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The Classical Limit of a Class of Quantum Dynamical Semigroups

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Abstract: The Ghirardi-Rimini-Weber (G.R.W.-) model is studied in the limit $\hbar \to 0$ and it is shown that a weak-coupling limit is needed in order to retain its dissipative character at the phase-space level. As a byproduct, solutions of the corresponding Chapman-Kolmogorov differential equation, with linear Liouville term, are provided explicitly.

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1. Introduction

The G.R.W.-model [1,2] consists of a dissipative modification of the unitary evolution of quantum states. The generator of the quantum motion is no longer given by the commutator with the Hamiltonian operator but contains a linear term of Lindblad type [3] which has the notable property of transforming pure states into mixtures. Before embarking on an outline of the physical aspects of the model, we introduce some of the notation we will use. We shall consider mainly a non-relativistic one-dimensional one-particle quantum system described by a state operator, or density matrix, $\hat{\rho}$ and a Hamiltonian $\hat{H}$ acting on a separale Hilbert space $\mathcal{H}$; $\hat{q}$, $\hat{p}$ will denote position and momentum operators respectively. We shall assume that the system, besides evolving accordingly to the Hamiltonian equation of motion, undergoes a localization process in position occurring with mean frequency λ and characterized by a coherence lenght $\frac{1}{\sqrt{\alpha}}$. At each point x along the real line the state $\hat{\rho}$ gets transformed into the new state

$$\left\{\mathrm{Tr}\left[\hat{\rho}\sqrt{\frac{\alpha}{\pi}}\exp\left(-\alpha(\hat{q}-x)^2\right)\right]\right\}^{-1}\sqrt{\frac{\alpha}{\pi}}\exp(-\frac{\alpha}{2}(\hat{q}-x)^2)\,\hat{\rho}\exp(-\frac{\alpha}{2}(\hat{q}-x)^2) \qquad (1.1)$$

by the localization process, the probability of its happening at the given point being

$$\mathrm{Tr}\left[\hat{\rho}\sqrt{\frac{\alpha}{\pi}}\exp(-\alpha(\hat{q}-x)^2)\right]$$

so that globally

$$\hat{\rho}\to\mathrm{T}\left[\hat{\rho}\right]=\sqrt{\frac{\alpha}{\pi}}\int_R\exp(-\frac{\alpha}{2}(\hat{q}-x)^2))\,\hat{\rho}\exp(-\frac{\alpha}{2}(\hat{q}-x)^2))\,dx. \qquad (1.2)$$

The Hamiltonian unitary evolution with superimposed localizing process occurring randomly with mean frequency λ ought to preserve proability and this is in turn accomplished by:

$$\partial_t\hat{\rho}_t=-i\left[\hat{H},\,\hat{\rho}_t\right]-\lambda\hat{\rho}_t+\lambda\mathrm{T}\left[\hat{\rho}_t\right], \qquad (1.3)$$

this is the evolution equation of the G.R.W.-model, in units in which $\hbar=1$.

We observe that the non-Hamiltonian term in (1.3) can be cast into the form:

$$\left\{\left[-\frac{\lambda}{2}\sqrt{\frac{\alpha}{\pi}}\int_R \exp\left(-\alpha(\hat{q}-x)^2\right)dx\right], \hat{\rho}_t\right\}$$
$$+\lambda\sqrt{\frac{\alpha}{\pi}}\int_R \exp(-\frac{\alpha}{2}(\hat{q}-x)^2)\,\hat{\rho}_t \exp(-\frac{\alpha}{2}(\hat{q}-x)^2)\,dx,$$

where $\left\{\cdot,\cdot\right\}$ is the anticommutator, and eventually rewritten [4] as:

$$-\frac{\lambda}{2}\left\{\sum_{n=0}^{+\infty}\hat{A}_n^\dagger\,\hat{A}_n\,,\,\hat{\rho}_t\right\}+\lambda\sum_{n=0}^{+\infty}\hat{A}_n\,\hat{\rho}_t\,\hat{A}_n^\dagger,$$
$$\hat{A}_n\;=\;\hat{A}_n^\dagger\;=\;\sqrt{(\frac{\alpha}{2})^k\frac{1}{k!}}\,\hat{q}^k\,\exp\left(-\frac{\alpha}{4}\hat{q}^2\right),$$

thus enabling us to identify the r.h.s. of (1.3) as the Lindblad type generator [3] of a quantum dynamical semigroup $\{\gamma_t\}_{t\geq 0}$, whose properties are the following [3,5]:

i) $\gamma_t : \mathrm{B}(\mathcal{H})_1^{\mathrm{s.a}} \to \mathrm{B}(\mathcal{H})_1^{\mathrm{s.a.}} \qquad \forall t \geq 0,$

$\mathrm{B}(\mathcal{H})_1^{\mathrm{s.a.}}$ being the Banach space of trace class self-adjoint operators on $\mathcal{H}$ with the norm:

$\|\hat{\rho}\|_1 = \mathrm{Tr}\sqrt{\hat{\rho}^2}$;

ii) γ_t is a completely positive contraction on $\mathrm{B}(\mathcal{H})_1^{\mathrm{s.a.}}$:

$\|\gamma_t\hat{\rho}\|_1 \leq \|\hat{\rho}\|_1 \qquad \forall t \geq 0;$

iii) $\{\gamma_t\}_{t\geq 0}$ is strongly continuos:

$\lim_{t\to 0^+}\|\gamma_t\hat{\rho}-\hat{\rho}\|_1 = 0 \qquad \forall\hat{\rho}\in \mathrm{B}(\mathcal{H})_1^{\mathrm{s.a.}}$.

The asserted localization properties of the map $T[\cdot]$ in (1.2) become apparent if we consider the position representation and work out:

$$< q|\,\mathrm{T}[\hat{\rho}]\,|\overline{q}>= \exp\left(-\frac{\alpha}{4}(q-\overline{q})^2\right) < q|\hat{\rho}|\overline{q}>, \tag{1.4}$$

the linear map $T[\cdot]$ on $B(H)_1^{s.a.}$ is introduced infact in [6] as a model of a position measuring gaussian device. From (1.4) we deduce that states which are largely delocalized with respect to $\frac{1}{\sqrt{\alpha}}$ have off-diagonal elements which are damped by the term: $-\lambda < q|\hat{\rho}|\overline{q} >$ in (1.3), while evolving, whereas those for which $|q-\overline{q}| \simeq \frac{1}{\sqrt{\alpha}}$ are nearly unaffected. This brings the model's aims to the fore in that it is capable of depressing quantum entanglement arising from the linear superposition principle which cannot be avoided with unitary evolutions.By using this disentangling mechanism of far-away localized states, the authors find an escape route from the puzzling situations connected with the broad concept of reduction of the wave packet[1,2,7].Indeed, it is a striking feature [1] of the model that, if a macrosystem is considered and the relative motion can be separated from that of the center of mass, then the localization process does not disturb the former,whereas amounts to a localization in the center of mass position governed by the same coherence length $\frac{1}{\sqrt{\alpha}}$ but occurring with a mean frequency proportional to the number of constituent particles. This very fact enables the authors [1] to choose the parameters α and λ so that the quantum mechanical properties of few particles are the usual ones up to enormous times on one hand and on the other disentanglement is provided for macrosystems like, e.g., crystals. They propose the Quantum Mechanics With Spontaneous Localizations as a building block in the attempt of constructing a quantum dynamics able to overcome the difficulties of a consistent micro-macro description of the physical world. All the developments of the theory [8,9,10] reduce to (1.3) when confronted with a non-relativistic one-dimensional one-particle system. It seems thus reasonable to study how the model behaves when pushed toward classical mechanics by letting the quantum of action go to zero, being aware that, for instance, in the case of linear equations of motions (quadratic Hamiltonians) the quantum dynamics is the classical one for mean values of positions and momenta. At most quadratic Hamiltonians have been used in [11,12] to give explicit solutions of the G.R.W-model and would then be studied first. It will be shown that, in order to save the stochastic behaviour inherent in the Lindblad generator of (1.3), a joint limit $\hbar \to 0$, $\alpha \to 0$ with $\alpha\hbar^2 =$ constant, is to be performed. A general result [13] will then be met, as the above amounts to be a weak-coupling limit. This paper will be divided as follows: in Chapter 2. a short account of classical limit techniques will be given. The G.R.W.-model with quadratic Hamiltonians is discussed in Chapter 3. and Chapter 4. is devoted to general (time-independent) Hamiltonians.

