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GAUGE FIXING, UNITARITY AND PHASE SPACE PATH INTEGRALS

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ABSTRACT. We analyse the extent to which path integral techniques can be used to directly prove the unitarity of gauge theories. After reviewing the limitations of the most widely used approaches, we concentrate upon the method which is commonly regarded as solving the problem, i.e., that of Fradkin and Vilkovisky. We show through explicit counterexamples that their main theorem is incorrect. A proof is presented for a restricted version of their theorem. From this restricted theorem we are able to rederive Faddeev's unitary phase space results for a wide class of canonical gauges, which includes the Coulomb gauge. However, we show that there are serious problems with the extensions of this argument to Landau gauge. We conclude that there does not yet exist any satisfactory path integral discussion of the covariant gauges.

1. Path integral approaches to gauge theories

This paper has two main aims, and possibly audiences, which are reflected in its structure. In the main body we develop, within the specific application to electrodynamics, a manifestly unitary path integral formalism for a wide class of gauges. This account will be essentially self contained and, we hope, transparent for the general physics community. The appendices are a completely different kettle of fish; here we develop the mathematical machinery needed for our purposes and derive general theorems appropriate for non-abelian theories.

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Path integrals provide a most attractive and efficient approach to quantisation. Their reliance on the classical action to weight the contributions of the various possible paths in the space of configurations of the system has an immediate, intuitive appeal. It comes as some surprise then to find that this simple transcription of a classical action into a quantum partition function appears to break down when gauge theories are investigated.

It is well known that for Yang-Mills theory the gauge invariance of the action implies that the kinetic energy term is degenerate. So we can expect that the classical action should have a gauge fixing term added in order to remove this degeneracy. However, a path integral based on such an action will not, in general, yield a unitary quantum theory unless additional, ghost fields are added [1, 2]. Since these ghosts do not correspond to physically allowed particles, and they are only needed at one loop and above, they are usually thought of as intrinsically quantum artifacts; obstructing the simple path integral relationship between a classical and a quantum theory. However, there is no denying the utility or necessity of such fields, and one can present a formal argument—the Faddeev-Popov trick—for the inclusion of an additional ghost term in the effective action such that the correct interaction terms and propagators for the ghosts can be derived using standard path integral techniques [3].

The Faddeev-Popov trick is based on the observation that gauge invariance implies a multiple counting of the physically relevant Yang-Mills configurations. The trick then is to remove some of this redundancy from the path integral. Let us recall the main steps in their argument. The path integral is

$$Z = \int dA \, \exp(iS(A)).$$

Here $S(A)$ is the Yang-Mills action and dA is a formal measure. The Faddeev-Popov trick then involves rewriting this expression as

$$Z = \int dg \int dA \, \Delta(A)\delta(\mathcal{F}(A)) \exp(iS(A)),$$

where $\int dg$ is the (infinite) volume of the group of all gauge transformations, $\Delta(A)$ is the Faddeev-Popov determinant and $\mathcal{F}(A)$ is some (local) gauge fixing condition. This expression for the path integral allows us to remove the redundant gauge group volume from Z. The determinant and gauge fixing terms in the measure are then exponentiated to give the full effective action with ghost fields.

This response to the multiple counting of physical states, due to the gauge symmetry, is at best a matter of taste, and certainly one that does not seem to be forced upon us. Indeed, in the lattice approach to such a problem one can just learn to live with the (now finite) multiplicity of states (see, for example, [4]). A more serious problem with this approach is that there seems to be little connection with unitarity. Indeed, here the form of the ghost interactions is limited by their role in exponentiating a determinant. Yet one knows that, in order to maintain unitarity, higher order ghost interactions are needed if nonlinear gauge fixing terms are used [5], or in more complicated theories [6].

Even though the above arguments are clearly limited, the identification of gauge invariance as the culprit for the problems with unitarity is clearly correct. However, the configuration space path integral formulation of the theory does not appear to be flexible enough to fully deal with the problems.

A more fundamental approach to path integrals in general, and gauge theories in particular, is to work within the phase space formulation. For Yang-Mills theory the conjugate momentum π^μ_a is introduced and gauge invariance is seen, as will be discussed in Sect. 2, to imply the existence of (first class) constraints. These constraints naturally split up into two first class subsets: the primary constraints, $\pi^0_a(x) = 0$, and the Gauss law constraint, $G_a(x) = 0$, where

$$G_a = -\partial_k \pi^k_a + f^c_{ab} \pi^k_c A^b_k, \tag{1.1}$$

and f^c_{ab} are the structure constants of some semi-simple Lie group.

The primary constraint tells us that A^a_0 and π^0_a are redundant conjugate variables. Hence, one usually takes as the configuration space the space $\mathcal{A}^{(3)}$ of vector potentials $\vec{A} \equiv A^a_i$. Gauss' law then tells us that the correct dynamical description takes place on the true degrees of freedom which, for this system, can be identified with the phase space constructed over the configuration space $\mathcal{A}^{(3)}/\mathcal{G}$ —spatial connections modulo those gauge transformations generated by the Gauss law constraint.

This global description of the true degrees of freedom is central to any non-perturbative analysis of gauge theories. However, within the perturbative framework implicit in the standard path integral formalism, a more useful characterization is to introduce a set of gauge fixing conditions, $\chi^a = \chi^a(\vec{A}, \vec{\pi})$, so that locally the true degrees of freedom are those states in the phase space that satisfy $\chi^a = G_a = 0$.

Unitarity, as it is used in this context (see, for example, [7]), is the condition that the unphysical states do not contribute to the S-matrix elements. So, the problem posed by a path integral formulation of this system is how can we write the physical partition function Z_{phys}, defined on the true degrees of freedom, in terms of a path integral over the extended variables $(\vec{A}, \vec{\pi})$. A solution to this was given by Faddeev [8] who showed that the (formal) Liouville measure $d\mu_{\text{phys}}$, on the true degrees of freedom, could be written as

$$d\mu_{\text{phys}} = d\vec{A} d\vec{\pi} \det |\{\chi^b, G_c\}| \prod_a \delta(G_a)\delta(\chi^a).$$

Hence the unitary path integral can be written as

$$Z_{\text{phys}} = \int d\vec{A} d\vec{\pi} \det |\{\chi^b, G_c\}| \prod_a \delta(G_a)\delta(\chi^a) \exp\left(i \int dt\, (\pi^k_b \dot{A}^b_k - H_0)\right), \tag{1.2}$$

where $H_0 = \frac{1}{2}\int d^3x\, (\pi^i_a \pi^a_i + B^i_a B^a_i)$, $\vec{B}$ is the magnetic field constructed out of the potentials $\vec{A}$ and for notational simplicity we have written the action term $\int d^3x\, \pi^k_b(x) \dot{A}^b_k(x)$ as $\pi^k_b \dot{A}^b_k$. (We will use the notation that $f(x)$ is the density corresponding to the local functional f throughout this paper.)

A serious limitation of this description of Z_{phys} is that we cannot now recover a manifestly Lorentz invariant form for the effective action since the A^a_0 component of the field has been removed. The obvious thing to do is to reinstate the primary constraint and hence insert into the measure the trivial term

$$dA_0^a d\pi_a^0 \prod_b \delta(A_0^b)\delta(\pi_b^0) \exp\left(i \int dt\, \pi_a^0 \dot{A}_0^a\right).$$

However, this does not yield the required form for the effective Hamiltonian since upon exponentiating the δ-functions in the measure we would get the effective action

$$\int dt\, (\pi_a^\mu \dot{A}_\mu^a - H_0 - \lambda_1^a G_a - \lambda_2^a \pi_a^0 - \lambda_a^3 \chi^a - \lambda_a^4 A_0^a),$$

with the λ's as multipliers and $\lambda_1^a G_a = \int d^3x\, (\lambda_1^a(x) G_a(x))$, etc. But this expression, with its multiplicity of multipliers, is not what is needed since due to the secondary nature of the Gauss law constraint it is actually the potential A_0^a that should be the multiplier for Gauss' law, i.e. the expected form for this effective action in, say, the Landau gauge would be

$$\int dt\, (\pi_a^\mu \dot{A}_\mu^a - H_0 - A_0^a G_a - \pi_a^0 \partial_i A_i^a),$$

So, in order to get the correct covariant form of the effective action directly from (1.2) we need to exponentiate Gauss' law with the field A_0^a and choose a gauge fixing condition of the form $\chi^a = \dot{A}_0^a - \varphi(\vec{A}, \vec{\pi})$; with $\varphi(\vec{A}, \vec{\pi})$ a canonical gauge fixing term. The fact that this gauge fixing condition has a time derivative in it does not, in itself, cause any real concern. The problem is that it is the time derivative of the multiplier field that enters. This will have the effect of elevating the multiplier field for the gauge fixing condition into the conjugate momentum to A_0^a; apparently contradicting the fact that the primary constraint told us that the conjugate momentum to A_0^a is constrained to be zero. We cannot, therefore, recast the above expressions in a way that permits comparison with (1.2). A more flexible approach is needed.

The two approaches to the path integral formulation of Yang-Mills theory discussed above have in common the emphasis on manipulating the (formal) measure to be used in the partition function. Now it is the lack of a well defined measure that lies at the heart of the problems with these functional methods. Thus, at least heuristically, a more attractive procedure would be to ignore the problems with the measure (i.e. just use the measure appropriate to an unconstrained system) and instead worry directly about the form of the effective Hamiltonian to use. As this is, in some sense, the classical input into the path integral formalism one could hope to be able to gain a better understanding of what is actually going on.

Such an approach to the path integral formulation of (first class) constrained systems was initiated by Fradkin and Vilkovisky (FV) [9]. Their main conclusion was that the following path integral is independent of the function Ψ (of ghost number minus one):

$$Z_\Psi = \int d\mu\, \exp(iS_{\text{eff}})\,, \tag{1.3}$$

where

$$d\mu = dA_i^a dA_0^a d\pi_a^i d\pi_a^0 dc^a d\bar{c}_a db^a d\bar{b}_a,$$

$$S_{\text{eff}} = \int dt\,(\dot{A}_i^a \pi_a^i + \dot{A}_0^a \pi_a^0 + \dot{\bar{c}}_a b^a + \dot{c}^a \bar{b}_a - H_{\text{eff}}), \tag{1.4}$$

$$H_{\text{eff}} = H_0 - \{\Omega, \Psi\}.$$

The fields $(c^a, \bar{b}_a)$ and $(b^a, \bar{c}_a)$ are the conjugate pairs of ghost variables associated with the Gauss law and primary constraints, respectively, and Ω is the $BRST$ charge given by

$$\Omega = \int d^3x\ \left(G_a(x)c^a(x) + \tfrac{1}{2}f_{bc}^a \bar{b}_a(x)c^b(x)c^c(x) - i\pi_a^0(x)b^a(x)\right). \tag{1.5}$$

Various specific choices for Ψ will then recover all the known forms of the effective action needed in the analysis of Yang-Mills theory. In particular, setting $\Psi = \int d^3x(\bar{b}_a A_0^a + \frac{i}{\beta}\bar{c}_a \partial_k A_k^a)$ will, for $\beta = 1$, yield the correct form of the effective action in the Landau gauge; the fields c being identified with the ghosts and $\bar{c}$ with the anti-ghosts. While, letting $\beta \to 0$, one can recover Faddeev's expression (1.2) for the physical partition function in the Coulomb gauge. Since the FV theorem tells us that Z_Ψ is independent of β, we can deduce from this the expected result that the partition function in the Landau gauge is equivalent to the physical partition function and hence manifestly unitary.

