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CHAOTIC BEHAVIOUR OF RENORMALISATION FLOW IN A COMPLEX MAGNETIC FIELD

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ABSTRACT

It is demonstrated that decimation of the one dimensional Ising model, with periodic boundary conditions, results in a non-linear renormalisation transformation for the couplings which can lead to chaotic behaviour when the couplings are complex. The recursion relation for the couplings under decimation is equivalent to the logistic map, or more generally the Mandelbrot map. In particular an imaginary external magnetic field gives chaotic trajectories in the space of couplings, not for all values of the field but for a set of values which is dense in the real numbers. The magnitude of the field must be greater than a minimum value which tends to zero as the critical point $T = 0$ is approached.

The renormalisation group has been developed into an immensely powerful tool for the analysis of physical theories near critical points and also for continuum field theories. There are by now various forms of "renormalisation group equation" which govern how physical amplitudes and couplings change under change of scale. Perhaps one of the most intuitively appealing is the version due to Wilson [1], motivated by a suggestion of Kadanov [2], involving "decimation".

In principle one can derive recursive formulae for the couplings of a theory, which dictate how they should change when the underlying lattice is decimated, so that the Hamiltonian involving the new couplings on the new lattice is the same as the Hamiltonian involving the old couplings on the old lattice, i.e. the partition function does not change under the simultaneous operations of decimation and redefinition of couplings. The recursive formulae for the couplings are in general non-linear (indeed they are not invertible, so the transformation involved here is not a group but a semi-group).

Non-linear recursive formulae are one of the central themes of study for chaos theory and one can pose the question - can the renormalisation transformation lead to chaotic behaviour in the space of couplings? It will be demonstrated, by explicit example, that the answer to this question is yes and, in the model examined (the one dimensional Ising model), the onset of chaotic behaviour appears to be associated with the second order phase transition at $T = 0$. It remains an open question as to whether this is a peculiarity of this model or is a more general feature.

One severe problem in extracting general features is the paucity of models for which the recursion relations can be obtained exactly, and the one dimensional Ising model - because of its simplicity - is one example for which progress can be made. Nevertheless, despite its simplicity, the results are startling enough to merit description.

It will be shown that the onset of chaotic behaviour is brought about by extending the couplings of the theory to the complex plain. This is not a new idea in the analysis of such theories. Dyson [3] pointed out that one could learn something about the structure of Quantum Electrodynamics by considering imaginary electric charges, so that $\alpha = e^2/\hbar c < 0$. Such a theory must be intrinsically unstable, and so amplitudes cannot be analytic at $\alpha = 0$, hence perturbation theory must diverge and expansions in α are, at best, asymptotic. These ideas have been further developed by making α complex and there is by now a whole literature on complex analyticity and Borel summability (e.g. [4]). In statistical mechanics, extending the couplings to the complex plane is a key step in solving many two-dimensional models [5] and has led to some beautiful results concerning the analyticity of the partition function [6].

In this paper yet another example of the fascination of complex variables will be exhibited - by allowing the couplings of the one dimensional Ising model to be complex, the recursive renormalisation transformations can become chaotic. To exhibit this phenomenon, some well known features of the one dimensional Ising model will be summarised and the recursion relations derived. It will then be shown that the recursion relation is nothing other than the logistic map, and chaos ensues!

Consider the one dimensional Ising model on a periodic latice of N sites [5]. The partition function is

$$Z_N = \sum_{\{\sigma\}} exp\left[K\sum_{j=1}^{N}\sigma_j\sigma_{j+1} + h\sum_{j=1}^{N}\sigma_j\right] \tag{1}$$

where $K = \frac{J}{kT}$ and $h = \frac{H}{kT}$, with J the spin coupling and H the external magnetic field, (periodic boundary conditions require $\sigma_{N+1} \equiv \sigma_1$). $Z_N(K,h)$ can be conveniently expressed in terms of the transfer matrix

$$V = \begin{pmatrix} V_{++} & V_{+-} \\ V_{-+} & V_{--} \end{pmatrix} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} \tag{2}$$

as $Z_N = TrV^N$.

