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$\mathcal{W}$-ALGEBRAS OF GENERALIZED TODA THEORIES*

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The Hamiltonian reduction of Wess-Zumino-Novikov-Witten (WZNW) theories to conformally invariant Toda theories is reviewed. The relationship between the WZNW and the Lax pair approaches to Toda theories is clarified. Extended conformal algebras associated to arbitrary embeddings of $sl(2)$ into the simple Lie algebras, and generalized Toda theories possessing these $\mathcal{W}$-algebras as their canonical symmetries are considered by using the WZNW framework.

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1. Introduction

Conformally invariant and affine Toda type systems are among the most studied models of the theory of integrable non-linear equations [1-7] and they are also important in two-dimensional conformal field theory [8-17]. The key feature of the conformal Toda theories is that they possess [3, 4, 10, 11, 13-17] interesting non-linear symmetry algebras. These symmetry algebras are polynomial extensions of the Virasoro algebra by chiral conformal primary fields. Extended conformal algebras of this kind are called $\mathcal{W}$-algebras, currently they are the subject of intense studies [18-23].

Two-dimensional Toda systems have been investigated earlier mainly by using the formalism of the Lax pair. In some recent papers [12-15], see also [16, 26], we proposed an alternative framework for describing the conformally invariant Toda systems. This approach is based on the observation that Toda systems can be viewed as Hamiltonian reductions of the Wess-Zumino-Novikov-Witten (WZNW) theory, which is essentially a free theory, whose main feature is that it provides a natural, canonical realization of affine Kac-Moody (KM) symmetries [24, 25]. The most important, qualitative conclusion of the investigations in [12-15, 26] is that the WZNW setting of the conformal Toda theories amounts to their linearization, which at the same time resolves their apparent singularities and makes their non-linear $\mathcal{W}$-symmetries manifest. In our opinion their embedding into the WZNW theory provides the natural global setting of the Toda theories.

To make contact with the Toda theories, we consider the WZNW theory

$$S_{\mathrm{WZ}}(g) = \frac{\kappa}{2}\int d^2x\,\eta^{\mu\nu}\,\mathrm{Tr}\,(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g) - \frac{\kappa}{3}\int_{B_3}\mathrm{Tr}\,(g^{-1}dg)^3\ , \tag{1.1}$$

for a simple, maximally non-compact, connected real Lie group G. In other words, we assume that the simple Lie algebra $\mathcal{G}$ of G allows for a Cartan decomposition over the field of real numbers. The field equation of the WZNW theory can be written in the equivalent forms*

$$\partial_- J = 0 \qquad \text{or} \qquad \partial_+ \tilde{J} = 0\ , \tag{1.2}$$

where

$$J = \partial_+ g\cdot g^{-1}\ , \quad \text{and} \quad \tilde{J} = -g^{-1}\partial_- g\ . \tag{1.3}$$

* We use the notation $\kappa = -\frac{k}{4\pi}$, where k is the KM level, and $\partial_\pm = \frac{\partial}{\partial x^\pm}$ with $x^\pm = \frac{1}{2}(x^0 \pm x^1)$. For notational simplicity we set $\kappa = 1$ from now on.

These equations express the conservation of the left- and right KM currents, J and $\tilde{J}$, respectively. The general solution of the WZNW field equation is given by the simple formula

$$g(x^+, x^-) = g_L(x^+) \cdot g_R(x^-) \ , \tag{1.4}$$

where g_L and g_R are arbitrary G-valued functions.

In their pioneering work [1, 3], Leznov and Saveliev proved the exact integrability of the conformal Toda systems by exhibiting chiral quantities by using the field equation and the special graded structure of the Lax pair $\mathcal{A}_\pm$, in terms of which the Toda equation takes the zero curvature form

$$[\partial_+ - \mathcal{A}_+ \, , \, \partial_- - \mathcal{A}_-] = 0 \ . \tag{1.5}$$

In our framework the exact integrability of Toda systems is seen as an immediate consequence of the obvious integrability of the WZNW theory, which survives the reduction to Toda theory. In our approach the chiral fields underlying the integrability of the Toda equation are available from the very beginning. Of course, they come from the fields entering the left $\times$ right decomposition of the general WZNW solution (1.4). Furthermore, the Toda Lax pair itself emerges naturally from the trivial, chiral Lax 'pair' of the WZNW theory. To this one first observes that the WZNW field equation is a zero curvature condition, since one can write for example the first equation in (1.2) as

$$[\partial_+ - J \, , \, \partial_- - 0] = 0 \ . \tag{1.6}$$

Using the constraints of the reduction, the Toda zero curvature condition (1.5) of [1, 3] arises from (1.6) by conjugating this equation by a certain field, defined in terms of a generalized Gauss decomposition of the WZNW field g [15].

The WZNW theory provides the most 'economical' realization of the KM symmetry, in the sense that the WZNW phase space is essentially (up to constraints coming from the boundary condition one imposes on $g(x^0, x^1)$) a direct product of the left $\times$ right KM phase spaces. The WZNW $\longrightarrow$ Toda symplectic reduction reduces the chiral KM phase spaces to phase spaces carrying the chiral $\mathcal{W}$-algebras as their Poisson bracket structure. Thus the $\mathcal{W}$-algebra is related to the pase space of the Toda theory in the same way as the KM algebra is related to the phase space of the WZNW theory. Before describing the generalized Toda theories, first we shall elaborate on the structure of the relevant $\mathcal{W}$-algebras in some detail.

