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On Some Hamiltonian Models of Brownian Motion.

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Abstract: The Langevin and Fokker-Planck structures of two phase-space gaussian Markov processes are investigated in terms of their algebraic properties.

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1.Introduction

The long-standing problem of deriving irreversible behaviour from reversible, Hamiltonian, laws of motion received a great impulse from the seminal work of Ford, Kac and Mazur [1] who:

i) were able to account for the Brownian behaviour of a subsystem by embedding it in a linear chain of altogether interacting harmonic oscillators, the whole system being in equilibrium at a given temperature T;

ii) because of linearity, opened in turn the way to the subsequent theory of Quantum Stochastic Processes, by exhibiting a Quantum Langevin Equation.

Two points emerged clearly: on one hand the need for an infinitely extended linear chain, otherwise quasiperiodicity would forbid decaying of correlations and on the other the difficulty of an harmonic oscillator interpretation of the heat bath that would oblige to consider infinitely strong couplings. The latter was not a serious drawback and, already repaired in part by the authors themselves, found a final and consistent adjustment in a subsequent development of those initial ideas [2,3] that showed how the previous picture were one among other possible realizations of a more general and far-reaching scheme: the construction of a linear Hamiltonian heat bath able to induce Brownian behaviour on one of its components, to which models like Ford, Kac and Mazur's, Lamb's, Schwabl-Thirring's and Planck's are proved to be be isomorphic [3]. It was at the level of giving a quantum version of such a linear Hamiltonian heat bath that the strategy as it stands met its limits [3,4]. Nevertheless some fundamental structures [5,6,7], already well-known in ergodic theory as Kolmogorov systems [8], made their appearance, and transported in a non-commutative frame, produced a first attempt at a definition of a quantum Kolmogorov-Sinai entropy [10] for general type III von Neumann algebras, line of thought that has been recently revived [11,10,13] and developed into a quantum ergodic theory. In this note we look at the classical-quantum differences how they show up within the above sketched approach with particular reference to the Langevin and Fokker-Planck descriptions which are showed to be equivalent classically and to depart quantum mechanically. To make the content self-consistent, we briefly review in passing the necessary background and results.

2. Brownian Motion, Hilbert Space Stochastic Processes and Kolmogorov Systems

2.1. Brownian Motion [14,15]

We shall be concerned with the following two classical Brownian particles described by the Langevin systems of stochastic differential equations:

$$L1: \begin{cases} \begin{cases} dq = \frac{p}{m} dt \\ dp = -m\omega^2 q dt - \frac{\gamma}{m} p dt + \sigma dW(t) \end{cases} \\ < W(s)W(t) >= min(s,t) \qquad \forall s,t \geq 0 \\ \sigma^2 = 2\gamma kT \end{cases}$$

$$L2: \begin{cases} \begin{cases} dq = -\gamma q dt + \sigma_q dW_q(t) \\ dp = -\gamma p dt + \sigma_p dW_p(t) \end{cases} \\ < W_q(s)W_p(t) >= 0 \\ < W_q(s)W_q(t) >=< W_p(s)W_p(t) >= min(s,t) \ldots s,t \geq 0 \\ \sigma_p^2 = 2\gamma mkT \\ \sigma_q^2 = \frac{2\gamma kT}{m\omega^2}. \end{cases}$$

Remarks 2.1.1.

1. $W(t)$, $W_q(t)$ and $W_p(t)$ are Wiener processes so that the corresponding stochastic forces $\sigma\frac{dW(t)}{dt}$, $\sigma_q\frac{dW_q(t)}{dt}$ and $\sigma_p\frac{dW_p(t)}{dt}$ have white noise time correlations. σ, σ_q and σ_p are chosen to satisfy, in equilibrium at temperature T, the principle of equipartition of energy.

2. L1 and L2 are equivalent to the Fokker-Planck equations

$$FP1: \partial_t \rho_t(q,p) = \{ -\frac{p}{m}\partial_q + m\omega^2 q\partial_p + \frac{\gamma}{m}\partial_p p + \gamma kT\partial_p^2 \} \rho_t(q,p)$$

$$FP2: \partial_t \rho_t(q,p) = \gamma\{\partial_p p + \partial_q q + mkT\partial_p^2 + \frac{kT}{m\omega^2}\partial_q^2 \} \rho_t(q,p).$$

These are evolution equations for probability density distributions on R^2 which have as stationary solutions the Gibbs measure at inverse temperature $\beta = \frac{1}{kT}$:

$$\rho_\beta(q,p)dqdp = \frac{\omega\beta}{2\pi}\exp\left(-\frac{\beta}{2}\left(\frac{p^2}{m} + m\omega^2 q^2\right)\right)dqdp.$$

FP1 and FP2, or, equivalently, L1 and L2, describe classical, stationary Markov processes, in particular, the r.h.s. of FP1 and FP2 are the generators of contraction Markov semigroups on the space of probability distributions over R^2.

3. All the properties of a given probability density distribution $\rho(q,p)$ are fixed by its characteristic function:

$$\rho\{\mathrm{W}(x,y)\} \equiv \int_{R^2} dqdp e^{i(xp+yq)}\rho(q,p),$$

which amounts to be the expectation of the exponential function $\mathrm{W}(x,y)(q,p)$ w.r.t. the given probability measure.

