] N.J. Burroughs, M.F. de Groot, T.J. Hollowood and J.L. Miramontes, Commun. Math. Phys. **153** (1993) 1879.

[6] F. Delduc and L. Fehér, J. Phys. **A 28** (1995) 5943.

[7] L. Fehér and I. Marshall, Commun. Math. Phys. **183** (1997) 423.

[8] L.A. Dickey, *Soliton Equations and Hamiltonian Systems*, Adv. Ser. Math. Phys., Vol. 12, World Scientific, Singapore, 1991.

[9] B.A. Kupershmidt, Commun. Math. Phys. **99** (1985) 51.

[10] W. Oevel and W. Strampp, Commun. Math. Phys. **157** (1993) 51.

[11] W. Oevel, Phys. Lett. **A 186** (1994) 79.

[12] M.A. Semenov-Tian-Shansky, Funct. Anal. Appl. **17** (1983) 259.

[13] V.G. Kac, *Infinite Dimensional Lie Algebras*, Cambridge University Press, Cambridge, 1985.

[14] L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. **222** (1992) 1.

[15] L. Fehér, talk at the Marseille 1995 conference on $\mathcal{W}$-algebras, hep-th/9510001.

[16] M. Adler, Invent. Math. **50** (1979) 219.

[17] P. Di Francesco, C. Itzykson and J.B. Zuber, Commun. Math. Phys. **140** (1991) 543.

[18] Q.P. Liu, preprint solv-int/9707012

[19] B.A. Kupershmidt and G. Wilson, Invent. Math. **62** (1981) 403.

[20] A.G. Reyman, J. Sov. Math. **19** (1982) 1507.

[21] K. Kiso, Prog. Theor. Phys. **83** (1990) 1108.

[22] B.A. Kupershmidt, Commun. Math. Phys. **167** (1995) 351.

[23] J.C. Shaw and M.H. Tu, J. Phys. **A 30** (1997) 4825.

[24] W.J. Huang, J.C. Shaw and H.C. Yen, J. Math. Phys. **36** (1995) 2959.

[25] F. Delduc and L. Gallot, to be published in Commun. Math. Phys.

[26] J.C. Shaw and M.H. Tu, preprint solv-int/9707014.

[27] L. Freidel and J.M. Maillet, Phys. Lett. **B 262** (1991) 278.

[28] Yu.B. Suris, Phys. Lett. **A 186** (1993) 419.

[29] L.C. Li and S. Parmentier, Commun. Math. Phys. **125** (1989) 545.

[30] W. Oevel and O. Ragnisco, Physica **A 161** (1989) 181.

[31] V.G. Kac, D.H. Peterson, pp. 276-298 in: Proc. of Symposium on Anomalies, Geometry and Topology, W.A. Bardeen, A.R. White (eds.), World Scientific, Singapore, 1985.

[32] Yi Cheng, J. Math. Phys. **33** (1992) 3774.