2. The Classical Limit

As we are more concerned in investigating the existence of a sensible stochastic process on the phase space corresponding to (1.3), rather than in facing pathological situations that can arise from the outset by allowing for poorly regular Hamiltonians, we shall take for granted all the assumptions available to give a meaning to what follows. The classical description is centered around

$$\begin{aligned}\partial_t\rho_t(q,p) &= \{H(q,p),\rho_t(q,p)\} \\ &= \partial_q H(q,p)\partial_p\rho_t(q,p) - \partial_p H(q,p)\partial_q\rho_t(q,p),\end{aligned} \tag{2.1}$$

namely the Liouville equation for a classical phase-space distribution ρ and a Hamiltonian function H(q,p). The r.h.s. of (2.1) is, a part for the sign, the Lie-derivative L_H of ρ_t along the Hamiltonian vector field $X_H = (\frac{\partial H}{\partial p}\,,\,-\frac{\partial H}{\partial q})$. Both X_H and ρ are to be considered regular enough on R^2 so that we can express the solution of (2.1) as:

$$\rho_t = e^{-tL_H}\rho = \sum_{k=0}^{+\infty}\frac{(-t)^k}{k!}L_H^k\rho, \tag{2.2}$$

where

$$L_H^0\rho = \rho\,,\; L_H^k\rho = L_H^{k-1}L_H\rho \quad \text{and} \quad L_H\rho = \{\rho, H\}.$$

Remarks 2.1

1. The underlying Hamiltonian dynamics is provided by the (local) flow of automorphisms $\{\Phi_t^H(q,p)\}$ on the phase-space R^2, generated by the vector field $X_H = (\frac{\partial H}{\partial p}\,,\,-\frac{\partial H}{\partial q})$ via the equations of motion $\frac{d}{dt}\Phi_t^H(q,p) = (X_H\cdot\Phi_t^H)(q,p)$. Where the flow of diffeomorphisms is defined, thus giveing rise to the trajectories $(q,p)\rightarrow(q_t,p_t) = \Phi_t^H(q,p)$, a group of automorphisms $\{\tau_t^H\}$ on the state-space of summable positive functions is set up according to:

$$\tau_t^H\rho = \rho\cdot\Phi_{-t}^H = e^{-tL_H}\rho.$$

Analyticity of both Hamiltonian vector field and distribution function provides for the power expansion (2.2).

2. It is well known that if the Hamiltonian function is at most quadratic and we consider the corresponding quantum operator

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$$

on the Hilbert space $\mathcal{H}$ (anyone of its self-adjoint extensions), then, for $\psi \in \mathcal{D}(\hat{H})$

$$\begin{cases} \frac{d}{dt} < \psi, \hat{q}_t \psi > = \frac{1}{m} < \psi, \hat{p}_t \psi > \\ \frac{d}{dt} < \psi, \hat{p}_t \psi > = < \psi, V'(\hat{q}_t) \psi > = V'(< \psi, \hat{q}_t \psi >) \end{cases}$$

are classical Hamiltonian equations corresponding to $H = \frac{1}{2m} p^2 + V(q)$ with initial conditions $< \psi, \hat{q}\psi >$, $< \psi, \hat{p}\psi >$. If the Hamiltonian equations of motions are not linear, then the limit $\hbar \to 0$ gives sensible results when it involves mean values of positions and momenta taken with respect to coherent states [14]. Indeed, we describe the quantum mechanical counterpart of the classical system by means of the Weyl-algebra $\mathcal{W}$ which is linearly spanned by the (bounded) Weyl-operators

$$\hat{\mathrm{W}}(-q, p) = \exp\left[i(p\hat{q} - q\hat{p})\right] \tag{2.3}$$

where $(q, p) \in R^2$

$$[\hat{q}\, , \hat{p}] = i \tag{2.4}$$

and

$$\hat{\mathrm{W}}(-q_1, p_1)\hat{\mathrm{W}}(-q_2, p_2) = \hat{\mathrm{W}}(-q_1 - q_2, p_1 + p_2)\exp[\frac{i}{2}(p_1 q_2 - q_1 p_2)] \tag{2.5}$$

We shall consider the strongly continuous irreducible Schrödinger representation of $\mathcal{W}$ on $\mathcal{H} = L^2(R)$ given by:

$$[\hat{\mathrm{W}}(-q, p)\psi](x) = \exp\left[i(px - \frac{pq}{2})\right]\psi(x - q) \tag{2.6}$$

Its strong Hilbert space closure amounts to the entire Banach space $\mathrm{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ [15,16]. It turns out that any $\hat{\mathrm{A}} \in \mathrm{B}(\mathcal{H})$ can be represented as

$$\hat{\mathrm{A}} = \int_{R^2} \widetilde{a}(q, p)\hat{\mathrm{W}}(-q, p)\, dq dp$$

on $\mathcal{H}$ [16].

In order to pursue the classical correspondence we follow [14] and introduce the symmetric representation

$$\begin{cases} \hat{q}_\hbar &= \sqrt{\hbar}\hat{q} \\ \hat{p}_\hbar &= \sqrt{\hbar}\hat{p} \end{cases}$$

where

$$\begin{cases} (\hat{q}\psi)(x) &= x\psi(x) \\ (\hat{p}\psi)(x) &= -i\psi'(x). \end{cases} \tag{2.7}$$

Given the gaussian state

$$\psi_0(x) = \sqrt[4]{\frac{1}{\pi}} \exp\left(-\frac{x^2}{2}\right),$$

we construct the overcomplete family of coherent states $|q,p>_{(q,p)\in R^2}$

$$|q,p> = \hat{W}(-q,p)|\psi_0> . \tag{2.8}$$

We have [16]:

$$< q_1, p_1 | q_2, p_2 > = \exp\left(-\frac{1}{4}[(q_1 - q_2)^2 + (p_1 - p_2)^2 - 2i(q_1 p_2 - p_1 q_2)]\right) \tag{2.9}$$

and

$$\int_{R^2} |q,p>< q,p| \frac{dqdp}{2\pi} = 1. \tag{2.10}$$

The following result is easily obtained:

Proposition 2.2

$$\lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | \hat{W}(-\sqrt{\hbar}\xi, \sqrt{\hbar}\pi) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > = \exp[i(\pi q - \xi p)]$$

Proof

By means of (2.5) and (2.9) we get:

$$< \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | \hat{W}(-\sqrt{\hbar}\xi, \sqrt{\hbar}\pi) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > = \exp i(\pi q - \xi p) \exp -\frac{\hbar}{4}(\xi^2 + \pi^2).$$

Remarks 2.3

1. According to [16] we term $\hat{W}(-\sqrt{\hbar}\xi, \sqrt{\hbar}\pi)$ a classical operator. Given the Weyl-representation of a bounded operator $\hat{A}$, see Remark 2.1.2, we can rescale q and p to get:
$$\hat{A}_\hbar = \int_{R^2} \widetilde{a}_\hbar(q,p)\hat{W}(-\sqrt{\hbar}q, \sqrt{\hbar}p).$$
Therefore, if $\lim_{\hbar\to 0} \widetilde{a}_\hbar(q,p)$ exists at least in a distributional sense, then we call $\hat{A}_\hbar$ a classical operator.

2. If $\hat{A}_\hbar$ and $\hat{B}_\hbar$ are classical operators, then we have:
$$a(q,p) = \lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}}|\hat{A}_\hbar|\frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > = \int_{R^2} \widetilde{a}(\xi,\pi) \exp i(q\pi - p\xi)\, d\xi d\pi$$
$$b(q,p) = \lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}}|\hat{B}_\hbar|\frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > = \int_{R^2} \widetilde{b}(\xi,\pi) \exp i(q\pi - p\xi)\, d\xi d\pi,$$
then it can be shown that [16]:
$$\lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}}| -\frac{i}{\hbar}[\hat{A}_\hbar, \hat{B}_\hbar]\,|\frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > = \{a,b\}(q,p) \;=\; (\partial_q a \partial_p b - \partial_p a \partial_q b)(q,p).$$

3. The Weyl operators are translation operators:
$$\hat{W}(\frac{q}{\sqrt{\hbar}}, -\frac{p}{\sqrt{\hbar}})(\hat{q}_\hbar, \hat{p}_\hbar)\hat{W}(-\frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}}) = (\hat{q}_\hbar + q, \hat{p}_\hbar + p)$$
A suitably regular Hamiltonian $\hat{H}_\hbar = \frac{1}{2m}\hat{p}_\hbar^2 + V(\hat{q}_\hbar)$ is a classical operator. Indeed,
$$< \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}}|\,\hat{H}_\hbar\,|\frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > = < \psi_0\,|\frac{1}{2m}(\hat{p}_\hbar + p)^2 + V(\hat{q}_\hbar + q)\,|\psi_0 > .$$
If, moreover, $\hat{A}_\hbar$ is a classical operator we arrive at:
$$\lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}}|\,\partial_t \hat{A}_\hbar|_{t=0} \; - \frac{i}{\hbar}[\hat{H}_\hbar, \hat{A}_\hbar]\,|\frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > =$$
$$= \partial_t a(q,p)|_{t=0} - \{a, H\}(q,p) = 0.$$

To probe further into the classical limit we consider the Cauchy problem originated by the Hamiltonian vector field $X_H = (\frac{p}{m}, -V'(q))$ on R^2:
$$\begin{cases} \dot{q}(t) = \dfrac{p(t)}{m} \\ \dot{p}(t) = -V'(q(t)), \end{cases}$$

with initial conditions

$$\begin{cases} q(0) = q \\ p(0) = p \end{cases}$$

Then the following result holds:

Theorem 2.4 [14]

As long as the local flow of diffeomorphisms $\{\Phi_t^H\}$ exists

$$s - \lim_{\hbar \to 0} \hat{W}(\frac{q}{\sqrt{\hbar}}, -\frac{p}{\sqrt{\hbar}}) \exp(\frac{i}{\hbar}\hat{H}_\hbar t)\hat{W}(-\sqrt{\hbar}\xi, \sqrt{\hbar}\pi) \exp(-\frac{i}{\hbar}\hat{H}_\hbar t)\hat{W}(-\frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}})$$
$$= \exp(i[q(t)\pi - p(t)\xi]). \tag{2.4.1}$$

Remark 2.5

The above theorem is a more general restatement of the well-known fact that the mean values $< \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | \hat{q}_\hbar(t) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} >$ and $< \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | \hat{p}_\hbar(t) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} >$ go into the classical solutions $\Phi_t^H(q,p)$ when $\hbar \to 0$. It has been proved for a potential V(q(t)) with δ-Hölder continuous second derivative ($\delta > 0$) about the classical trajectory $\Phi_t^H(q,p) = (q(t), p(t))$.

