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THE RADIATION CLASS: A NEW SET OF TEMPORAL GAUGES

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Abstract We present in a path integral framework a new class of gauges for QED that are both temporal and yet do not display the notorious singularity of the naive temporal gauge. These gauges follow from a generalised radiation gauge, where the Coulomb gauge fixing is "smeared" out. We show that the use of two gauge fixings necessitates the incorporation of gauge dependent Coulomb interactions. The correctness of our theory is demonstrated in two ways: we can reduce to the true degrees of freedom and we show that it reproduces electron-positron scattering in lowest order perturbation theory. Although Landshoff's α prescription for the temporal gauge can be understood as a limit of our class, extra terms also appear. It is seen that these terms are necessary to obtain the periodic Wilson loop at finite temperature correctly.

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1. Introduction

In recent years there has been much interest in non-covariant gauges[1] and, in particular, in the temporal gauge[2]. In the normal path integral approach to this gauge one introduces the following gauge fixing Lagrangian:

$$\mathcal{L}_{\rm gf} = -\frac{1}{2\xi}(\eta.A)^2 \ , \tag{1.1}$$

where η is a vector $(1,0,0,0)$. This leads (in the limit $\xi \to 0$) to the vector propagator

$$D_{\mu\nu}(k^2, k.\eta^2) = -\frac{i}{k^2}\left[g_{\mu\nu} - \frac{k_\mu\eta_\nu + \eta_\mu k_\nu}{k.\eta} + \frac{k_\mu k_\nu}{k.\eta^2}\right] \ . \tag{1.2}$$

This propagator is truely temporal, i.e. $D_{0\mu} = D_{\mu 0} = 0$. Furthermore, from the Faddeev-Popov trick one readily sees for non-abelian theories that the ghost-gluon vertex is proportional to η_μ, and hence one observes that the ghosts decouple. These simplifications are the main reasons for the interest in temporal (axial) gauges. However, the naive application of these Feynman rules is not possible because of the $\frac{1}{k.\eta}$ singularity in the vector propagator. A regulator must be introduced.

It was long thought that the principal value (PV) prescription on the non-zero part of the propagator, i.e. D_{ij}, was the correct way to regulate the above longitudinal singularity until Caracciolo, Curci and Menotti[3] showed that this fails to reproduce the Feynman gauge result for a rectangular Wilson loop in order g^4. These authors proposed a temporal propagator which however, violated time translational invariance. This and similar propagators have now been derived many times[4]. However, since they have no straightforward momentum space representation they are not very tractable.

Various possible regularisations of the D_{ij} of (1.2) were then suggested more or less ad hoc by various authors[1,5]. A systematic approach was introduced by Cheng and Tsai[6], who pointed out that regulating the propagator necessitates regulating the entire $D_{\mu\nu}$ structure and the Faddeev-Popov determinant if one wants to retain gauge invariance. They concluded from diagrammatic arguments that if the vector propagator is

$$D_{\mu\nu} = -\frac{i}{k^2}\left[g_{\mu\nu} - a_\mu(k)k_\nu + k_\mu a_\nu(-k)\right] \ , \tag{1.3a}$$

and the product of the outgoing ghost propagator and the ghost-vector vertex is

$$\frac{1}{k^2}\left[(a.k-1)k^\mu - k^2 a^\mu\right] \ , \tag{1.3b}$$

then the on-shell scattering amplitude and the matrix elements of gauge invariant operators are independent of the choice of a_μ. It is easily seen that all standard gauges, e.g. the Lorentz class, can be expressed in this way. Using this theorem for the PV prescription one sees that there are corresponding regularisations of both temporal gluons and of the ghost-vector vertex. Although these corrections naively vanish as the regulator (ϵ) is removed, loop integrals yield $\frac{1}{\epsilon}$ terms, and it is imperative to keep ϵ finite until the very end of the calculation. Cheng and Tsai showed that if this is done the Wilson loop can indeed be calculated with the PV prescription. Indeed their formalism opens a floodgate of possible, equivalent regularisations. (A Faddeev-Popov formalism which leads to the Feynman rules (1.3) can be found in Ref. 7.)