Diagonalising V gives the eigenvalues

$$\lambda_{\pm} = e^K\left\{\cosh h \pm \sqrt{\sinh^2 h + e^{-4K}}\right\}. \tag{3}$$

Thus

$$Z_N = \lambda_+^N\left[1+\left(\frac{\lambda_-}{\lambda_+}\right)^N\right]. \tag{4}$$

The recursive renormalisation transformation is obtained by asking: can one find new couplings K' and h' such that

$$Z_{\frac{N}{2}}(K',h') = A^N Z_N(K,h) \tag{5}$$

gives the same physical amplitudes? (A is a normalisation factor.) The number of lattice sites is here being reduced by a factor of two, so perhaps this should be called semimation rather than decimation.

Equation (5) is easily satisfied by demanding

$$\begin{pmatrix} e^{K'+h'} & e^{-K'} \\ e^{-K'} & e^{K'-h'} \end{pmatrix} = A^2 \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}^2 \tag{6}$$

giving the recursive formulae:

$$e^{2h'} = e^{2h}\frac{\cosh(2K+h)}{\cosh(2K-h)} \tag{7}$$

$$e^{4K'} = \frac{\cosh(4K) + \cosh(2h)}{2\cosh^2(h)}. \tag{8}$$

The normalisation factor A is unimportant for the present analysis.

The combination $e^{4K'}\sinh^2(h') = e^{4K}\sinh^2(h)$ is a renormalisation transformation invariant. This is not unexpected since the magnetisation per site, in the thermodynamic limit, is

$$M_{N\to\infty} = -\frac{\partial\, ln\, Z_N}{N\partial h} = -\frac{e^{2k}\sinh(h)}{\sqrt{1 + e^{4K}\sinh^2(h)}} \tag{9}$$

which is a physical quantity.

All these facts about the one dimensional Ising model are well known [5] and are included only for completeness.

It will now be shown that the recursion relations (7) and (8) are equivalent to the logistic map and, for certain (complex) values of couplings gives rise to chaotic behaviour.

Define

$$m = 1 + e^{4K}\sinh^2(h) \tag{10}$$

which is a renormalisation transformation invariant, $m = m'$. It is now only necessary to consider one of equations (8) and (7) as the existence of the invariant, m, makes one of them redundant.

Eliminating h from (8) using (10) gives

$$e^{4K'} - 1 = \frac{1}{4}\frac{(e^{4k} - 1)^2}{[(e^{4K} - 1) + m]}. \tag{11}$$

Now replace K with a new variable

$$x = -\frac{m}{(e^{4K} - 1)} \tag{12}$$

with -$\infty < x < 0$ for $m > 0$ and $K > 0$. The recursion relation (11) now becomes

$$x' = 4x(1 - x) \tag{13}$$

which is the logistic map.

For $0 < x < 1$, the recursion relation (13) leads to chaotic behaviour, as is easily seen by defining $x = \sin^2(\pi\psi)$, $0 < \psi < \frac{1}{2}$ (see e.g. [7]) giving

$$\sin(\pi\psi') = \sin(2\pi\psi) \tag{14}$$

Writing ψ in binary form, we see that the iterative map merely shifts all bits one step to the left and throws away the integral part, leaving the fractional part behind. For an initial value of ψ which is rational this will lead to a periodic orbit, but for a starting value of ψ which is irrational, the process never repeats and ψ jumps around chaotically. Since the irrationals are dense in the reals, almost all initial values lead to chaotic motion.

For real values of the couplings, $m > 1$ and -$\infty < x < 0$. Chaotic trajectories require

$$m = 1 + e^{4K} \sinh^2(h) < 0. \tag{15}$$

For example, if $K > 0$ is real and h is pure imaginary ($h = i\theta$), then $m < 0$ for $\sin^2\theta > e^{-4K}$ and $x = \frac{e^{4K}\sin^2\theta - 1}{e^{4K}-1}$ lies between 0 and 1 for $e^{4K} > \frac{1}{\sin^2\theta}$.

The region of chaotic flow is shown in Figure 1. Note that not all points above the line $x = 0$ lead to chaotic trajectories, only those with irrational ψ. Indeed there is an infinite number of periodic trajectories as well as chaotic ones. For example there are lines of unstable fixed points (period one) at $x = 0$ and $x = 3/4$ and the values $x = (5 \pm \sqrt{10})/8$ give orbits of period two , etc.

