A Study of a
90^o Vortex-Vortex Scattering Process

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Abstract

Following Ruback, we discuss the evidence for scattering at right angle of two vortices in a head-on collision. The evidence is given in terms of the approximate solutions of the equations of motion. This makes it possible to extend the analysis to the case of a small net repulsive force between the corresponding static vortex configurations.The ordinary differential equations, which result from the ansatz for the approximate solutions, are solved by Taylor series at the origin and by asymptotic series at infinity.

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1 Introduction

Over the years, soliton and soliton-like solutions of nonlinear partial differential equations have been studied in great detail. One of the most important results of these studies was the discovery of the unusual behaviour of solitons in a scattering process. In recent years, mainly based on an idea by Manton [1], results for the scattering of soliton-like objects, like magnetic monopoles [2], CP^1 skyrmions [3] and cosmic strings or vortices [4] have been obtained. Important numerical work has also been done for example on cosmic strings or vortices [5] and skyrmions in $(2+1)$ dimensions [6] . We consider the work on the scattering of vortices to be of particular importance because, unlike the other soliton-like objects mentioned, vortices can be produced in the laboratory and with conventional techniques [7], it may be possible to study their collisions experimentially.

The theoretical predictions for the scattering of soliton-like objects are very exciting. The scattering of slowly moving monopoles, for example, shows an extremely rich structure which is partly due to inner degrees of freedom, internal phases. For static vortices the only degrees of freedom are the positions of the vortices, and any unusual behaviour would hence be due to their soliton-like nature. Left-right symmetry in head-on collision would only allow scattering at an angle of $0^o, 90^o$ or 180^o. For slowly moving vortices at the point between type I and type II superconductivity there is in fact analytic evidence for scattering at right angle [4]. If the repulsion between the vortices increases and they cannot come very close anymore, we would expect to see a switch over to back scattering at a certain value of the repulsion. There is numerical evidence [8] that for fixed repulsion an increase in the velocity can bring the vortices close enough together again to produce scattering at right angles. Another parameter that could be changed is the angle between the vortices or cosmic strings, which are the objects one would have in mind in this type of problem. When we turn the strings out of their parallel position we would expect to see not just scattering but intercommutation of the strings [5].

Of the three parameters, strength of repulsion, velocity and relative angle, in this article we only change the first one. We review the evidence for scattering at right angle of slowly-moving vortices at values of the parameters for which the net force between static vortices is zero. The ansatz used leads to ordinary differential equations which we solve by Taylor series at the origin and by asymptotic series at infinity. When we turn on a small repulsion between the vortices we find that they still scatter at right angle, which, of course, is essential if we want to see 90^o scattering experimentially.

2 The Approximate Solution

The Ginzburg-Landau model of a superconductor in a magnetic field in direction z is given by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)(D^\mu\phi)^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8}\lambda(\phi\phi^* - 1)^2. \tag{1}$$

ϕ is the complex Higgs field, and $D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in terms of the gauge potentials $A_\mu, \mu, \nu = 0, 1, 2$. The metric is $g = diag(+1, -1, -1)$. The Euler-Lagrange equations are

$$D_\mu D^\mu\phi + \frac{1}{2}\lambda\phi(\phi\phi^* - 1) = 0,$$

$$\partial_\mu F^{\mu\nu} + \frac{i}{2}[\phi^* D^\nu \phi - \phi(D^\nu \phi)^*] = 0. \tag{2}$$

For all λ the Euler-Lagrange equations have static, finite energy n-vortex solutions of the form [9]

$$\begin{aligned} \phi(r,\theta) &= e^{in\theta} f(r), & A_0 &= 0, \\ A_i(r,\theta) &= -\epsilon_{ij} x_j \frac{n}{r} a(r), & i,j &= 1,2, \end{aligned} \tag{3}$$

where

$$\begin{aligned} r(rf')' - n^2 f(a-1)^2 - \frac{1}{2} r^2 \lambda f(f^2-1) &= 0, \\ (\frac{a'}{r})' - \frac{f^2}{r}(a-1) &= 0, \end{aligned} \tag{4}$$

and

$$f(0) = a(0) = 0, \qquad \lim_{r\to\infty} f(r) = \lim_{r\to\infty} a(r) = 1. \tag{5}$$

In the special case $\lambda = 1$, it can be shown [10] that the solutions actually satisfy the first-order Bogomolnyi equations [11], and f and a satisfy

$$rf' - n(1-a)f = \frac{2n}{r} a' + f^2 - 1 = 0. \tag{6}$$

In this case, there exists also a 2n-parameter family of static n-vortex solutions describing vortices located at arbitrary positions [10]. The reason for its existence is the fact that for $\lambda = 1$ the net force between static vortices is zero.