2. $\mathcal{W}$-algebras for arbitrary $sl(2)$ embeddings

Let us consider a real, split (maximally non-compact), simple Lie algebra $\mathcal{G}$ together with an $sl(2)$ subalgebra $\mathcal{S}$ spanned by standard generators $M_\pm$, M_0 subject to

$$[M_0\,,\,M_\pm]=\pm M_\pm\ ,\qquad\qquad [M_+\,,\,M_-]=2M_0\ . \tag{2.1}$$

We are interested in a certain reduction of the KM phase space associated to $\mathcal{G}$, which carries the standard Poisson bracket

$$\{\langle u, J(x)\rangle\,,\,\langle v\,,\,J(y)\rangle\}=\langle [u,v]\,,\,J(x)\rangle\delta(x-y)+\kappa\langle u\,,\,v\rangle\delta'(x-y)\ , \tag{2.2}$$

where $\langle\ ,\ \rangle$ is the Cartan-Killing form. The $sl(2)$ subalgebra $\mathcal{S}$ defines a grading of $\mathcal{G}$, that is we have

$$\mathcal{G}=\mathcal{G}_+ +\mathcal{G}_0+\mathcal{G}_-\ ,\qquad\qquad \mathcal{G}_\pm=\sum_{m=\frac{1}{2}}^{N}\mathcal{G}_{\pm m}\ , \tag{2.3}$$

where $\mathcal{G}_0$ and $\mathcal{G}_{\pm m}$ are eigenspaces of ad_{M_0} with eigenvalues 0 and $\pm m$, respectively. The reduced phase space we consider is identified with the set of KM currents of the following special form:

$$J^{\rm red}(x)=M_- + j^{\rm red}(x)\ ,\qquad j^{\rm red}(x)\in {\rm Ker}(M_+)\ . \tag{2.4}$$

In other words, $j^{\rm red}(x)$ is restricted to be an arbitrary linear combination of the highest weight vectors of $\mathcal{S}$ in the adjoint of $\mathcal{G}$. We denote this set of currents as $M_{\mathcal{S}}$. The property of $M_{\mathcal{S}}$ which makes it interesting to consider is that its induced Poisson bracket structure can be regarded as a classical $\mathcal{W}$-algebra. To be more precise, let us introduce an angular momentum basis $T^i_{l,m}$ in $\mathcal{G}$, such that

$$[M_0\,,\,T^i_{l,m}]=m\,T^i_{l,m}\ , \tag{2.5}$$

with i running from 1 to the multiplicity of the spin l representation, for any l occuring in the decomposition of the adjoint of $\mathcal{G}$ under $\mathcal{S}$. By convention, here we take $T^1_{1,\pm1}\equiv M_\pm$. By using this basis, we can write

$$J(x)=\sum_{l,m,i}U^{l,m}_i(x)T^i_{l,m} \tag{2.6}$$

and

$$J^{\rm red}(x)=M_- +\sum_{l,i}U^{l,l}_i(x)T^i_{l,l}\ . \tag{2.7}$$

The induced Poisson bracket algebra carried by $M_{\mathcal{S}}$, which can be specified by second class constraints, is given by the Dirac bracket algebra $\{\ ,\ \}^*$ of the components $U_i^{l,l}(x)$. The crucial properties of this Dirac bracket are the following:

(i) The Dirac brackets of the $U_i^{l,l}(x)$ close on polynomials in their derivatives and δ-distributions.

(ii) The quantity $L_{\rm red}(x) = \frac{1}{2}\langle J^{\rm red}(x)\,,\, J^{\rm red}(x)\rangle$ generates a Virasoro subalgebra, and, with the exception of the M_+-component $U_1^{1,1}$, the components $U_i^{l,l}$ are conformal primary fields of conformal weight $(1+l)$ with respect to this Virasoro subalgebra. (It is worth noting that in general $L_{\rm red}(x)$ is a linear combination of the M_+-component $U_1^{1,1}(x)$ and a quadratic expression in the singlet components $U_i^{0,0}(x)$, it reduces to $U_1^{1,1}(x)$ when there are no singlets of $\mathcal{S}$ in the adjoint of $\mathcal{G}$.)

This means that the Dirac bracket algebra of all the differential polynomials of the components of $J^{\rm red}(x)$ indeed qualifies as a classical $\mathcal{W}$-algebra, since it is a polynomial extension of the Virasoro algebra by conformal primary fields. We denote this $\mathcal{W}$-algebra as $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$. Below we explain how to establish properties (i) and (ii) of the chiral algebras $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$.

There are in principle two ways of investigating the reduced phase space $(M_{\mathcal{S}}\ ,\{\ ,\ \}^*)$. The first, direct approach [26] is based on looking at $M_{\mathcal{S}}$ as the submanifold in the KM phase space given by the following equations

$$U_1^{1,-1}(x) - 1 = 0\ , \quad \text{and} \quad U_i^{l,m}(x) = 0 \quad \forall\, l,\, i,\, m \neq l\ . \tag{2.8}$$

In this approach one investigates the standard explicit formula of the Dirac bracket $\{\ ,\ \}^*$, which contains the inverse $C^{-1}_{a,b}(z,w)$ of

$$C^{a,b}(x,y) \equiv \{U^a(x)\,,\, U^b(y)\}\ , \tag{2.9}$$

where $U^a(x)$ denotes the collection of the constraints given by (2.8). By making use of the grading structure, it is possible to show that $C^{a,b}(x,y)$ is indeed invertibe on $M_{\mathcal{S}}$ and that the inverse is a differential polynomial in the surviving components $U_i^{l,l}(x)$ and in $\delta(x-y)$. This then implies property (i) of the Dirac bracket, and (ii) can also be checked by inspection.