4. From a physical point of view, whenever $\rho(q,p)$ is absolutely continuous w.r.t. the equilibrium measure, we speak of a local perturbation of the equilibrium state.If it is then let evolve according to FP1,2, it stays absolutely continuous, returns back to the unperturbed state and the system exhibites tendency to equilibrium. This is best seen in terms of the positivity of entropy production, or monotonicity of the relative entropy which seems indeed to be a key word even in a quantum setting [13,16,17]:

$$\begin{cases} S(\rho_\beta,\rho_t) \equiv \int_{R^2} dqdp\rho_t(q,p)\, ln\left\{\frac{\rho_t(q,p)}{\rho_\beta(q,p)}\right\} \geq 0, \\ = 0 \quad \text{iff} \quad \rho_t(q,p) = \rho_\beta(q,p) \text{ a.e.}, \\ \frac{d}{dt}S(\rho_\beta,\rho_t) \leq 0. \end{cases}$$

To account for the type of convergence of perturbed states to equilibrium, we make an algebraic choice, namely, we consider the abelian von Neumann algebra $L^\infty(R^2,\rho_\beta)$ formed by the ρ_β-essentially bounded functions on R^2 that act as multiplication operators on the Hilbert space $L^2(R^2,\rho_\beta)$.This algebra is linearly generated by the exponential functions by taking the operator strong closure on the Hilbert space. This gives us means to pass from a Schrödinger like picture, in which the state ρ evolves into ρ_t, to a Heisenberg like one in which the operators $W(x,y)$ evolve into $W_t(x,y)$. Obviously the state ρ on $L^\infty(R^2,\rho_\beta)$ is given by integration w.r.t. the probability measure and the corresponding duality

$$\rho_t\{W(x,y)\} = \rho\{W_t(x,y)\}$$

allows us to deduce the Heisenberg equations of motion:

$$H1: \partial_t W_t(x,y) = \{-\frac{\gamma}{m}x\partial_x - m\omega^2 x\partial_y + \frac{y}{m}\partial_x - kT\gamma x^2\}\, W_t(x,y)$$
$$H2: \partial_t W_t(x,y) = -\gamma\{x\partial_x + y\partial_y + mkTx^2 + \frac{kT}{m\omega^2}y^2\}\, W_t(x,y).$$

2.2.Hilbert Space Stochastic (H-) Processes.

The strategy in the construction of a heat bath that may lead to L1 and L2, is to seek a reversible evolution over a larger system which reproduce by restriction to the subsystem embedded in it a semigroup distinguished by a strong damping term. Embedding and conditioning are indeed the leading concepts that come up in the dilation theory of contraction semigroups on Hilbert spaces [3,7,18].

Definition 2.2.1 [3,7]

A K-based, stationary, Markov H-process in a real Hilbert space **H** equipped with a one parameter group of orthogonal transformations $\{U_t\}_{t\in R}$, is a strongly continuous family $\{j_t\}_{t\in R}$ of isometries from a real Hilbert space **K** onto subspaces of **H** such that:

$$\begin{cases} i) & j_t : \mathbf{K} \to \mathbf{H}_t \equiv j_t[\mathbf{K}] \subseteq \mathbf{H}, \\ ii) & j_t = U_t j_o, \\ iii) & S_{t-s} \equiv j_s^* j_t = j_o^* U_{t-s} j_0 \quad \text{is its covariance and satisfies} \\ iv) & S_s S_t = S_{t+s} \quad \text{on } \mathbf{K} \quad \text{for s,t} \geq 0. \end{cases}$$

The process is called regular if $\{S_t\}_{t\geq 0}$ contracts strongly to zero on **K**.

It turns out that the converse is also true.

Theorem 2.2.2.

Given a strongly continuous one-parameter semigroup $\{\mathbf{S}_t\}_{t\geq 0}$, strongly contracting to zero on a real Hilbert space $\mathbf{K}$, then there exists a regular, stationary, Markov H-process based on $\mathbf{K}$, in a real Hilbert space $\mathbf{H}$ which has $\mathbf{S}_t$ as its covariance when $t \geq 0$.

Proof

This is Theorem 3.13 in [18].

In our examples

$$\mathbf{S}_t = \begin{cases} \exp\left\{t\begin{pmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & -\frac{\gamma}{m}\end{pmatrix}\right\} \\ \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & e^{-\gamma t}\end{pmatrix} \end{cases}$$

are strongly contracting on R^2 equipped with the energy norm

$$||\mathbf{k}(x,y)||_E^2 = m\omega^2 x^2 + \frac{y^2}{m}$$

which will be the space $\mathbf{K}$. Because of Kolmogorov theory of Positive Definite Kernels [18, Chapt.1], we know that the various possible triples $(\mathbf{H}, \{\mathbf{j}_t\}_{t\in R}, \{\mathbf{U}_t\}_{t\in R})$ that decompose the kernel $< \mathbf{k}, \mathbf{S}_{t-s}\mathbf{k}' >_E$ are unitarily equivalent [18, Theorem 3.15] and one of them is the following [3, Lemma 2.4]:

$$D1: \begin{cases} \mathbf{H} = L^2(R \to R, dx) \\ (\mathbf{U}_t\psi)(x) = \psi(x-t)\ \forall\psi \in \mathbf{H} \\ \left[\mathbf{j}_t\mathbf{k}(x,y)\right](s) = \Theta(t-s)\frac{\sqrt{2\gamma}}{m}\left[\mathbf{S}_{t-s}\mathbf{k}(x,y)\right]_2\ \forall\mathbf{k}(x,y) \in \mathbf{K}, \forall t \geq 0; \end{cases}$$

($[\cdot]_2$ means the vector second component)

$$D2: \begin{cases} \mathbf{H} = L^2(R \to \mathbf{K}, dx) \\ (\mathbf{U}_t\psi)(x) = \psi(x-t)\ \forall\psi \in \mathbf{H} \\ \left[\mathbf{j}_t\mathbf{k}(x,y)\right](s) = \Theta(t-s)e^{-\gamma(t-s)}\mathbf{k}(x,y)\ \forall\mathbf{k}(x,y) \in \mathbf{K}, \forall t \geq 0. \end{cases}$$

So far we have accomodated the semigroup but the very same dilation technique provides us with the route to stochasticity (diffusive term in L1, L2).