Now the scattering of slowly-moving vortices during the time from shortly before to shortly after the collision is studied. We set

$$\begin{aligned} \phi(t,\vec{x}) &= \hat{\phi}(\vec{x}) + \tilde{\phi}(t,\vec{x}), \\ A_i(t,\vec{x}) &= \hat{A}_i(\vec{x}) + \tilde{A}_i(t,\vec{x}), \\ A_0(t,\vec{x}) &= 0, \end{aligned} \tag{7}$$

and assume that the solution is not too different from the configurations (3) of vortices sitting on top of each other. In fact, $(\hat{\phi}, \hat{A}_i)$ is taken to be the solution (3) for $\lambda = 1$ and $n = 2$, and

$$\begin{aligned} \tilde{\phi}(t,\vec{x}) &= \tilde{\lambda}\varphi(\vec{x}) + t\xi(\vec{x}), \\ \tilde{A}_i(t,\vec{x}) &= \tilde{\lambda} a_i(\vec{x}) + tB_i(\vec{x}) \end{aligned} \tag{8}$$

where $\lambda = 1 + \tilde{\lambda}, 0 < \tilde{\lambda} \ll 1, t \in (-\epsilon, \epsilon), \epsilon \ll 1$,and $(\hat{\phi} + \tilde{\lambda}\varphi, \hat{A}_i + \tilde{\lambda} a_i)$ satisfies the static equations of motion linearized in $\tilde{\lambda}$. Under these assumptions it is possible to linearize the equations of motion (2) in $\tilde{\phi}, \tilde{A}_i$ and $\tilde{A}_0$. This leads to the equations

$$\begin{aligned} \hat{D}_i\hat{D}_i\tilde{\phi} - 2i\tilde{A}^i\hat{D}_i\hat{\phi} - i\hat{\phi}\partial^i\tilde{A}_i + \frac{1}{2}\tilde{\phi}(|\,\hat{\phi}\,|^2 - 1) + \frac{1}{2}\hat{\phi}[\hat{\phi}\tilde{\phi}^* + \hat{\phi}^*\tilde{\phi}] + \frac{1}{2}\tilde{\lambda}\hat{\phi}(|\,\hat{\phi}\,|^2 - 1) &= 0, \\ \partial^i\tilde{F}_{ij} + \tilde{A}_j\,|\,\hat{\phi}\,|^2 + \frac{i}{2}[\tilde{\phi}^*\hat{D}_j\hat{\phi} - \tilde{\phi}(\hat{D}_j\hat{\phi})^* + \hat{\phi}^*\hat{D}_j\tilde{\phi} - \hat{\phi}(\hat{D}_j\tilde{\phi})^*] &= 0, \\ \partial^i\partial_0\tilde{A}_i + \frac{i}{2}[\hat{\phi}^*\partial_0\tilde{\phi} - \hat{\phi}\partial_0\tilde{\phi}^*] &= 0, \end{aligned} \tag{9}$$

where $\hat{D}_i = \partial_i - i\hat{A}_i$ and $\tilde{F}_{ij} = \partial_i \tilde{A}_j - \partial_j \tilde{A}_i$.