However, as stressed by Cheng and Tsai, longitudinal gluons must be included in the Feynman rules for all the regulators they consider and, in practically every case, ghosts do not decouple. Hence they cannot be considered temporal gauges in any traditional sense. A prescription which offers all that one could want of temporal gauge perturbation theory was presented by Landshoff[8], and called by him the α prescription. Here the vector propagator is

$$D_{\mu\nu} = -\frac{i}{k^2}\left[g_{\mu\nu} - \frac{(k_\mu\eta_\nu + \eta_\mu k_\nu)k.\eta - k_\mu k_\nu + \alpha^2\eta_\mu\eta_\nu}{k.\eta^2 + \alpha^2}\right], \qquad (1.4a)$$

which is truely temporal. It may therefore be rewritten as

$$D_{ij} = \frac{i}{k^2}\left[\delta_{ij} - \frac{k_i k_j}{k.\eta^2 + \alpha^2}\right], \qquad D_{0\mu} = 0\,. \qquad (1.4b)$$

Ghost fields are neglected in this prescription. Using (1.4) and taking $\alpha \to 0$ first at the very end of the calculation[3], Landshoff has been able to rederive the correct result for the Wilson loop to order g^4.

This propagator is very similar to that for free photons in the radiation gauge[9] where two gauge fixings are present ($A_0 = \partial_i A^i = 0$)

$$D^{\mathrm{Rad}}_{\mu\nu} = -\frac{i}{k^2}\left[g_{\mu\nu} - \frac{(k_\mu\eta_\nu + \eta_\mu k_\nu)k.\eta - k_\mu k_\nu - k^2\eta_\mu\eta_\nu}{k.\eta^2 - k^2}\right], \qquad (1.5)$$

However, the inclusion of matter fields or interactions is incompatible with the standard derivation of (1.5).

In fact this prescription remains unproven. It is clear that it cannot fit into the class of gauges proposed by Cheng and Tsai – the $\eta_\mu\eta_\nu$ tensor structure in (1.4) cannot

[3] If the Feynman ϵ is taken to zero before α the wrong result is obtained.

be fitted into (1.3) for any a_μ. Also following Ref. 7, it can be seen that such a tensor structure cannot follow from the Faddeev-Popov trick. Steiner's attempt[10] to derive the α prescription founders on this: his result for the total propagator (before he divides it up) does not contain this tensor structure and it is *not* clear that one can, essentially, add them by hand.

It should also perhaps be mentioned that this $\eta_\mu \eta_\nu$ tensor structure is essential for the temporal nature of Landshoff's proposal. Trivial algebra suffices to show that the only exactly temporal propagator (i.e. $D_{0\mu} = D_{\mu 0} = 0$) which one can obtain from (1.3) or from the Faddeev-Popov trick is just (1.2) — the naive, unregulated propagator. (This remains the case even if, in the spirit of the currently fashionable Leibbrandt-Mandelstam prescriptions[5,11], a_μ is allowed to depend on extra vectors.)

We also stress that in the spirit of Cheng and Tsai all prescriptions, including that of Landshoff, must be viewed as coming from some classes of gauges and that as a consequence of the singularity, the gauge parameter may only be taken to the limit which yields the naive temporal gauge at the end of the calculations.

To reiterate: attempts to put D_{ij} into any form suggested in the literature must necessarily be accompanied by the appearance of longitudinal gluons and, possibly, ghosts if one wants to *derive* the Feynman rules from either the Faddeev-Popov or Cheng and Tsai approaches. It remains in principle possible that for some so derivable set of Feynman rules the ghosts and the non-temporal gluons could have zero contributions to all quantities when the regulator was taken to zero at the end of the calculation: in this case a truely temporal prescription would exist. However, to the best of our knowledge no *demonstration* of such a state of affairs has been made, although the work of Cheng and Tsai has provided many counterexamples.

In this letter we will discuss a new class of gauges with two gauge fixings for QED. We call it, for reasons that will become apparent, the radiation class. Such gauges clearly cannot be directly fitted into the approaches discussed above and we therefore have to demonstrate the physical nature of our theory. To this aim we discuss in Sect. 2 the physical content of QED: specifically we show how to reduce to the true degrees of freedom of the theory. In Sect. 3 we write down the generating functional for the class of radiation gauges and show that it can be reduced to the physical theory. We further show that in lowest order perturbation theory it yields the correct results for electron-positron scattering. This class is truely temporal and it is seen to include the α prescription propagator. The radiation gauges are also seen to require the introduction of extra Coulomb interactions, not present in the original α prescription. Finally in Sect. 4 we discuss our results, their relationship to other work and the extension to non-abelian theories.