So the FV theorem provides a systematic way to generate all the well tested forms for the effective action of Yang-Mills theory. This is well and good but, even within the accepted vagaries of functional methods, there are several unsatisfactory aspects to both the statement and proof of this important result. For example, it is clear that the choice $\Psi = 0$, or more generally of the coboundary form $\Psi = \{\Omega, \zeta\}$, leads to an ill defined expression; yet the theorem as stated does not notice this. Also, there are other choices for Ψ that lead to unacceptable results. An example of this being

$$\Psi = \int d^3x \left(\bar{b}_a A_0^a + \frac{i\bar{c}_a}{\beta}\left(A_0^a + \frac{\alpha}{2\beta}\pi_a^0\right)\right), \tag{1.6}$$

which gives, in the limit as β goes to zero, the effective action appropriate to the temporal gauge. But, after integrating out the A_0^a field, we do not recover an expression equivalent to the physical partition function. (See [10] for more such elementary examples, more subtle problems with the FV theorem are discussed in [11].) Another point of unease with the FV theorem is the complete absence of any reference to gauge fixing; all that is required is that Ψ has ghost number minus one. This is very much at odds with the central role gauge fixing played in both the Faddeev-Popov trick and Faddeev's phase space path integral, and with any geometric intuition we have developed for the description of the physical partition in terms of the extended variables of the system.

The above points clearly show that the FV theorem as stated is wrong. Its use in developing new gauged fixed actions is highly suspect and it should not be thought of as anything more than a trick.

The argument used to derive this result is that if one performs the *non-canonical* change of variables

$$\mu \to \tilde{\mu} = \mu + \{\mu, \Omega\}\tau, \tag{1.7}$$

with

$$\tau = i \int dt(\Psi' - \Psi),$$

then the effective action is invariant and, due to the field dependence in τ, the change in the measure is given by

$$d\mu = d\tilde{\mu}(1 - i \int dt\{\Psi' - \Psi, \Omega\}).$$

It is then argued that for small $\Psi' - \Psi$ this immediately gives $Z_{\Psi'} = Z_{\Psi}$ (see [12] for a detailed account of this).

From the above discussion it is clear that this argument cannot always hold. Indeed, if one has ψ and ψ' such that $\psi - \psi' = \epsilon t$, then we can satisfy the condition that the difference is small for any finite t, yet τ diverges! More generally, one cannot see how the parameter τ can be calculated and controlled until the full history of the fields are known; which in turn requires knowledge of the form of the effective Hamiltonian and hence τ itself!

Highly non-trivial, non-canonical transformations like (1.7) cannot, as stated above, be controlled. The natural class of transformations one would think of applying to a phase space path integral is the canonical one, since they are invariant under such transformations and one has good control over them. Now it is well known that not all canonical transformations can be elevated into the quantum theory—but one can at least deal with, say, the point transformations. Our point of view, then, will be that a phase space path integral argument based on the effect of (point) canonical transformations is a good one, and we should have confidence in the conclusions of such an argument. Within this framework we would like to redevelop the *FV* path integral formulation. In particular, we would like to understand what are the restrictions on Ψ that guarantee that the path integral (1.4) is just the unitary, physical partition function.

The plan of this paper is as follows. In Sect. 2 the dynamical aspects of electrodynamics will be reviewed, including an account of the classical role of ghost variables for this system. Then, in Sect. 3, a restricted version of the *FV* theorem will be developed for this theory and it will be shown how to recover Faddeev's canonical gauge fixing result (1.2) within this approach. These techniques will then be applied in Sect. 4 to the Landau gauge, where we shall show that the unitarity of such a covariant gauge cannot be deduced within the Fradkin-Vilkovisky formalism. We will then conclude the main body of this paper in Sect. 5 with a discussion of these results and of the possible extensions of the approach used here. There will then be two appendices where some general results on graded phase spaces and the restricted *FV* theorem will be collected together.

2. Electrodynamics and ghost variables

In this section we will review the dynamical aspects of electrodynamics and show how gauge fixing in conjunction with ghost variables can be used in the classical theory to isolate the true degrees of freedom.

The configuration space variables in electrodynamics are the vector potentials $A_\mu(x)$. Their conjugate momentum are written as $\pi^\mu(x)$ and the basic Poisson bracket between them is $\{A_\mu(x), \pi^\nu(y)\} = \delta^\nu_\mu \delta(x-y)$. We denote the phase space of these variables by $P^{(4)}$. In order to relate these momentum variables to the velocities we use the standard action with Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. The momentum is then given by

$$\pi^\mu := \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu},$$

which implies the primary constraint

$$\pi^0(x) = 0. \tag{2.1}$$

The canonical Hamiltonian derived from the above action is then

$$H_c = H_0 - \int d^3x\, A_0 \partial_i \pi_i, \tag{2.2}$$

where $H_0(x) = \frac{1}{2}(\vec{\pi}^2(x) + \vec{B}^2(x))$ and $\vec{B}$ is the magnetic field. The requirement that the primary constraint is preserved in the time evolution generated by this Hamiltonian implies the secondary, Gauss law, constraint

$$G(x) := -\partial_i \pi_i(x) = 0. \tag{2.3}$$

There are no more constraints in this system and it is clear that the primary and Gauss law constraints are first class, i.e. $\{\pi^0(x), G(y)\} = 0$.

The easiest constraint to deal with is the primary one. This simply tells us that A_0 is a redundant dynamical variable and that we can reduce to the phase space $P^{(3)}$ with canonical variables (A_i, π_i), Hamiltonian H_0 and the constraint (2.3).

Gauss' law is almost as easy to deal with in this abelian system. We make the familiar decomposition of variables into transverse and longitudinal degrees of freedom i.e., $\pi_i = \pi_i^T + \pi_i^L$ where $\operatorname{div} \vec{\pi}^T = 0$ and $\operatorname{curl} \vec{\pi}^L = 0$ (with similar formulas for A_i). Since π_i^L has zero curl, we know that there is a scalar field $p(x)$ such that $\pi_i^L = \partial_i p$. Gauss' law then becomes the statement that $-\nabla^2 p = 0$, which implies that $p = 0$. The conjugate configuration space variable to the longitudinal mode p is simply $q = \partial_i A_i = \partial_i A_i^L$. Hence Gauss' law tells us that the true degrees of freedom for this system are the transverse fields (A_i^T, π_i^T). The physical Hamiltonian, H_{phys}, is then

$$H_{\text{phys}} = H_0|_{G=0} = \frac{1}{2}\int d^3x \left((\vec{\pi}^T)^2 + \vec{B}^2\right), \tag{2.4}$$

since $\vec{B} = \vec{B}(A_i^T)$.

The Batalin-Fradkin-Vilkovisky [9, 13] approach to this system requires the introduction of a ghost variable, and its conjugate momentum, for each first class constraint. So for the theory defined on the phase space $P^{(3)}$, with the Gauss law constraint, this involves the extension of the phase space to a graded, or super, phase space $\mathcal{P}^{(3)}$ via the addition of the ghost variable $c(x)$ and its conjugate $\bar{b}(x)$. As discussed in Appendix B, these odd variables satisfy the basic Poisson algebra

$$\{c(x), \bar{b}(y)\} = \{\bar{b}(y), c(x)\} = -\delta(x-y).$$

Central to this approach to constrained dynamics is the *BRST* charge Ω, which is a function of ghost number (plus) one that is abelian, $\{\Omega, \Omega\} = 0$, and is defined by

$$\Omega = \int d^3x\, c(x) G(x). \tag{2.5}$$

This allows us to construct the *BRST* operator δ acting on functions on $\mathcal{P}^{(3)}$ via

$$\delta f = \{\Omega, f\}. \tag{2.6}$$

Then Ω being abelian implies that $\delta^2 = 0$.

General results then tell us that the physical observables (and hence states) of this system can be characterized as the set of all (local) functions on $\mathcal{P}^{(3)}$, of ghost number zero, that are *BRST* invariant, $\delta f = 0$, but not trivially so, i.e. $f \neq \delta g$ for some function g of ghost number minus one [12, 14].

A direct way to see the isolation of the physical states and observables within this formalism is to use gauge fixing conditions to define the (symplectic) dual, $\bar{\Omega}$, to the *BRST* charge. This is an abelian function of ghost number minus one which, for the Coulomb gauge, is given by

$$\bar{\Omega} = \int d^3x\, \bar{b}(x) q(x). \tag{2.7}$$

The dual version, $\bar{\delta}$, of the *BRST* operator δ is then given by

$$\bar{\delta} f = \{\bar{\Omega}, f\}. \tag{2.8}$$

Again, $\bar{\Omega}$ being abelian implies that $\bar{\delta}^2 = 0$.

The requirement that the observables are both δ and $\bar{\delta}$ invariant *almost* fixes them to be physical (see [15] for a full discussion of this point). For states, though, this is enough to pick-out the physical transverse states from the graded phase space $\mathcal{P}^{(3)}$. It is instructive to see how this construction works.

On the phase space $P^{(3)}$ the (pure) states are simply the points on the manifold $P^{(3)}$. The problem we face then is how to adapt this view of states to the graded phase space $\mathcal{P}^{(3)}$ where, as we discuss in Appendix A, geometric concepts such as points on $\mathcal{P}^{(3)}$ are at best hard to grasp.

An equivalent, but less geometric, description of the states on $P^{(3)}$ is to follow the approach taken in statistical mechanics and define the states as objects dual to a suitable class of functions on $P^{(3)}$ (we refer to [16] for a more precise definition of these terms). So given a function f on $P^{(3)}$, we define $\langle f \rangle_s$, the value of f in the state s, by

$$\langle f \rangle_s = \int dA_i d\pi_i\, s(A_i, \pi_i) f(A_i, \pi_i). \tag{2.9}$$

For the pure states on $P^{(3)}$, $\langle f \rangle_s$ is simply the function f evaluated at the point in $P^{(3)}$ corresponding to the state s. So the pure state concentrated at the point (A_i^*, π_i^*) is given by

$$s(A_i, \pi_i) = \prod_i \delta(A_i - A_i^*)\delta(\pi_i - \pi_i^*).$$

and from (2.9) we see that $\langle f \rangle_s = f(A_i^*, \pi_i^*)$ as expected.