Proposition 2.2.3.

The family $\{\mathrm{j}_t\}_{t\in R}$ fulfils the Hilbert space Langevin equation:

$$[\mathrm{j}_t\mathbf{k}](x) - [\mathrm{j}_o\mathbf{k}](x) =$$

$$\begin{cases} \int_0^t ds\,[\mathrm{j}_s\mathbf{G}\mathbf{k}](x) + \frac{\sqrt{2\gamma}}{m}[\mathbf{k}]_2\chi_{[o,t]}(x) & \mathbf{G} = \begin{pmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & -\frac{\gamma}{m} \end{pmatrix} \\ \int_0^t ds\,[\mathrm{j}_s\mathbf{G}\mathbf{k}](x) + \mathbf{k}\chi_{[o,t]}(x) & \mathbf{G} = -\gamma\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

Proof

This is Theorem 3.15 in [18].

Diffusion enters the picture through Minlos'Theorem, once we have observed that:

$$\int_{-\infty}^{+\infty} dx\chi_{[o,s]}(x)\chi_{[o,t]}(x) = min(s,t)\ \forall s,t \geq 0.$$

Indeed we have:

Theorem 2.2.4. [3,19,20,21]

It is possible to embed isometrically $\mathbf{H}$ into a Hilbert space $L^2(\Omega,\mu_\beta)$ of square-summable functions over a probability space Ω with probability measure μ_β such that, if

$$\mathbf{\Phi} : \mathbf{H} \ni \psi \to \mathbf{\Phi}_\psi(\omega) \in L^2(\Omega,\mu_\beta),$$

and

$$\mathbf{W}(\psi) \equiv \exp\left(i\mathbf{\Phi}_\psi(\cdot)\right) \in L^2(\Omega,\mu_\beta),$$

then

$$\mu_\beta\left[\mathbf{W}(\psi)\right] = \exp\left(-\frac{||\psi||^2}{2\beta}\right).$$

The net result following from above is that $\chi_{[o,t]}(\cdot)$ goes into a gaussian stochastic variable $w(t) \equiv \mathbf{\Phi}_{\chi_{[o,t]}}(\cdot)$ and $\mu_\beta[w(s)w(t)] = \frac{1}{\beta}min(s,t)\,\forall s,t \geq 0$. So we get

Proposition 2.2.5.

The couple of orthogonal vectors in $\mathbf{K}$

$$\begin{cases} \mathbf{e}_1 \equiv \mathbf{k}(-\frac{1}{m\omega^2}, 0) \\ \mathbf{e}_2 \equiv \mathbf{k}(0, m) \end{cases}$$

is isometrically mapped into gaussian stochastic variables on (Ω, μ_β),

$$\begin{cases} \mathbf{Q}_t \equiv \Phi_{\mathbf{j}_t \mathbf{e}_1} \\ \mathbf{P}_t \equiv \Phi_{\mathbf{j}_t \mathbf{e}_2} \end{cases}$$

satisfying L1 and L2, when the family of isometries $\{\mathbf{j}_t\}_{t\in R}$ is given according to D1 and D2 respectively.

Proof

The choice of $\mathbf{e}_1$ and $\mathbf{e}_2$ produces the right linear dependence between them and $\mathbf{G}\mathbf{e}_{1,2}$ so that the Liouville part of the r.h.s. of L1 and L2 is achieved.On the other hand the Wiener processes

$$\begin{cases} \mathbf{w}(t) \equiv \Phi_{\chi_{[o,t]}\sqrt{2\gamma}} \\ \mathbf{w}_q(t) \equiv \Phi_{\chi_{[o,t]}\sqrt{2\gamma}\mathbf{e}_1} \\ \mathbf{w}_p(t) \equiv \Phi_{\chi_{[o,t]}\sqrt{2\gamma}\mathbf{e}_2} \end{cases}$$

are easily verified to obey the conditions required in L1,L2.

Minlos' Theorem gives rise to the Hilbert space $L^2(\Omega, \mu_\beta)$ as a Fock like space over the generating gaussian subspace $\Phi[\mathbf{H}]$ [19,20], the translation operator $\mathbf{U}_t$ provides an orthogonal operator on $L^2(\Omega, \mu_\beta)$ that sends $\mathbf{W}[\psi]$ into $\mathbf{W}[\mathbf{U}_t\psi]$ and which, moreover, preserves the gaussian measure.What is showed in [3] is that a symplectic structure of an infinite Hamiltonian system in equilibrium at temperature T can be constructed so that it fits into the above infinite dimensional Hilbert space with the given invariant measure μ_β to which all the other linear models like Ford,Kac and Mazur's, Lamb's, Scwhabl-Thirring's and Planck's isomorphically correspond (see [2] and [3] for a detailed account of the models). Needless to say that it is rather intriguing that an infinite linear chain of coupled harmonic oscillators, under certain conditions, those giving rise to Brownian behaviour, appears physically equivalent to a semi-infinite string with an harmonic oscillator attached to the constrained end as is the case in the Lamb model.