For finite K there is a gap and a small imaginary magnetic field is not sufficient to induce chaos, but as K increases ($T \to 0$) this gap reduces to zero. Following Baxter [5], define $t = e^{-2K}$ with $t \to 0$ being the critical point, then near $t = 0$ the line separating chaotic from regular flow is given by

$$t^2 \sim \theta^2 \quad \Rightarrow \quad \theta \sim t. \tag{16}$$

If we define a critical exponent, ξ, such that the critical value of θ is

$$\theta \sim t^{\xi}, \tag{17}$$

then it has been shown that $\xi = 1$ for the one dimensional Ising model.

The partition function has interesting properties for $h = i\theta$. The eigenvalues of the transfer matrix become

$$\lambda_{\pm} = e^K \cos\theta \pm e^{-K}\sqrt{m} \tag{18}$$

These are complex for $m < 0, \lambda_{\pm} = e^K \cos\theta \pm ie^{-K} \mid m \mid^{\frac{1}{2}}$ with $\mid \lambda_+ \mid^2 \; = \; \mid \lambda_- \mid^2$ $= 2\sinh 2K$. On the critical line $e^{4K}\sin^2\theta = 1$, $m = 0$ and $\lambda_+ = \lambda_- = e^K \cos\theta$ giving

$$Z_N = (2e^K \cos\theta)^N = (2\sinh 2K)^{N/2} \tag{19}$$

and the partition function vanishes at $\theta = \pi/_2$. One can obtain further insights by allowing K to become complex. Define $x = -\frac{z}{4} + \frac{1}{2}$ in equation (13) to give

$$z' = z^2 - 2. \tag{20}$$

This is the Mandelbrot map $z' = z^2 + c$ for complex z, with $c = -2$. One can have divergence or convergence depending on c and the initial choice of z. The values of c for which the iterates of the starting point $z = 0$ stay bounded is the Mandelbrot set and clearly $c = -2$ is an element of this set. The Julia set for a given value of c is the set of points in the complex z-plane which stay within a bounded region upon repeated iteration of the Mandelbrot map (strictly speaking this is the filled in Julia set J_c - the Julia set is actually the boundary of this set). The set J_c is generated by the inverse set of the unstable fixed points. For $c = -2$ these are $z = -1$ and $z = 2$ and so the earlier analysis of the logistic map tells us that the inverse iterates generate the segment of the real axis lying between -2 and $+2$ (this corresponds to $0 < x < 1$ in the previous notation). Thus the whole Julia set is just this segment, which is equal to its boundary since it is one dimensional. This analysis shows that the forward iterations send $|z|$ to infinity if the temperature has an imaginary component or if the magnetic field has both real and imaginary parts non-zero (this latter possibility would result in m having non-zero imaginary part and thus so would z). Thus chaotic trajectories occur only for real K and pure imaginary h. The behaviour of the Julia set under iteration is shown in Figure 2 for $c = -2$.

An obvious question is: how generic is this behaviour? For a general Hamiltonian when is it possible to obtain chaotic behaviour in some region of (complex) coupling space? For the moment this question must remain unanswered, but a few comments should be made. Feigenbaum was aware of the universality in chaos [8]. Near an extremum any non-linear map (with non-vanishing second derivative) can be put into the form (13) with the number 4 replaced, in general, by a parameter, λ.

Thus

$$x' = \lambda x(1 - x) \tag{21}$$

is generic, but whether or not one has chaotic behaviour depends on the value of λ and the initial value of x. I do not know of any reason why λ has the rather special value of 4 for the one dimensional Ising model. More generally, Feigenbaum has shown [8] that the properties of a general non-linear map

$$x' = \lambda f(x) \tag{22}$$

are independent of the exact form of $f(x)$ near a maximum. For any particular model, the value of λ would have to be calculated ab initio and I know of no way of deciding in advance whether or not chaotic flow would result, though it is interesting to speculate that chaotic motion may be related to second order phase transitions. Unfortunately the number of models for which the recursion relations are known exactly is rather few.

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Figure Captions

Figure 1. Critical line in the K-θ plane. The renormalisation flow is regular below the critical line $x = 0$ and chaotic above it for all values of ψ which are irrational ($x = \sin^2(\pi\psi)$).