The solution (3) to the equations (2) are analytic in $\tilde{\lambda}$ [12], and can be expanded $\phi = \hat{\phi} + \tilde{\lambda}\varphi, A_i = \hat{A}_i + \tilde{\lambda}a_i$. Hence, $(\tilde{\lambda}\varphi, \tilde{\lambda}a_i)$ is a solution of the inhomogeneous system of equations (9). The homogeneous system is the one which had to be solved in the case $\lambda = 1$. In this case, Ruback[4], extending work by Weinberg[13], found the solutions

$$\xi = 2(\alpha + i\beta)e^{i\theta}f'(r),$$
$$B_1 + iB_2 = \frac{2}{rf^2}(\beta - i\alpha)(rff'' - rf'^2 + ff'), \tag{10}$$

and

$$\xi = 2f(r)k(r),$$
$$B_1 + iB_2 = \frac{-2i}{r}e^{-i\theta}[rk'(r) + 2k], \tag{11}$$

and

$$\xi = 2if(r)k(r),$$
$$B_1 + iB_2 = \frac{2}{r}e^{-i\theta}[rk'(r) + 2k], \tag{12}$$

where k(r) satisfies the equation

$$\frac{-1}{r}(rk')' + (f^2 + \frac{4}{r^2})k = 0. \tag{13}$$

The solutions (10) lead to fields $(\hat{\phi} + t\xi, \hat{A}_i + tB_i)$ which are of the form

$$\hat{\phi}(\vec{x}) + t\xi(\vec{x}) = e^{i\chi}\hat{\phi}(\vec{x} + \vec{a}),$$
$$\hat{A}_i(\vec{x}) + tB_i(\vec{x}) = \hat{A}_i(\vec{x} + \vec{a}) + \partial_i\chi, \tag{14}$$

where

$$\chi = t(\frac{2f'}{f} - \frac{4}{r})(\beta\cos\theta - \alpha\sin\theta) \tag{15}$$

and $\vec{a} = 2t(\alpha, \beta)$, to first order in t. Hence, these solutions describe overall translations.

In the next section, we discuss (13) in detail and show that it has solutions with asymptotic behaviour $k \sim ce^{-r}, c \neq 0$, at infinity and $k \sim c_1 r^{-2} + c_2 r^2, c_1 \neq 0$, at the origin, and that all other non-zero solutions have asymptotic behaviour $k \sim c_3 e^{-r} + c_4 e^{r}, c_4 \neq 0$, at infinity. Without loss of generality we can set $c = 1$. That $c_1 = 0$ cannot hold can be seen as follows: Asymptotic behaviour e^{-r} at infinity and $c_2 r^2$ at the origin would imply the existence of a point $r = r_0$ for which $k(r_0) > 0, k'(r_0) = 0$ and $k''(r_0) \leq 0$. This is not consistent with (13). Therefore, k is strictly monotonic decreasing from infinity to zero, as r increases, and there exists a point $r = \rho > 0$ such that $k(\rho) = 1/(2 \mid t \mid)$. For the solution (11) we have

$$\mid \phi \mid^2 = f^2(1 + 4tk\cos 2\theta + 4t^2k^2) \geq f^2(1 - 2 \mid t \mid k)^2. \tag{16}$$

The zeros of the Higgs field, which give the locations where the magnetic field penetrates the superconductor, are therefore at $r = \rho$, $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ for $t > 0$, and at $r = \rho$, $\theta = 0$ and $\theta = \pi$ for $t < 0$. This solution describes 90^o scattering.

Futhermore, for the solution (11) the energy density reads

$$\begin{aligned}\mathcal{E}(r,\theta) &= \frac{1}{8}(f^2-1)^2 + \frac{4f^2(1-a)^2}{r} + 8(\frac{akft}{r})^2 + 16akt(\frac{f}{r})^2(a-1)\cos(2\theta) \\ &+ 2t^2(k'f + \frac{2kf}{r}(1-a))^2 + 2t^2f^2(k' + \frac{2k}{r})^2(1+4kt\cos(2\theta) + (2kt)^2) \\ &- \frac{8}{r}\cos(2\theta)tf^2(k' + \frac{2k}{r})[(a-1) + 2kat\cos(2\theta) - tk\cos(2\theta) + 2ak^2t^2] \\ &- 4t^2k'f^2(k' + \frac{2k}{r})\sin^2(2\theta) + f^2kt\cos(2\theta)(2\cos(2\theta)f^2kt + f^2 - 1) \\ &+ \frac{1}{8}(f^2 + 4ktf^2\cos(2\theta) + (2kft)^2 - 1)^2. \end{aligned} \tag{17}$$

By investigating the terms, one finds that the energy for a function k is finite if k has asymptotic behaviour $k \sim e^{-r}$ at infinity, or infinite if k has asymptotic behaviour $k \sim e^{r}$ at infinity. In Fig.1, the energy density (17) has been plotted for $t = \frac{-1}{2}, t = 0$ and $t = \frac{+1}{2}$, respectively. The plot shows how the scattering process proceeds: As the two vortices approach each other, their energy densities form a cylinder-like structure from which two vortices emerge at right angle for $t > 0$.