The single particle description is now in terms of the gaussian stochastic variables $\mathbf{Q}, \mathbf{P}$ or of the exponential functions

$$\mathbf{W}(x,y) \equiv \begin{cases} e^{i(x\mathbf{Q}+y\mathbf{P})} = \\ \mathbf{W}\big[y\mathbf{j}_o\mathbf{e}_1 + x\mathbf{j}_o\mathbf{e}_2\big] = \\ \mathbf{W}\big[\mathbf{j}_o\mathbf{D}\mathbf{k}(x,y)\big] \quad \mathbf{D} = \begin{pmatrix} 0 & -\frac{1}{m\omega^2} \\ m & 0 \end{pmatrix} \end{cases}$$

We are thus led to

Proposition 2.2.6.

Given the abelian von Neumann algebra $L^\infty(\Omega, \mu_\beta) \equiv \mathbf{M}$ of μ_β-essentially bounded functions on Ω acting as multiplication operators on $L^2(\Omega, \mu_\beta)$, we have:

i) the abelian von Neumann algebra $\mathbf{M}_o \subset \mathbf{M}$ generated by $\{\mathbf{W}[\psi] : \psi \in \mathbf{j}_o\mathbf{K}\}$ is isomorphic to $L^\infty(R^2, \rho_\beta)$;

ii) the evolution given on $L^2(\Omega, \mu_\beta)$ provides a group of automorphisms $\alpha_t : \mathbf{M} \to \mathbf{M}$ that preserves the state ω_β given on $\mathbf{M}$ by integration w.r.t. μ_β;

iii) the abelian von Neumann algebra $\mathbf{M}_o$ is not invariant under α_t and

$$\alpha_t\big[\mathbf{W}(x,y)\big] = e^{i(x\mathbf{P}_t + y\mathbf{Q}_t)},$$

where $\mathbf{P}_t$ and $\mathbf{Q}_t$ satisfy the Langevin equations L1, L2.

Proof

We only note that the correspondence

$$L^\infty(R^2, \rho_\beta) \ni \mathrm{W}(x,y) \longleftrightarrow \mathbf{W}\big[\mathbf{j}_o\mathbf{D}\mathbf{k}(x,y)\big] \in \mathbf{M}_o$$

is such that

$$\begin{aligned} \rho_\beta\big[\mathrm{W}(x,y)\big] &= \exp\Big(-\frac{kT}{2\omega^2}[m\omega^2 x^2 + \frac{y^2}{m}]\Big) \\ &= \exp\Big(-\frac{1}{2\beta}||\mathbf{D}\mathbf{k}(x,y)||_E^2\Big) \\ &= \mu_\beta\big\{\mathbf{W}\big[\mathbf{j}_o\mathbf{D}\mathbf{k}(x,y)\big]\big\} \end{aligned}$$

Once the Langevin picture is obtained, we still need the Fokker-Planck equation for the theory to be complete: the latter will be provided by a system of conditional expectations: we have infact seen that the single particle subalgebra M is not α_t-invariant.

2.3. Kolmogorov (K-)Systems.

We shall formulate algebraically some well-known classical ergodic concepts [5,6,8,10,12,13].

Definition 2.3.1.

A dynamical system consists of a triple $(\mathrm{M}, \{\alpha_t\}_{t\in R}, \omega)$, where:

$$\begin{cases} \mathrm{M} \quad \text{is a von Neumann algebra ;} \\ \{\alpha_t\}_{t\in R} \quad \text{is a group of automorphisms of}\, \mathrm{M}; \\ \omega \quad \text{is an invariant state on}\, \mathrm{M}. \end{cases}$$

Definition 2.3.2.

The dynamical system $(\mathrm{M}, \{\alpha_t\}_{t\in R}, \omega)$ is a K-system whenever a subalgebra $\mathrm{M}_{o]}$ exists such that:

$$\begin{cases} i)\ \mathrm{M}_{s]} \equiv \alpha_s(\mathrm{M}_o) \subseteq \mathrm{M}_{t]} \equiv \alpha_t\{\mathrm{M}_{o]}\}\ \forall s \leq t; \\ ii)\ \bigvee_{s\geq 0} \mathrm{M}_{s]} = \mathrm{M} \quad (\bigvee \text{ means algebraic generation}); \\ iii)\ \bigwedge_{s\leq 0} \mathrm{M}_{s]} = \lambda I \quad (\bigwedge \text{ means algebraic intersection}). \end{cases}$$

Proposition 2.3.3.

The dynamical system $(\mathrm{M} = L^\infty(\Omega, \mu_\beta), \{\alpha_t\}_{t\in R}, \omega_\beta)$ of Proposition 2.2.6. is a K-system.

Proof

Given the isometries $\mathrm{j}_t : \mathrm{K} \to \mathrm{H}$, we have the orthogonal projections

$$\Pi_{t]} \equiv \bigvee_{s\leq t} \mathrm{j}_s\mathrm{j}_s^* : \mathrm{H} \to \mathrm{H}_{t]} \equiv \bigvee_{s\leq t} \mathrm{j}_s[\mathrm{K}]$$

such that

$$\Pi_{t]} = U_t \Pi_{o]} U_{-t}\,,$$

as $j_t = U_t j_o$.

