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Wu-Yang Fields

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Abstract

We generalise the definition of a Wu-Yang field in $\mathbb{R}_3$, to the generic case in $\mathbb{R}_d$, with the exceptions of d = 2,4.

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We start by defining what we mean by Wu-Yang (WY) fields in general. The nomenclature stems from the case of the spherically symmetric SU(2) Yang-Mills (YM) field on $\mathbb{R}_3$, which asymptotically coincides with the U(1) gauge field on the S^2 at infinity. The importance of this WY field on S^2 derives from its coincidence with the asymptotic YM field of the 't Hooft-Polyakov[2)] monopole. The latter is well known to be a nonsingular finite action (energy) solution of the YM-Higgs (YMH) model inter SU(2) gauge field, and adjoint representation Higgs field. The most succinct illustration of the gauge symmetry breakdown from SU(2) in $\mathbb{R}_3$, to U(1) or the large S^2, is afforded by Dirac's string-gauge[3)] formulation of the monopole field. This involves the gauge transformation of the asymptotic Higgs field

$$\Phi^{\infty} = \eta\, \hat{x}_i\, \sigma_i \tag{1}$$

to the covariantly constant field in the string gauge

$$^{s}\Phi = g\, \Phi^{\infty}\, g^{-1} = i\, \eta\, \sigma_3 \tag{2}$$

where η is the usual Higgs vacuum expectation value. The corresponding gauge connection $^{s}A_\mu = g\, A_\mu^{\infty}\, g^{-1} + g\, \partial_\mu\, g^{-1}$ is given[3)] both in cartesian coordinates of $\mathbb{R}_3$ and the coordinates of S^2 as

$$^{s}A_i = -\frac{i}{2r}\, \frac{\varepsilon_{ij}\, \hat{x}_j}{(1 + \cos\theta)}\, \sigma_3 \;\;,\;\; {}^{s}A_3 = 0 \tag{3a}$$

$$^{s}A_\varphi = -\frac{i}{2}(1 - \cos\theta)\sigma_3 \;\;,\; {}^{s}A_\theta = {}^{s}A_r = 0, \tag{3b}$$

where $\mu = (i,3)$, $i = 1,2$. In (3a), the string singularity along $\cos\theta = -1$ is obvious. The corresponding curvature field strength is given on $\mathbb{R}_3$ and S^2 again by

$$^{s}F_{\mu\nu} = \frac{1}{r^2}\, \varepsilon_{\mu\nu\lambda}\, \hat{x}_\lambda \tag{4a}$$

$$^{s}F_{\theta\varphi} = \frac{1}{2}\, \sigma_3 \sin\theta \tag{4b}$$

where the regularity of F_μ is conspicuous. This is not surprising since the monopole solution of the YMH system is regular and has a gauge convariant quantity cannot develop any singularities by a singular gauge transformation.

We note that the asymptotic fields (4) are U(1) valued. Indeed, the topological flux density $\hat{x}_\lambda \varepsilon_{\lambda\mu\nu}$ tr $^{s}\Phi^{s}F_{\mu\nu}$, calculated in the string gauge on the S^2 at infinity, is nothing but the U(1) magnetic flux.

The above is a description of the salient properties of the usual WY field in $\mathbb{R}_3$. We exploit these properties to give a generic definition for a WY field on $\mathbb{R}_d$. To qualify as a WY field, the following criteria must be satisfied :

a) For an SO(d) gauge field on $\mathbb{R}_d$, interacting minimally with a Higgs field in the *vector* representation, there must exist a regular finite action topologically stable spherically symmetric solution. We shall refer to such solutions as 'instantons', even though the 't Hooft-Polyakov 'monopole' is the d = 3 case.
b) The Higgs-independent term in the Lagrangian of the model in question must decay sufficiently slowly, so that in the string-gauge, it does not vanish on the infinite S^{d-1} surface in $\mathbb{R}_d$. This criterion is *satisfied* by the 'monopole' field for d = 3, and is *not satisfiedl* by the vortex[4)] field for d = 2, in which case the curvature 2-form necessarily vanishes on S^1.