The second, indirect approach [13, 16, 14-15, 26] is based on realizing $M_{\mathcal{S}}$ as the space of orbits obtained in reducing the KM phase space by a convenient system of linear, conformally invariant *first class* constraints. This Hamiltonian reduction of the KM phase space generalizes the one used, for example, by Drinfeld and Sokolov in their study of generalized KdV hierarchies [5]. Before turning to the general situation, now we recall this approach in the simple case of an *integrally embedded* $sl(2)$ subalgebra, which does not require practically any modification of the standard construction [5, 13].

The first step in this reduction procedure is to impose a first class system of constraints by requiring the constrained current, J^c, to satisfy

$$\pi_{\leq -1}(J^c(x)) = M_- \ , \tag{2.10}$$

where $\pi_{\leq -1}$ is the projection operating according to the decomposition $\mathcal{G} = \mathcal{G}_{\leq -1} + \mathcal{G}_0 + \mathcal{G}_{\geq 1}$, provided by the *integral* spectrum of $\mathcal{S}$. It follows that the constraint surface, consisting of currents of the form

$$J^c(x) = M_- + j(x) \ , \qquad j(x) \in \mathcal{G}_{\geq 0} \ , \tag{2.11}$$

is invariant under the gauge transformations

$$J^c(x) \longrightarrow A(x) \cdot J^c(x) \cdot A^{-1}(x) + A'(x) \cdot A^{-1}(x) \tag{2.12a}$$

for any

$$A(x) = \exp[a(x)] \ , \qquad a(x) \in \mathcal{G}_{\geq 1} \ . \tag{2.12b}$$

Of course, these gauge transformations are the canonical transformations generated by the constraints themselves. One can then associate to these constraints the classical $\mathcal{W}$-algebra given by the KM Poisson bracket algebra of all the differential polynomials of the constrained current invariant under (2.12). The construction automatically yields the canonical Virasoro subalgebra given by the gauge invariant polynomial

$$L_{\mathcal{S}}(x) \equiv L_{\mathrm{Sugawara}}(x) - \langle M_0 \ , J'(x) \rangle \ . \tag{2.13}$$

A family of very convenient complete gauge fixings is provided by the so called Drinfeld-Sokolov (DS) gauges [5, 13], which are defined by further restricting the current to be of the form

$$J^{\mathrm{DS}}(x) = M_- + j^{\mathrm{DS}}(x) \ , \qquad j^{\mathrm{DS}}(x) \in V_{\mathrm{DS}} \ , \tag{2.14}$$

where V_{DS} is any graded vector space specifying a direct sum decomposition of the type

$$\mathcal{G}_{\geq 0} = [M_- \, , \mathcal{G}_{\geq 1}] + V_{\mathrm{DS}} \ . \tag{2.15}$$

The crucial point is that every DS gauge defines a basis of the $\mathcal{W}$-algebra in a natural way, namely by requiring the base elements to reduce to the independent current components in the DS gauge. This implies that the $\mathcal{W}$-algebra spanned by the gauge invariant differential polynomials is isomorphic to the Dirac bracket algebra of the current components surviving the DS gauge fixing. Taking the highest weight DS gauge, that is the special case

$$V_{\rm DS} \equiv {\rm Ker}(M_+) \tag{2.16}$$

in (2.15), one identifies the $\mathcal{W}$-algebra as $\mathcal{W}^{\mathcal{G}}_S$. Plainly, this construction immediately yields property (i) of the Dirac bracket algebra. To establish property (ii) one observes that $L_S(x)$ reduces to $L_{\rm red}(x)$ in the highest weight gauge and investigates the formula of the induced conformal transformations:

$$\delta^*_Q\, J^{\rm red}(x) \equiv -\{Q\,,\, J^{\rm red}(x)\}^*\ , \qquad Q = \int dx\, a(x) L_S(x)\ , \tag{2.17}$$

for any test function $a(x)$. By using the method of [13, 15], it is easy to verify that

$$\delta^*_Q\, J^{\rm red}(x) = \partial_x K(x) + [K(x)\,,\, J^{\rm red}(x)]\ , \tag{2.18a}$$

with

$$K(x) = a(x)J^{\rm red}(x) + M_0\partial_x a(x) - \frac{1}{2}M_+\partial_x^2 a(x)\ , \tag{2.18b}$$

which implies property (ii) of $\mathcal{W}^{\mathcal{G}}_S$.

Now we turn to the non-trivial case of a *half-integral* $sl(2)$ subalgebra for which we have

$$\mathcal{G} = \mathcal{G}_{\leq -1} + \mathcal{G}_{-\frac{1}{2}} + \mathcal{G}_0 + \mathcal{G}_{\frac{1}{2}} + \mathcal{G}_{\geq 1}\ . \tag{2.19}$$

We would like to impose first class constraints completing (2.10) in such a way that the reduced phase space consists of currents of the form (2.4). A simple counting argument tells us that for this the candidate for the 'gauge algebra', $\Gamma \subset \mathcal{G}$, which was $\mathcal{G}_{\geq 1}$ in the previous case, has to satisfy the equality

$$\dim({\rm Ker}\, M_+) = \dim(\mathcal{G}) - 2\dim(\Gamma)\ . \tag{2.20}$$

From this, by taking into account the $sl(2)$ property $\dim({\rm Ker}\, M_+) = \dim(\mathcal{G}_0) + \dim(\mathcal{G}_{\frac{1}{2}})$, we obtain

$$\dim(\Gamma) = \dim(\mathcal{G}_{\geq 1}) + \frac{1}{2}\dim(\mathcal{G}_{\frac{1}{2}})\ . \tag{2.21}$$