We are now able to prove the following two straightforward propositions.

Proposition 2.6

Given the classical phase-space distribution $\rho(q,p) \in L^1(R^2)$, the state operator

$$\hat{\rho}_\hbar = \int_{R^2} d\xi d\pi\, \rho(\xi,\pi) | \frac{\xi}{\sqrt{\hbar}}, \frac{\pi}{\sqrt{\hbar}} >< \frac{\xi}{\sqrt{\hbar}}, \frac{\pi}{\sqrt{\hbar}} | \tag{2.6.1}$$

gives rise to a classical operator as follows:

$$\lim_{\hbar \to 0} < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | \frac{\hat{\rho}_\hbar}{2\pi\hbar} | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > = \rho(q,p). \tag{2.6.2}$$

Moreover:

$$\frac{\hat{\rho}_\hbar}{2\pi\hbar} = \int_{R^2} d\xi d\pi\, \tilde{\rho}_\hbar(\xi,\pi)\hat{W}(-\sqrt{\hbar}\xi, \sqrt{\hbar}\pi) \tag{2.6.3}$$

where

$$\begin{aligned}\widetilde{\rho}_\hbar(\xi,\pi) &= \frac{1}{(2\pi)^2}\int_{R^2} d\bar{\xi}d\bar{\pi}\,\rho(\bar{\xi},\bar{\pi})\exp\left(-\frac{\hbar}{4}(\xi^2+\pi^2)\right)\exp\left(i[\xi\bar{\pi}-\pi\bar{\xi}]\right)\\ &\equiv \widetilde{\rho}(\xi,\pi)\exp\left(-\frac{\hbar}{4}(\xi^2+\pi^2)\right).\end{aligned} \tag{2.6.4}$$

Proof

(2.6.2) comes from (2.9) and

$$\lim_{\hbar\to 0}\frac{1}{2\pi\hbar}\exp\left(-\frac{1}{2\hbar}[(q-\xi)^2+(p-\pi)^2]\right)=\delta(q-\xi)\delta(p-\pi).$$

(2.6.4) is in turn obtained by equating the mean values of both expressions (2.6.1) and (2.6.3) and by Fourier transforming.

Proposition 2.7

With $\hat{\rho}^\hbar$ as in Proposition (2.6) and $\hat{\mathrm{H}}_\hbar$ as in Remark (2.5), we have:

$$\begin{aligned}&\lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}|\exp\left(-\frac{i}{\hbar}\hat{\mathrm{H}}_\hbar t\right)\frac{\hat{\rho}^\hbar}{2\pi\hbar}\exp\left(\frac{i}{\hbar}\hat{\mathrm{H}}_\hbar t\right)|\frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}>\\ &= \int_{R^2} d\xi d\pi\,\widetilde{\rho}(\xi,\pi)\exp i(\pi q(t)-\xi p(t))\\ &= (\rho\cdot\Phi^H_{-t})(q,p).\end{aligned} \tag{2.7.1}$$

Proof: this comes about from Theorem 2.4 and Proposition 2.6.

Remarks 2.8

1. The fact that the classical distribution arises from $\frac{\hat{\rho}^\hbar}{2\pi\hbar}$ and not from $\hat{\rho}^\hbar$ is due to the finite size of the elementary cells of the phase-space associated with the quantum system which makes $2\pi\hbar$ the right normalization factor [17].This can also be seen along the lines followed in [13]. Namely, given a density matrix $\hat{\rho}^\hbar$, $\mathrm{Tr}\,\hat{\rho}^\hbar=1$, the mapping:

$$\hat{\rho}^\hbar \to < \frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}|\frac{\hat{\rho}^\hbar}{2\pi\hbar}|\frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}>,$$

gives an $\hbar$-dependent phase space distribution. Indeed, because of 2.10,

$$\int_{R^2}\frac{dqdp}{2\pi\hbar} < \frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}|\hat{\rho}^\hbar|\frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}> = \mathrm{Tr}\hat{\rho}^\hbar.$$

2. The symmetric representation of $\hat{q}_\hbar$ and $\hat{p}_\hbar$ amounts to going from the units in which $\hbar = 1$ to the pysical ones. In the latter the Hamiltonian $\hat{\mathrm{H}} = H(\hat{q}, \hat{p})$ has to be replaced by $\hat{\mathrm{H}}_\hbar = \frac{1}{\hbar}\hat{\mathrm{H}}(\hat{q}_\hbar, \hat{p}_\hbar)$.

3. Another kind of rescaling is possible [16] by allowing for $\hat{q}_\hbar$ and $\hat{p}_\hbar$ to be replaced by

$$\hat{q}_g = g\hat{q}_\hbar \quad , \quad \hat{p}_g = g\hat{p}_\hbar$$

and $\frac{1}{\hbar}\hat{\mathrm{H}}_\hbar$ by

$$\frac{1}{g^2\hbar}\hat{\mathrm{H}}_g = \frac{1}{g^2\hbar}\hat{\mathrm{H}}(\hat{q}_g, \hat{p}_g),$$

respectively. In this system of units momentum and position get dilated by a factor $\frac{1}{g}$ with respect to the physical units, $\hbar$ is replaced by $g^2\hbar$ and the classical limit is now performed by letting $g^2 \to 0$, which can be interpreted as a rescaling of the unit of time by a factor g^2 and a subsequent joint limit $g^2 \to 0$ $t \to +\infty$, with $g^2 t = const.$. The latter is known as van Howe or weak-coupling limit.

3. G.R.W.-Model: Quadratic Hamiltonians

We go back now to equation (1.3) which is actually written in units such that $\hbar = 1$. As we are to consider the limit $\hbar \to 0$ and interested in those features of the localization mechanism which survive it, we make the $\hbar$-dependence explicit by rewriting:

$$\partial_t \hat{\rho}_t^\hbar = -\frac{i}{\hbar}\left[\hat{\mathrm{H}}_\hbar, \hat{\rho}_t^\hbar\right] - \lambda\hat{\rho}_t^\hbar + \lambda\sqrt{\frac{\alpha}{\pi}}\int_R dx\, \exp(-\frac{\alpha}{2}(\hat{q}_\hbar - x)^2)\,\hat{\rho}_t^\hbar\,\exp(-\frac{\alpha}{2}(\hat{q}_\hbar - x)^2). \tag{3.1}$$

We notice that the coherence length $\frac{1}{\sqrt{\alpha}}$ is now the physical one and is measured in centimeters. By using (1.4) we can check that the map $T[\cdot]$ in (1.2), i.e. the third term in (3.1), can be reformulated [11] as:

$$\begin{aligned} \mathrm{T}[\hat{\rho}_t^\hbar] &= \int_R dy\, \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp(-\frac{y^2}{\alpha\hbar^2})\, \exp(-\frac{i}{\hbar}y\hat{q}_\hbar)\,\hat{\rho}_t^\hbar\,\exp(\frac{i}{\hbar}y\hat{q}_\hbar) \\ &= \int_R dy\, \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp(-\frac{y^2}{\alpha\hbar^2})\,\hat{\mathrm{W}}(0, -\frac{y}{\sqrt{\hbar}})\,\hat{\rho}_t^\hbar\,\hat{\mathrm{W}}(0, \frac{y}{\sqrt{\hbar}}) \end{aligned} \tag{3.2}$$

in terms of the Weyl operators introduced in (2.3).