The regularity of the semigroup $S_t = j_o^* U_t j_o$ on K is equivalent to $\bigwedge_{t \le 0} j_t j_t^* = 0$ (Lemma 2.3 in [3]), thence:

$$\begin{cases} i)\ \Pi_{s]} \le \Pi_{t]} \quad \forall s \le t, \\ ii)\ \bigvee_{s \ge 0} \Pi_{s]} = I, \\ iii)\ \bigwedge_{s \le 0} \Pi_{s]} = 0. \end{cases}$$

Let $M_{o]}$ the subalgebra of M generated as strong closure on $L^2(\Omega, \mu_\beta)$ by

$$\{W[\psi] \ : \psi \in \Pi_{o]} H\}$$

then the family $M_{t]} \equiv \alpha_t(M_{o]})$ of subalgebras generated by

$$\{W[\psi] \ : \psi \in \Pi_{t]} H\}$$

fulfils the conditions for giving M the properties of a K-system.

Remarks 2.3.4.

1. Associated with the family $\{M_{t]}\}$ there are conditional expectations $E_{t]} : M \to M_{t]}$, that satisfy:

$$i)\ E_{t]}\Big[W[\psi]\,W[\phi]\Big] = W[\psi]\,E_{t]}\Big[W[\phi]\Big]\ \forall \psi \in \Pi_{t]}H,\ \phi \in H,$$

$$ii)\ E_{t]} = \alpha_t \cdot E_{o]} \cdot \alpha_{-t},$$

$$iii)\ E_{s]} \cdot E_{t]} = E_{s]}\ \forall s \le t,$$

$$iv)\ \mu_\beta \cdot E_{t]} = \mu_\beta\ \forall t,$$

$$v)\ E_{t]}\Big[W[\psi]\Big] = W[\Pi_{t]}\psi] \cdot \mu_\beta\{W[(I - \Pi_{t]})\psi]\}.$$

They correspond indeed to Hilbert space projectors onto $L^2(\Omega_{t]}, \mu_\beta)$ where the latter is the Fock like space which has $\Pi_{t]}H$ as a generating subspace [19,20].

2. The following natural semigroup $\{\Gamma_t\}_{t\geq 0}$ on $\mathbf{M}$ arises [7,13]:

$$\Gamma_t = \mathbf{E}_{o]} \cdot \alpha_t.$$

which can be also viewed as a semigroup on $L^2(\Omega, \mu_\beta)$ because of previous remark.It is then worth noticing that as such the semigroup appears to be the covariance of a stationary, regular, Markov $L^2(\Omega, \mu_\beta)$-process based on $L^2(\Omega_{o]}, \mu_\beta)$ and acting on the orthogonal complement of the constant functions in $L^2(\Omega, \mu_\beta)$, the Kolmogorov decomposition being given by

$$\begin{cases} \mathbf{j}_t \ : L^2(\Omega_{t]}, \mu_\beta) \hookrightarrow L^2(\Omega, \mu_\beta) \\ \mathbf{j}_t^* = \mathbf{E}_{t]} \ : L^2(\Omega, \mu_\beta) \rightarrow L^2(\Omega_{t]}, \mu_\beta). \end{cases}$$

The presence of this state-preserving, contraction semigroup is quite independent from any other considerations but the existence of a Kolmogorov structure within the underlying algebra or Hilbert space. Our a priori knowledge that the structure has come up from dilating the semigroup $\mathbf{S}_t$ gives:

Proposition 2.3.5.

Let $\mathbf{E}_o$ be the expectation $\mathbf{M} \rightarrow \mathbf{M}_o$ and $\mathbf{i}_o$ the corresponding embedding $\mathbf{M}_o \hookrightarrow \mathbf{M}$, then

$$\gamma_t \equiv \mathbf{E}_o \cdot \Gamma_t \cdot \mathbf{i}_0 \ : \mathbf{M}_o \rightarrow \mathbf{M}_o \ \forall t \geq 0$$

is the Markov semigroup associated with the Fokker-Planck equations FP1 and FP2.

Proof

It is immediately verified that $\{\gamma_t\}_{t\geq 0}$ is indeed a semigroup, by taking into account that

$$a) \ \mathbf{M}_o \subseteq \mathbf{M}_{o]} \Rightarrow \mathbf{E}_o \cdot \mathbf{E}_{o]} = \mathbf{E}_o$$

$$\begin{aligned} b) \ \gamma_t\{\mathbf{W}(x,y)\} &= \mathbf{E}_o\{\mathbf{W}[\mathbf{U}_t\mathbf{j}_o\mathbf{D}\mathbf{k}(x,y)]\} = \\ &= \mathbf{W}[\mathbf{j}_o\mathbf{S}_t\mathbf{D}\mathbf{k}(x,y)] \exp\{-\frac{1}{2\beta}||\mathbf{D}\mathbf{k}(x,y)||_E^2 - ||\mathbf{S}_t\mathbf{D}\mathbf{k}(x,y)||_E^2]\} = \\ &= \mathbf{W}(\mathbf{S}_t^*\mathbf{k}(x,y)) \exp\{-\frac{1}{2\beta\omega^2}[||\mathbf{k}(x,y)||_E^2 - ||\mathbf{S}_t^*\mathbf{k}(x,y)||_E^2]\} \end{aligned}$$

where

$$\mathbf{D}\mathbf{S}_t^* = \mathbf{S}_t\mathbf{D} \quad \mathbf{D} = \begin{pmatrix} 0 & -\frac{1}{m\omega^2} \\ m & 0 \end{pmatrix},$$

see discussion after Proposition 2.2.5.. By appealing to Proposition 2.2.6. and a time derivative we can check that H1 and H2 are satisfied.