Furthermore, it is clear that if we require the complete system of first class constraints to contain the ones given by (2.10) then we must have

$$\Gamma = \mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1}\ , \tag{2.22}$$

where $\mathcal{P}_{\frac{1}{2}}$ is some subspace of $\mathcal{G}_{\frac{1}{2}}$ of dimension $\frac{1}{2}\dim(\mathcal{G}_{\frac{1}{2}})$. Our candidate for the complete system of first class constraints is then given as

$$\pi_{\leq -1}(J^c(x)) = M_- \quad \text{and} \quad \langle u \,,\, J^c(x)\rangle = 0 \quad \forall\, u \in \mathcal{P}_{\frac{1}{2}} \;. \tag{2.23}$$

Introducing the orthogonal subspace $\mathcal{P}_{\frac{1}{2}}^{\perp} \subset \mathcal{G}_{-\frac{1}{2}}$ by the definition

$$\mathcal{P}_{\frac{1}{2}}^{\perp} = \{ v \in \mathcal{G}_{-\frac{1}{2}} \,|\, \langle v \,,\, u\rangle = 0 \quad \forall\, u \in \mathcal{P}_{\frac{1}{2}} \} \;, \tag{2.24}$$

we can write the constrained current $J^c(x)$ as

$$J^c(x) = M_- + j(x) \;, \qquad j(x) \in (\mathcal{P}_{\frac{1}{2}}^{\perp} + \mathcal{G}_{\geq 0}) \;. \tag{2.25}$$

Our problem now is to find a subspace $\mathcal{P}_{\frac{1}{2}} \subset \mathcal{G}_{\frac{1}{2}}$, with the right dimension, such that the system of constraints (2.23) is indeed first class. To this it turns out to be useful to consider the Kostant-Kirillov 2-form associated to the constant matrix M_- entering the constraints. This 2-form ω is defined on $\mathcal{G}$ by the formula

$$\omega(u \,,\, v) \equiv \langle M_- \,,\, [u \,,\, v]\rangle \;. \tag{2.26}$$

This 2-form is an useful tool in our context because, as follows from (2.2), the first class nature of the constraints (2.23) requires its vanishing on the gauge algebra Γ. In our case, for grading reasons, this is automatic apart from the non-trivial condition on the $\mathcal{G}_{\frac{1}{2}}$ part of Γ:

$$\omega_{\frac{1}{2}}(u \,,\, v) = 0 \;, \qquad \forall\, u,\, v \,\in \mathcal{P}_{\frac{1}{2}} \;, \tag{2.27}$$

where $\omega_{\frac{1}{2}}$ is the restriction of ω to $\mathcal{G}_{\frac{1}{2}}$. In fact, equation (2.27) guarantees the first class nature of the constraints (2.23). To find subspaces qualifying as $\mathcal{P}_{\frac{1}{2}}$, we first point out that $\omega_{\frac{1}{2}}$ is a symplectic, i. e. non-degenerate, 2-form on $\mathcal{G}_{\frac{1}{2}}$. Indeed, as follows from the invariance of the Cartan-Killing form and from the fact that $[M_- \,,\, \mathcal{G}_{\frac{1}{2}}] = \mathcal{G}_{-\frac{1}{2}}$, $\omega_{\frac{1}{2}}(u \,,\, \mathcal{G}_{\frac{1}{2}}) = 0$ is equivalent to $\langle u \,,\, \mathcal{G}_{-\frac{1}{2}}\rangle = 0$, which implies the vanishing of $u \in \mathcal{G}_{\frac{1}{2}}$. Note that $\omega_{\frac{1}{2}}$ being symplectic implies in particular that the dimension of $\mathcal{G}_{\frac{1}{2}}$ is even, which was implicitly assumed in the above. As for any symplectic form, there exists a (non-unique) decomposition of the space $\mathcal{G}_{\frac{1}{2}}$ which brings $\omega_{\frac{1}{2}}$ to standard form. This means that we can find subspaces $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$ of dimension $\frac{1}{2}\dim(\mathcal{G}_{\frac{1}{2}})$ such that

$$\mathcal{G}_{\frac{1}{2}} = \mathcal{P}_{\frac{1}{2}} + \mathcal{Q}_{\frac{1}{2}} \;, \tag{2.28}$$

and $\omega_{\frac{1}{2}}$ vanishes on these subspaces separately. (The subspaces $\mathcal{Q}_{\frac{1}{2}}$, and $\mathcal{P}_{\frac{1}{2}}$ are the analogues of the usual 'coordinate' and 'momentum' subspaces of analytic mechanics.) This way we proved the existence of a subspace $\mathcal{P}_{\frac{1}{2}}$ of the right dimension for which (2.23) is indeed a first class system of constraints.

The choice of the constraint surface (2.23) is not unique, because of the non-uniqueness in choosing $\mathcal{P}_{\frac{1}{2}}$, but the current can always be brought to the highest weight gauge (2.4) by a gauge transformation of the form (2.12a), where now

$$A(x) = \exp[a(x)] \ , \qquad \text{with} \qquad a(x) \in (\mathcal{P}_{\frac{1}{2}} + \mathcal{G}_{\geq 1}) \ . \tag{2.29}$$

(It should be noted that the gauge transformation bringing J^c to the highest weight gauge is a unique differential polynomial in J^c.) This means that the ambiguity in choosing $\mathcal{P}_{\frac{1}{2}}$ is immaterial as far as the reduced phase space is concerned, the Poisson bracket algebra of the gauge invariant differential polynomials yields $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ in every case.