Figure 2. A representation of the Julia set in the complex z-plane for $c = -2$. The Julia set itself is the real line segment $-2 < \mathrm{Re}(z) < 2$ which is the width of the diagram. The contours depict the rate at which a point is repelled from the Julia set, the darker the contour the more rapid the expulsion. The picture was generated use the program FRACTINT produced by the Stone Soup Group.

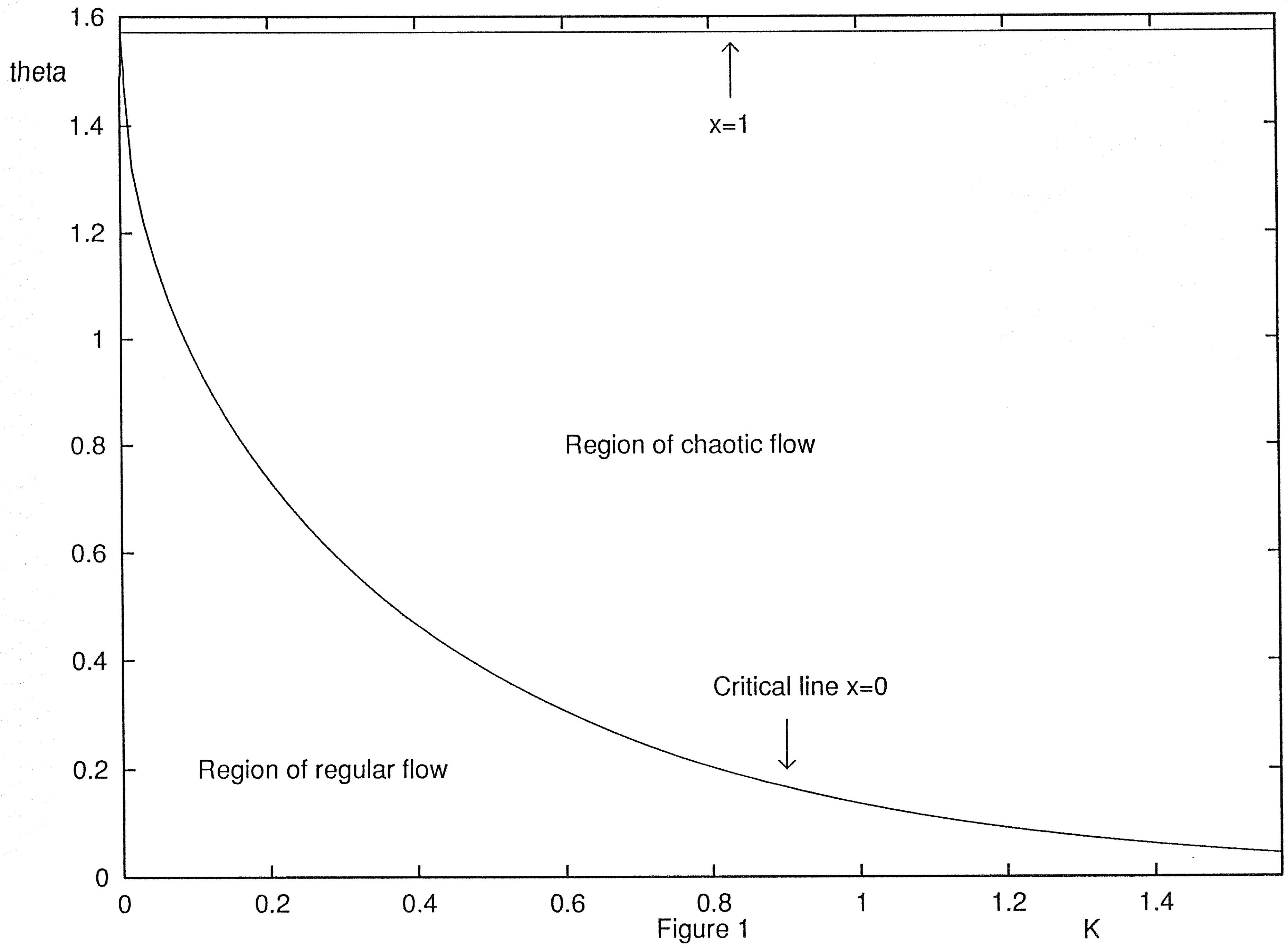
theta
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
x=1
Region of chaotic flow
Critical line x=0
Region of regular flow
0
0.2
0.4
0.6
0.8
1
1.2
1.4
K