3. Heat Baths Quantized.

3.1. Quantum Kolmogorov Systems.

Definitions 2.3.1. and 2.3.2. are well suited to embody some kind of non-commutative ergodicity for they are formulated in the most general algebraic terms.Nevertheless we encounter a first basic departure from the classical case.

Proposition 3.1.1. [13]

Given a dynamical K-system $(\mathbf{M}, \alpha, \omega)$ and the G.N.S. triple $(\mathbf{H}_\omega, \mathbf{U}_\omega, \mathbf{\Omega}_\omega)$ based on it, there are two natural semigroups

$$\begin{cases} \mathbf{\Gamma_t} = \mathbf{E}_{o]} \cdot \alpha_t & \text{on } \mathbf{M} \\ \mathbf{\Gamma}_t^* = \mathbf{\Pi}_{o]} \cdot \mathbf{U}_\omega^t & \text{on } \mathbf{H}_\omega \end{cases}$$

associated with the Kolmogorov structure $\{\mathbf{M}_{t]}\}_{t\in R}$. The two coincide iff the conditional expectations $\mathbf{E}_{t]} : \mathbf{M} \to \mathbf{M}_{t]}$ are canonical, that is $\omega \cdot \mathbf{E}_{t]} = \omega \ \forall t$.

Remarks 3.1.2.

1. In the classical case of an abelian von Neumann algebra $(\mathbf{H}_\omega, \mathbf{U}_\omega, \mathbf{\Omega}_\omega)$ is unitarily equivalent to $(L^2(\Omega, \mu_\beta), \mathbf{U}_t, \mu_\beta)$ and as any conditioning is canonical no differences arise. Quantum mechanically things change in that a canonical conditional expectation (projection of norm one) exists if and only if the algebra onto which it focuses is invariant under the modular automorphism, σ_ω, relative to the state ω, supposed to be faithful [22].Thus, if σ_ω^t happens to coincide with α_t there is no chance how could $\mathbf{\Gamma_t}$ describe an irreversible evolution of $\mathbf{M}$ that has ω as an equilibrium state.Moreover, irreversibility at the algebraic level seems to be somewhat disconnected from its Hilbert space counterpart whereas they coincide classically.

2. There is to mention a second point at which classical and quantum Kolmogorov systems differ from each other. Classically the algebraic structure is equivalent to strict positivity of the Kolmogorov-Sinai(K-S-)dynamical entropy which is a measure of the predictability of the future stages in the history of a dynamical system, whenever a finite grained knowledge of all its past is given [8,12]. Strict positivity of the K-S-entropy implies complete memory loss of the initial conditions and if we use the recently developed concept of non -commutative dynamical entropy [11] to extend this fact to give an entropic characterization of quantum K-systems [12] we see that a distinction has to be made between Algebraic Quantum K-Systems and Entropic Quantum K-Systems [23].
3. From the point of view of tendency to equilibrium, the properties of Algebraic Quantum K-systems can find a nice analog to what discussed in Remark 2.1.1.4. concerning the monotonic behaviour of quantum relative entropy and entropy production [13,16,17], whereas stronger decorrelations (clustering properties) seem to be offered by Entropic Quantum K-systems [24].

Remark 3.1.2.1 is of the most importance in outlining the properties of Hamiltonian models of heat baths. If we want to quantize any of the available linear models, what we have to do is to construct a representation of the Canonical Commutation Relations over the symplectic infinite linear structure made fit in the infinite dimensional Hilbert space $\mathbf{H}$, that be compatible with equilibrium conditions at temperature T. This has been done in [3,25] by using the CCR representation based on a quasifree state, KMS w.r.t. the translation operator $\mathbf{U}_t$ on $\mathbf{H}$.
There are two main intriguing features and corresponding drawbacks in the ensuing boson system: a quantum Langevin equation is constructed for quantum position and momentum operators and tendency to equilibrium indeed show up to a certain extent [3,25]. On the other hand the momentum observable exhibits wild fluctuations and the quantum noise as well so that they cannot fit into the algebra of observables and point instead to a renormalized theory. Moreover, according to Remark 3.1.2.1., there is no way to obtain a Quantum Markov semigroup that might have the KMS-state as

an invariant state, by using canonical expectations. Indeed, the equilibrium state has, as its own modular automorphism, the one which arises from the translation operator $\mathbf{U}_t$ on the basic space $\mathbf{H}$ and, therefore, will move any of the $\mathbf{M}_{t]}$ out of itself by definition. Embedding and conditioning have been the key tools in the construction of a general heat bath able to characterize the irreversible behaviour of a single particle system in the classical setting. Embedding a semigroup of contractions in a unitary group on a larger space provides the Kolmogorov structure of the latter and in turn that of the ensuing algebra of observables, the noise in the Langevin equations being the result of the existence of the family of moving subalgebras $\{\mathbf{M}_{[t]}\}_{t\in R}$ [19]. The very same Kolmogorov structure, on the other hand, which plays a fundamental role in this context, is at the basis of the Fokker-Planck description that arises through conditioning with respect to the past history of the sytem. This twofold consistent picture of irreversibility breaks up in a quantum setting where embedding works, apart from divergences, but conditioning does not. The only way out is to consider sytems, utterly different, in spirit and construction, in which the equilibrium state is translation-invariant, but KMS w.r.t. another evolutions that leaves, instead, all the subalgebras $\mathbf{M}_{t]}$ invariant and makes things look classical.