Remarks 3.1

1. If we consider a density matrix as in (2.6.1), we get:

$$\lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | \mathrm{T}[\frac{1}{2\pi\hbar}\hat{\rho}^{\hbar}] | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} >$$

$$= \lim_{\hbar\to 0} \int_R dy \left\{ \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp(-\frac{y^2}{\alpha\hbar^2}) \right.$$

$$\left. < \psi_0 | \hat{\mathrm{W}}(\frac{q}{\sqrt{\hbar}}, -\frac{p+y}{\sqrt{\hbar}}) \frac{1}{2\pi\hbar}\hat{\rho}^{\hbar}\, \hat{\mathrm{W}}(-\frac{q}{\sqrt{\hbar}}, \frac{p+y}{\sqrt{\hbar}}) | \psi_0 > \right\}$$

$$= \rho(q,p)$$

Thus the whole dissipative mechanism

$$-\lambda \frac{1}{2\pi\hbar}\hat{\rho}^{\hbar} + \lambda \mathrm{T}[\frac{1}{2\pi\hbar}\hat{\rho}^{\hbar}]$$

would disappear in the classical limit and the Hamiltonian term only would contribute. It is apparent though that the Dirac's δ at the origin can be avoided if both limits $\hbar \to 0$ and $\alpha \to 0$, with $\hbar^2\alpha = \beta$ kept constant, are performed at the same time. The result would be:

$$\lim_{\substack{\hbar\to 0\\ \alpha\to+\infty\\ \alpha\hbar^2=\beta}} < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | \mathrm{T}[\frac{1}{2\pi\hbar}\hat{\rho}^{\hbar}] | \frac{q}{\sqrt{\hbar}}, p_{\hbar} >$$

$$= \int_R dy \frac{1}{\sqrt{\beta\pi}} \exp(-\frac{y^2}{\beta}) \rho(q, p+y)$$

2. We notice that the indeterminacy principle associates with the coherence length $\frac{1}{\sqrt{\alpha}}$ a typical momentum $\hbar\sqrt{\alpha}$. To understand the joint limit we have to study how the classical limit interferes with the diagonalizing properties of the map $T[\cdot]$. This is done in Appendix 1., the result being that the limit $\hbar \to 0$ is by itself a diagonalizing process of some sort and therefore only a corresponding scaling of the coherence length can make the process $T[\cdot]$ be felt by the system in the classical limit.

3. Equation (3.1) has been explicitly solved [11] for Hamiltonians like

$$\hat{H}(\hat{q},\hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{q})$$

with $V(\hat{q})$ at most quadratic, the result being:

$$\hat{\rho}_t^\hbar = \frac{1}{(2\pi\hbar)^2}\int_{R^4} dx dy d\xi d\pi \Big\{ e^{-\frac{i}{\hbar}(y\xi + x\pi)} F(\alpha,\xi,\pi,t) \\ \hat{W}(\frac{x}{\sqrt{\hbar}},\frac{y}{\sqrt{\hbar}}) \exp(-\frac{i}{\hbar}\hat{H}_\hbar t)\hat{\rho}^\hbar \exp(\frac{i}{\hbar}\hat{H}_\hbar t)\hat{W}(-\frac{x}{\sqrt{\hbar}},-\frac{y}{\sqrt{\hbar}})\Big\} \tag{3.3}$$

where

$$F(\alpha,\xi,\pi,t) = \exp\Big\{ -\lambda t + \lambda\int_0^t ds \exp(-\frac{\alpha}{4}\xi_{-s}^2(\xi,\pi))\Big\} \tag{3.4}$$

and $\xi_{-s}(\xi,\pi)$ is the position at time -s as it develops from the initial conditions $(\xi_0(\xi,\pi) = \xi, \pi_0(\xi,\pi) = \pi)$ through the linear Hamiltonian equations of motion. Hence:

$$\xi_{-s}(\xi,\pi) = a(-s)\xi + b(-s)\pi.$$

The integral in (3.3) it has been obtained [11] from a uniformly convergent series in $B(\mathcal{H})_1^{s.a.}$, see beginning of the next section, by virtue of the linearity of the equations of motion, thus it is a well defined mathematical object to work with.

We rewite (3.3) as:

$$\hat{\rho}_t^\hbar = \frac{1}{(2\pi)^2}\int_{R^4} dx dy d\xi d\pi \Big\{ e^{i(\xi y - \pi x)} F(\alpha,\hbar\xi,\hbar\pi,t) \\ \hat{W}(\frac{x}{\sqrt{\hbar}},-\frac{y}{\sqrt{\hbar}}) \exp(-\frac{i}{\hbar}\hat{H}_\hbar t)\hat{\rho}^\hbar \exp(\frac{i}{\hbar}\hat{H}_\hbar t)\hat{W}(-\frac{x}{\sqrt{\hbar}},\frac{y}{\sqrt{\hbar}})\Big\} \tag{3.5}$$

where now, because of the linearity of the equations of motion,

$$F(\alpha,\hbar\xi,\hbar\pi,t) = \exp\Big\{ -\lambda t + \lambda\int_0^t ds \exp(-\frac{\hbar^2\alpha}{4}\xi_{-s}^2(\xi,\pi))\Big\} \tag{3.6}$$

If we choose $\hat{\rho}^\hbar$ as in (2.6.1) and apply Theorem 2.4 to (3.5), we get:

$$\begin{aligned}\rho_t^\beta(q,p) &= \lim_{\substack{\hbar\to 0\\ \alpha\to+\infty\\ \hbar^2\alpha=\beta}} < \frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}|\frac{1}{2\pi\hbar}\hat{\rho}_t^\hbar|\frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}} > \\ &= \frac{1}{(2\pi)^2}\int_{R^4} dx dy d\xi d\pi\, e^{i(\xi y - \pi x)} F(\beta,\xi,\pi,t)\,(\rho\cdot\Phi_{-t}^H)(q+x,p+y)\end{aligned} \tag{3.7}$$

Proposition 3.2

ρ_t^β in (3.7) is the solution of the differential Chapman-Kolmogorov equation

$$\partial_t \rho_t^\beta(q,p) = \Big\{H, \rho_t\Big\}(q,p) - \lambda \rho_t(q,p) + \frac{\lambda}{\sqrt{\beta\pi}} \int_R dy\, e^{-\frac{y^2}{\beta}} \rho_t(q, p+y)$$

with initial condition $\rho_0(q,p) = \rho(q,p)$.

Proof

As $\xi\, y \;-\; \pi\, x$ is a symplectic form for and the volume element dxdy is invariant under the flow of diffeomorphisms $\{\Phi_t^H\}$ generated by the, at most quadratic, Hamiltonian H, we can rewrite (3.7) as:

$$\rho_t^\beta(q,p) = \frac{1}{(2\pi)^2} \int_{R^4} dx dy d\xi d\pi\; e^{i(\pi_t\, q \,-\, \xi_t\, p)}\; e^{i(y\, \xi \,-\, x\, \pi)}\; F(\beta, \xi_t, \pi_t, t)\, \rho(x,y)\,,$$

where

$$\begin{aligned}
&(\xi_t, \pi_t) = \Phi_t^H(\xi, \pi),\\
&F(\beta, \xi_t, \pi_t, t) = \exp\Big\{\lambda t + \lambda \int_0^t ds \exp\big(-\frac{\beta}{4}\xi_{-s}^2(\xi_t, \pi_t)\big)\Big\}\\
&\quad = \exp\Big\{ -\lambda t + \lambda \int_0^t ds\, \exp\big(-\frac{\beta}{4}\xi_s^2(\xi, \pi)\big)\Big\}.
\end{aligned}$$

Now:

$$\partial_t\, e^{i(\pi_t\, q \,-\, \xi_t\, p)} = \Big\{\frac{p^2}{2m} + V(q),\; e^{i(\pi_t\, q \,-\, \xi_t\, p)}\Big\}$$

and

$$\begin{aligned}
\partial_t F \;&=\; -\lambda F + \lambda\, F\, \exp\big(-\frac{\beta}{4}\xi_t^2(\xi, \pi)\big)\\
&= -\lambda F + \frac{\lambda}{\sqrt{\beta\pi}} F \int_R dy\, e^{-\frac{y^2}{\beta}}\; e^{-i\, y\, \xi_t(\xi,\pi)}.
\end{aligned}$$

Thus the result easily follows. Moreover:

$$\rho_0(q,p) = \frac{1}{(2\pi)^2} \int_{R^4} dx dy d\xi d\pi e^{i\, \pi(q-x) \,-\, i\, \xi(p-y)} \rho(x,y) = \rho(q,p).$$

Remarks 3.3

1. $\rho_t^\beta(q,p)$ is the phase-space probability distribution associated with the homogeneous Markov-process whose transition probability $P(t,q,p|t_0,q_0,p_0)$ is determined by a forward differential Chapman-Kolmogorov equation comprising a deterministic (linear) Liouville term and a jump process:

$$\int_{R^2} d\overline{q} d\overline{p} \Big\{ W(q,p|\overline{q},\overline{p})\, P(t,\overline{q},\overline{p}|t_0,q_0,p_0) - W(\overline{q},\, \overline{p}|q,p)\, P(t,q,p|t_0,q_0,p_0)\Big\}$$

with

$$W(q,p|\overline{q},\overline{p}) = \frac{\lambda}{\sqrt{\pi\beta}}\delta(q-\overline{q})\exp\left(-\frac{(p-\overline{p})^2}{\beta}\right).$$

We notice that:

$$\rho_t^{\beta}(q,p) = \int_{R^2} d\overline{q}d\overline{p}\, P(t,q,p|0,\overline{q},\overline{p})\rho(\overline{q},\overline{p})$$

and

$$P(t,q,p|0,\overline{q},\overline{p}) = \frac{1}{(2\pi)^2}\int_{R^2} d\xi d\pi e^{i(\pi_t\, q\,-\,\xi_t\, p)+i(\xi\overline{p}-\pi\overline{q})}\, F(\beta,\xi_t,\pi_t,t).$$