Figure 1

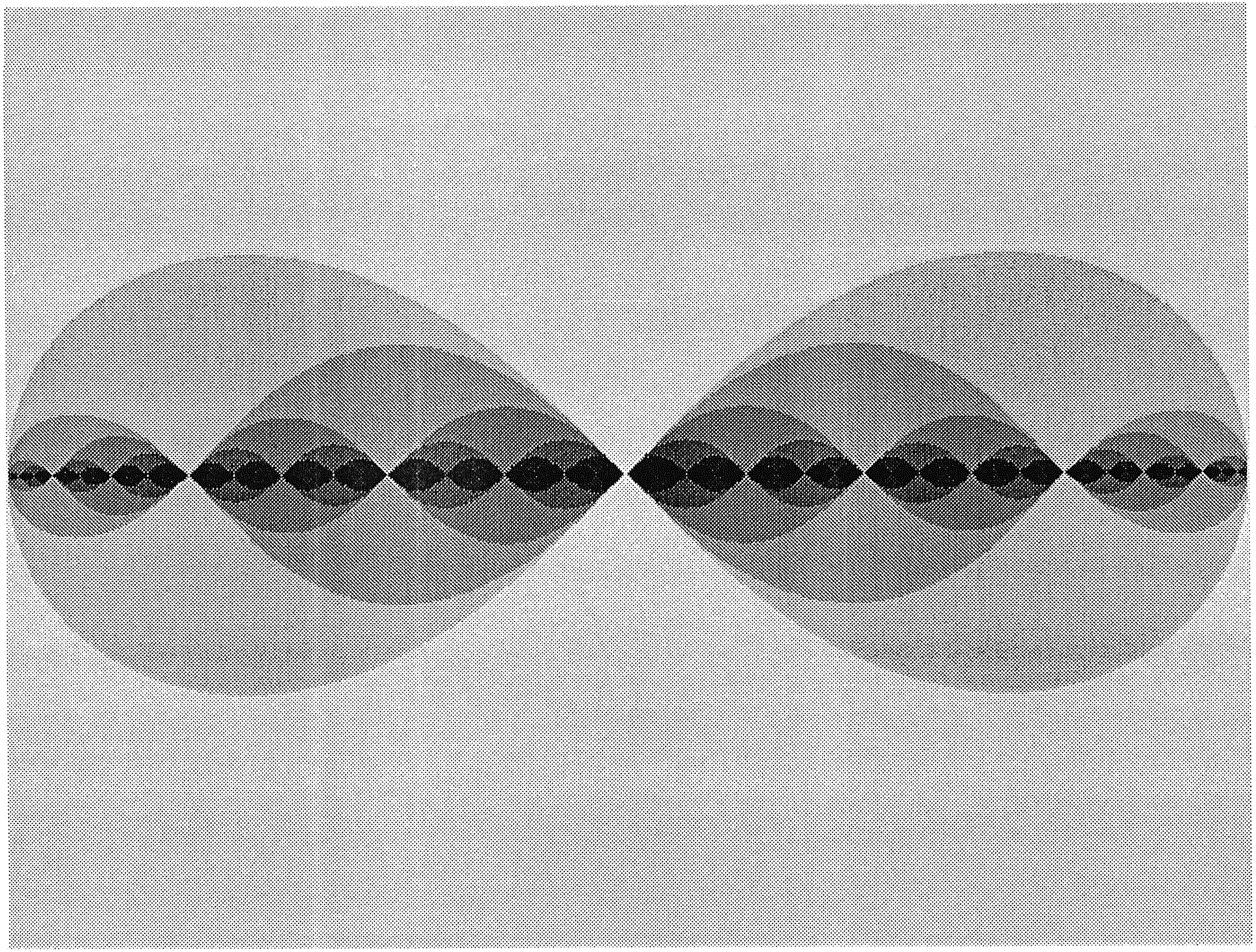

FIGURE 2.