In the above we proposed a 'halving procedure' for constructing first class constraints allowing for realizing $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ as an algebra of gauge invariant differential polynomials. The algebra $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ has been considered recently in [16] in the special case of $\mathcal{G} = sl(n)$. However, in [16] $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ was constructed by considering a mixed system of constraints imposed on the KM phase space, which was obtained by complementing (2.10) with *the second class* constraints

$$\langle u \, , \, J^c(x) \rangle = 0 \ , \qquad \text{for} \quad \forall \ u \in \mathcal{G}_{\frac{1}{2}} \ . \tag{2.30}$$

Eventually, this system of constraints also results in the $\mathcal{W}$-algebra $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$, but the corresponding reduction of the WZNW theory to a generalized Toda theory is much easier to find by using first class constraints only. Actually in [16] the authors were also led to replacing (2.30) by a first class system of constraints, in order to be able to consider the BRST quantization of the theory. They in fact constructed a first class system of constraints by introducing unphysical 'auxiliary fields' and thus resolved (2.30) in an extended phase space. In that construction one has to check that the auxiliary fields finally disappear from the physical quantities. One of the advantages of the 'halving procedure' proposed here is that it renders the use of any such auxiliary fields completely unnecessary, since one can start by imposing a complete system of first class constraints on the KM phase space from the very beginning.

In the above we outlined the KM $\longrightarrow \mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ reduction by using the language of first class constraints. For completeness, we now briefly explain how this construction is interpreted in terms of the standard terminology of symplectic reduction [27]. To this we consider the subgroup G_Γ of G obtained by exponentiating $\Gamma = (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}})$ associated to the $sl(2)$ subalgebra $\mathcal{S}$. The corresponding loop-group $\hat{G}_\Gamma$ acts on the KM phase space, that is on the dual of the affine KM Lie algebra, in the usual way:

$$(\, J(x)\,,\, k\,) \longrightarrow (\, A(x)\cdot J(x)\cdot A^{-1}(x) + \kappa A'(x)\cdot A^{-1}(x)\,,\, k\,)\,, \quad \forall\, A(x)\in \hat{G}_\Gamma\,. \tag{2.31}$$

This action preserves the Lie-Poisson bracket (2.2), and it is in fact a Hamiltonian action possessing a moment map. To describe the moment map, first we note the obvious fact that the dual-space $\hat{\Gamma}^*$ of the loop-algebra $\hat{\Gamma}$ can be identified with the space of the Γ^* valued loops. Furthermore, the space Γ^*, which is the dual of Γ, can be identified with an arbitrarily chosen complement of $\Gamma^\perp$ in $\mathcal{G}$. In fact, for any direct sum decomposition of the form

$$\mathcal{G} = \Gamma^\perp + \Gamma^*\,, \tag{2.32}$$

the Cartan-Killing form provides a non-degenerate pairing between Γ and Γ^*. By using the above identifications, one finds that the moment map Φ generating the Hamiltonian action (2.31) is given by the formula

$$\Phi\,:\,(\, J(x)\,,\, k\,) \longrightarrow \pi_{\Gamma^*}(J(x)) \in \hat{\Gamma}^*\,, \tag{2.33}$$

where the projection operator π_{Γ^*} is defined by the decomposition (2.32). The Lie algebra element M_- defines a linear functional on Γ by means of the Cartan-Killing form and thus it determines a unique element M_-^* of $\Gamma^* \subset \hat{\Gamma}^*$. Our constraints (2.23) exactly corresponds to restricting the current to lie on the level surface $C_{M_-^*}$ of the moment map specified as follows:

$$C_{M_-^*} \equiv \{\,(\, J(x)\,,\, k\,)\mid \Phi((J(x)\,,\, k)) = M_-^*\,\}\,. \tag{2.34}$$

It is easy to see that the isotropy group of M_-^* in the co-adjoint representation of $\hat{G}_\Gamma$ is the full group $\hat{G}_\Gamma$. Therefore the reduced phase space belonging to the Hamiltonian reduction is the factor space $C_{M_-^*}/\hat{G}_\Gamma$, which is nothing but the space of orbits we considered previously. Of course, the induced Poisson bracket carried by the reduced phase space of the Hamiltonian reduction is identified with the Dirac bracket $\{\ ,\ \}^*$, by parametrizing the factor space $C_{M_-^*}/\hat{G}_\Gamma$ by a section of the orbits. In our case the most convenient such section is provided by the highest weight gauge.

3. Generalized Toda theories

The field equation of the standard conformal Toda field theories is given by

$$\partial_+\partial_-\varphi_i + \exp[\sum_{j=1}^{l} K_{ij}\varphi_j] = 0 \ , \tag{3.1}$$

where K_{ij} is the Cartan matrix of $\mathcal{G}$. It has been realized recently [4, 10, 11, 13, 17] that these theories possess the chiral algebra $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}} \times \tilde{\mathcal{W}}_{\mathcal{S}}^{\mathcal{G}}$ as canonical symmetries, where $\mathcal{S}$ is the principal $sl(2)$ subalgebra of $\mathcal{G}$. The natural explanation of this result is provided by the fact that the Toda field theories are nothing but reduced WZNW theories, which result from the WZNW theory by imposing KM constraints of the type discussed in the previous chapter. In fact it has been demonstrated in [12] that the standard Toda theory can be obtained from the WZNW theory by imposing the constraints

$$\pi_{\leq -1}(J) = M_- \qquad \text{and} \qquad \pi_{\geq 1}(\tilde{J}) = -M_+ \ , \tag{3.2}$$