2. Physical QED

In any discussion of electrodynamics we start with the action

$$S = \int d^4x \quad -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi \,, \tag{2.1}$$

where $D_\mu = \partial_\mu + igA_\mu$. This action describes the interaction of electrons with the two physical components of the photon. How to isolate these two components is reasonably well understood; however, one cannot simply identify the fields ψ above with physical electrons[12]. This is because the field ψ only creates the particle and not its associated electric field. To distinguish the physical fields ψ_{phys} it is necessary to use a phase space formalism.

In order to derive the phase space version of (2.1) we need to introduce momenta. For the Dirac field it is extremely simple since only one time derivative enters this action and so it is already cast in Hamiltonian form. One so sees that the momentum conjugate to ψ is $i\psi^\dagger$ and that the fundamental Poisson bracket is

$$\{\psi^\dagger(x), \psi(y)\} = -i\delta(x-y) \,. \tag{2.2}$$

(Note that henceforth we will drop the explicit spatial dependence.) The Dirac Hamiltonian including minimal coupling is given by

$$H_{\text{Dirac}} = -\bar{\psi}(i\gamma_i D^i - m)\psi \,. \tag{2.3}$$

Gauge invariance raises its head when we try[13] to construct the momenta, π_μ, conjugate to the electromagnetic potentials, A^μ. One sees that $\pi_\mu = -F_{0\mu}$, which implies the primary constraint

$$\pi_0 = 0 \,, \tag{2.4}$$

and the electromagnetic Hamiltonian

$$H_{\text{em}} = \tfrac{1}{2}(\pi^2 + B^2) - A_0 \partial_i \pi^i \,. \tag{2.5}$$

To preserve (2.4) under time evolution generated by the combined Hamiltonian, H ($= H_{\text{em}} + H_{\text{Dirac}}$), we find the constraint

$$-\partial_i \pi^i + gJ_0 = 0\,, \tag{2.6}$$

which is just Gauss' law. No further constraints arise.

Physical fields and their momenta must preserve both constraints. For the electromagnetic field we introduce the decomposition into transverse and longitudinal fields

$$A_i = A_i^T + A_i^L\,, \quad A_i^L = -\frac{\partial_i \partial_j}{\nabla^2} A^j\,, \quad q = -\partial^i A_i^L\,, \tag{2.7}$$

and similarly

$$\pi_i = \pi_i^T + \pi_i^L\,, \quad \text{with} \quad \pi_i^L = \partial_i p\,, \tag{2.8}$$

where $\nabla^2 = -\partial_i \partial^i$. (Note that the Poisson bracket of the longitudinal variables is $\{q, p\} = 1$ and that both p and q Poisson commute with the transverse fields.) It is now clear, since (2.6) may be rewritten as $\nabla^2 p + gJ_0 = 0$, that the transverse fields and their conjugate momenta are the physical components of the electromagnetic fields.

Additionally it becomes evident that ψ is not physical; $\{\psi, \nabla^2 p + gJ_0\} = -ig\psi$. Defining,

$$\psi_{\text{phys}} = \exp\left(\frac{igq}{\nabla^2}\right)\psi \quad \text{and} \quad \psi^\dagger_{\text{phys}} = \exp\left(-\frac{igq}{\nabla^2}\right)\psi^\dagger \tag{2.9}$$

we obtain the desired Poisson brackets for our physical fermions. We note that the Dirac Hamiltonian is already in physical form! We see that we can rewrite

$$H_{\text{Dirac}} = -\bar{\psi}_{\text{phys}}(i\gamma_i D_T^i - m)\psi_{\text{phys}}\,. \tag{2.10}$$

where D_T^i is the covariant derivative with only the transverse components of the gauge fields, $D_T^i = \partial^i + igA_T^i$.