The extension of this definition of states to the graded phase space $\mathcal{P}^{(3)}$ is now quite straightforward. Restricting attention to the even functions on $\mathcal{P}^{(3)}$ i.e., $f = f_1 + f_2 c\bar{b}$, with f_1 and f_2 functions on $P^{(3)}$, we see that the (normalised) states on $\mathcal{P}^{(3)}$ are of the form $s = \alpha s_1 + \beta s_2 c\bar{b}$, with s_1 and s_2 states on $P^{(3)}$. The constants α and β are determined by working with states normalised to one, i.e. $\langle 1 \rangle_s = 1$, where, in general,

$$\begin{aligned} \langle f \rangle_s &= \int dA_i d\pi_i dc d\bar{b}\, s f \\ &= \langle f_1 \rangle_{\beta s_2} + \langle f_2 \rangle_{\alpha s_1} . \end{aligned}$$

From this it is clear that the pure states on $\mathcal{P}^{(3)}$ are naturally identified with those states s on $\mathcal{P}^{(3)}$ with s_1 and s_2 pure states on $P^{(3)}$ and where $\beta = 1$.

We have defined the action of the *BRST* and dual-*BRST* operators on functions on $\mathcal{P}^{(3)}$. The action of these operators on the states is then simply given by the relations

$$\langle \delta f \rangle_s = \langle f \rangle_{\delta s} \quad \text{and} \quad \langle \bar{\delta} f \rangle_s = \langle f \rangle_{\bar{\delta} s} . \tag{2.10}$$

From this definition and equations (2.6) and (2.8) we deduce that if the pure state s_{phys} on $\mathcal{P}^{(3)}$ satisfies $\delta s_{\mathrm{phys}} = \bar{\delta} s_{\mathrm{phys}} = 0$, then s_{phys} must be of the form

$$s_{\mathrm{phys}} = \prod_i \delta(A_i^T - A_i^*)\delta(\pi_i^T - \pi_i^*)\delta(p)\delta(q)c\bar{b}. \tag{2.11}$$

This is clearly the natural embedding of the physical, pure, transverse states into the graded phase space $\mathcal{P}^{(3)}$.

We have seen in the above discussion how ghost variables can be used in the classical theory to isolate the physical degrees of freedom. The rest of this paper will be concerned with the use of ghost variables in the path integral description of quantum electrodynamics. Before starting the quantum theory we conclude, for completeness, with a discussion on what happens in the classical theory when matter is present.

The minimal coupling of the above system to (fermionic) matter involves adding to the phase space the conjugate variables $\psi(x)$ and $\psi^\dagger(x)$. The canonical Hamiltonian, (2.2), then becomes

$$H_c = H_0 + H_{\mathrm{Dirac}} + \int d^3x\, A_0(-\partial_i \pi_i - g\psi^\dagger\psi), \tag{2.12}$$

where the Dirac Hamiltonian density is $H_{\mathrm{Dirac}}(x) = i\bar{\psi}(x)(\gamma^i\partial_i - g\gamma^i A_i + m)\psi(x)$. There is still the primary constraint (2.1) which now induces the Gauss constraint

$$G(x) = -\partial_i \pi_i(x) - gj_0(x) = 0, \tag{2.13}$$

where the charge density $j_0(x) = \psi^\dagger(x)\psi(x)$.

The reduction to the true degrees of freedom follows in much the same way as above. The only new point is that the fermionic fields ψ and $\psi^\dagger$ are not physical since they do not even weakly preserve the Gauss law constraint. Rather, the physical matter fields are given by

$$\psi^* = e^{-g\frac{q}{\nabla^2}}\psi \quad \text{and} \quad \bar{\psi}^* = e^{g\frac{q}{\nabla^2}}\bar{\psi}. \tag{2.14}$$

The physical Hamiltonian density is then given by

$$H_{\rm phys}(x) = \tfrac{1}{2}\left((\vec{\pi}^T)^2(x) + \vec{B}^2(x)\right) + i\bar{\psi}^*(x)(\gamma^i\partial_i - g\gamma_i A_i^T(x) + m)\psi^*(x) + H_{j_0j_0}(x), \tag{2.15}$$

where we have used the result that $\partial_i\psi^* = -gA_i^L\psi^* + e^{-gq/\nabla^2}\partial_i\psi$. The Coulomb term

$$H_{j_0j_0}(x) = \tfrac{1}{2}g^2\int dy\,\frac{j_0(y)j_0(x)}{|x-y|},$$

comes from the longitudinal momentum term in $H_0(x)$.

The $BRST$ charge Ω is constructed as in (2.5) but now using expression (2.13) for the Gauss law constraint. No change is needed in the dual charge $\bar{\Omega}$.

3. A restricted Fradkin-Vilkovisky theorem for electrodynamics

In this section we shall present a restricted versions of the Fradkin -Vilkovisky theorem for electrodynamics: a version that will allow us to deal with a wide class of canonical gauge fixing conditions. The strategy we shall follow is to state the theorem in some generality and refer to Appendix B for the formal proof. However, for clarity and in order to develop the key arguments needed in Sect. 4, we shall present the details of the proof as it applies to the Coulomb gauge.

Recall that on the phase space $\mathcal{P}^{(3)}$ Gauss' law, in conjunction with the Coulomb gauge fixing condition $q = \partial_i A_i = 0$, allows us to isolate the physical transverse fields. The additional unphysical degrees of freedom are the longitudinal variables $(q(x), p(x))$ and the ghost fields $(c(x), \bar{b}(x))$. The physical partition function is given by

$$Z_{\rm phys} = \int dA_i^T d\pi_i^T \exp\left(i\int dt(\pi_i^T\dot{A}_i^T - H_{\rm phys})\right), \tag{3.1}$$

where $H_{\rm phys}$ is given in (2.4).

Theorem 2 in Appendix B (with the notational shifts on the ghost variables $c \to \eta$ and $\bar{b} \to \rho$) shows that we can write this path integral on $\mathcal{P}^{(3)}$ as

$$Z_{\rm phys} = \int dA_i d\pi_i dc d\bar{b} \exp\left(i\int dt(\dot{A}_i\pi_i + \dot{c}\bar{b} - H_{\rm eff})\right), \tag{3.2}$$

where $H_{\text{eff}} = H_{\text{phys}} - \{\Omega, \Psi\}$ and Ψ a ghost number minus one function corresponding to a first class gauge fixing condition; by this we simply mean that Ψ is of the form $\Psi = \int d^3x\, \chi \bar{b} + \cdots$ where χ is a gauge fixing term for the Gauss law constraint, and Ψ is abelian, i.e., $\{\Psi, \Psi\} = 0$. The simplest example of such a Ψ is the one appropriate to the Coulomb gauge with

$$\Psi = \int d^3x\, q(x)\bar{b}(x), \tag{3.3}$$

which we note from (2.7) is equal to the dual *BRST* charge. The reason why this result is true for this specific choice of Ψ is quite straightforward: Under the even canonical transformation:

$$\begin{aligned}
A^T &\rightarrow A^T, \\
\pi^T &\rightarrow \pi^T, \\
q &\rightarrow q^\beta, \\
p &\rightarrow \tfrac{1}{\beta} q^{1-\beta} p + \tfrac{\beta-1}{\beta} q^{-\beta} c\bar{b}, \\
c &\rightarrow \beta q^{\beta-1} c, \\
\bar{b} &\rightarrow \tfrac{1}{\beta} q^{1-\beta} \bar{b},
\end{aligned}$$

the *BRST* charge Ω is preserved while the Ψ given in (3.3) is transformed into

$$\Psi_\beta = \tfrac{1}{\beta} \int d^3x\, q(x)\bar{b}(x). \tag{3.4}$$

(We refer to the arguments in Appendix B that show why this is a canonical transformation with the stated properties.)

Since this is a canonical transformation we can deduce that expression (3.2) is independent of β. From the above explicit form of the transformations it is clear that some care is needed in the limit as $\beta \rightarrow 0$. However, after performing the non-canonical (see Appendix B) change of variables: $q \rightarrow \beta q$; $\bar{b} \rightarrow \beta \bar{b}$, the path integral is seen to be well defined in this limit. Indeed one finds that

$$Z_{\text{phys}} = \int dA_i^T d\pi_i^T dq\, dp\, dc\, d\bar{b} \exp\left(i \int dt (\dot{A}_i^T \pi_i^T - H_{\text{phys}} + q\nabla^2 p - (\nabla^2 c)\bar{b}) \right).$$

The longitudinal and ghost integrals are clearly trivial; yielding the expression (3.1) for the physical partition function. Thus we have explicitly verified the restricted Fradkin-Vilkovisky theorem for the Coulomb gauge.

It is interesting to note that in this argument the gauge fixing condition $q \equiv \partial_i A_i$ is essentially playing the role of a multiplier for the Gauss law constraint. Although this gives an interesting class of expressions for the physical partition function over $\mathcal{P}^{(3)}$, we would like to see the gauge fixing condition and constraints treated in a more democratic fashion. In particular, we would like to see Faddeev's result (1.2) emerge.

It is somewhat surprising to find that in order to derive Faddeev's description of the physical partition function on $\mathcal{P}^{(3)}$ we need to extend our phase spaces in a dynamically acceptable way; introducing fields that will become, in some limit, the multiplier fields for both the Coulomb term and Gauss' law.

Clearly there is a natural set of additional variables that we can introduce into the above discussion. Reinstating the primary constraint (2.1), its conjugate variable A_0 and their ghost fields $(-ib, i\bar{c})$, the $BRST$ charge becomes

$$\Omega = \int d^3x\,(G(x)c(x) - i\pi^0(x)b(x)), \tag{3.5}$$

and the direct extension of the above theorem tells us that on $\mathcal{P}^{(4)}$ we can write the physical partition function as

$$Z_{\text{phys}} = \int dA_i d\pi_i dA_0 d\pi_0 dc\, d\bar{b}\, db\, d\bar{c}\, \exp\left(i\int dt(\dot{A}_i\pi_i + \dot{A}_0\pi_0 + \dot{c}\bar{b} + \dot{\bar{c}}b - H_{\text{eff}})\right), \tag{3.6}$$

where now $H_{\text{eff}} = H_{\text{phys}} - \{\Omega, \Psi\}$ and Ψ is a first class gauge fixing term for the constraints $\pi^0(x) = 0$ and $G(x) = -\nabla^2 p(x) = 0$. (We recall again from Appendix B that Ψ being first class simply means that $\{\Psi, \Psi\} = 0$.) An example of such a Ψ is given by

$$\Psi_{\text{Fad}} := \int d^3x\,(A_0\bar{b} + iq\bar{c}). \tag{3.7}$$

The proof of this theorem for this specific Ψ would first (following the argument presented in Appendix B) involve performing a $BRST$ charge preserving rotation into $\Psi^r_{\text{Fad}} = \int d^3x\,(q\bar{b} + iA_0\bar{c})$. Then one could repeat the simple rescaling of the gauge fixing terms given above to reduce to the physical partition function (full details of this type of argument are given in Appendix B).