The arguments we gave for the solution (11) can easily be repeated to also show that the solution (12) describes a 90^o scattering process. We have therefore four linearly independent, gauge inequivalent approximate solutions whose superposition describes 90^o scattering plus translation. Since the parameter space for two static vortices is four- dimensional we do not expect more solutions for slowly-moving vortices. No special initial conditions are therefore required in an experiment. Head-on collision of slowly-moving vortices should always lead to 90^o scattering. An important point made in this section is that all the above arguments hold for $\lambda = 1 + \tilde{\lambda} > 1$, $\tilde{\lambda} \ll 1$.

Finally, we have to address the problem that the approximate solution for $t \in (-\epsilon, \epsilon)$ (which we used to discriminate against 0^o and 180^o scattering in favour of 90^o scattering) is not a scattering solution. However, we can take the configuration for $t = 0$ as initial data for a solution for $t \in (-\infty, \infty)$, which we know exists [14]. For $t \in (-\epsilon, \epsilon), \epsilon \ll 1$, the linearization which leads to eqs.(9) should be justified. Therefore, the solutions we discussed should be approximations for $t \in (-\epsilon, \epsilon)$ to the scattering solution for $t \in (-\infty, \infty)$, although we have not rigously proven this. In fact, we are not aware of any rigorous proof which establishes that any of the configurations discussed in the literature are approximate solutions.

3 Series Solutions

In this section, we solve the equations (6) and (13) near the origin and use the equations to find asymptotic expansiond near infinity. The technique used has been developed for the Euler-Lagrange equations (4) [15]. First, we study the equations for small r. The equation (6) can be rewritten as

$$\begin{aligned} f(r) &= \gamma_0 r^2 - 2r^2 \int_0^r \frac{a(s)f(s)}{s^3} ds, \\ a(r) &= \frac{1}{8}r^2 - \frac{1}{4}\int_0^r sf^2(s)ds. \end{aligned} \tag{18}$$

A Green's function for the linear equation $r(rk')' = 4k$ is

$$g(r,\rho) = H(r-\rho)(\frac{-r^2}{4\rho} - \frac{\rho^3}{4r^2})$$

which leads to the integral equation

$$k(r) = \frac{c_1}{r^2} + c_2 r^2 + \int_0^r (\frac{r^2}{4\rho} - \frac{\rho^3}{4r^2}) f^2(\rho) k(\rho) d\rho. \tag{19}$$

Given the behaviour of f and a at the origin [9], where $f \sim \gamma_0 r^2$ and $a \sim \frac{1}{8} r^2$, we can prove by induction that f, a and k are of the form

$$\begin{aligned} f(r) &= \sum_{n=1}^{\infty} f_n r^{2n}, \\ a(r) &= \sum_{n=1}^{\infty} a_n r^{2n}, \\ k(r) &= \sum_{n=-1}^{\infty} k_n r^{2n}, \qquad k_o = 0, \end{aligned} \tag{20}$$

This leads to the following recurrence relations for $n \geq 2$:

$$\begin{aligned} f_n &= \frac{1}{1-n} \sum_{n_1,n_2=1}^{\infty} a_{n_1} f_{n_2} \delta_{n,n_1+n_2}, \\ a_n &= \frac{-1}{8n} \sum_{n_1,n_2=1}^{\infty} f_{n_1} f_{n_2} \delta_{n,n_1+n_2+1}, \\ k_n &= \frac{1}{4(n^2-1)} \sum_{n_1,n_2,n_3=1}^{\infty} f_{n_1} f_{n_2} k_{n_3} \delta_{n,n_1+n_2+n_3+1}. \end{aligned} \tag{21}$$