If criteria (a) and (b) are satisfied, then the WY field can be identified with the SO(d-1) asymptotic gauge field on the infinite S^{d-1}. In the d = 3 example, it is the Abelian gauge field on S^2.

Note that criterion (b) disqualifies the YM instantons[5)] as well as all instanton solutions of the scale-invariant generalised YM (GYM) models[6)] on $\mathbb{R}_{4p}$ as WY fields, as all these solutions[5,7)] are asymptotically pure-gauge.

Having defined our criteria, we proceed as follows : We consider the gauge-Higgs *models* which are capable of supporting 'instanton' solutions, and then disuss the relevant qualitative properties of their *solutions*. This supplices the requisites of criterion (a). Then we perform the gauge transformation to the *string-gauge,* which enables us to impose the requirement of criterion (b), and to make the definition for a WY field. Finally we check our results by performing the exercise of calculating the topological *flux density* in the string-gauge.

The models

The first ingrediants we need are some gauge field-Higgs, or non Abelian Higgs models, generalising the YMH model, which can support regular, finite action, topologically stable and localised solutions on $\mathbb{R}_d$ analogous to the Monopole in d = 3. Recently, such models were found[8)]. These are obtained by subjecting the p-th member of the hierarchy of scale invariant GYM systems[6)]

$$\mathcal{L}^{(p)} = \mathrm{tr}\, F\,(2p)^2 \tag{5}$$

defined on a 4p-dimensional manifold, to dimensional reduction[9)]. F(2p) in (5) is the p-fold totally antisymmetrised product of the curvature 2 form $F(2) = F_{\mu\nu}$. It is known from the work of Refs. [6 ,9], that these models support finite action, topologically stable (instanton) solutions, which are regular. These solutions are localised to the absolute scale stemming from the "radius" of the compact coset space employed in the dimensional reduction [9)]. Also, with the exception of the hierarchty of models [10)] on $\mathbb{R}_{4p-1}$ and $\mathbb{R}_2$, descended from $\mathbb{R}_{4p-1} \times S^1$ and $\mathbb{R}_2 \times S^{4p-2}$, the solutions are non-minimal, because of the overdetermined[11)] nature of the corresponding selfduality equations.

We do not give explicit expressions for the residual Lagrangians $\mathcal{L}_{res}[A_\mu, \Phi]$ which are obtained by the above described process of dimensional reduction [4], as examples of these are given elsewhere[8, 10, 11]. We do however give a qualitative property of these Lagrangians, $\mathcal{L}_{res}$, in the generic case, which will concern us below. These are obtained by imposing the symmetry appropriate to the dimensional reduction [9] over $\mathbb{R}_d \times S^q$, on the fields $\widehat{A}_M = (\widehat{A}_\mu, \widehat{A}_m)$, where $x_\mu \in \mathbb{R}_d$ and $x_m \in S^q$, and, integrating out the coordinates on S^q. The result is

$$\mathcal{L}_{res}[A_\mu, \Phi] = \int_{S^q} d^q x \, \mathcal{L}^{(p)}[\widehat{A}_\mu, \widehat{A}_m] \tag{6}$$

where $\mathcal{L}^{(p)}$ is given by (5), and $d + q = 4p$

What interests us here is which if any, term in $\mathcal{L}_{res}[A_\mu, \Phi]$ is Φ independent ? Since $\mathcal{L}^{(p)}$ involves the 2p-form fieldstrength, then, provided that $d = 4p - q \geq 2p$, there will be *one* such term, namely $F(2p)^2$, in $\mathcal{L}_{res}$. When $d < 2p$, then $F(2p)$ on $\mathbb{R}_d$ vanishes identically and there will be *no* Φ-independent term, so

$$\mathcal{L}_{res}[A_\mu, \Phi] = \alpha_d \,\mathrm{tr}\, F(2p)^2 + \Phi\text{- dependent terms} \; ; \; \alpha_d = 0 \text{ for } d < 2p \tag{7}$$