where M_+ and M_- are the step-operators of the principal $sl(2)$ subalgebra of $\mathcal{G}$ [28], which specifies the grading used in defining the projections in (3.2). The grading defined by the principal $sl(2)$ is an integral grading, the spins occuring in the decomposition of the adjoint of $\mathcal{G}$ under the principal $sl(2)$ are the exponents of $\mathcal{G}$. To derive the Toda equation (3.1) from the WZNW equation (1.2), one uses the generalized Gauss decomposition $g = g_+ \cdot g_0 \cdot g_-$ of the WZNW field g, where $g_{0,\pm}$ are from the subgroups $G_{0,\pm}$ of G with Lie algebras $\mathcal{G}_{0,\pm}$, respectively. In this framework the Toda fields φ_i are given by the gauge invariant middle-piece of the Gauss decomposition, $g_0 = \exp[\sum_{i=1}^{l} \varphi_i H_i]$. Note that $\mathcal{G}_0$ is a Cartan subalgebra in this case, and the H_i are chosen to be the Cartan generators associated to some system of simple roots. The above derivation of the standard Toda theory from the WZNW theory offers some natural generalizations [14-16, 26], which we now briefly review.

3. 1. Generalized Toda theories associated with integral gradings

Let us consider a diagonalizable element $H \in \mathcal{G}$ whose spectrum in the adjoint of $\mathcal{G}$ consists of integer numbers and includes ± 1. By means of its eigenspaces, such an element defines an integral grading of $\mathcal{G}$:

$$\mathcal{G} = \mathcal{G}_+^H + \mathcal{G}_0^H + \mathcal{G}_-^H \ , \qquad \mathcal{G}_\pm^H = \sum_{n=1}^{N} \mathcal{G}_{\pm n}^H \ . \tag{3.3}$$

As a rather immediate generalization of (3.2), we can impose the following first class constraints on the WZNW phase space:

$$\pi^H_{\leq -1}(J) = M_- \qquad \text{and} \qquad \pi^H_{\geq 1}(\tilde{J}) = -M_+ \ , \tag{3.4}$$

where now $M_\pm$ are some arbitrary but non-zero elements from $\mathcal{G}^H_{\pm 1}$. The reduced theory turns out to be an integrable field theory for a local field b varying in the little group G^H_0 of H in G. The field b is defined through the generalized Gauss decomposition $g = g^H_+ \cdot g^H_0 \cdot g^H_-$, namely $b \equiv g^H_0$. The reduced field equation is the zero curvature condition of the Lax potential

$$\mathcal{A}_+ = \partial_+ b \cdot b^{-1} + M_- \ , \qquad \mathcal{A}_- = -bM_+b^{-1} \ , \tag{3.5}$$

and the effective action is given as

$$S^H_{\text{Toda}}(b) = S_{\text{WZ}}(b) - \int d^2x \, \text{Tr}\,(bM_+b^{-1}M_-) \ . \tag{3.6}$$

These generalized, or non-Abelian, Toda theories have been first considered by Leznov and Saveliev [3]. In their approach the Lax pair (3.5) is obtained by 'specializing' a pure-gauge potential $\mathcal{B}_\pm \equiv \partial_\pm \hat{g} \cdot \hat{g}^{-1}$, $\hat{g} \in G$, by making use of an integral grading. In comparison, in the WZNW framework the Lax pair is derived by conjugating the chiral zero curvature equation (1.6) by the upper triangular piece g^H_+ of the Gauss decomposition of g, and making use of the constraints (3.4) of the Hamiltonian reduction. Clearly, these two methods are different but closely related. We note that in the WZNW approach the effective action (3.6) can also be derived in a natural way, from the Lagrangian, gauged WZNW version of the Hamiltonian reduction [13, 26]. One of the important advantages of the WZNW setting is that it provides a global description of the Toda type theories. The point is that the WZNW description remains valid also in the sectors where the Gauss decomposition of g breaks down, which is reflected in the existence of apparent, non-physical singularities for a class of solutions of the local Toda system given by (3.6), see [12, 13].

It was shown in [3], in the special case when H and $M_\pm$ span an integral $sl(2)$ subalgebra of $\mathcal{G}$, that the non-Abelian Toda equation allows for conserved chiral currents which underlie its exact integrability. These currents generate chiral $\mathcal{W}$-algebras of the type discussed here in Chapter 2 [4, 14-16].

In [15] we investigated the chiral algebras of a wider class of non-Abelian Toda systems, namely we considered the Toda systems obtained by constraints of the type (3.4) satisfying the following chiral *non-degeneracy conditions*:

$$\text{Ker}(\text{ad}_{M_\mp}) \cap \mathcal{G}^H_\pm = \{0\} \ . \tag{3.7}$$

We proved that in this case the left $\times$ right chiral algebra is isomorphic to $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}_-} \times \tilde{\mathcal{W}}^{\mathcal{G}}_{\mathcal{S}_+}$, where $\mathcal{S}_-$ ($\mathcal{S}_+$) is an $sl(2)$ subalgebra of $\mathcal{G}$ containing the nilpotent generator M_- (M_+), respectively. Such $sl(2)$ subalgebras exist by some powerful theorems of Morozov, Jacobson and Kostant on $sl(2)$ embeddings [28]. This structure of the chiral algebra can be established by exhibiting highest weight DS gauges for the chiral, left- and right KM parts of the constraint surface, (3.4). We note that, although we took H to be an integral grading operator, $\mathcal{S}_\pm$ are not necessarily integrally embedded $sl(2)$ subalgebras, but in general not every half-integral $sl(2)$ subalgebra of $\mathcal{G}$ is obtained in the above manner [26].