To prove the convergence of the Taylor series we show by induction that

$$| f_n | \leq \frac{M^n}{(n+1)^2}, | a_n | \leq \frac{M^n}{(n+1)^2}, | k_n | \leq \frac{M^n}{(n+1)^2} \tag{22}$$

hold for sufficiently large n and $M \geq 1$. The estimate for f_n we need for the proof, is of the form

$$\begin{aligned} | \sum_{n_1,n_2=1} a_{n_1} f_{n_2} \delta_{n,n_1+n_2} | &\leq M^n \sum_{n_1=1}^{n-1} \frac{1}{(n_1+1)^2} \frac{1}{(n-n_1+1)^2} \\ &\leq M^n \int_{\frac{1}{2}}^{n-\frac{1}{2}} \frac{1}{(1+x)^2(n-x+1)^2} dx \\ &= \frac{4M^n}{(n+2)^2} [\frac{1}{3} - \frac{1}{2n+1} + \frac{1}{n+2} \log \frac{2n+1}{3}] \\ &\leq \frac{M^n}{(n+2)^2} O(1). \end{aligned} \tag{23}$$

Using similar estimates for a_n and twice for k_n we can complete the induction proof for the inequalities (22). These inequalities imply that the series (20) converge and solve (6) and (13) for $r < 1/\sqrt{M}$. Analyticity of the solutions discussed in section 2 then implies that they are included in the set of series solutions (20).

For large r, f and a have the following asymptotic behaviour [9]:

$$\begin{aligned} f &= 1 + f_1(r)e^{-r} + F_2(r), \\ a &= 1 + a_1(r)e^{-r} + A_2(r), \end{aligned} \tag{24}$$

where $f_1(r)$ and $a_1(r)$ are polynomially bounded, and $F_2(r)$ and $A_2(r)$ approach zero faster than $r^m e^{-r}$ for any power of m. This implies that, to leading order, k satisfies

$$r^2 k'' + rk' - (r^2 + 4)k = 0. \tag{25}$$

To this order,

$$k(r) = c_1 H_2^{(1)}(ir) + c_2 H_2^{(2)}(ir), \tag{26}$$

where $H_\nu^{(i)}$ are Hankel functions. Their asymptotic behaviour [16] and the finite energy condition require $c_2 = 0$. Therefore,

$$k(r) = k_1(r)e^{-r} + K_2(r), \tag{27}$$

where $k_1(r)$ is polynomially bounded and $K_2(r)$ approachs zero faster than $r^m e^{-r}$ for any power of m.

We now prove by induction that

$$\begin{aligned} f(r) &= \sum_{n=0}^{\infty} f_n(r)e^{-nr} =: \sum_{n=0}^{\infty} F_n(r), \\ a(r) &= \sum_{n=0}^{\infty} a_n(r)e^{-nr} =: \sum_{n=0}^{\infty} r\tilde{A}_n(r) =: \sum_{n=0}^{\infty} r\tilde{a}_n(r)e^{-nr}, \\ k(r) &= \sum_{n=0}^{\infty} k_n(r)e^{-nr} =: \sum_{n=0}^{\infty} K_n(r), \end{aligned} \tag{28}$$

where f_n, a_n and k_n are polynomially bounded. Equations (6) imply for $f = 1 + F$ and $a = 1 + r\tilde{A}$

$$F'' + \frac{1}{r}F' - F = \frac{3}{2}F^2 + 4\tilde{A}^2 + \frac{1}{2}F^3 + 4\tilde{A}^2 F \tag{29}$$

and

$$\begin{aligned} F_n'' + \frac{1}{r}F_n' - F_n &= [\frac{3}{2} \sum_{n_1,n_2=1}^{\infty} f_{n_1} f_{n_2} \delta_{n,n_1+n_2} \\ &+ 4 \sum_{n_1,n_2=1}^{\infty} \tilde{a}_{n_1}\tilde{a}_{n_2}\delta_{n,n_1+n_2} + \frac{1}{2} \sum_{n_1,n_2,n_3=1}^{\infty} f_{n_1} f_{n_2} f_{n_3} \delta_{n,n_1+n_2+n_3} \\ &+ 4 \sum_{n_1,n_2,n_3=1}^{\infty} \tilde{a}_{n_1}\tilde{a}_{n_2} f_{n_3}\delta_{n,n_1+n_2+n_3}]e^{-nr} \\ &=: \alpha_n(r)e^{-nr}. \end{aligned} \tag{30}$$