The 'instanton' solutions

Briefly, the topological stability of these solutions is assured by the inequality[5]

$$\int_{\mathbb{R}_d \times S^q} \mathrm{tr}\, \widehat{F}(2p)^2 \geq \int_{\mathbb{R}_d \times S^q} \mathrm{tr}\, \widehat{F}\wedge\widehat{F}\wedge\ldots\wedge\widehat{F}, \tag{8}$$

with $d + q = 4p$. Performing the dimensional reduction by imposing symmetries and integrating out the coordinates on S^q, yields the residual inequality

$$\int_{\mathbb{R}_d} \mathcal{L}_{res}[A, \Phi] \geq \int_{\mathbb{R}_d} \partial_\mu \Omega_\mu [A, \Phi] \tag{9}$$

That the residual integrand on the right hand side of (7), under a suitable mode of descent is a total divergence, follows form the results of the work of Ref. [9]. Thus, the positive definite action integral on the left hand side of (9) is bounded from below by a surface integral which can take non-trival values if

$$|\Phi|^2 \xrightarrow[r \to \infty]{} \eta^2 \tag{10}$$

This topological lower bound then guarantees the existence of the 'instanton' solution, *provided* that $\mathcal{L}_{res}$ features no constant term, namely that the descent procedure results in a Higgs selfinteraction potential whose minimum is equal to zero.

Finally, we give the spherically symmetric 'instanton' solutions on $\mathbb{R}_d$, by means of which we shall subsquently define the WY fields. We know from the work of Refs [10] and [6], for odd and even d respectively, that we obtain 'instanton' solutions when the residual gauge groups are SO(d) for odd d, and SO(d) x $U_1 \approx SO_+(d)$ x $SO_-(d)$ x U(1) for even d. In particular, the representations of SO(d) *must* be the spinor represention*s*, with $SO_\pm(d)$ being the chiral represention*s*. It is of material importance to specify the representations, since $\mathcal{L}_{res}$ [A, Φ] involves for example, four-form curvatures, whose values are not restricted to the algebra of the gauge group and hence are representation dependent.

The sphesically symmetric 'instanton' fields then are

$$A_\mu = \frac{1}{r}(1 + f(r))\Gamma_{\mu\nu}\hat{x}_\nu \ , \quad A_\mu = \frac{1}{r}(1 + f(r))\begin{bmatrix} \Sigma^{(+)}_{\mu\nu} & \\ & \Sigma^{(-)}_{\mu\nu} \end{bmatrix}\hat{x}_\nu = \frac{1}{r}(1 + f(r))\Gamma_{\mu\nu}\hat{x}_\nu \tag{11 a,b}$$

$$\Phi = \eta h(r)\, i\, \Gamma_\mu \hat{x}_\mu, \quad \Phi = \eta h(r)\, \Gamma_{d+1}\, \Gamma_\mu \hat{x}_\mu, \tag{12 a,b}$$

with (11a) and (12a) pertaining to odd d, and (11b) and (12b) to even d. Here $\Gamma_{\mu\nu} = -\frac{1}{4}[\Gamma_\mu, \Gamma_\nu]$, Γ_μ are the γ-matrices in d-dimensions, and Γ_{d+1} is the chiral matrix. We note also, that the U(1) gauge field in (11b) vanishes due to the spherical symmetry constraint, except when d = 2, when (11b) becomes Abelian. The following boundary conditions are satisfied by these solutions

$$-1 \xleftarrow[r\to 0]{} f(r) \xrightarrow[r\to\infty]{} 0,\ 0 \xleftarrow[r\to\infty]{} h(r) \xrightarrow[r\to\infty]{} 1 \tag{13 a,b}$$