The physical Hamiltonian is therefore given by

$$H_{\text{phys}} = \tfrac{1}{2}(\pi_T^2 + B^2) - \bar{\psi}_{\text{phys}}(i\gamma_i D_T^i - m)\psi_{\text{phys}} - \tfrac{1}{2}g^2 J_0 \frac{1}{\nabla^2} J_0\,, \tag{2.11}$$

where the final term (the Coulomb interaction) comes from (2.5) using Gauss' law.

The physical partition function from this Hamiltonian is then defined as

$$Z_{\mathrm{phys}} = \int d\pi_T^i dA_i^T d\bar{\psi}_{\mathrm{phys}} d\psi_{\mathrm{phys}} \ \exp\left(S_{\mathrm{phys}}\right) ,$$
$$S_{\mathrm{phys}} = \int d^4x \ \pi_T^i \dot{A}_i^T + i\psi_{\mathrm{phys}}^\dagger \dot{\psi}_{\mathrm{phys}} - H_{\mathrm{phys}} . \tag{2.12}$$

With this knowledge of the physical version of QED we are ready to study the radiation class.

3. The Radiation Class

The radiation class is defined by

$$S_{\mathrm{eff}} = \int d^4x \ \ -\tfrac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi - \tfrac{1}{2\lambda}q^2 - \tfrac{1}{2\xi}A_0^2 + g^2 J_0 \tfrac{1}{2\Lambda} J_0 \tag{3.1}$$

where Λ will be determined below in two ways. This is clearly a theory with two gauge fixings: the Coulomb (recall the definition of q) and the temporal. Note that both of these conditions can be smeared out and that we allow for extra Coulomb terms in this interacting theory.

We first reduce to the physical degrees of freedom. To do this recall that we can write the electromagnetic part of the action as

$$-\tfrac{1}{4}F^2 \rightarrow \pi_T \partial_0 A_T - \tfrac{1}{2}(\pi_T^2 + B^2) + \tfrac{1}{2}q\tfrac{\partial_0^2}{\nabla^2}q - q\partial_0 A^0 - \tfrac{1}{2}A_0 \nabla^2 A^0 \tag{3.2}$$

where we have integrated out the longitudinal momentum, p. Also recollect that

$$\bar{\psi}(i\not{D} - m)\psi = i\psi_{\mathrm{phys}}^\dagger \dot{\psi}_{\mathrm{phys}} - \bar{\psi}_{\mathrm{phys}}(i\gamma_i D_T^i - m)\psi_{\mathrm{phys}} - gq\frac{\partial_0}{\nabla^2}J_0 \tag{3.3}$$

Here we have used the important identity

$$i\psi^\dagger \dot{\psi} = i\psi_{\mathrm{phys}}^\dagger \dot{\psi}_{\mathrm{phys}} - gq\tfrac{\partial_0}{\nabla^2}J_0 \tag{3.4}$$

and furthermore we have here dropped the $J_0 A^0$ term because we want to specialise to the temporal sector (i.e. $\xi \rightarrow 0$) of the full radiation class[4].

In this sector we can trivially perform the A_0 integral and the q integral, which is a Gaussian, may also be easily carried out. This yields an action which is just the physical

[4] It is straightforward, if not terribly interesting, to extend our discussion to arbitrary ξ.

one (2.12) plus extra terms of a Coulomb type. When we require that these vanish so as to reproduce $S_{\rm phys}$ we obtain

$$\Lambda = -\frac{1}{\lambda\partial_0^2 - \nabla^2} \tag{3.5}$$

which means that the effective action in the radiation class must be

$$S_{\rm rad} = \int d^4x \;\; -\tfrac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi - \tfrac{1}{2\lambda}q^2 - \tfrac{1}{2\xi}A_0^2 - g^2 J_0 \tfrac{1}{2(\lambda\partial_0^2 - \nabla^2)} J_0 \,, \tag{3.6}$$

if it is to reproduce physical quantities. The naive temporal gauge is yielded by the (nontrivial![14]) limit $\lambda \to \infty$.