Therefore

$$F_n(r) = \frac{i\pi}{4}\int_r^\infty \rho[H_o^{(2)}(i\rho)H_o^{(1)}(ir) - H_o^{(1)}(i\rho)H_o^{(2)}(ir)]\alpha_n(\rho)e^{-n\rho}d\rho. \tag{31}$$

Substituting for $\alpha_n(\rho)$ and calculating the integral leads to terms of the form $f_n(r)e^{-nr}$, where f_n is polynomially bounded. Solutions of the homogeneous equation have the wrong asymptotic behaviour and cannot be added to (31).

To prove that a and k are of the form (28) we use

$$\tilde{A}_n(r) = -\frac{1}{r}\int_r^\infty \rho\beta_n(\rho)e^{-n\rho}d\rho, \tag{32}$$

where

$$\beta_n(r) = -\frac{r}{2}f_n - \frac{1}{4}\sum_{n_1,n_2=1}^{\infty} f_{n_1}f_{n_2}\delta_{n,n_1+n_2}. \tag{33}$$

One also finds

$$K_n(r) = \frac{i\pi}{4}\int_r^\infty \rho[H_2^{(2)}(i\rho)H_2^{(1)}(ir) - H_2^{(1)}(i\rho)H_2^{(2)}(ir)]\gamma_n(\rho)e^{-n\rho}d\rho, \tag{34}$$

where

$$\gamma_n(r) = 2\sum_{n_1,n_2=1}^{\infty} f_{n_1}k_{n_2}\delta_{n,n_1+n_2} + \sum_{n_1,n_2,n_3=1}^{\infty} f_{n_1}f_{n_2}k_{n_3}\delta_{n,n_1+n_2+n_3}. \tag{35}$$

To prove convergence of the series, we assume there exist numbers M and R such that

$$\begin{aligned} sup_{r>R}|rf_n(r)e^{-\frac{nr}{2}}| &< \frac{M^n}{(n+1)^2}, \\ sup_{r>R}|r\tilde{a}_n(r)e^{-\frac{nr}{2}}| &< \frac{M^n}{(n+1)^2}, \\ sup_{r>R}|rk_n(r)e^{-\frac{nr}{2}}| &< \frac{M^n}{(n+1)^2}, \end{aligned} \tag{36}$$

for large enough n. Taking R large enough, we can bound $|H_\nu^{(1,2)}(ir)|$ by $e^{\pm r}$ and derive the estimate

$$sup_{r>R}|rf_N(r)e^{-Nr/2}| < \frac{N\pi}{N^2-1}sup_{r>R}|r^2\alpha_N(r)e^{-Nr/2}|. \tag{37}$$

The induction hypothesis leads to

$$\begin{aligned} sup_{r>R}|rf_N(r)e^{-Nr/2}| < \frac{NM^N\pi}{N^2-1}\Big[\frac{11}{2}\sum_{n_1=1}^{N-1}\frac{1}{(n_1+1)^2}\frac{1}{(N-n_1+1)^2} \\ + \frac{9}{2}\sum_{n_1=1}^{n-2}\frac{1}{(n_1+1)^2}\sum_{n_2=1}^{N-n_1-2}\frac{1}{(n_2+1)^2}\frac{1}{N-n_1-n_2+1)^2}\Big]. \end{aligned} \tag{38}$$

The inequality in (23) then completes the induction proof for $f_n(r)$. Similar arguments for $\tilde{a}_n(r)$ and $k_n(r)$ establish all inequalities (36). Thus, the series (28) converge for $r > R$ and $r > 2\log M$.

4 References

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Figure Caption

Figure 1: The energy density (17) for $t = \frac{-1}{2}, t = 0$ and $t = \frac{+1}{2}$.