The string-gauge : Wu-Yang fields

Second, the regular, finite action, spherically symmetric solutions (11) and (12) will be employed below to define the WY fields, by first calculating their asymptotic values

$$A^\infty_\mu = \frac{1}{r}\,\Gamma_{\mu\nu}\,\hat{x}_\nu, \tag{14}$$

$$\Phi^\infty = \eta\, \Gamma_{d+1}\, \Gamma_\mu \hat{x}_\mu, \ \Phi^\infty = \eta\, i\, \Gamma_\mu \hat{x}_\mu \tag{15a,b}$$

in the string-gauge. The required SO(d) gauge group element g, in the sprink representation

$$g = \frac{1}{\sqrt{2(1+\hat{x}_d)}}[(1+\hat{x}_d)\,1 + \Gamma_d\,\Gamma_i\hat{x}_i], \tag{16}$$

is constructed to satisfy the analogue of (2), ${}^s\Phi = g\Phi\, g^{-1}$, with

$$ {}^s\Phi = \eta\, i\, \Gamma_d \ , {}^s\Phi = \eta\ \Gamma_{d+1}\ \Gamma_d \tag{17a,b}$$

In (16), $\hat{x}_d = (x_d/r) = \cos\psi$, where ψ is the 'last' polar angle in $\mathbb{R}_d$.
The corresponding action on $A^{\infty}_{\mu} \rightarrow g\, A^{\infty}_{\mu}\, g^{-1} + g\, \partial_{\mu}\, g^{-1}$, yields

$$ {}^sA_i = \frac{1}{r\,(1+\hat{x}_d)}\ \Gamma_{ij}\ \hat{x}_j\ , {}^sA_d = 0, \tag{18}$$

where $\mu = (i,d)$, $i = 1,\dots,(d-1)$, and hence Γ_{ij} spans the algebra of SO(d-1). For d = 3, (18) is simply the cartesian version of (3), exhibiting the string singularity along $\hat{x}_d = -1$. In that case $\Gamma_{ij} = -\frac{i}{2}\,\varepsilon_{ij}\,\sigma_3$, which expresses the Abelian nature of ${}^sA_{\mu}$. Next, we compute the curvature field strength in the string gauge, and readily obtain

$$ {}^sF_{ij} = -\frac{1}{r^2}\,[\Gamma_{ij} + \frac{1}{1+\hat{x}_d}\ \hat{x}_{[i}\ \Gamma_{j]k}\ \hat{x}_k], \tag{19a}$$

$$ {}^sF_{i4} = \frac{1}{r^2}\ \Gamma_{ij}\ \hat{x}_j. \tag{19b}$$

Again, it is straightfroward to check that for d = 3, (19) reduces to

$$ {}^sF_{\mu\nu} = \frac{i}{2r^2}\ \varepsilon_{\mu\nu\lambda}\ \hat{x}_{\lambda}\ \sigma_3, \tag{20}$$

which is regular unlike ${}^sA_{\mu}$ in (18), and is the only member of the hierarchy of curvature fieldstrengths (19) in the string-gauge, which is endowed with SO(d) covariance.

Before completing the definition of a generic WY field, we remark that the fieldstrengths (19a,b) in the string gauge consist of (d-1) ! / 2! (d-3)! components, and not d! / 2! (d-2)!. In other words, only the components of the fieldstengths labeled by the (polar) angular coordinates on S^{d-1} are nonvanishing. This means that the asymptotic fields (19a,b) are defined effectively on a (d-1) dimensional manifold, while the dynamics is defined on the d dimensional space $\mathbb{R}_d$. These is also, attendendant to this descent in dimensionality, a breakdown of the gauge symmetry SO(d) on $\mathbb{R}_d$., to SO (d-1) on S^{d-1}. For example for d = 3, there is only *one* nonvanishing component of ${}^sF(2)$, namely the U(1) fieldstrength $F_{\theta\varphi}$ given by (4b), and for d = 4, one can readily compute the *three* SO(3) valued nonvanishing components of ${}^sF(2)$:

$$ F_{\psi\theta} = \sin\psi\ (-\cos\varphi\ \gamma_{31}\ +\ \sin\varphi\ \gamma_{23}) \tag{21a}$$

$$ F_{\psi\varphi} = \sin\psi\ \sin\theta\ [-\sin\theta\ \ \gamma_{12} + \cos\theta\ (\sin\varphi\ \ \gamma_{31} + \cos\varphi\ \gamma_{23})] \tag{21b}$$

$$F_{\theta\varphi} = \sin^2\psi \sin\theta\,[\cos\theta\ \gamma_{12} + \sin\theta\ (\sin\varphi\ \gamma_{31} + \cos\varphi\ \gamma_{23})] \qquad (21c)$$

where ψ and θ are the polar angles and φ the asymuthal angle in $\mathbb{R}_4$, and, $\gamma_{ij} = -\frac{1}{4}[\gamma_i, \gamma_j]$, i,j, = 1, 2, 3, in terms of the usual Dirac matrices.