Concentrating further on the non-degenerate case, let us consider some, say left moving, gauge invariant differential polynomial of the constrained KM current, $W(J)$. Note that the components of the Toda field b form a complete system of gauge invariant quantities in the Gauss decomposable sector of the reduced theory defined by (3.4). It follows that for any $W(J)$ there exists a unique function $W_{\rm Toda}(b)$ such that

$$W_{\rm Toda}(b) = W(J) \ . \tag{3.8}$$

We know that

$$\mathcal{A}_+ = g_+^{-1} J g_+ + \partial_+ g_+^{-1} \cdot g_+ \ , \tag{3.9}$$

and that the W's are form-invariant differential polynomials under 'upper triangular' gauge transformations. Combining these, we see that $W_{\rm Toda}$ depends on b only through $\mathcal{A}_+$ and that $W_{\rm Toda}(\mathcal{A}_+)$ is obtained by simply substituting $\mathcal{A}_+$ for the argument of the differential polynomial W. In other words, we have

$$W_{\rm Toda}(b) = W(\mathcal{A}_+) \ . \tag{3.10}$$

In ref. [3, 4] the conserved currents of the Toda theory are constructed by directly solving the 'characteristic equation'

$$\partial_- W_{\rm Toda}(b) = 0 \tag{3.11}$$

for $W_{\rm Toda}$. The arguments sketched above [15, 29] yield the translation between the Lax pair [3, 4, 11, 17] and the constrained KM descriptions of the $\mathcal{W}$-algebra of the Toda system (3.6), for non-degenerate $M_\pm$.

3. 2. Generalized Toda theory for arbitrary half-integral $sl(2)$

In this final section we briefly describe a new class of generalized Toda theories, which are studied in detail in [26]. These theories are constructed in such a way that they possess the left $\times$ right chiral algebras $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}} \times \tilde{\mathcal{W}}^{\mathcal{G}}_{\mathcal{S}}$ for some arbitrarily chosen *half-integral* $sl(2)$ subalgebra $\mathcal{S} = \{M_0,\ M_\pm\}$.

Clearly, if we impose first class constraints of the type (2.23) on the currents of the WZNW theory then the reduced theory will have the chiral algebra we require. There is a large freedom in this construction arising from the non-uniqueness of the, say, left-moving constraints and from the fact that one can in principle choose the left- and right-moving constraints quite independently. The procedure we adopt here is that we take the left-moving constraints to be given by (2.23) according to some decomposition of the type (2.28), and then choose the right-moving constraints to be obtained by 'transposing' the left-moving ones. See [26] for arguments in favour of this left-right related choice of the constraints.

Turning to the details of the reduction, first we choose a decomposition of $\mathcal{G}_{\frac{1}{2}}$ of the type (2.28) and then define the *induced decomposition* of $\mathcal{G}_{-\frac{1}{2}}$,

$$\mathcal{G}_{-\frac{1}{2}} = \mathcal{P}_{-\frac{1}{2}} + \mathcal{Q}_{-\frac{1}{2}}\ , \tag{3.12a}$$

to be given by the subspaces

$$\mathcal{Q}_{-\frac{1}{2}} \equiv \mathcal{P}_{\frac{1}{2}}^{\perp} = [M_-\,,\,\mathcal{P}_{\frac{1}{2}}] \quad \text{and} \quad \mathcal{P}_{-\frac{1}{2}} \equiv \mathcal{Q}_{\frac{1}{2}}^{\perp} = [M_-\,,\,\mathcal{Q}_{\frac{1}{2}}]\ . \tag{3.12b}$$

The right-moving constraints then read as

$$\pi_{\geq 1}(\tilde{J}(x^-)) = -M_+ \quad \text{and} \quad \langle u\,,\,\tilde{J}(x^-)\rangle = 0 \quad \forall\, u \in \mathcal{P}_{-\frac{1}{2}}\ . \tag{3.13}$$

The total system of constraints is first class, as a consequence of the left-moving ones being first class.

Now we shall give the effective field equation of the reduced theory. We start by assuming that the WZNW field g is given by the generalized Gauss decomposition defined by the $sl(2)$ subalgebra. This means that we have

$$g(x^+, x^-) = g_+(x^+, x^-)\cdot g_0(x^+, x^-)\cdot g_-(x^+, x^-)\ , \tag{3.14}$$

where g_0 is from the little group of M_0 and $g_\pm$ are from the subgroups $G_\pm$ of G obtained by exponentiating the Lie algebras $\mathcal{G}_\pm$, defined by the grading (2.3). Therefore we can parametrize $g_\pm$ as

$$g_+ = \exp[r_{\geq 1} + p_{\frac{1}{2}} + q_{\frac{1}{2}}] \quad \text{and} \quad g_- = \exp[r_{\leq -1} + p_{-\frac{1}{2}} + q_{-\frac{1}{2}}]\ . \tag{3.15}$$

Here the subscript indicates the grade of the variables, and $p_{\pm\frac{1}{2}}(x^+,x^-) \in \mathcal{P}_{\pm\frac{1}{2}}$, $q_{\pm\frac{1}{2}}(x^+,x^-) \in \mathcal{Q}_{\pm\frac{1}{2}}$. The gauge transformations belonging to our first class constraints act on the field g according to

$$g(x^+,x^-) \longrightarrow e^{\gamma_+(x^+)} \cdot g(x^+,x^-) \cdot e^{\gamma_-(x^-)} , \tag{3.16}$$