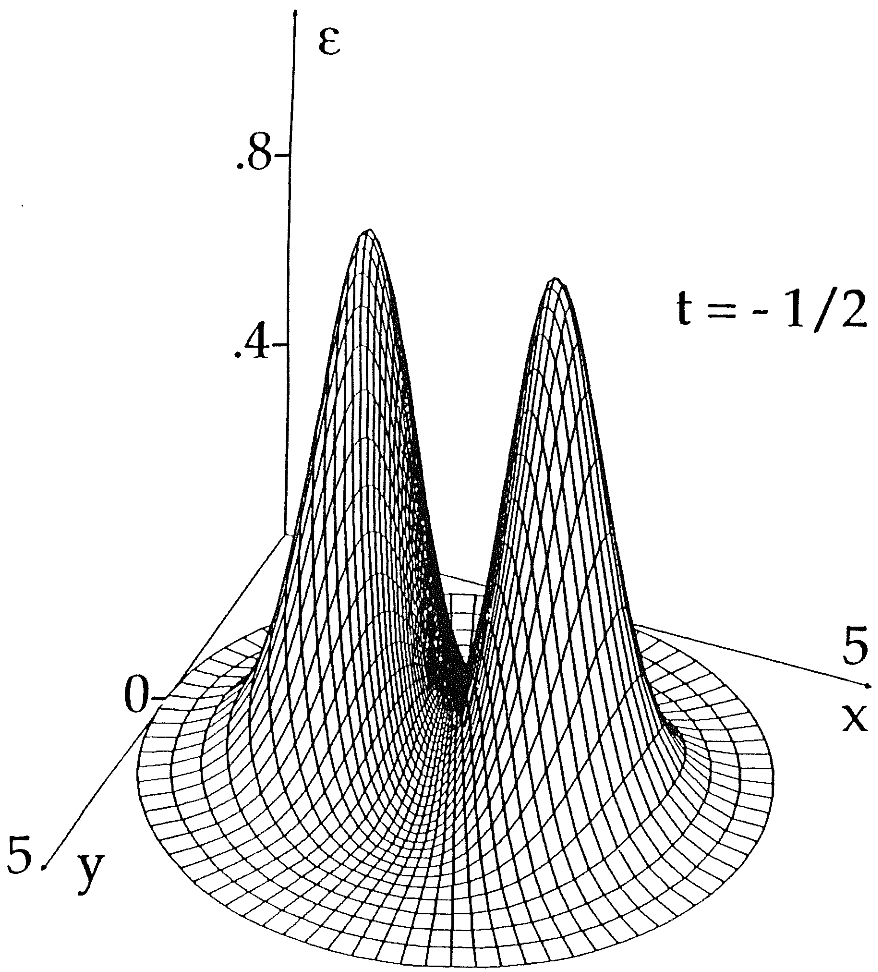

Figure 1 (i)