We are now in a position to define the generic WY fields according to the criteria (a) and (b) given above. We proceed as follows : We examine the gauge-Higgs models obtained from each member (p = 1, 2, 3...) of the GYM hierarchy on $\mathbb{R}_d \times S^{4p-d}$, by dimensional reduction. Then we identify the models pertaining to each d, for which WY fields occur.

p=1. This is the usual YM model, and the residual model on $\mathbb{R}_3$ prossesses the usual WY field[1)]. In the corresponding residual model on $\mathbb{R}_2$, these are vortex solutions, but we know there is no WY field, since there is no F(2) curvature on the asymptotic S^1.

p = 2. Here, the residual models obtained from the dimension reduction of the system(5) over $\mathbb{R}_d \times S^{8-d}$ support 'instanton' solutions, but those with $d \leq 4$ *do not* have WY fields. This is because in these cases the fieldstrengths (19) in the string-gauge, have vanishing F(4) curvature on the asymptotic S^{d-1}, since ${}^sF_{r\psi} = {}^sF_{r\theta} = {}^sF_{r\varphi} = 0$ on S^3, ${}^sF_{r\theta} = {}^sF_{r\varphi} = 0$ or S^2, and ${}^sF_{r\varphi} = 0$ on S^1, respectively. This violates criterion (b), which requires that the Φ-independent term in (7) must not vanish on S^3. Thus the WY fields here occur in d = 5, 6 and 7.

p = 3. Analogous to the p = 2 case, here the residual models with $d \leq 6$ *do not* have WY fields, since the F(6) curvature in (7) vanishes on the asymptotic, S^5,....,S^1, respectively. The WY in this case occur in d = 7, 8,..., 11.

It becomes clear at this stage that there is no need to list the WY fields that occur for subsequent members (with p > 3) of this hierarchy, since these start occuring in overlapping dimensions. Thus there are two distinct WY fields in d = 7, each pertaining to the residual model of p = 2 and p = 3 respectively. This qualitative aspect persists for increasing p. Since we are not here examining the detailed dynamical features of each WY field, we shall stop the above list at p = 3.

We note however, an apparently interesting feature of this list : There occur WY fields, pertaining to models with increasing p, in all dimensions d, with the notable exceptions of d = 2 and d = 4.

Topological flux densities

As a check for our formulas (19), as well as for its intrinsic interest, we calculate the flux density of the topological charge which bounds from below the action of the 'instantons' discussed above, in the string-gauge.

We shall restrict our computations below to the hierarchy member p = 2, and further, to the odd dimensions d = 5 and d = 7. The first restriction leads to no loss of qualitative features, and is made merely for simplicity. The second restriction however is based on the fact that the topological flux density in odd dimensions happens to be a *gauge-invariant* quantity, and hence it is trivially adapted to working in any gauge - in this case the string-gauge. This is known from the work of Refs. [9, 8], from which we also know that the topological flux densities in even dimensions are composed of the usual *gauge-variant* Chern-Simons densities, augmented by Higgs-dependent terms whose contributions to the surface integrals vanish. Since it is more difficult to work with *gauge-variant* flux densities in different gauges, we shall opt to restrict to the odd dimensional examples here, relegating the even dimensional case of d = 6 to a future, more detailed work.