An alternative and, as we shall see in the next section, more useful approach to showing the above theorem with $\Psi = \Psi_{\text{Fad}}$ is to construct a canonical transformation which preserves Ω and takes Ψ_{Fad} to $\Psi_{\text{Fad}}(\beta)$, where

$$\Psi_{\text{Fad}}(\beta) = \int d^3x\,(A_0\bar{b} + \frac{i}{\beta}q\bar{c}). \tag{3.8}$$

Such a transformation can be constructed with the help of Example 4 in Appendix B. There is, however, a slight complication in directly applying that example to this situation. The problem is that the Gauss law constraint is $-\nabla^2 p = 0$ and not $p = 0$. We can overcome this rescaling of the constraint by splitting our canonical transformation into three steps: first we rescale $c \to c/k^2$ and $\bar{b} \to k^2\bar{b}$ (where we have taken the Fourier transform of Ψ_{Fad}); then we apply Example 4 (with the identifications $\pi^0 \to p_1$, $A_0 \to q^1$, $p \to p_2$, $q \to q^2$, $-ib \to \eta^1$, $i\bar{c} \to \rho_1$, $c \to \eta^2$ and $\bar{b} \to \rho_2$); finally we recover the correct form for Gauss' law by rescaling $c \to k^2 c$ and $\bar{b} \to \bar{b}/k^2$.

The conclusion of this is that the generating function (see Appendix A for full definitions)

$$\mathcal{F} = A_i^T\tilde{\pi}_i^T + f^1\tilde{\pi}^0 + f^2\tilde{p} - \left(\frac{\partial f^1}{\partial A_0}b + ik^2\frac{\partial f^1}{\partial q}c\right)\tilde{\bar{c}} - \left(\frac{-i}{k^2}\frac{\partial f^2}{\partial A_0}b + \frac{\partial f^2}{\partial q}c\right)\tilde{\bar{b}},$$

with

$$
\begin{aligned}
f^1 &= f_+^{\sqrt{\beta}} - f_-^{\sqrt{\beta}} \\
f^2 &= \sqrt{k^2}\left(f_+^{\sqrt{\beta}} + f_-^{\sqrt{\beta}}\right)
\end{aligned}
\tag{3.9}
$$

and

$$
f_\pm = \tfrac{1}{2}\left(\frac{1}{\sqrt{k^2}}q \pm \sqrt{\beta}A_0\right),
$$

generates an Ω-preserving even canonical transformation which maps $\Psi_{\mathbf{Fad}}$ to $\Psi_{\mathbf{Fad}}(\beta)$.

The details of this claim are quite straightforward: Under the above transformation we find that

$$
\begin{aligned}
A_0 &\rightarrow f^1, \\
q &\rightarrow f^2, \\
c &\rightarrow -\frac{i}{k^2}\frac{\partial f^2}{\partial A_0}b + \frac{\partial f^2}{\partial q}c, \\
b &\rightarrow \frac{\partial f^1}{\partial A_0}b + ik^2\frac{\partial f^1}{\partial q}c, \\
\bar{c} &\rightarrow \frac{1}{J}\frac{\partial f^2}{\partial q}\bar{c} + \frac{i}{Jk^2}\frac{\partial f^2}{\partial A_0}\bar{b}, \\
\bar{b} &\rightarrow -i\frac{k^2}{J}\frac{\partial f^1}{\partial q}\bar{c} + \frac{1}{J}\frac{\partial f^1}{\partial A_0}\bar{b},
\end{aligned}
$$

while

$$
\begin{aligned}
p \rightarrow \frac{1}{2\sqrt{\beta}}&\left(f_+^{1-\sqrt{\beta}} + f_-^{1-\sqrt{\beta}}\right)p + \frac{1}{2\beta\sqrt{k^2}}\left(f_+^{1-\sqrt{\beta}} - f_-^{1-\sqrt{\beta}}\right)\pi^0 \\
&+ \frac{(\sqrt{\beta}-1)}{8\sqrt{\beta k^2}}\left(f_+^{\sqrt{\beta}-2}f_-^{2(1-\sqrt{\beta})} + f_+^{-\sqrt{\beta}} + 2f_+^{2(1-\sqrt{\beta})}f_-^{\sqrt{\beta}-2}\right)b\bar{c} \\
&+ \frac{i(\sqrt{\beta}-1)}{8k^2}\left(-2f_+^{-\sqrt{\beta}} + f_-^{-\sqrt{\beta}} - f_+^{2(1-\sqrt{\beta})}f_-^{\sqrt{\beta}-2} - 2f_+^{1-\sqrt{\beta}}f_-^{-1}\right)b\bar{b} \\
&+ \frac{i(\sqrt{\beta}-1)}{4\beta}\left(f_+^{-\sqrt{\beta}} - f_+^{1-\sqrt{\beta}}f_-^{-1}\right)c\bar{c} \\
&+ \frac{(\sqrt{\beta}-1)}{4\sqrt{\beta k^2}}\left(f_+^{-\sqrt{\beta}} - f_+^{1-\sqrt{\beta}}f_-^{-1}\right)c\bar{b},
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
\pi^0 \to \frac{1}{2\beta}\left(f_+^{1-\sqrt{\beta}} + f_-^{1-\sqrt{\beta}}\right)\pi^0 + \frac{\sqrt{k^2}}{2\sqrt{\beta}}\left(f_+^{1-\sqrt{\beta}} - f_-^{1-\sqrt{\beta}}\right)p \\
+ \frac{(\sqrt{\beta}-1)}{8\sqrt{\beta}}\left(f_+^{2(1-\sqrt{\beta})} f_-^{\sqrt{\beta}-2} + 2f_+^{-\sqrt{\beta}} - f_-^{-\sqrt{\beta}} - 2f_+^{1-\sqrt{\beta}} f_-^{-1}\right) b\bar{c} \\
+ \frac{i(\sqrt{\beta}-1)}{8\sqrt{k^2}}\left(-f_+^{-\sqrt{\beta}} + f_+^{\sqrt{\beta}-2} f_-^{2(1-\sqrt{\beta})} - 3f_+^{-1} f_-^{1-\sqrt{\beta}} - 3f_+^{1-\sqrt{\beta}} f_-^{-1}\right) b\bar{b} \\
+ \frac{i\sqrt{k^2}(\sqrt{\beta}-1)}{4\beta}\left(f_+^{-\sqrt{\beta}} + f_+^{1-\sqrt{\beta}} f_-^{-1}\right) c\bar{c} \\
+ \frac{(\sqrt{\beta}-1)}{4\sqrt{\beta}}\left(f_+^{-\sqrt{\beta}} + f_+^{1-\sqrt{\beta}} f_-^{-1}\right) c\bar{b},
\end{aligned} \tag{3.11}
$$

where

$$
J := \frac{\partial f^1}{\partial A_0}\frac{\partial f^2}{\partial q} - \frac{\partial f^1}{\partial q}\frac{\partial f^2}{\partial A_0} = \beta\sqrt{\beta}(f_+ f_-)^{\sqrt{\beta}-1}.
$$

The Ω preserving part of the claim can either be seen through explicit (albeit tedious) calculations using the above transformations, or by appealing to Theorem 1 in Appendix B; taking into account the initial and final k^2 rescaling discussed above.

Acting on Ψ_{Fad} we find that

$$
\Psi_{\mathrm{Fad}} \to \int d^3x\, \frac{1}{J}\left(\left(f^2\frac{\partial f^2}{\partial q} - k^2 f^1 \frac{\partial f^1}{\partial q}\right) i\bar{c} + \left(-\frac{f^2}{k^2}\frac{\partial f^2}{\partial A_0} + f^1 \frac{\partial f^1}{\partial A_0}\right)\bar{b}\right),
$$

and the result then follows from the identities

$$
f^2\frac{\partial f^2}{\partial q} - k^2 f^1\frac{\partial f^1}{\partial q} = \frac{J}{\beta}q,
$$

and

$$
-\frac{f^2}{k^2}\frac{\partial f^2}{\partial A_0} + f^1\frac{\partial f^1}{\partial A_0} = JA_0.
$$

The above argument tells us that substituting $\Psi = \Psi_{\mathrm{Fad}}(\beta)$ into expression (3.6) yields a path integral that is independent of β. Now if we perform the non-canonical change of variables

$$
\pi^0 \to \beta\pi^0 \qquad \bar{c} \to \beta\bar{c} \tag{3.12}
$$

and take the limit as $\beta \to 0$ (and integrating out A_0, π^0 and the ghost fields) we deduce that

$$
Z_{\mathrm{phys}} = \int dA_i d\pi_i \delta(G)\delta(q)\, \exp\left(i\int dt\, (\pi_i \dot{A}_i - H_{\mathrm{phys}})\right). \tag{3.13}
$$

Which, *in this limit*, yields Faddeev's results (1.2) since $H_{\mathrm{phys}} = H_0|_{G=0}$.

Clearly, using the results from Appendix B, we can extend the above analysis to recover Faddeev's result for more general, first class gauge fixing conditions.

4. The Landau gauge

In this section we discuss the difficulties encountered in applying the Fradkin-Vilkovisky formalism to the Landau gauge. It is pointed out that this type of gauge fixing is not covered by our restricted version of their theorem and that there appears to be no possibility of making the argument used by them to show that the Landau gauge (and more generally the Lorentz class) is equivalent to the canonical Coulomb gauge precise.

The results discussed in the previous section show that the restricted version of the *FV* theorem has many of the attractive features of the original theorem and overcomes the criticism that gauge fixing did not play a role in its formulation. This version also excludes the coboundary Ψ's, since such objects are not first class gauge fixing terms. However, this large machinery has only really been used to rederive Faddeev's path integral expression; which was already known to be valid for this class of gauge fixing conditions and can be derived using much more elementary techniques [8]. As we discussed in Sect. 1, the most impressive aspect of the original *FV* approach to these path integrals was their elementary argument that the covariant gauges would yield unitary S-matrix elements. Our aim now is to see the effect of our restricted version of their theorem on this argument.

Applying their theorem (see Eq. (1.4)) to this abelian situation, Fradkin and Vilkovisky argued that using the effective Hamiltonian

$$H_{\text{eff}} = H_0 - \{\Omega, \Psi\}, \tag{4.1}$$

with $\Psi = \int d^3x\,(\bar{b}A_0 + \frac{i}{\beta}\bar{c}q)$, will yield, for $\beta = 1$, the correct form of the effective action in the Landau gauge. Then letting $\beta \to 0$, after making the familiar change of variables given by (3.12), Faddeev's canonical expression (1.2) is derived in the Coulomb gauge. Thus showing that the original covariant gauge fixed expression yields a unitary theory since their theorem states that the S-matrix will not depend on the value of β.

At first this seems to be exactly the argument we used in Sect. 3 to derive Faddeev's result (3.13) using the restricted *FV* theorem. Indeed, their choice of Ψ is precisely $\Psi_{\text{Fad}}(\beta)$, given in (3.8). Hence we might expect expression (3.6) to also give the correct path integral expression in the Landau gauge; this is *not* the case though!