If we set the initial condition at $t=t_0$ we get:

$$\Phi_t^H(\xi,\pi) = (\xi_t,\pi_t) \rightarrow \Phi_{t-t_0}^H(\xi,\pi) = (\xi_{t-t_0}\,,\pi_{t-t_0})$$

$$F(\beta,\xi_t,\pi_t,t) \rightarrow F(\beta,\xi_{t-t_0},\pi_{t-t_0},t-t_0) =$$

$$= e^{-\lambda(t-t_0)}\exp\left\{\lambda\int_0^{t-t_0} ds\, \exp\left(-\frac{\beta}{4}\xi_s^2(\xi,\pi)\right)\right\}.$$

One can thus verify that the transition probabilities $P(t,q,p|t_0,q_0,p_0)$, which have actually a distributional meaning, satisfy the Chapman-Kolmogorov equation:

$$P(t,q,p|t_0,q_0,p_0) = \int_{R^2} dq_1dp_1\, P(t,q,p|t_1,q_1,p_1)\, P(t_1,q_1,p_1|t_0,q_0,p_0)$$

and depend only on time differences as homogeneity requires.

2. As mentioned in Remark 2.8.2, by using a unit system in which the physical ones are dilated by a factor g^{-1}, the classical limit can be replaced by the weak-coupling limit $g^2 \rightarrow 0$. In the new units:

$$\begin{cases} \hat{q}_\hbar \rightarrow \hat{q}_g = g\hat{q}_\hbar \\ \hat{p}_\hbar \rightarrow \hat{p}_g = g\hat{p}_\hbar \\ \lambda \rightarrow \lambda \\ \hbar \rightarrow \hbar g^2 \\ \alpha \rightarrow \alpha_g = \alpha\, g^{-2} \end{cases}$$

and hence (3.1) and (3.5) read:

$$\partial_t\hat{\rho}_t^g = -\frac{i}{\hbar g^2}\left[\hat{H}_g,\hat{\rho}_t^g\right] - \lambda\hat{\rho}_t^g$$

$$+\frac{\lambda}{\sqrt{\alpha\hbar^2 g^2}}\int_R dy\, e^{-\frac{y^2}{\alpha\hbar^2 g^2}}\hat{\mathrm{W}}(0,-\frac{y}{g\sqrt{\hbar}})\hat{\rho}_t^g\hat{\mathrm{W}}(0,\frac{y}{g\sqrt{\hbar}})$$

and

$$\hat\rho_t^g = \frac{1}{(2\pi)^2}\int_{R^4} dx dy d\xi d\pi \Big\{ e^{i(\xi y - \pi x)} F(\alpha\hbar^2 g^2, \xi, \pi, t)$$

$$\hat{\mathrm{W}}(\frac{x}{g\sqrt{\hbar}}, -\frac{y}{g\sqrt{\hbar}}) \exp(-\frac{i}{\hbar g^2}\hat H_g t)\hat\rho_t^g \exp(\frac{i}{\hbar g^2}\hat H_g t)\hat{\mathrm{W}}(-\frac{x}{g\sqrt{\hbar}}, \frac{y}{g\sqrt{\hbar}})\Big\},$$

respectively.

The choice of the scaling factor $g = \sqrt{\frac{\beta}{\alpha}\frac{1}{\hbar}}$ and the limit $\alpha \to +\infty$ will then get:

$$\rho_t^\beta(q,p) = \lim_{\alpha\to+\infty} < q\sqrt{\frac{\alpha}{\beta\hbar}}, p\sqrt{\frac{\alpha}{\beta\hbar}} | \hat\rho_t^{\sqrt{\frac{\beta}{\alpha}\frac{1}{\hbar}}} | q\sqrt{\frac{\alpha}{\beta\hbar}}, p\sqrt{\frac{\alpha}{\beta\hbar}} >.$$

It is to be noticed that, as the modified quantum mechanics is characterized by two parameter λ and α, the coresponding classical stochastic process is governed by λ and β, β weighing a phase-space jump process affecting the momentum, which occurs with mean frequency λ.

4. G.R.W.-Model: General Hamiltonians.

Equation (3.1) can be transformed into an integral one to be solved, formally, by iteration [11]:

$$\begin{aligned}\hat\rho_t^\hbar &= e^{-\lambda t} U_\hbar{}^\dagger(t)\Big\{\hat\rho^\hbar + \lambda\int_0^t ds\, U_\hbar(s)\mathrm{T}[\hat\rho_s^\hbar]U_\hbar{}^\dagger(s)\Big\}U_\hbar(t) \\ &= e^{-\lambda t}U_\hbar{}^\dagger(t)\Big\{\sum_{k=0}^{+\infty}\lambda^k\int_0^t ds_k\, \tau_{s_k}^k[\hat\rho^\hbar]\Big\}U_\hbar(t),\end{aligned} \tag{4.1}$$

where:

$$\begin{aligned} U_\hbar(t) &= \exp(\frac{i}{\hbar}\hat{\mathrm{H}}_\hbar t) \\ \mathrm{T}[\hat\rho_s^\hbar] &= \frac{1}{\sqrt{\hbar^2\alpha\pi}}\int_R dy\, e^{-\frac{y^2}{\hbar^2\alpha}} e^{-\frac{i}{\hbar}\hat q_\hbar y}\hat\rho_s^\hbar e^{\frac{i}{\hbar}\hat q_\hbar y} \\ \int_0^t ds_0\, \tau_{s_0}^0[\hat\rho^\hbar] &= \hat\rho^\hbar \\ \tau_{s_k}^k[\hat\rho^\hbar] &= \int_0^{s_k} ds_{k-1}\, U_\hbar(s_k)\mathrm{T}[U_\hbar{}^\dagger(s_k)\,\tau_{s_{k-1}}^{k-1}[\hat\rho^\hbar]\,U_\hbar(s_k)]U_\hbar{}^\dagger(s_k) \\ \| \int_0^t ds_k\, U_\hbar{}^\dagger(t)\,\tau_{s_k}^k[\hat\rho^\hbar]\,U_\hbar(t)\|_1 &\le \frac{t^k}{k!}. \end{aligned} \tag{4.2}$$

If $\hat\rho^\hbar$ is a density matrix, then each integrand in (4.1) is a density matrix and the sequence of partial sums converges to $\hat\rho^\hbar_t$ in the trace-norm. Convergence then holds for mean values so that:

$$< \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | \hat\rho^\hbar_t | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} >= \\ = e^{-\lambda t}\Big\{ \sum_{k=0}^{+\infty} \lambda^k \int_0^t ds_k < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | U_\hbar{}^\dagger(t)\, \tau^k_{s_k} [\hat\rho^\hbar]\, U_\hbar(t) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} >\Big\}. \tag{4.3}$$

As we are interested in density matrices as the one in (2.6.1) and hence in $\frac{1}{2\pi\hbar}\hat\rho^\hbar$ as far as the classical limit is concerned, we notice that:

$$< \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | \frac{1}{2\pi\hbar}\hat\rho^\hbar | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > \quad \leq \quad \mathrm{Tr}\,\hat\rho^\hbar = 1$$

because of coherent states' completeness, eqn.(2.10). Thus the convergence of the partial sums in (4.3) is uniform with respect to $\hbar$ and α. In Appendix A.2 it is shown that, with the choice

$$\frac{1}{2\pi\hbar}\hat\rho^\hbar = \int_{R^2} d\xi d\pi\, \widetilde\rho_\hbar(\xi,\pi)\, \hat{\mathrm{W}}(-\sqrt{\hbar}\xi, \sqrt{\hbar}\pi)$$

$$\hat\rho^\hbar(\xi,\pi) = e^{-\frac{\hbar}{4}(\xi^2+\pi^2)}\, \widetilde\rho(\xi,\pi)$$

$$\widetilde\rho(\xi,\pi) = \frac{1}{(2\pi)^2}\int_{R^2} d\bar\xi\, d\bar\pi\, e^{i(\xi\bar\pi - \pi\bar\xi)}\, \rho(\bar\xi,\bar\pi)$$

(see Proposition 2.6), the weak-coupling limit is needed to keep the stochastic process working and that the following proposition holds:

Proposition 4.1

Let $\Phi^p_y(q,p) = (q, p+y)$ be the phase-space flow of momentum translations generated by the Lie-derivative

$$L_q[\cdot] = \{\cdot\,, q\}$$

$$\rho\cdot\Phi^p_y = \exp\left(y\{q,\cdot\}\right)\rho\,,$$

then:

$$\lim_{\substack{\hbar\to 0\\ \alpha\to+\infty\\ \hbar^2\alpha=\beta}} \int_0^t ds_k < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | U_\hbar{}^\dagger(t)\, \tau^k_{s_k}[\frac{1}{2\pi\hbar}\hat\rho^\hbar] U_\hbar(t) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} >$$

$$= \int_0^t ds_k \int_0^{s_k} ds_{k-1}\dots\int_0^{s_2} ds_1 \int_R dy_k \dots \int_R dy_1 \left\{ \frac{\exp\left(-\frac{1}{\beta}\sum_{l=1}^k y_l^2\right)}{(\sqrt{\beta\pi})^k} \right. \tag{4.1.1}$$

$$\left.\left[\rho\cdot\left(\prod_{l=1}^k \Phi^H_{-s_l}\cdot\Phi^p_{y_l}\cdot\Phi^H_{s_l}\right)\cdot\Phi^H_{-t}\right](q,p)\right\}$$

with

$$\prod_{l=1}^{k} \Phi^H_{-s_l} \cdot \Phi^p_{y_l} \cdot \Phi^H_{s_l} = \left[\prod_{l=1}^{k-1} \Phi^H_{-s_l} \cdot \Phi^p_{y_l} \cdot \Phi^H_{s_l} \right] \Phi^H_{-s_k} \cdot \Phi^p_{y_k} \cdot \Phi^H_{s_k}.$$

From above we derive:

Proposition 4.2

$$\rho^\beta_t(q,p) = e^{-\lambda t} \Bigg\{ \sum_{k=0}^{+\infty} \lambda^k \int_0^t ds_k \ldots \int_0^{s_2} ds_1 \int_R dy_k \ldots \int_R dy_1 \, \frac{\exp\left(-\frac{1}{\beta}\sum_{l=1}^k y_l^2\right)}{(\sqrt{\beta\pi})^k}$$
$$\left[\rho \cdot \left(\prod_{l=1}^{k} \Phi^H_{-s_l} \cdot \Phi^p_{y_l} \cdot \Phi^H_{s_l} \right) \cdot \Phi^H_{-t} \right](q,p) \Bigg\}$$

is the solution of the differential Chapman-Kolmogorov equation

$$\partial_t \rho^\beta_t(q,p) = \left\{ H, \rho^\beta_t \right\}(q,p) - \lambda\, \rho^\beta_t(q,p) + \frac{\lambda}{\sqrt{\beta\pi}} \int_R dy\, e^{\frac{-y^2}{\beta}}\, \rho^\beta_t(q, p+y)$$

with initial condition $\rho^\beta_0(q,p) = \rho(q,p)$.

Proof

It is easily seen that the above series solves the equivalent integral equation:

$$\rho^\beta_t(q,p) = e^{-\lambda t} \left\{ (\rho \cdot \Phi^H_{-t})(q,p) + \lambda \int_0^t ds\, e^{\lambda s} \int_R \frac{e^{\frac{-y^2}{\beta}}}{\sqrt{\beta\pi}} \left(\rho^\beta_s \cdot \Phi^p_y \cdot \Phi^H_{-t+s} \right)(q,p) \right\}.$$

Remarks 4.3

1. The process T[.] can be formally rewritten as follows:

$$\begin{aligned}
\mathrm{T}[\hat\rho^\hbar] &= \int_R dy \, \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp\left(-\frac{y^2}{\alpha\hbar^2}\right) e^{-\frac{i}{\hbar}\hat q_\hbar y} \hat\rho^\hbar \, e^{\frac{i}{\hbar}\hat q_\hbar y} \\
&= \int_R dy \, \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp\left(-\frac{y^2}{\alpha\hbar^2}\right) \left\{ \exp\left(-\frac{i}{\hbar} y \left[\hat q_\hbar, \cdot\right]\right) \hat\rho^\hbar \right\} \\
&= \sum_{k=0}^{+\infty} \left\{ \left(-\frac{i}{\hbar}\right)^k \frac{1}{k!} \int_R dy \, \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp\left(-\frac{y^2}{\alpha\hbar^2}\right) y^k \underbrace{\left[\hat q_\hbar, \left[\hat q_\hbar, \ldots \left[\hat q_\hbar, \hat\rho^\hbar\right] \ldots\right]\right]}_{\text{k times}} \right\} \\
&= \left\{ \exp\left(-\frac{\alpha}{4}\left[\hat q_\hbar, \left[\hat q_\hbar, \cdot\right]\right]\right) \hat\rho^\hbar \right\}
\end{aligned}$$

As in the classical limit we expect $-\frac{i}{\hbar}[\hat{q}_\hbar, \cdot]$ to go over into $\left\{q, \cdot\right\}$, the classical Poisson bracket, we see another reason why the joint limit is needed so that $\hbar^2\alpha = \beta$ and

$$\exp\left(-\frac{\alpha}{4}\left[\hat{q}_\hbar, \left[\hat{q}_\hbar, \cdot\right]\right]\right) \underset{\substack{\hbar\to 0 \\ \alpha\to+\infty \\ \hbar^2\alpha=\beta}}{\longrightarrow} \exp\left(\frac{\beta}{4}\left\{q, \left\{q, \cdot\right\}\right\}\right).$$

The right hand side of the latter expression must be understood as the operator on the classical phase-space distributions which give rise to the term

$$\int_R dy\, \frac{e^{\frac{-y^2}{\beta}}}{\sqrt{\beta\pi}}\, \rho(q, p+y)$$

in the differential Chapman-Kolmogorov equation.

2. If we expand the jump process in powers of β, we get the Kramers-Moyal expansion whose first two terms yeld the following Fokker-Planck equation:

$$\partial_t \rho_t^\beta(q,p) = \left\{H, \rho_t^\beta\right\}(q,p) + \frac{\lambda\beta}{4}\partial_p^2 \rho_t^\beta(q,p).$$

The corresponding quantum evolution would then be given by:

$$\begin{aligned}\partial_t \hat{\rho}_t^\hbar &= -i\hbar\left[\hat{H}_\hbar, \hat{\rho}_t^\hbar\right] - \frac{\alpha\lambda}{4}\left[\hat{q}_\hbar, \left[\hat{q}_\hbar, \hat{\rho}_t^\hbar\right]\right] \\ &= -\frac{i}{\hbar}\left[\hat{H}_\hbar, \hat{\rho}_t^\hbar\right] - \frac{\alpha\lambda}{4}\left\{\hat{q}_\hbar^2, \hat{\rho}_t^\hbar\right\} + \frac{\alpha\lambda}{2}\hat{q}_\hbar\, \hat{\rho}_t^\hbar\, \hat{q}_\hbar,\end{aligned}$$

with $\left\{\cdot, \cdot\right\}$ the anticommutator.

5. Conclusions

In [13] the general problem of studying the classical limit of quantum dynamical semigroups has been addressed by considering a generator

$$L[\cdot] = \frac{1}{\lambda^2} L_o[\cdot] + L_d[\cdot]. \tag{5.1}$$

and the corresponding, λ-dependent, one-parameter semigroup

$$\gamma_t^\lambda = \exp\left(t(\frac{1}{\lambda^2} L_o[\cdot] + L_d[\cdot])\right) \tag{5.2}$$

on the state-space $B(H)_1^{s.a.}$ and by taking the limit $\lambda \to 0$ in, accordingly rescaled, vector states, see also [18]. The classical limit does then amount to a weak-coupling limit in which the generator $L_o[\cdot]$ of the group of isometries, the Hamiltonian evolution in our case, is rescaled and long time behaviour is sought after. The G.R.W-model we have investigated, belongs to a particular class of quantum dynamical semigroups in which the scaling parameter, unlikely in (5.1) and (5.2), appears in the dissipative term as well as in the Hamiltonian one. We have then showed that keeping the stochastic properties throughout the classical limit requires a joint limit and eventually a weak-coupling limit. We have been forced to do so in order that the localizing properties of the evolution equation (1.3) be felt on the background of $\hbar \to 0$ which itself tends to suppress coherence (see Appendix A.1).From a conceptual point of view the G.R.W.-model would hint at a physically meaningful and powerful modification of the unitary, Hamiltonian quantum evolution, which, preserving practically unaltered usual atomic physics on one hand, on the other suppresses quantum entanglement as far as macroobjects are concerned, thus paving the way toward a unification of micro and macrophysics [1,2]. Because of its inherent stochasticity, as embodied in the dissipative term which gives rise to the spontaneous localizations, it seemed reasonable to try to preserve this fundamental feature at the phase-space level too. It is the universe, as a kind of reservoir, that, ultimately, acts as a source of stochasticity, this being true at the quantum as well as at the classical level.

Appendix A.1.