where $\gamma_\pm(x^\pm)$ are arbitrary elements from the left- and rigth gauge algebras $\Gamma_+ = (\mathcal{G}_{\geq 1} + \mathcal{P}_{\frac{1}{2}})$ and $\Gamma_- = (\mathcal{G}_{\leq -1} + \mathcal{P}_{-\frac{1}{2}})$, respectively. One sees that the quantities $g_0(x^+,x^-)$ and $q_{\pm\frac{1}{2}}(x^+,x^-)$ are invariant under (3.15), and they in fact form a complete set of gauge invariant local fields parametrizing the reduced phase space. We would like to obtain the effective field equation in terms of these variables, by projecting the WZNW field equation to the reduced phase space. As usual, this can be achieved by conjugating the WZNW field equation (1.6) by the field $g_+^{-1}(x^+,x^-)$ defined by (3.14). Indeed, this way we obtain the equivalent equation $[\partial_+ - \mathcal{A}_+ \, , \, \partial_- - \mathcal{A}_-] = 0$, with

$$\mathcal{A}_+ = g_+^{-1} J g_+ + \partial_+ g_+^{-1} \cdot g_+ \, , \quad \text{and} \quad \mathcal{A}_- = \partial_- g_+^{-1} \cdot g_+ \, , \tag{3.17}$$

and the point is that $\mathcal{A}_\pm(x^+,x^-)$ become gauge invariant quantities upon imposing the constraints. Thus the effective field equation derived in the above manner is automatically in the zero curvature form.

To display the result of the analysis detailed in [26], we have to introduce some notation. First let us consider the operator Ad_{g_0}, which, in particular, maps the space $\mathcal{G}_{-\frac{1}{2}}$ to itself. By writing the general element u of $\mathcal{G}_{-\frac{1}{2}}$ as a two-component column vector $u = (u_1 \; u_2)^t$ with $u_1 \in \mathcal{P}_{-\frac{1}{2}}$ and $u_2 \in \mathcal{Q}_{-\frac{1}{2}}$, we can designate this operator as a 2×2 matrix:

$$\mathrm{Ad}_{g_0 | \mathcal{G}_{-\frac{1}{2}}} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} , \tag{3.18}$$

where, for example, $X_{11}(g_0)$ and $X_{12}(g_0)$ are linear operators mapping $\mathcal{P}_{-\frac{1}{2}}$ and $\mathcal{Q}_{-\frac{1}{2}}$ to $\mathcal{P}_{-\frac{1}{2}}$, respectively. We note that the operator $X_{11}(g_0)$ is certainly invertible in a neighbourhood of the identity, since $\mathrm{Ad}_e = 1$. Analogously, we introduce the notation

$$\mathrm{Ad}_{g_0^{-1} | \mathcal{G}_{\frac{1}{2}}} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} , \tag{3.19}$$

which corresponds to writing the general element of $\mathcal{G}_{\frac{1}{2}}$ as a column vector, the upper and lower components of which belong to $\mathcal{P}_{\frac{1}{2}}$ and $\mathcal{Q}_{\frac{1}{2}}$, respectively. It is not too hard to show that on the constraint surface specified by equations (2.23) and (3.13) the quantities $\mathcal{A}_\pm$ in (3.17) can be written as follows:

$$\begin{aligned} \mathcal{A}_+ &= M_- + \partial_+ g_0 \cdot g_0^{-1} + g_0 \cdot \mathcal{E}_{-\frac{1}{2}} \cdot g_0^{-1} \, , \\ \mathcal{A}_- &= - g_0 M_+ g_0^{-1} + \mathcal{E}_{+\frac{1}{2}} \, , \end{aligned} \tag{3.20a}$$

where

$$\begin{aligned}\mathcal{E}_{-\frac{1}{2}}(g_0, q_{\pm\frac{1}{2}}) &= \partial_+ q_{-\frac{1}{2}} - X_{11}^{-1}(g_0) \cdot \left\{ [q_{\frac{1}{2}}\,,\, M_-] + X_{12}(g_0) \cdot \partial_+ q_{-\frac{1}{2}} \right\} , \\ \mathcal{E}_{+\frac{1}{2}}(g_0, q_{\pm\frac{1}{2}}) &= -\partial_- q_{\frac{1}{2}} + Y_{11}^{-1}(g_0) \cdot \left\{ [M_+\,,\, q_{-\frac{1}{2}}] + Y_{12}(g_0) \cdot \partial_- q_{\frac{1}{2}} \right\} ,\end{aligned} \tag{3.20b}$$

The effective field equation of our generalized Toda theory is the zero curvature condition of this Lax pair, which reduces to (3.5) in the case of an integral $sl(2)$ subalgebra, as it should. The above arguments clearly show the WZNW origin of the Lax pair of the Toda type systems.

It is shown in [26] that, as a consequence of the obvious integrability of the WZNW field equation, the field equations of the reduced theory (3.20) are integrable by quadrature. As far as we know the integrable non-linear equations specified by the Lax pair (3.20) have not been considered before. The effective action underlying these theories, and the relationship between these theories and the ones given by the simpler Lax pair (3.5) and action (3.6) are discussed in detail in [26], where some examples are also worked out.

We end this report by pointing out the relationship between the generalized Toda systems (3.20) and certain non-linear, integrable equations which have been associated to the half-integral $sl(2)$ subalgebras of the simple Lie algebras by Leznov and Saveliev. (See equation (1.24) in the review paper in *J. Sov. Math.* referred to in [3].) For this purpose we note that by using our method one can in principle impose only the obvious, *non-maximal* constraints $\pi_{\leq -1}(J) = M_-$ and $\pi_{\geq 1}(\tilde{J}) = -M_+$ on the WZNW theory even in the case when M_- and M_+ belong to a half-integral $sl(2)$ sublagebra. The point we make is that the Lax pair of the reduced theory which one obtains in this case exactly coincides with the one proposed by Leznov and Saveliev. Thus their system is in some sense lies between the WZNW theory and our generalized Toda system, which has been obtained by imposing a *maximal* set of first class KM constraints on the WZNW phase space. In other words, our system can also be thought of as a reduction of theirs.

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