3.2.2. Modified Quantum Heat Baths.

In this last paragraph we follow the theory of Dilations of Quasi-Free Dynamical Semigroups [18, Chapt. 10,11,12] and apply it to the cases considered.

Proposition 3.2.1. [18, Chapt.12]

Let $\{\mathbf{S}_t : t \geq 0\}$ be a strongly continuous semigroup of contractions on a Hilbert space $\mathbf{K}$ and

$$(\mathbf{H}, \{\mathbf{j}_t\}_{t\in R}, \{\mathbf{U}_t\}_{t\in R})$$

the triple arising from its dilation. For each $\lambda \geq 1$ there is a strongly continuous semigroup

$$\{\hat{\mathbf{W}}_\lambda(\mathbf{S}_t)\}_{t\geq 0}$$

of completely positive contraction maps on the CCR representation over $\mathbf{H}$, $\mathbf{W}_\lambda(\mathbf{H})$, such that:

$$\begin{cases} 1)\ \hat{\mathbf{W}}_\lambda(\mathbf{S}_t) = \hat{\mathbf{W}}_\lambda(\mathbf{j}_o^*)\hat{\mathbf{W}}_\lambda(\mathbf{U}_t)\hat{\mathbf{W}}_\lambda(\mathbf{j}_o)\ \forall t \geq 0 \\ 2)\ \hat{\mathbf{W}}_\lambda(\mathbf{j}_o) \text{ embeds the CCR algebra over } \mathbf{K} \text{ into } \mathbf{W}_\lambda(\mathbf{H}) \text{ and} \\ 3)\ \hat{\mathbf{W}}_\lambda(\mathbf{j}_o^*) \text{ is the corresponding expectation} \\ 4)\ \hat{\mathbf{W}}_\lambda(\mathbf{S}_t)\Big\{\hat{\mathbf{W}}_\lambda\big[\mathbf{j}_o\mathbf{k}\big]\Big\} = \hat{\mathbf{W}}_\lambda\big[\mathbf{j}_0\mathbf{S}_t\mathbf{k}\big]\exp\big\{-\frac{\lambda}{4}\big[||\mathbf{k}||_\mathbf{K}^2 - ||\mathbf{S}_t\mathbf{k}||_\mathbf{K}^2\big]\big\}\ \forall \mathbf{k} \in \mathbf{K}. \end{cases}$$

The above CCR representation is determined by the positive definite functional

$$\mu_\lambda\Big[\hat{\mathbf{W}}_\lambda[\psi]\Big] = e^{-\lambda||\psi||^2}$$

over the Weyl operators $\big\{\hat{\mathbf{W}}_\lambda[\psi] : \psi \in \mathbf{H}\big\}$ satisfying:

$$\hat{\mathbf{W}}_\lambda[\psi]\hat{\mathbf{W}}_\lambda[\phi] = \hat{\mathbf{W}}_\lambda[\psi + \phi]\, e^{\frac{i}{2}\mathrm{Im}<\psi|\phi>}$$

[18, Chapt.7]. With the choice

$$\begin{cases} \coth\dfrac{\hbar\omega\beta}{2},\ \beta = \dfrac{1}{kT} \\ \psi = \mathbf{j}_o\mathbf{D}\mathbf{k}(x,y),\ \mathbf{D} = \sqrt{\dfrac{\omega}{\hbar}}\begin{pmatrix} 0 & -\frac{1}{m\omega^2} \\ m & 0 \end{pmatrix} \\ \mathbf{k}(x,y) \in \mathbf{K} \equiv \big\{R^2, ||\cdot||_E\big\} \end{cases}$$

(see Proposition 2.2.5,6.), we have:

$$\begin{aligned} \omega_\beta\Big\{\hat{\mathbf{W}}_\lambda\big[\mathbf{j}_0\mathbf{D}\mathbf{k}(x,y)\big]\Big\} &\equiv \mu_\lambda\Big\{\hat{\mathbf{W}}_\lambda\big[\mathbf{j}_o\mathbf{D}\mathbf{k}(x,y)\big]\Big\} \\ &= \exp\big\{-\frac{1}{4\hbar\omega}\coth\frac{\hbar\omega\beta}{2}||\mathbf{k}(x,y)||_E^2\big\} \end{aligned}$$

which is the expectation of the one-particle Weyl operator

$$\hat{\mathrm{W}}(x,y) = e^{\frac{i}{\hbar}(x\hat{p}_\hbar + y\hat{q}_\hbar)}$$

in the Gibbs state

$$\hat{\rho}_\beta = \frac{\exp\big\{-\beta\big[\frac{\hat{p}_\hbar^2}{2m} + \frac{m\omega^2}{2}\hat{q}_\hbar^2\big]\big\}}{Tr\ \exp\big\{-\beta\big[\frac{\hat{p}_\hbar^2}{2m} + \frac{m\omega^2}{2}\hat{q}_\hbar^2\big]\big\}}.$$