In this appendix it will be discussed how the clasical limit $\hbar \to 0$ and the localizing process $T[\cdot]$ interfere with each other. Let

$$|\psi> = C_\psi\{\hat{W}(-\frac{q_1}{\sqrt{\hbar}},0) + \hat{W}(-\frac{q_2}{\sqrt{\hbar}})\}|\psi_0>$$
$$C_\psi{}^{-2} = 2\{1 + \exp(-\frac{1}{\hbar}[q_1 - q_2]^2)\}$$

be a linear superposition of two coherent states with mean positions $\frac{q_1}{\sqrt{\hbar}}$ and $\frac{q_2}{\sqrt{\hbar}}$ respectively and zero mean momenta, then

$$\frac{|\psi><\psi|}{2\pi\hbar}$$

and

$$\frac{1}{2\pi\hbar}\hat{\rho}^\hbar = \frac{1}{2\pi\hbar}\left\{\frac{1}{2}\sum_{i=1}^{2}\hat{W}(-\frac{q_i}{\sqrt{\hbar}},0)|\psi_0><\psi_0|\hat{W}(\frac{q_i}{\sqrt{\hbar}},0)\right\}$$

are two classical operators as defined in Chapter 2. Remark 2.4.1. Indeed,

$$\lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}|\frac{1}{2\pi\hbar}\hat{\rho}_\psi|\frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}} > = \frac{1}{2}\delta(p)[\delta(q-q_1)+\delta(q-q_2)],$$

on the other hand

$$\begin{aligned}
&\frac{1}{2\pi\hbar}| < \frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}|\psi > |^2 \\
&= \frac{|C_\psi|^2}{2\pi\hbar}| < \frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}|\frac{q_1}{\sqrt{\hbar}},0 > + < \frac{q}{\sqrt{\hbar}},\frac{p}{\sqrt{\hbar}}|\frac{q_2}{\sqrt{\hbar}},0 > |^2 \\
&= \frac{|C_\psi|^2}{2\pi\hbar}\left\{\exp(-\frac{p^2}{2\hbar})\left[\exp(-\frac{1}{2\hbar}(q-q_1)^2) + \exp(-\frac{1}{2\hbar}(q-q_2)^2)\right.\right. \\
&\left.\left. + 2\mathrm{Re}\exp(-\frac{i}{2\hbar}(q_1-q_2))\exp(-\frac{1}{4\hbar}[(q-q_1)^2+(q-q_2)^2])\right]\right\}.
\end{aligned}$$

Coherent and incoherent superpositions like the above ones are therefore indistinguishable in the classical limit. On the other hand the process $T[\cdot]$ operates a suppression of those off-diagonalities in position which exceed the coherence length $\frac{1}{\sqrt{\alpha}}$. It should then be expected that the coherence length ought to vanish in order that the process be felt on the background of $\hbar \to 0$. The right scaling is found by comparing the two diagonalizing mechanisms and this is better done by studying:

$$\mathrm{Tr}\Big\{\mathrm{T}\,[\,|\psi><\psi|]\,\hat{\mathrm{W}}(-\sqrt{\hbar}\xi,\sqrt{\hbar}\pi)\Big\}$$
$$= \int_R dy \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp\,(-\frac{y^2}{\alpha\hbar^2}) \,<\psi|\hat{\mathrm{W}}(0,\frac{y}{\sqrt{\hbar}})\hat{\mathrm{W}}(-\sqrt{\hbar}\xi,\sqrt{\hbar}\pi)\hat{\mathrm{W}}(0,-\frac{y}{\sqrt{\hbar}})|\psi>$$
$$= \int_R dy \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp\,(-\frac{y^2}{\alpha\hbar^2})\, e^{i\,y\,\xi} <\psi|\hat{\mathrm{W}}(-\sqrt{\hbar}\xi,\sqrt{\hbar}\pi)|\psi>$$
$$= |C_\psi|^2 \exp\,(-\frac{\hbar^2\alpha\xi^2}{4})\Big\{\exp\,(-\frac{\hbar}{4}[\xi^2+\pi^2])\Big[e^{i\,\pi\,q_1}+e^{i\,\pi\,q_2}\Big]$$
$$+\exp\,(-\frac{\hbar\pi^2}{2}+\frac{i}{2}(q_1+q_2))\Big[\exp\,(-\frac{1}{4\hbar}[\hbar\xi+(q_1-q_2)]^2)$$
$$+\exp\,(-\frac{1}{4\hbar}[\hbar\xi+(q_1-q_2)]^2)\Big]\Big\}$$

The classical limit would give:

$$\lim_{\hbar\to 0}\mathrm{Tr}\Big\{\mathrm{T}\,[\,|\psi><\psi|]\,\hat{\mathrm{W}}(-\sqrt{\hbar}\xi,\sqrt{\hbar}\pi)\Big\}$$
$$=\frac{1}{2}\Big[e^{i\,\pi q_1}+e^{i\,\pi q_2}\Big]$$
$$=\lim_{\hbar\to 0}\mathrm{Tr}\Big\{\rho_\psi\hat{\mathrm{W}}(-\sqrt{\hbar}\xi,\sqrt{\hbar}\pi)\Big\}$$

and, in agreement with what previously obtained, no trace of the process $T[\cdot]$ would be left. If, on the other hand, $\hbar$ is kept fixed and ξ is chosen such that $\hbar\xi \approx \frac{1}{\sqrt{\alpha}}$, then the off-diagonal elements are depressed if $|q_1-q_2| >> \frac{1}{\sqrt{\alpha}}$, thus revealing the localizing action of T[.]. The latter remains none the less significant only if $\hbar^2\alpha \approx$ const. when the classical limit is considered.

Appendix A.2.

In this last appendix we shall prove by induction Proposition 4.1. We restrict ourselves to studying the limit $\hbar \to 0$ for the moment, which we can do by considering

$$\lim_{\hbar\to 0} \int_0^t ds_k < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | U_\hbar{}^\dagger(t)\, \tau^k_{s_k}[\frac{1}{2\pi\hbar}\hat{\rho}^\hbar]U_\hbar(t) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} >.$$

The choice of the initial density matrix is chosen, as in Proposition 4.1, to be:

$$\begin{aligned} \frac{1}{2\pi\hbar}\hat{\rho}^\hbar &= \int_{R^2} d\xi d\pi\, \widetilde{\rho}_\hbar(\xi,\pi)\, \hat{\mathrm{W}}(-\sqrt{\hbar}\xi, \sqrt{\hbar}\pi) \\ \widetilde{\rho}^\hbar(\xi,\pi) &= e^{-\frac{\hbar}{4}(\xi^2+\pi^2)}\, \widetilde{\rho}(\xi,\pi) \\ \widetilde{\rho}(\xi,\pi) &= \frac{1}{(2\pi)^2}\int_{R^2} d\bar{\xi}\, d\bar{\pi}\, e^{i(\xi\bar{\pi}-\pi\bar{\xi})}\, \rho(\bar{\xi},\bar{\pi}). \end{aligned}$$

The term with k=0 gives $(\rho\cdot\Phi^H_{-t})(q,p)$, see Proposition 2.7. The term with k=1 is more interesting, indeed

$$\begin{aligned} &\int_0^t ds_1\, U_\hbar{}^\dagger(t)\, \tau^1_{s_1}[\frac{1}{2\pi\hbar}\hat{\rho}^\hbar]U_\hbar(t) = \\ &= \int_0^t ds_1 \int_R dy_1 \Bigg\{ \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp(-\frac{y_1^2}{\alpha\hbar^2}) \\ &U_\hbar{}^\dagger(t-s_1)\hat{\mathrm{W}}(0,-\frac{y_1}{\sqrt{\hbar}})U_\hbar{}^\dagger(s_1)\,\frac{1}{2\pi\hbar}\hat{\rho}^\hbar\, U_\hbar(s_1)\, \hat{\mathrm{W}}(0,\frac{y_1}{\sqrt{\hbar}})\, U_\hbar(t-s_1)\Bigg\}, \end{aligned}$$

and $U_\hbar{}^\dagger(s_1)\frac{1}{2\pi\hbar}\hat{\rho}^\hbar U_\hbar(s_1)$ is a classical operator, in the sense that:

$$\lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | U_\hbar{}^\dagger(s_1)\frac{1}{2\pi\hbar}\hat{\rho}^\hbar U_\hbar(s_1) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > = (\rho\cdot\Phi^H_{-s_1})(q,p).$$

Hence we can write:

$$U_\hbar{}^\dagger(s_1)\frac{1}{2\pi\hbar}\hat{\rho}^\hbar U_\hbar(s_1) = \int_{R^2} d\xi_1 d\pi_1\, \widetilde{\rho}_\hbar(s_1,\xi_1,\pi_1)\, \hat{\mathrm{W}}(-\sqrt{\hbar}\xi_1, \sqrt{\hbar}\pi_1),$$

where

$$\lim_{\hbar\to 0}\widetilde{\rho}_\hbar(s_1,\xi_1,\pi_1) = \frac{1}{(2\pi)^2}\int_{R^2} d\bar{\xi}_1 d\bar{\pi}_1\, e^{i(\xi_1\bar{\pi}_1+\pi_1\bar{\xi}_1)}\, (\rho\cdot\Phi^H_{-s_1})(\bar{\xi}_1,\bar{\pi}_1).$$

Thus:

$$
\begin{aligned}
&< \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | U_\hbar{}^\dagger(t)\, \tau^1_{s_1} [\, \frac{1}{2\pi\hbar} \hat{\rho}^\hbar\,] U_\hbar(t) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > \\
&= \int_{R^3} dy_1\, d\xi_1\, d\pi_1 \left\{ \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp\,(-\frac{y_1^2}{\alpha\hbar^2})\, e^{-i\, y_1\, \xi_1} \right. \\
&\left. < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | U_\hbar{}^\dagger(t-s_1) \hat{\mathrm{W}}(-\sqrt{\hbar}\xi_1, \sqrt{\hbar}\pi_1) U_\hbar(t-s_1) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > \right\}.
\end{aligned}
$$