The difference between the Fradkin-Vilkovisky argument and our discussion in Sect. 3 is that we defined the effective Hamiltonian by

$$H_{\text{eff}} = H_{\text{phys}} - \{\Omega, \Psi_{\text{Fad}}(\beta)\}. \tag{4.2}$$

The use of the physical Hamiltonian in (4.2) was central to the derivation of the restricted *FV* theorem. However, the difference between (4.1) and (4.2), $H_0 - H_{\text{phys}}$, is simply $\frac{1}{2}k^2p^2$ which can be written as $-\{\Omega, \int d^3x\, \frac{1}{2}p\bar{b}\}$ and hence absorbed into the form of Ψ. So we can put the effective Hamiltonian appropriate to the Landau gauge, (4.1) with $\beta = 1$, into the form used in the restricted *FV* theorem by writing

$$H_{\text{eff}} = H_{\text{phys}} - \{\Omega, \Psi_{\text{Lan}}\}, \tag{4.3}$$

where

$$\Psi_{\text{Lan}} = \int d^3k\,(\bar{b}(A_0 + \tfrac{1}{2}p) + i\bar{c}q). \tag{4.4}$$

The problem with this choice of Ψ in (3.6) now becomes apparent; the restricted FV theorem required Ψ to be a first class gauge fixing term, in particular, Ψ had to be abelian. But Ψ_{Lan} is *not* abelian, indeed

$$\{\Psi_{\mathrm{Lan}}, \Psi_{\mathrm{Lan}}\} = i \int d^3k \, \bar{c}\bar{b} \neq 0. \tag{4.5}$$

Thus there is no guarantee that the resulting partition function is the physical one.

Although, within this formalism, we cannot give general arguments for the acceptability of this, "non-abelian" type of gauge fixing, we might hope to be able to follow the FV argument, outlined above, for this specific choice of $\Psi = \Psi_{\mathrm{Lan}}$. So our strategy is to apply the canonical transformation (3.9) to Ψ_{Lan} and determine its β dependence. If all goes well and we do not pick up any nasty $1/\beta$ terms in the effective Hamiltonian, then we might be able to recover the statement that the Landau gauge is equivalent to the Coulomb gauge and hence unitary.

The new term in the effective Hamiltonian (4.3) is $\frac{1}{2}k^2p^2$. Using (3.11) we see that under the canonical transformation (3.9) we get

$$\begin{aligned} p^2 \to & \frac{1}{4\beta}\left(f_+^{1-\sqrt{\beta}} + f_-^{1-\sqrt{\beta}}\right)^2 p^2 + \frac{1}{4\beta^2k^2}\left(f_+^{1-\sqrt{\beta}} - f_-^{1-\sqrt{\beta}}\right)^2 \pi_0^2 \\ & + \frac{1}{2\beta^{3/2}\sqrt{k^2}}\left(f_+^{2(1-\sqrt{\beta})} - f_-^{2(1-\sqrt{\beta})}\right)^2 p\pi^0 + \text{ higher order ghost terms.} \end{aligned}$$

Now letting $\pi^0 \to \beta\pi^0$ ($\bar{c} \to \beta\bar{c}$) we get three terms (ignoring the ghost terms):

$$\begin{aligned} & \frac{1}{4\beta}\left(f_+^{1-\sqrt{\beta}} + f_-^{1-\sqrt{\beta}}\right)^2 p^2 + \frac{1}{4k^2}\left(f_+^{1-\sqrt{\beta}} - f_-^{1-\sqrt{\beta}}\right)^2 \pi_0^2 \\ & \quad + \frac{1}{2\sqrt{\beta}\sqrt{k^2}}\left(f_+^{2(1-\sqrt{\beta})} - f_-^{2(1-\sqrt{\beta})}\right)^2 p\pi^0. \end{aligned}$$

In order to recover (3.13) we must at least show that these terms are well behaved as $\beta \to 0$. But, even from the first term we see that this is not the case. Indeed, using (3.9) we see that

$$\frac{1}{4\beta}\left(f_+^{1-\sqrt{\beta}} + f_-^{1-\sqrt{\beta}}\right)^2 p^2 \approx \frac{1}{4\beta k^2} q^2 p^2,$$

which blows up as $\beta \to 0$. Hence we cannot derive the statement that the Landau gauge is unitary within this restricted Fradkin-Vilkovisky formalism.

Actually, what we have shown is that there are no general arguments for supposing that the Landau gauge is unitary within this approach, and that the blind application of the methods we developed in Sect. 3 to this case does not work. It might be thought, though, that with a bit of luck another, more complicated, canonical transformation could be found that did recover the steps in the Fradkin-Vilkovisky argument outlined above. However, we feel that, apart from being completely impractical, this is also not possible. In order to understand our reason for this statement let us recall some properties of the Lorentz class of gauges.

The Lorentz class of gauges are generated by $\Psi_{\rm Lor}(\lambda)$ where

$$\Psi_{\rm Lor}(\lambda) = \int d^3k\,(\bar{b}(A_0 + \tfrac{1}{2}p) + i\bar{c}(q + \frac{\lambda}{2}\pi^0)), \tag{4.6}$$

and λ is the gauge parameter. It is straightforward to see that

$$\{\Psi_{\rm Lor}(\lambda), \Psi_{\rm Lor}(\lambda)\} = i(1-\lambda)\int d^3k\,\bar{c}\bar{b}. \tag{4.7}$$

So for $\lambda = 1$, the Feynman gauge, we have $\Psi_{\rm Lor}(1)$ abelian. We still cannot appeal to our restricted version of the FV theorem to show unitarity for this gauge since we have only discussed point transformations, and with such transformations we clearly cannot get from $\Psi_{\rm Fad}$ to $\Psi_{\rm Lor}(1)$. However, because it is abelian, we can now determine the general way β has to enter this expression in order to recover the argument of Fradkin and Vilkovisky for the Feynman gauge.

In order to show unitarity, they required that $\Psi_{\rm Lor}(1)$ had the following dependency on β:

$$\Psi_{\rm Lor}(1,\beta) = \int d^3k\,(\bar{b}(A_0 + \tfrac{1}{2}p) + \frac{i}{\beta}\bar{c}(q + \frac{\lambda}{2\beta}\pi^0)). \tag{4.8}$$

However, since $\Psi_{\rm Lor}(1,1)$ was abelian, this β dependency cannot come from a canonical transformation. Instead we must have at least that

$$\Psi_{\rm Lor}(1,\beta) = \int d^3k\,(\bar{b}(A_0 + \frac{1}{2\beta}p) + \frac{i}{\beta}\bar{c}(q + \frac{\lambda}{2\beta}\pi^0)). \tag{4.9}$$

Hence the effective Hamiltonian will pick up a p^2/β term, as before, and we cannot show unitarity.

5. Conclusions

It is commonly thought that one can argue within the path integral formalism for the unitarity of, say, the Landau gauge, thus avoiding the complicated, perturbative demonstrations of this result. The Faddeev-Popov trick is recognized by many people as an instructive, but heuristic, path integral account of this. It is widely believed that the Fradkin-Vilkovisky theorem for phase space path integrals provides a, non-diagrammatic, proof of unitarity for such gauges. We have show in this paper that this is not the case and that the method of Fradkin and Vilkovisky, as it is applied to, for example, covariant gauges, is no more than a trick to recover the well know and well tested perturbative results.

Our arguments are based on the philosophy that, within the phase space path integral formalism, one should restrict attention to transformations that are canonical. From this point of view we have been able to derive a restricted version of the FV theorem, thus giving us confidence that this is the correct approach to take to these problems.

It is clear that the key obstacle we face in extending our restricted version of the FV theorem to more interesting gauge fixing conditions is the procedure of lumping the constraints into the $BRST$ charge and the gauge fixing conditions into Ψ, since we are then not able to take advantage of the general phase space result that locally any second class set of constraints can be made to look like the trivial set (q^a, p_a). This suggests that a more fruitful approach to these systems might be to start off by treating them as a second class system and then add ghosts for them all. Such an approach is under investigation.

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Appendix A. Some general results about graded phase spaces

In this appendix we wish to collect together the main results about graded phase spaces used in this paper. We shall restrict the discussion to finite dimensional systems; the assumption being that, with care, these results will go over unchanged into the field theoretic situation of interest to us. To a large extent this is just going to be a matter of making clear our conventions and keeping track of potentially errant signs. However, on top of this, we shall present a detailed analysis of the even point transformations. In order to do this we shall find it useful to work with the description of the Poisson brackets in terms of a symplectic form. As this is not a wholly familiar approach we shall briefly review the construction for a bosonic phase space before generalizing to the graded situation.

The important new perspective on constrained dynamics that emerges from the work of Batalin, Fradkin and Vilkovisky [9, 13] rests on the radical step of extending the classical dynamical arena from a phase space[3] $P = T^*Q$ to a graded, or super, phase space $\mathcal{P} = T^*\mathcal{Q}$; constructed over a graded extension, $\mathcal{Q}$, of the bosonic configuration space Q. Now a graded manifold is usually thought of as a space on which some of the coordinates commute and the others anti-commute. Geometrically this definition is quite impenetrable. However, one usually sidesteps this conceptual hurdle by restricting attention to the algebra of functions on the manifold. Thus, from our experience with exterior algebras, we are quite content to talk about grading functions as either even or odd and using the product rule for monomials:

$$f_1 f_2 = (-1)^{f_1 f_2} f_2 f_1 .$$

Here we follow the convention that $(-1)^f = \pm 1$, depending on whether f is even or odd. The graded extension of an algebra is then constructed by applying the heuristic rule of signs [17]: *If in some formula of usual algebra there are monomials with interchanged terms, then in the corresponding formula in a graded algebra every interchange of neighbouring terms, say f_1 and f_2, is accompanied by the multiplication of the monomial by the factor $(-1)^{f_1 f_2}$.*

A direct application of this rule to the Poisson algebra of functions on a phase space gives us a graded Poisson bracket which satisfies the algebraic relations:

(i) $\{f_1, f_2\} = (-1)^{f_1 f_2}\{f_2, f_1\}$;
(ii) $\{f_1 f_2, f_3\} = f_1\{f_2, f_3\} + (-1)^{f_2 f_3}\{f_1, f_3\} f_2$;
(iii) $\{f_1, \{f_2, f_3\}\} + (-1)^{f_1(f_2+f_3)}\{f_2, \{f_3, f_1\}\} + (-1)^{f_3(f_1+f_2)}\{f_3, \{f_1, f_2\}\} = 0$.

The canonical transformations are then the transformations on the graded phase space that preserves the graded Poisson bracket algebra. As we would expect, the infinitesimal canonical transformations are determined by the functions on the graded phase space. But, in the applications we have in mind, we really want to describe, in the most economic manner possible, the large canonical transformations. The most efficient way to do this in a normal phase space is to construct the *generating function* associated to any canonical transformation. The existence and basic properties of such a function follow most easily

[3]We adopt the useful notation that the symbol T^*Q simply denotes the natural phase space with configuration space Q; little of the geometry of such cotangent bundles will be needed in this paper.

if one works with the object dual to the Poisson bracket—the symplectic form. (For an excellent account of all such matters see [18].) Clearly we should try to follow a similar tack in this graded phase space context.