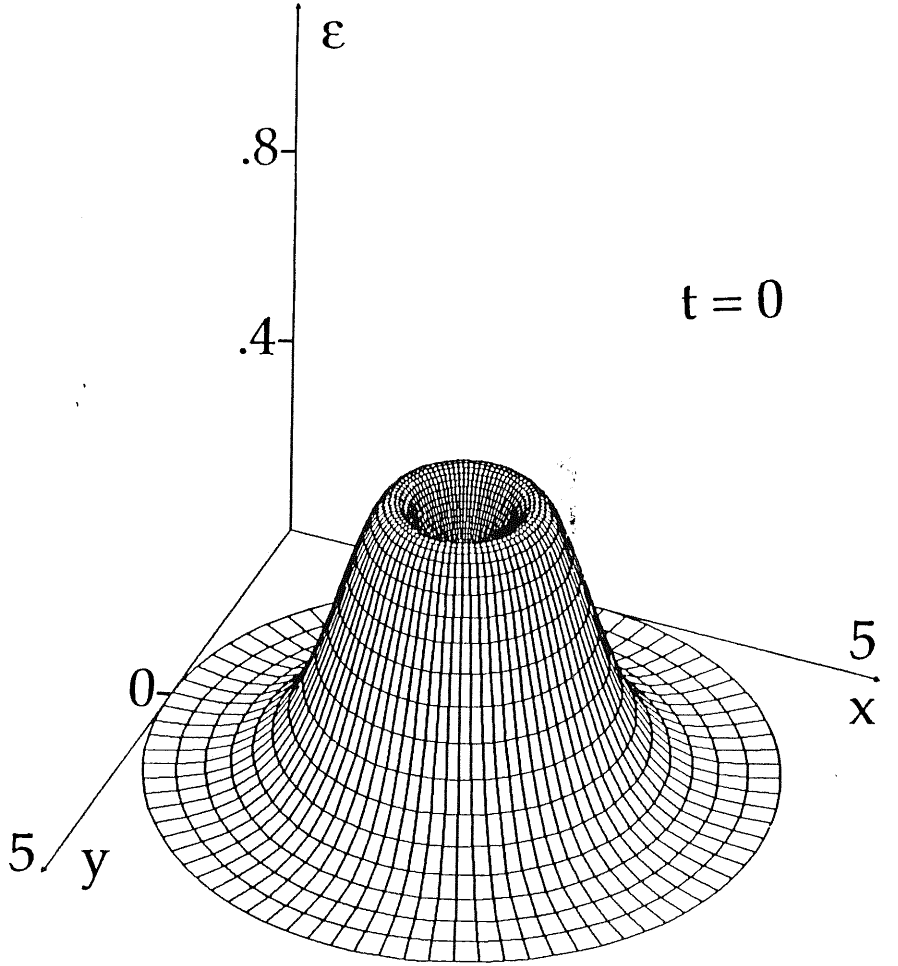

Figure 1 (ii)

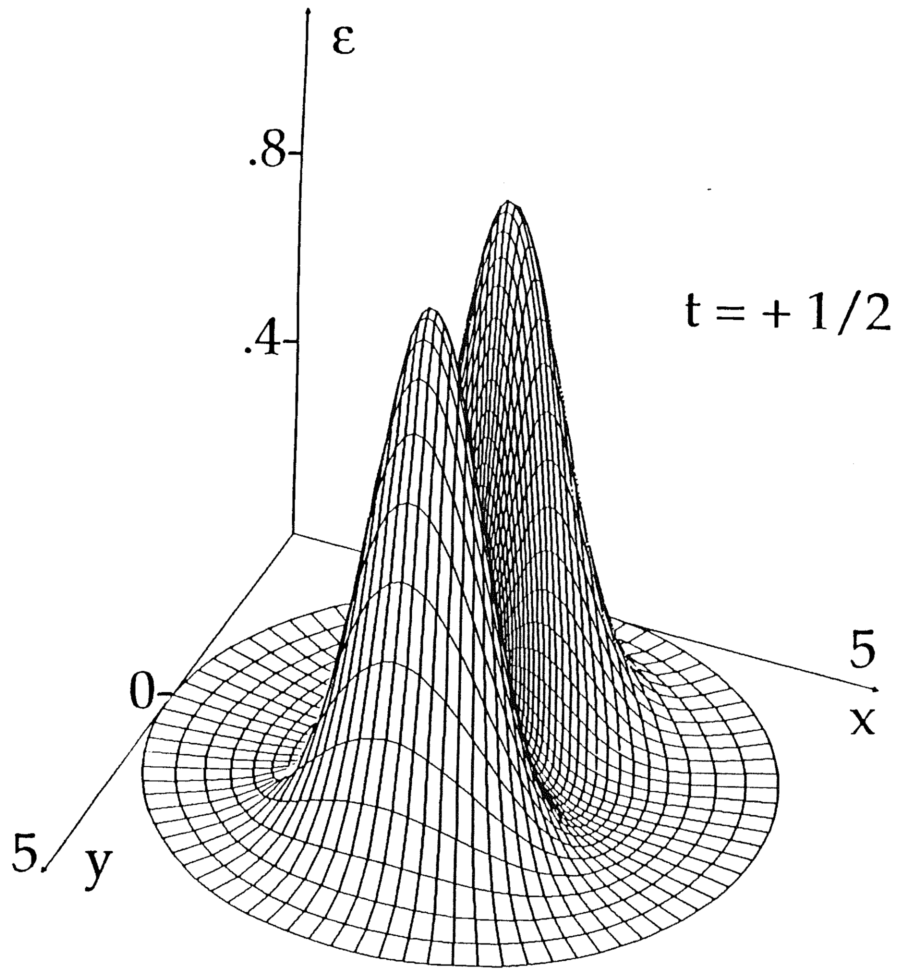

Figure 1 (iii)