Before proceeding with the p = 2 exemples, let us recall the well known p = 1 case, namely the WY field[1)] on S^2. The quantity in question is the magnetic-monopole flux density

$$\Omega = \hat{x}_\rho \; \varepsilon_{\rho\mu\nu} \; \mathrm{tr}\, \Phi \, F_{\mu\nu}, \tag{22}$$

which has to be integrated with respect to the surface element $r^2 \sin\theta \, d\theta \, d\varphi$. Since the expression (22) is gauge-invariant, we can directly replace $(F_{\mu\nu,} \Phi)$ by $({}^sF_{\mu\nu,} {}^s\Phi)$, the latter pair being given by (20) and (2). The result is simply $\Omega = 2\eta \, r^{-2}$, so that the flux integral reduces (up to the factor 2η) to the *angular integral* over S^2. This is as expected, and we shall reproduce this situation below in two exercises, with p = 2 models in dimensions 5 and 7.

i) p = 2, d = 5 : The residual Lagrangian density on $\mathbb{R}_5$ is

$$\mathcal{L}_{res} = \mathrm{tr}\,[4\, F_{\mu\nu\rho\sigma}{}^2 + 3\, \{F_{[\mu\nu}, D_{\rho]}\, \Phi\}^2 + 6\, (\{S, F_{\mu\nu}\} + [D_\mu \Phi,\; D_\nu \Phi])^2 + 27\, \{S, D_\mu \Phi\}^2], \tag{23}$$

where $F_{\mu\nu\rho\sigma} = F(4)$, and $S = -\eta^2 - \Phi^2$. Its topological flux density as defined by (9), is

$$\Omega = \hat{x}_\lambda \, \varepsilon_{\mu\nu\rho\sigma\lambda} \; \mathrm{tr}\, (3\eta^2 \Phi \, F_{\mu\nu} \, F_{\rho\sigma} + \Phi^3 \, F_{\mu\nu} F_{\rho\sigma} \; - \; \Phi [D_\mu \Phi, D_\nu \Phi] \, F_{\rho\sigma}) \tag{24}$$

The integral of (23) over $\mathbb{R}_5$ yields a finite action for the 'instanton' fields, and is bounded from below by the surface integral of the flux (24). The latter takes a finite value since the 'instanton' solution satisfies the asymptotic condition (10). The 'instanton' in this case is a non-minimal [11)], i.e. non-selfdual solution of the Enler-Lagrange equations.

As in the (p = 1, d = 3) case above, where the magnetic flux density (22) was gauge-invariant, here the topological flux density (24) is also gauge-invariant. We can therefore calculate this quantity in the string-gauge by simply substituting into it $({}^sF_{\mu\nu}, {}^s\Phi)$ from (19) and (15a). The computation is simplified by noting that the third term in (24) is equal to zero because of the *covariant constancy* of Φ in the string-gauge. After a straightforward computation, we find that $\Omega = 32\eta^3 r^{-4}$. The integral of this Ω with respect to the surface element $r^4 \sin^3\psi_2 \sin^2\psi_1 \sin\theta d\psi_1 d\psi_2 d\theta$ $d\varphi$ then reduces to the *angular integral* over S^5. This is the required result, and is a verification for the correctness of our rotationally non-covariant formulas (19).

ii) $p = 2$, $d = 7$: The residual Lagrangian density on $\mathbb{R}_7$ is

$$\mathcal{L}_{res} = \mathrm{tr}\,(F_{\mu\nu\rho\sigma}{}^2 + 4\,\{F_{[\mu\nu}, D_{\rho]}\Phi\}^2). \tag{25}$$

with topological flux density

$$\Omega = \hat{x}_\kappa\ \varepsilon_{\mu\nu\rho\sigma\tau\lambda\kappa}\,\mathrm{tr}\,\Phi\, F_{\mu\nu}\, F_{\rho\sigma}\, F_{\tau\lambda} \tag{26}$$

Following the same procedure as in (i) above, we find that in this case $\Omega = 144\eta^5$ r^6,which is exactly what is needed to render the surface integral with respect to $r^6 \sin^5\psi_4 ... \sin^2\psi_1$, $\sin\theta\, d\psi_4 ... d\psi_1\, d\theta\, d\varphi$, to an *angular integral* over S^6.

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