Such a point of view, though, does usually call for a more geometric description of the phase space since the symplectic form is tied-up with the differentiable structure of the space. Now it is possible to make precise the definition of a graded manifold (for an attractive approach to this see, for example, [17]). Rather than work in such generality, though, we shall content ourselves with the situation where the bosonic phase space, P, is simply the vector space[4] $T^*\mathbb{R}^n = \mathbb{R}^{2n}$. The advantage to working with such a phase space is that we can essentially replace symplectic geometry with symplectic algebra and hence appeal to the above rule of signs in order to construct its graded extension.

We recall [18] that a *symplectic form* ω on the vector space P is a nondegenerate, closed two form on P. By nondegenerate we simply mean that $\omega^n := \omega \wedge \cdots \wedge \omega$ (n-times) is a volume form on P; and we recall that ω is said to be closed if $d\omega = 0$, where d is the exterior derivative. Note that manifolds of the form T^*Q come equipped with a natural symplectic form and hence can always be given the structure of a phase space. From now on ω will denote this canonical symplectic form on P. For example, if we take (q^A), $A = 1, \ldots, n$, to be the standard cartesian coordinates on the configuration space $\mathbb{R}^n$, then there exists a canonical coordinate system (q^A, p_A) on $P = T^*\mathbb{R}^n$ such that the canonical symplectic form on P is given by

$$\omega = -dp_A \wedge dq^A. \tag{A.1}$$

Another way to understand the nondegeneracy of ω is to note that being a two-form on P simply means that given two vector fields X and Y on P then $\omega(X, Y) = -\omega(Y, X)$ is a function on P. Hence, given ω, we can construct a mapping from vector fields on P to 1-forms on P via $X \mapsto \omega(X, \ )$. Nondegeneracy then means that this is an invertible mapping. Our aim now is to show how these structures defined on the phase space can be used to construct the Poisson bracket.

Given a function f on P, df is a 1-form such that its inner product $df(X)$, with the vector field X, is the directional derivative of f in the direction X. The *Hamiltonian vector field* X_f, associated to f, is then defined by

$$\omega(X_f, Y) = df(Y), \tag{A.2}$$

for arbitrary vector field Y. The nondegeneracy of ω guarantees that such a vector field exists for all but the dynamically irrelevant constant functions. In terms of the canonical coordinates it is straightforward to see that

$$X_f = \frac{\partial f}{\partial p_A}\frac{\partial}{\partial q^A} - \frac{\partial f}{\partial q^A}\frac{\partial}{\partial p_A}. \tag{A.3}$$

The Poisson bracket $\{f, g\}$ between two functions f and g is then defined by

$$\begin{aligned}\{f, g\} &= -X_f(g) \\ &\equiv \omega(X_f, X_g).\end{aligned} \tag{A.4}$$

[4] Indeed, locally all $2n$-dimensional phase spaces look like this one; so this is no great restriction as far as our applications to gauge fixing is concerned.

The defining properties of the Poisson bracket then follow from the properties of the symplectic form; in particular, the Jacobi identity is seen to be equivalent to the closure of ω. Using (A.3) we see that this definition of the Poisson bracket agrees is the familiar one, and we recover the fundamental Poisson brackets $\{q^A, q^B\} = \{p_A, p_B\} = 0$ and $\{q^A, p_B\} = \delta^A_B$.

Canonical transformations are now seen to be those mappings (changes of coordinates) on P which preserve the symplectic form ω. An important class of such canonical transformations are the *point transformations*. These are defined as the canonical transformations induced by a change of coordinates on the configuration space. Thus if $(q^A, p_A) \to (\tilde{q}^A, \tilde{p}_A)$ then ω changes to $\tilde{\omega} = -d\tilde{p}_A \wedge d\tilde{q}^A$. In order for this to equal to ω we must have that

$$\tilde{p}_A = \frac{\partial q^B}{\partial \tilde{q}^A} p_B. \tag{A.5}$$

Since then

$$\tilde{\omega} = -\frac{\partial q^B}{\partial \tilde{q}^A} dp_B \wedge d\tilde{q}^A - p_B d\left(\frac{\partial q^B}{\partial \tilde{q}^A}\right) \wedge d\tilde{q}^A.$$

The first term is simply ω while the second is zero since

$$d\left(\frac{\partial q^B}{\partial \tilde{q}^A}\right) \wedge d\tilde{q}^A = d\left(\frac{\partial q^B}{\partial \tilde{q}^A} d\tilde{q}^A\right) = d(dq^B) = 0$$

using $d^2 = 0$.

In order to extend the above description of the Poisson bracket to a graded phase space $\mathcal{P}$ we need to agree on the conventions to be followed when taking derivatives: We shall use the right Grassmann derivative through out this paper. Thus, if η^a is an odd variable then

$$\frac{\partial \eta^a}{\partial \eta^b} = \delta^a_b$$

and

$$\frac{\partial}{\partial \eta^a}(f_1 f_2) = \frac{\partial f_1}{\partial \eta^a} f_2 + (-1)^{f_1} f_1 \frac{\partial f_2}{\partial \eta^a}.$$

If τ_1 is a graded r-form and τ_2 a graded s-form then the graded wedge product satisfies

$$\tau_1 \wedge \tau_2 = (-1)^{rs}(-1)^{\tau_1 \tau_2} \tau_2 \wedge \tau_1.$$

If we collectively denote the coordinate functions on $\mathcal{P}$ by (x^μ), then we define the exterior derivative by

$$d = dx^\mu \wedge \frac{\partial}{\partial x^\mu}.$$

This takes graded r-forms to graded $(r+1)$-forms; satisfies $d^2 = 0$ and the Leibniz rule

$$d(\tau_1 \wedge \tau_2) = (d\tau_1) \wedge \tau_2 + (-1)^r \tau_1 \wedge d\tau_2.$$

Finally, the innerproduct of a vector field $Y = Y^\mu \partial/\partial x^\mu$ with the 1-form dx^μ is given by

$$dx^\mu(Y) = (-1)^{Y^\mu} Y^\mu.$$

As we shall see in Appendix B, the graded extension to P that we wish to consider are those phase spaces constructed over a graded configuration space. Thus we now take as our configuration space the graded vector space $\mathcal{Q}$ with coordinate functions (q^A, η^a), $a = 1, \ldots, k \leqslant n$. Then the associated graded phase space is $\mathcal{P} = T^*\mathcal{Q}$, with canonical coordinates $(q^A, p_A, \eta^a, \rho_a)$. On $\mathcal{P}$ there is a natural graded symplectic form, which we shall also denote by ω, given by

$$\omega = -dp_A \wedge dq^A - d\rho_a \wedge d\eta^a. \tag{A.6}$$

The Hamiltonian vector field X_f is defined as in (A.2) but with this symplectic form. Hence we deduce that

$$X_f = \frac{\partial f}{\partial p_A}\frac{\partial}{\partial q^A} - \frac{\partial f}{\partial q^A}\frac{\partial}{\partial p_A} - (-1)^f \frac{\partial f}{\partial \rho_a}\frac{\partial}{\partial \eta^a} - (-1)^f \frac{\partial f}{\partial \eta^a}\frac{\partial}{\partial \rho_a}. \tag{A.7}$$

The graded Poisson bracket is then given by substituting this expression into (A.4). One can then directly verify the relation (i), (ii) and (iii) given above for this bracket. It is clear to see from this that the odd variables have the basic non-vanishing Poisson bracket $\{\eta^a, \rho_b\} = -\delta^a_b$.

Our aim now is to get some precise information about the structure of the generalized point transformations on this graded phase space; where, as we would expect, the canonical transformations are simply those transformations on $\mathcal{P}$ which preserve the graded symplectic form (A.6). In this analysis we shall restrict ourselves to even point transformations; i.e., the canonical transformations induced in $\mathcal{P}$ from the change of coordinates on $\mathcal{Q}$ which preserves the grading of the coordinate functions. So on $\mathcal{P}$ we have the change of coordinates

$$(q^A, p_A, \eta^a, \rho_a) \to (\tilde{q}^A, \tilde{p}_A, \tilde{\eta}^a, \tilde{\rho}_a),$$

and in order for this to be a canonical transformation we must have $\omega = \tilde{\omega}$.

We note from (A.6), and its tildered version, that $\omega = d\theta_1$ and $\tilde{\omega} = d\theta_2$, where the even 1-forms θ_1 and θ_2 are given by

$$\theta_1 = -p_A dq^A - \rho_a d\eta^a, \tag{A.8}$$

$$\theta_2 = \tilde{q}^A d\tilde{p}_A - \tilde{\eta}^a d\tilde{\rho}_a. \tag{A.9}$$

Hence the requirement that $\omega = \tilde{\omega}$ becomes

$$d(\theta_2 - \theta_1) = 0. \tag{A.10}$$

Which implies that

$$\theta_2 - \theta_1 = d\mathcal{F} \tag{A.11}$$

for some even function $\mathcal{F}$ on $\mathcal{P}$.

Since we are interested in point transformations we know that, in a neighbourhood of some point $x_0 = (q_0^A, p_A^0, \eta_0^a, \rho_a^0)$, we can take $(q^A, \tilde{p}_A, \eta^a, \tilde{\rho}_a)$ as independent coordinates on $\mathcal{P}$; i.e,

$$\operatorname{Ber}\left(\frac{\partial(\tilde{p}, \tilde{\rho})}{\partial(p, \rho)}\right)\bigg|_{x_0} \neq 0. \tag{A.12}$$

Here Ber is the Berezinian, or superdeterminant, of a graded matrix which is defined by

$$\operatorname{Ber} B = \det(B_1 - B_2 B_4^{-1} B_3) \det B_4^{-1}$$

where the matrix B of an even transformation is written in the standard form

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

with B_1 and B_4 even, and B_2 and B_3 odd entries.

Thus in (A.11) we can write $\mathcal{F} = \mathcal{F}(q^A, \tilde{p}_A, \eta^a, \tilde{\rho}_a)$ and expand out the right hand side to deduce that

$$\begin{aligned} p_A &= \frac{\partial \mathcal{F}}{\partial q^A} & \tilde{q}^A &= \frac{\partial \mathcal{F}}{\partial \tilde{p}_A} \\ \rho_a &= -\frac{\partial \mathcal{F}}{\partial \eta^a} & \tilde{\eta}^a &= \frac{\partial \mathcal{F}}{\partial \tilde{\rho}_a} \end{aligned} \tag{A.13}$$

Since we are considering point transformations, these equations tell us that $\mathcal{F}$ must be linear in the momenta.