The Feynman rules for this theory are as follows. The propagator is truely temporal

$$D_{0\mu} = 0\,, \quad D_{ij} = \frac{i}{k^2}\left[\delta_{ij} - \frac{k_i k_j(1-\lambda)}{k_0^2\lambda - \vec{k}^2}\right]\,, \tag{3.7a}$$

which displays a spurious singularity at $\lambda = \frac{\vec{k}^2}{k_0^2}$. The fermion-photon coupling is the standard one, but there is an extra Coulomb term of the form

$$-g^2 J_0 \tfrac{1}{(\lambda\partial_0^2 - \nabla^2)} J_0 \,. \tag{3.7b}$$

One regains the α prescription propagator for $\lambda = (k^2 + \alpha^2)/\alpha^2$.

A simple, but nontrivial, perturbative check of our arguments is provided by electron scattering at the tree level. All Feynman rules of the form suggested by Cheng and Tsai ((1.3a) for an abelian theory) or, equivalently, by the Faddeev-Popov trick[7] yield the same answer for this ($J_\mu J^\mu/k^2$) since only the $g_{\mu\nu}$ term in the propagator survives ($k_\mu J^\mu = 0$ on shell). For the radiation class the situation is rather more subtle: the $\eta_\mu\eta_\nu$ structure also yields extra terms and these must be cancelled by the extra Coulomb terms[5]. That this is indeed the case is quickly seen. This provides strong confirmation of the correctness of our action (3.6).

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[5] In Landshoff's α prescription without Coulomb terms the correct result is only obtained in the $\alpha \to 0$ limit, i.e. the naive temporal gauge.

4. Discussion

The radiation class offers a set of gauges which have truely temporal propagators, are without the longitudinal singularity of the naive temporal gauge. They have been directly demonstrated to be physical in a nonperturbative manner. They feature Coulomb terms, which must be taken into account if one wants to calculate physical quantities correctly. This has also been checked in perturbation theory.

It is clearly of interest to extend these considerations directly to other non-covariant gauges and to so see what additional interactions are forced upon us when we try to retain the properties that have originally drawn attention to the naive gauges[15]. Similarly, we have here focussed upon Landshoff's prescription; one could try to derive a modified radiation class where the D_{ij} propagator was that of some other (e.g. the Leibbrandt-Mandelstam) prescription. We note also that it is possible to recast our Coulomb terms as a purely temporal propagator D_{00} and in this way one gains a direct link to the Cheng and Tsai approach[16]. There is in other words a choice between Coulomb terms and temporal gluons in this *abelian* theory.

The direct extension of our approach to non-abelian theories would be to say the least highly nontrivial: the physical degrees of freedom become highly complicated and non-Gaussian integrals appear. Hence it is necessary to find a systematic method to deal with these constrained systems. Such an approach has been partially developed in Ref. 17 and is currently being extended to deal with arbitrary linear gauges.

What can one say from the Feynman rules (3.7) about the α prescription? It is evident that in the tree level process we consider it is safe to take the limit $\alpha \to 0$ at the start of the calculation and so it is here safe to neglect the Coulomb terms or, equivalently, temporal gluons. Perturbative studies at higher order are needed to see when these terms decouple in the naive temporal limit.

The periodic Wilson loop, W_R, at finite temperature (in the imaginary time formalism) implies however, that such terms are needed. To see this it is simplest to recast the Coulomb terms into a temporal propagator. The gauge-invariant, W_R, is given to leading order in any gauge by[18]

$$W_R = 1 + \frac{(N_c^2-1)g^2}{2N_cT}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,[1-\cos(\mathbf{k}.\mathbf{R})]\,D_{00}(k_0=0,\mathbf{k}) + \cdots\,, \tag{4.1}$$

where $\mathbf{R}$ is the spatial extent of the loop. Hence in the naive temporal gauge or in the α prescription where $D_{00}(k)$ vanishes exactly there is clearly a problem. Regulating the propagator might be thought not to solve this since the temporal propagator is naively of order ϵ^2. However, the propagator $D_{00}(k_0 = 0, \mathbf{k})$ that enters (4.1) is not only ϵ independent, but identical to that in the Lorentz class. Indeed a glance at (1.3a) reveals that $D_{00}(k_0 = 0, \mathbf{k})$ is independent of the choice of $a_\mu(k)$. We conclude that the α prescription must in general be extended as discussed above.

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