The limit $\hbar \to 0$ would get:

$$
\begin{aligned}
&\int_{R^2} d\xi_1\, d\pi_1\, \widetilde{\rho}(s_1, \xi_1, \pi_1)\, \exp\,[i(\pi_1\, q_{s_1 - t}(q,p) - \xi_1\, p_{s_1 - t}(q,p))] \\
&= (\rho \cdot \Phi^H_{-s_1})(\Phi^H_{s_1 - t}(q,p)) = (\rho \cdot \Phi^H_{-t})(q,p)
\end{aligned}
$$

and no trace of the localizing process would be left. From the joint limit $\hbar \to 0$ and $\alpha \to +\infty$, with $\hbar^2 \alpha = \beta$, we obtain:

$$
\begin{aligned}
&\int_{R^3} dy_1\, d\xi_1\, d\pi_1 \Big\{ \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp\,(-\frac{y_1^2}{\alpha\hbar^2}) \\
&\exp\,[i(\pi_1\, q_{s_1 - t}(q,p) - \xi_1 (y_1 + p_{s_1 - t}(q,p))] \widetilde{\rho}(s_1, \xi_1, \pi_1) \Big\} \\
&= \int_R dy_1\, \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp\,(-\frac{y_1^2}{\alpha\hbar^2})\, (\rho \cdot \Phi^H_{-s_1})(q_{s_1 - t}(q,p), p_{s_1 - t}(q,p) + y_1).
\end{aligned}
$$

We can use a more compact and promising notation, by introducing the flow of phase-space momentum translations

$$
\Phi^p_y(q,p) = (q, p+y)
$$

generated by the Lie-derivative

$$
\begin{aligned}
L_p[\cdot] &= \{q, \cdot\} \\
\rho \cdot \Phi^p_y &= \exp\,(y\{q, \cdot\}).
\end{aligned}
$$

The result is:

$$
\begin{aligned}
&\lim_{\substack{\hbar \to 0 \\ \alpha \to +\infty \\ \hbar^2 \alpha = \beta}} \int_0^t ds_1 < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | U_\hbar{}^\dagger(t)\, \tau^1_{s_1} [\, \frac{1}{2\pi\hbar} \hat{\rho}^\hbar\,] U_\hbar(t) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} > \\
&= \int_0^t ds_1 \int_R dy_1\, \frac{1}{\sqrt{\alpha\pi\hbar^2}} \exp\,(-\frac{y_1^2}{\alpha\hbar^2})\, (\rho \cdot \Phi^H_{-s_1} \cdot \Phi^p_{y_1} \cdot \Phi^H_{s_1 - t})(q,p).
\end{aligned}
$$

We can prove by induction that:

$$\lim_{\substack{\hbar\to 0\\ \alpha\to+\infty\\ \hbar^2\alpha=\beta}} \int_0^t ds_k < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | U_\hbar{}^\dagger(t)\, \tau^k_{s_k}[\frac{1}{2\pi\hbar}\hat{\rho}^\hbar] U_\hbar(t) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} >$$

$$= \int_0^t ds_k \int_0^{s_k} ds_{k-1} \cdots \int_0^{s_2} ds_1 \int_R dy_k \cdots \int_R dy_1 \left\{ \frac{\exp(-\frac{1}{\beta}\sum_{l=1}^k y_l^2)}{(\sqrt{\beta\pi})^k} \right.$$

$$\left. \left[\rho \cdot \left(\prod_{l=1}^k \Phi^H_{-s_l} \cdot \Phi^p_{y_l} \cdot \Phi^H_{s_l}\right) \cdot \Phi^H_{-t}\right](q,p) \right\}$$

with

$$\prod_{l=1}^k \Phi^H_{-s_l} \cdot \Phi^p_{y_l} \cdot \Phi^H_{s_l} = \left[\prod_{l=1}^{k-1} \Phi^H_{-s_l} \cdot \Phi^p_{y_l} \cdot \Phi^H_{s_l}\right] \Phi^H_{-s_k} \cdot \Phi^p_{y_k} \cdot \Phi^H_{s_k}.$$

In fact (4.4.1) has been proved true for k=1, if it is supposed to hold for k=n, then:

$$\lim_{\substack{\hbar\to 0\\ \alpha\to+\infty\\ \hbar^2\alpha=\beta}} \int_0^{s_{n+1}} ds_n < \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} | U_\hbar{}^\dagger(s_{n+1})\, \tau^n_{s_n}[\frac{1}{2\pi\hbar}\hat{\rho}^\hbar] U_\hbar(s_{n+1}) | \frac{q}{\sqrt{\hbar}}, \frac{p}{\sqrt{\hbar}} >$$

$$= \int_0^{s_{n+1}} ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \int_R dy_n \cdots \int_R dy_1 \left\{ \frac{\exp(-\frac{1}{\beta}\sum_{l=1}^n y_l^2)}{(\sqrt{\beta\pi})^n} \right.$$

$$\left. \left[\rho \cdot \left(\prod_{l=1}^n \Phi^H_{-s_l} \cdot \Phi^p_{y_l} \cdot \Phi^H_{s_l}\right) \cdot \Phi^H_{-s_{n+1}}\right](q,p) \right\}$$

and we can write the density matrix (divided by $2\pi\hbar$), whose mean value is considered in the joint limit, as

$$U_\hbar{}^\dagger(s_{n+1})\, \tau^n_{s_n}[\frac{1}{2\pi\hbar}\hat{\rho}^\hbar] U_\hbar(s_{n+1}) = \int_{R^2} d\xi d\pi\, \tilde{f}_{\alpha,\hbar}(s_{n+1}, s_n, \xi, \pi)\, \hat{W}(-\sqrt{\hbar}\xi, \sqrt{\hbar}\pi),$$

where:

$$\lim_{\substack{\hbar\to 0\\ \alpha\to+\infty\\ \hbar^2\alpha=\beta}} \tilde{f}_{\alpha,\hbar}(s_{n+1}, s_n, \xi, \pi) =$$

$$\frac{1}{(2\pi)^2}\int_{R^2} d\bar{\xi} d\bar{\pi} e^{i(\xi\bar{\pi}-\pi\bar{\xi})} \int_0^{s_{n+1}} ds_n \cdots \int_0^{s_2} ds_1 \int_R dy_n \cdots \int_R dy_1 \left\{ \frac{\exp(-\frac{1}{\beta}\sum_{l=1}^n y_l^2)}{(\sqrt{\beta\pi})^n} \right.$$

$$\left. \left[\rho \cdot \left(\prod_{l=1}^n \Phi^H_{-s_l} \cdot \Phi^p_{y_l} \cdot \Phi^H_{s_l}\right) \cdot \Phi^H_{-s_{n+1}}\right](\bar{\xi},\bar{\pi}) \right\}.$$

By repeating the argument used for k=1 and the uniform convergence of the series in (4.3) with respect to $\hbar$ and α, (4.4.1) follows immediately.

References

[1] G.C.Ghirardi, A.Rimini, T.Weber, Phys.Rev.D **34**(1986), 470

[2] J.S.Bell, in *Schrödinger: Centenary Celebration of a Polymath* (C.W.Kilmister Ed.),Cambridge University Press, Cambridge, 1987

[3] G.Lindblad, Commun.Math.Phys.**48**(1976), 119

[4] F.Benatti, G.C.Ghirardi, A.Rimini, T.Weber, Il Nuovo Cim.**101B**(1988), 333

[5] E.B.Davies,*Quantum Theory of Open Systems*, Academic Press, London, New York, San Francisco, 1976

[6] G.W.Ford, J.T.Lewis, in *Probability, Statistical Mechanics and Number Theory,* Adv.in Math. Suppl. Stud.(G.C.Rota Ed.), Academic Press, New York, 1986

[7] F.Benatti, G.C.Ghirardi, A.Rimini, T.Weber, Il Nuovo Cim.**100B**(1987), 27

[8] P.Pearle, Phys.Rev.A**39**(1989), 2277

[9] G.C.Ghirardi, P.Pearle, A.Rimini, Phys.Rev.A **42**(1990),78

[10] G.C.Ghirardi, R.Grassi, P.Pearle, Found. of Phys.**20**(1990), 1271

[11] F.Benatti, J.Math.Phys.**31**(1990), 2399

[12] F.Benatti, T.Weber, Il Nuovo Cim.**103B**(1989), 511

[13] E.B.Davies, Commun.Math.Phys.**49**(1976), 113

[14] K.Hepp, Commun.Math.Phys.**35**(1974), 265

[15] J.R.Klauder, Phys.Rev.A**29**(1984), 2036

[16] L.G.Yaffe, Rev.Mod.Phys.**54**(1982), 407

[17] A.Wehrl, Rev.Mod.Phys.**50**(1978), 221

[18] J.V.Pulé, A.Verbeure, J.Math.Phys.**20**(1979), 733