Using the graded version of the implicit function theorem it is straightforward to reverse this construction. That is, if we are given $\mathcal{F}(q^A, \tilde{p}_A, \eta^a, \tilde{\rho}_a)$, an even function linear in the momenta, for which

$$\operatorname{Ber}\begin{pmatrix} \dfrac{\partial^2 \mathcal{F}}{\partial \tilde{p}\, \partial q} & \dfrac{\partial^2 \mathcal{F}}{\partial \tilde{\rho}\, \partial q} \\ \dfrac{\partial^2 \mathcal{F}}{\partial \tilde{p}\, \partial \eta} & \dfrac{\partial^2 \mathcal{F}}{\partial \tilde{\rho}\, \partial \eta} \end{pmatrix}\Bigg|_{x_0} \neq 0. \tag{A.14}$$

Then this function is the generating function of some even point transformation on $\mathcal{P}$.

As a simple application of the above result on generating functions consider the following:

Example 1. Let us take

$$\mathcal{F} = f^A(q)\tilde{p}_A - \eta^a \tilde{\rho}_a \quad \text{with } \det\left(\frac{\partial f^A}{\partial q^B}\right) \neq 0.$$

Then this generates the identity mapping on the odd variables and from (A.13) we recover the point transformation (A.5) on the even ones.

Example 2. A much more interesting transformation is generated by the function

$$\mathcal{F} = q^A \tilde{p}_A - \Lambda(q)^b_a \eta^a \tilde{\rho}_b, \tag{A.15}$$

where, from condition (A.14), we must have that Λ^b_a is an invertible matrix of functions of q^A. Then we find that this generates a point transformation with

$$\begin{aligned}
\tilde{q}^A &= q^A, \\
\tilde{p}_A &= p_A + (\Lambda^{-1})^b_{a,A} \Lambda^a_c \eta^c \rho_b, \\
\tilde{\eta}^a &= \Lambda^a_b \eta^b, \\
\tilde{\rho}_a &= (\Lambda^{-1})^b_a \rho_b.
\end{aligned}$$

This is in some sense complementary to Example 1; now we have essentially the identity mapping on the even variables and a simple rescaling of the odd ones. The existence of such a class of canonical transformations is central to the use of ghost variables in constrained dynamics [19].

More examples of generating functions will be given in Appendix B where we shall analyse the interplay of point transformations with the *BRST* charge and gauge fixing conditions.

The final topic we want to discuss in this appendix is the one clear place where graded manifolds differ from their even counterparts . For the bosonic phase space we have seen that the non-degeneracy of the symplectic form can be understood as the condition that ω^n is a volume for the phase space P. Indeed this is just the familiar Liouville volume element. It is now trivial that this volume element is invariant under canonical transformations since, by definition, they must preserve ω.

For the graded phase space the non-degeneracy of the graded symplectic form (A.6) can only mean that the induced mapping from graded vector fields to graded 1-forms on $\mathcal{P}$ is non-degenerate; it makes no sense to relate powers of ω to a volume since on a graded space there is no top differential form!

Berizin has introduced a set of formal rules for integrating functions on a graded space (see [17]). The natural application of his rules to the graded phase space $\mathcal{P}$ is to associate to the canonical coordinates $(q^A, p_A, \eta^a, \rho_a)$ the symbolic measure

$$d\mu = dq^A \, dp_A \, d\eta^a \, d\rho_a. \tag{A.16}$$

We stress, this is not directly related to the graded symplectic form (A.6). Thus the effect of canonical transformations on this symbolic measure needs to be separately determined.

Under a change of variables $d\mu$ is *defined* to transform in much the same way as a normal measure. Thus we have

$$d\mu \to d\tilde{\mu} = \mathrm{Ber}\left(\frac{\partial(\tilde{q}, \tilde{p}, \tilde{\eta}, \tilde{\rho})}{\partial(q, p, \eta, \rho)}\right) d\mu. \tag{A.17}$$

So, in order to show that this symbolic measure is invariant under a canonical transformation we need to show that the (point) canonical transformations have a graded Jacobian equal to 1. This is not too difficult to show either by direct calculation (combining the transformations given in Examples 1 and 2 above) or by using the graded version of the standard trick (see, for example, [20]) that the canonical transformations preserve the form of Hamiltons equations. We will omit the details of this straightforward calculation.

Appendix B. A restricted Fradkin-Vilkovisky theorem

In this appendix we wish to present a restricted form of the original Fradkin-Vilkovisky theorem. That is, we shall construct a general expression for the partition function of a first class constrained system which has an arbitrary function of a specific type in it. As we discussed in the introduction, in the original Fradkin-Vilkovisky theorem the arbitrary function had no restrictions on it. Here we shall show that in this approach it has to be a "first class gauge fixing term". We shall argue that this is the best one can do, with confidence, within the Fradkin-Vilkovisky approach to such systems.

We start with a general set of first class constraints on the phase space $P = T^*\mathbb{R}^n$. The first class constraints are a set of independent functions ϕ_a, $a = 1, \ldots, k \leqslant n$, on P which have the Poisson algebra

$$\{\phi_a, \phi_b\} = C^c_{ab}\phi_c, \tag{B.1}$$

for some structure functions C^c_{ab}. We shall actually restrict attention to constraints linear in momentum[5], hence $\phi_a = \phi^A_a(q)p_A$.

The kinematical step in the Batalin-Fradkin-Vilkovisky approach to such a system is to add a ghost variable η^a, and its conjugate ρ_a, for each constraint. Hence we enlarge the phase space from P to the graded phase space $\mathcal{P}$ discussed in Appendix A.

The functions on $\mathcal{P}$ have an additional integer grading given by their ghost number: A function f has ghost number n if $\{N, f\} = nf$, where $N = \rho_a\eta^a$.

On $\mathcal{P}$ we define the *BRST* charge Ω to be a function of ghost number 1 which is abelian; $\{\Omega, \Omega\} = 0$, and is of the form $\Omega = \phi_a\eta^a$+(higher order ghost terms). Although this does not fix Ω uniquely, there is a smallest Ω associated to the constraints (B.1) given by

$$\Omega = \phi_a\eta^a + \tfrac{1}{2}C^c_{ab}\rho_c\eta^a\eta^b. \tag{B.2}$$

The *BRST* charge Ω can be used to give a complete description of the physical observables for this system [12, 14].

Recall that we are interested in the situation where the constraints are linear in momenta. Given such a set of constraints then an equivalent set is obtained by rescaling $\phi_a \to \tilde{\phi}_a = \Lambda^b_a\phi_b$, for any invertible function $\Lambda^b_a(q)$ of the configuration space variables. It is a standard result that there exists a rescaling matrix Λ such that, locally, the new

[5] All constraints associated to a gauge symmetry are of this type. A much more difficult class of constraints arise when reparameterization invariance is relevant; then the constraints can be quadratic in the momenta. We have nothing to say in this paper about this important class of systems.

constraints are abelian and thus can, via a canonical transformation, be made into a set of momentum coordinates, p_a. Although this ability to trivialise the constraints seems at first sight very attractive, it has not had much of an impact on constrained systems since the initial step of rescaling is *not* a canonical transformation on P. (This is easy to see since, as we have stated, constraint rescaling can take us from a non-abelian set of constraints to an abelian set. But, canonical transformations must preserve the Poisson brackets and hence, would not allow this.)

One of the most attractive features of the Batalin-Fradkin-Vilkovisky approach to constrained systems is that on the extended graded phase space $\mathcal{P}$ there exists a wider class of even canonical transformations and, in particular, such rescalings can now be handled. The reason why this works is that now it is the *BRST* charge that characterises the constrained system and this charge is *always* abelian. Hence, there is no algebraic obstruction to a canonical change in Ω which mimics a rescaling of the constraints. As far as the *BRST* charge, (B.2), is concerned, a rescaling of the constraints can be reinterpreted as a rescaling of the ghost. Now, from Example 2 in Appendix A, we know how to construct such a transformation. Thus we are free to initially consider the situation where the constraints are of the pure momenta type, $p_a = 0$; having a *BRST* charge $\Omega = p_a\eta^a$.

Given that we have now trivialised Ω, another important class of point transformations on $\mathcal{P}$ are those that will preserve this simple form for the *BRST* charge. Consider the generating function

$$\mathcal{F} = g^i\tilde{p}_i + f^a\tilde{p}_a - \Lambda_a^b\eta^a\tilde{\rho}_b,$$

where the functions g^i, f^a and Λ_a^b are functions of the even configuration space variables q^A and we have denoted the additional, true degrees of freedom on P by (q^i, p_i), $i = k+1, \ldots, n$. Then $\Omega \to \tilde{\Omega} = \tilde{p}_a\tilde{\eta}^a$ where from (A.13) we have

$$\tilde{p}_a\tilde{\eta}^a = \left(\frac{\partial f^a}{\partial q^c}\right)^{-1}\Lambda_b^a p_c\eta^b - \left(\frac{\partial f^a}{\partial q^c}\right)^{-1}\frac{\partial g^i}{\partial q^c}\Lambda_b^a\tilde{p}_i\eta^b + \left(\frac{\partial f^a}{\partial q^c}\right)^{-1}\Lambda_e^a\frac{\partial\Lambda_b^d}{\partial q^c}\left(\Lambda^{-1}\right)_d^f\eta^b\eta^e\rho_f.$$

So, in order for $\Omega = \tilde{\Omega}$, we must have that the g^i's are just functions of the physical configurations g^i and, from the first and last terms, $\Lambda_a^b = \partial f^b/\partial q^a$. We recognise the first term in $\mathcal{F}$ as the generating function for point transformations on the true degrees of freedom. This discussion allows us to deduce the following important results:

Theorem 1. *The (trivialised) Ω-preserving point transformations on $\mathcal{P}$ have generating function*

$$\mathcal{F} = g^i(q^i)\tilde{p}_i + f^a(q)\tilde{p}_a - \frac{\partial f^b}{\partial q^a}\eta^a\tilde{\rho}_b.$$

The idea is to use this result in our description of the *FV* theorem to see, starting from some acceptable Ψ, what are the allowed variations in this "gauge fixing term". However, in order to get some feel for these transformations, and for the detailed applications to electrodynamics considered in this paper, we will give various examples of Ω-preserving point transformations and their effect on a specific Ψ. For this we shall restrict ourselves to just two constraints. Then the content of Theorem 1 can be summarised as:

Corollary. *If there are just two constraints then the Ω-preserving point transformations on the unphysical variables are given by*

$$\tilde{q}^a = f^a,$$
$$\tilde{\eta}^a = \frac{\partial f^a}{\partial q^b}\eta^b,$$

$$\tilde{\rho}_1 = \frac{1}{J}\frac{\partial f^2}{\partial q^2}\rho_1 - \frac{1}{J}\frac{\partial f^2}{\partial q^1}\rho_2, \qquad \tilde{\rho}_2 = -\frac{1}{J}\frac{\partial f^1}{\partial q^2}\rho_1 + \frac{1}{J}\frac{\partial f^1}{\partial q^1}\rho_2,$$
$$\tilde{p}_1 = \frac{1}{J}\frac{\partial f^2}{\partial q^2}\check{p}_1 - \frac{1}{J}\frac{\partial f^2}{\partial q^1}\check{p}_2, \qquad \tilde{p}_2 = -\frac{1}{J}\frac{\partial f^1}{\partial q^2}\check{p}_1 + \frac{1}{J}\frac{\partial f^1}{\partial q^1}\check{p}_2.$$