The state ω_β on $\mathbf{W}_\lambda(\mathbf{H})$ defined by the functional μ_λ is the equilibrium state at temperature T w.r.t. the evolution σ_t given by:

$$\sigma_t\Big\{\hat{\mathbf{W}}_\lambda[\psi]\Big\} = \hat{\mathbf{W}}_\lambda\big[e^{i\omega t}\psi\big]$$

i.e. the modular automorphism of ω_β [9,10]. On the other hand we have on $\mathbf{H}$ the translation operator $\mathbf{U}_t$ that gives $\mathbf{H}$ its Kolmogorov structure.
The latter can be lifted to $\mathbf{W}_\lambda(\mathbf{H})$, as done in Proposition 2.3.3.for the classical case, providing the family of subalgebras $\mathbf{W}_{t]} \equiv \mathbf{W}(\mathbf{H}_{t]})$ fulfilling the conditions for a K-system and that, above all, are left globally invariant by σ_t. Thence, the expectations $\hat{\mathbf{E}}_{t]} : \mathbf{W}_\lambda(\mathbf{H}) \to \mathbf{W}_{t]}$ generate a state-preserving semigroup, in agreement with Remark 3.2.1.:

$$\begin{aligned}\hat{\mathbf{\Gamma}}_t\Big\{\hat{\mathbf{W}}_\lambda\big[\psi\big]\Big\} &= \hat{\mathbf{E}}_{t]} \cdot \alpha_t\Big\{\hat{\mathbf{W}}_\lambda\big[\psi\big]\Big\} = \hat{\mathbf{E}}_{t]}\Big\{\hat{\mathbf{W}}_\lambda\big[\mathbf{U}_t\psi\big]\Big\} \\ &= \hat{\mathbf{W}}_\lambda\big[\mathbf{\Pi}_{t]}\psi\big]\omega_\beta\Big\{\hat{\mathbf{W}}_\lambda\big[(\mathbf{I}-\mathbf{\Pi}_{t]})\psi\big]\Big\}.\end{aligned}$$

The subalgebra $\mathbf{W}_\lambda(\mathbf{j}_o\big[\mathbf{K}\big])$, describing the one-particle system, is left invariant as well and, upon identification of $\hat{\mathrm{W}}(x,y)$ with $\hat{\mathbf{W}}_\lambda\big[\mathbf{j}_o\mathbf{D}\mathbf{k}(x,y)\big]$, we eventually get:

$$\begin{aligned}\hat{\gamma}_t\{\hat{\mathrm{W}}(x,y)\} &= \hat{\mathrm{W}}(\mathbf{S}_t^*\mathbf{k}(x,y))\cdot \\ &\cdot \exp\Big\{-\frac{1}{4\hbar\omega}\coth\frac{\hbar\omega\beta}{2}\big[||\mathbf{k}(x,y)||_E^2 - ||\mathbf{S}_t^*\mathbf{k}(x,y)||_E^2\big]\Big\}\end{aligned}$$

where $\mathbf{S}_t\mathbf{D} = \mathbf{D}\mathbf{S}_T^*$ and $\mathbf{S}_t^*$ is the same of Proposition 2.3.5..

Remarks 3.2.2.

1. If $\mathrm{S}_t = e^{-\gamma t}$, then

$$\hat{\mathrm{W}}(x,y) \equiv \hat{\gamma}_t\{\hat{\mathrm{W}}(x,y)\} = \hat{\mathrm{W}}(e^{-\gamma t}(x,y))\cdot\exp\Big\{-\frac{1}{4\hbar\omega}||\mathbf{k}(x,y)||_E^2\big[1-e^{-2\gamma t}\big]\Big\}$$

which is studied in [9,10] as a model of quantum diffusion in a harmonic well.

2. It is straightforward to connect $\hat{\gamma}_t$ with Γ_t of proposition 2.3.5. by letting $\hbar \to 0$. We follow [26] and introduce the symmetric representation

$$\begin{cases} \hat{q}_\hbar = \sqrt{\hbar}\hat{q}, & (\hat{q}\psi)(x) = x\psi(x) \\ \hat{p}_\hbar = \sqrt{\hbar}\hat{p}, & (\hat{p}\psi)(x) = -i\psi'(x)\end{cases}$$

on $L^2(R, dx)$, and the overcomplete family of coherent states

$$\big\{|q,p> \equiv e^{i(p\hat{q}+q\hat{p})}|0>\big\}_{(q,p)\in R^2}$$

$$<x|0> = \frac{e^{-\frac{x^2}{2}}}{\sqrt[4]{\pi}}.$$

After setting $(x(t), y(t)) = \mathbf{S}_t^*(x,y)$, we find

$$\lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}}\,\frac{p}{\sqrt{\hbar}}|\hat{\gamma}_t\{\hat{\mathrm{W}}(\hbar x, \hbar y)\}|\frac{q}{\sqrt{\hbar}}\,\frac{p}{\sqrt{\hbar}} > =$$

$$= \left\{ \lim_{\hbar\to 0} < \frac{q}{\sqrt{\hbar}}\,\frac{p}{\sqrt{\hbar}}|e^{i\sqrt{\hbar}\left[x(t)\hat{p}+y(t)\hat{q}\right]}|\frac{q}{\sqrt{\hbar}}\,\frac{p}{\sqrt{\hbar}} > \right\}.$$

$$\cdot \exp\left\{ -\frac{\hbar}{4\omega}\coth\frac{\hbar\omega\beta}{2}\left[||\mathbf{k}(x,y)||_E^2 - ||\mathbf{S}_t^*\mathbf{k}(x,y)||_E^2\right]\right\} =$$

$$= \mathrm{W}((\mathbf{S}_t^*\mathbf{k}(x,y)) \cdot \exp\left\{ -\frac{kT}{2\omega^2}\left[||\mathbf{k}(x,y)||_E^2 - ||\mathbf{S}_t^*\mathbf{k}(x,y)||_E^2\right]\right\}.$$

(see Proposition 2.3.5.).

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