Where

$$J = \frac{\partial f^1}{\partial q^1}\frac{\partial f^2}{\partial q^2} - \frac{\partial f^1}{\partial q^2}\frac{\partial f^2}{\partial q^1}$$

and

$$\check{p}_a = p_a + \frac{\partial^2 f^c}{\partial q^a\,\partial q^b}\eta^b\tilde{\rho}_c.$$

Example 1. If we take

$$f^1 = q^1 - \frac{1}{q^2} \qquad f^2 = q^1 + \frac{1}{q^2}$$

in the above, then $\Psi = q^2\rho_1 + q^1\rho_2$ is "rotated" into $\Psi^r = q^1\rho_1 + q^2\rho_2$. The inverse transformation taking $\Psi^r = q^1\rho_1 + q^2\rho_2$ to $\Psi = q^2\rho_1 + q^1\rho_2$ is given by

$$f^1 = \tfrac{1}{2}(q^1 + q^2) \qquad f^2 = \frac{2}{q^2 - q^1}.$$

Example 2. If we take

$$f^1 = (q^1)^{\sqrt{\beta}} \qquad f^2 = (q^2)^{\sqrt{\beta}}$$

then we can boost $\Psi^r = \alpha q^1\rho_1 + \gamma q^2\rho_2$ into Ψ^r_{boost}, where

$$\Psi^r_{\text{boost}} = \tfrac{1}{\sqrt{\beta}}\alpha q^1\rho_1 + \tfrac{1}{\sqrt{\beta}}\gamma q^2\rho_2.$$

Example 3. We can generalise Example 1 by taking

$$f^1 = Aq^1 - \frac{B}{q^2} \qquad f^2 = Cq^1 + \frac{D}{q^2}$$

with A, B, C and D constants. Acting on $\Psi_{\alpha\gamma} = \alpha q^2\rho_1 + \gamma q^1\rho_2$ we get $\tilde{\Psi}_{\alpha\gamma} = \alpha f^2\tilde{\rho}_1 + \gamma f^1\tilde{\rho}_2$ where

$$\tilde{\rho}_1 = \frac{D}{CB+DA}\rho_1 + \frac{C}{CB+DA}\rho_2,$$
$$\tilde{\rho}_2 = \frac{B}{CB+DA}\rho_1 - \frac{A}{CB+DA}\rho_2.$$

Therefore $\tilde{\Psi}_{\alpha\gamma}$ is of the form $\tilde{\Psi}_{\alpha\gamma} = Eq^1\rho_1 + Fq^2\rho_2$ if and only if

$$\alpha D^2 = \gamma B^2 \quad \text{and} \quad \alpha C^2 = \gamma A^2.$$

In which case $E = F = \dfrac{\alpha CD + \gamma AB}{CB + DA}$. An example of such a generalised rotation is given by choosing $A = B = \sqrt{\alpha}$ and $C = D = \sqrt{\gamma}$. Then we get the transformaton

$$\alpha q^2\rho_1 + \gamma q^1\rho_2 \to \sqrt{\alpha\gamma}q^1\rho_1 + \sqrt{\alpha\gamma}q^2\rho_2.$$

The inverse transformaton to this is given by taking

$$\begin{aligned} f^1 &= \tfrac{1}{2}\left(\frac{1}{\sqrt{\alpha}}q^1 + \frac{1}{\sqrt{\gamma}}q^2\right), \\ f^2 &= \frac{2\sqrt{\alpha\gamma}}{\sqrt{\alpha}q^2 - \sqrt{\gamma}q^1}. \end{aligned}$$

Example 4. In the applications to electrodynamics we need a canonical transformation which preserves the trivialised $BRST$ charge and takes us from $\Psi = q^2\rho_1 + k^2q^1\rho_2$ to $\Psi_\beta = \frac{1}{\beta}q^2\rho_1 + k^2q^1\rho_2$, where β and k^2 are constants. This is actually quite difficult to do directly . Instead, we shall split the transformation up into three steps, summarised by the following diagram:

$$\begin{array}{ccc} \Psi = q^2\rho_1 + k^2q^1\rho_2 & \xrightarrow{\text{rotate}} & \Psi^r = \sqrt{k^2}q^1\rho_1 + \sqrt{k^2}q^2\rho_2 \\ & & \Big\downarrow \text{boost} \\ \Psi_\beta = \frac{1}{\beta}q^2\rho_1 + k^2q^1\rho_2 & \xleftarrow[\text{rotate}]{} & \Psi^r_{\text{boost}} = \frac{\sqrt{k^2}}{\sqrt{\beta}}q^1\rho_1 + \frac{\sqrt{k^2}}{\sqrt{\beta}}q^2\rho_2. \end{array}$$

Then, combining the previous three examples we see that taking

$$f^1 = \left(\tfrac{1}{2}(\frac{1}{\sqrt{k^2}}q^2 + \sqrt{\beta}q^1)\right)^{\sqrt{\beta}} - \left(\tfrac{1}{2}(\frac{1}{\sqrt{k^2}}q^2 - \sqrt{\beta}q^1)\right)^{\sqrt{\beta}}$$

and

$$f^2 = \sqrt{k^2}\left(\tfrac{1}{2}(\frac{1}{\sqrt{k^2}}q^2 + \sqrt{\beta}q^1)\right)^{\sqrt{\beta}} + \sqrt{k^2}\left(\tfrac{1}{2}(\frac{1}{\sqrt{k^2}}q^2 - \sqrt{\beta}q^1)\right)^{\sqrt{\beta}}$$

gives us a canonical transformation which preserves the $BRST$ charge and takes us from $\Psi = q^2\rho_1 + k^2q^1\rho_2$ to $\Psi_\beta = \frac{1}{\beta}q^2\rho_1 + k^2q^1\rho_2$ as required.

Our aim now is to use the above results to develop a restricted FV theorem. The first step in the analyse is to use the ability to trivialise the constraints using canonical transformations. We will then construct an expression for the physical partition function appropriate to such an abelian system. Finally, we will undo the constraint trivialisation step to recover an expression for this partition function in terms of the original, nonabelian constraints.

So, after trivialisation, we have a system with pure momentum constraints $p_a = 0$; the physical dynamics takes place on the phase space with true degrees of freedom (q^i, p_i) and with Hamiltonian $H_{\text{phys}}(q^i, p_i)$. Hence the physical partition function, Z_{phys}, is given by

$$Z_{\text{phys}} = \int dq^i dp_i \exp i \int dt \, (\dot{q}^i p_i - H_{\text{phys}}). \tag{B.3}$$

What we now want is a method for writing this expression as a partition function defined over the graded phase space $\mathcal{P}$ with canonical coordinates $(q^A, p_A, \eta^a, \rho_a)$. To see how this can be achieved we initially study the following path integral expression defined on the unphysical degrees of freedom:

$$I(\beta) = \int dq^a dp_a d\eta^a d\rho_a \exp i \int dt \, (\dot{q}^a p_a + \dot{\eta}^a \rho_a + \{\Omega, \Psi_\beta\}), \tag{B.4}$$

where the $BRST$ charge $\Omega = p_a \eta^a$ and $\Psi_\beta = \frac{1}{\beta} q^a \rho_a$.

This phase space path integral is formally invariant under canonical transformations. Extending Example 2 above, we see that the point transformation with generating function

$$\mathcal{F} = (q^a)^{\beta'/\beta} \tilde{p}_a - \tfrac{\beta'}{\beta} (q^a)^{\beta'/\beta - 1} \eta^a \tilde{\rho}_a,$$

will take $I(\beta)$ to $I(\beta')$. Hence, we deduce that $I(\beta)$ is independent of β. In order to evaluate what $I(\beta)$ is we make the change of variables

$$q^a \to \beta q^a, \qquad \rho_a \to \beta \rho_a,$$

which has super Jacobian equal to one. Note that this is the only place in our argument where we use a change of variables that is not canonical. Clearly, though, this simple rescaling is more benign than the class of non-canonical transformations used by Fradkin and Vilkovisky. As this is not a canonical transformation the Poisson bracket $\{\Omega, \Psi_\beta\}$ must be evaluated before making this change of variables. Then, in the limit as β goes to zero, we can perform the intergral to get the result that $I(\beta) = I(0) = 1$.

Inserting this expression into (B.3) yields the result

$$Z_{\text{phys}} = \int dq^A dp_A d\eta^a d\rho_a \exp i \int dt \, (\dot{q}^A p_A + \dot{\eta}^a \rho_a - H_{\text{phys}} + \{\Omega, \Psi_\beta\}). \tag{B.5}$$

It is clear that we can now apply general Ω-preserving point transformations to this result. The net effect of which will be to allow us to replace Ψ_β by a general Ψ of the form $\Psi = \chi^a \rho_a$, where $\chi^a(q)$ is a set of gauge fixing functions for the constraints p_a, i.e., a set of functions such that $\det |\partial \chi^a / \partial q^b| \neq 0$. Since this change in Ψ comes about through a canonical transformation, it will still be the case that Ψ will be abelian; i.e, $\{\Psi, \Psi\} = 0$. We call such a function a *first class gauge fixing term.*

Under such a transformation there will, in general, also be a change in the form of H_{phys}—reflecting the effect of gauge fixing on the form of the physical Hamiltonian. However, the new expression for H_{phys} will still satisfy the basic identities $\{\Omega, H_{\text{phys}}\} = 0$ and $\{\Psi, H_{\text{phys}}\} = 0$.

Finally, we can undo the constraint trivialisation step to arrive at the following restricted version of the FV theorem:

Theorem 2. *The physical partition function can be written as*

$$Z_{\text{phys}} = \int dq^A dp_A d\eta^a d\rho_a \exp i \int dt \, (\dot{q}^A p_A + \dot{\eta}^a \rho_a - H_{\text{eff}}),$$

where $H_{\text{eff}} = H_{\text{phys}} - \{\Omega, \Psi\}$, Ψ *is a first class set of gauge fixing conditions and* H_{phys} *satisfies* $\{\Omega, H_{\text{phys}}\} = \{\Psi, H_{\text{phys}}\} = 0$ *and* Ω *is given by (B.2).*

As we see from the applications to electrodynamics, such first class gauge fixing conditions do not cover all the types of gauge fixing needed in practice. However, this is the best one can do if the constraints and gauge fixing conditions are treated in this separate manner and we are relying on Ω-preserving point transformations. An alternative approach to this problem would be to lump both the constraints and gauge fixing conditions together into a second class set of constraints and then add ghosts for them all [15]. This will be discussed elsewhere.

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