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## A Remark on the Twin “Paradox”.

M.A. Vandyck<sup>(1)</sup>

Physics Department, University College, Cork, Ireland

### Abstract

The Twin “Paradox” is investigated using a method that emphasises the symmetry between the twins. All the calculations are performed using the Schwarzschild metric, making the result valid in General Relativity as well in the special case of Minkowski space.

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(1) Permanent address: Cork Regional Technical College, Rossa Avenue, Bishopstown, Co. Cork, Ireland and Dublin Institute for Advanced Studies, Dublin 4, Ireland.

## 1. Introduction

Most textbooks on Special Relativity contain an analysis of the well-known Twin Paradox. (See, e.g. [1-14].) The discussion is usually formulated as follows:

Let  $K$  be the frame in which Twin 1 is at rest at the origin  $x = 0$ . At  $t = 0$ , Twin 2 leaves the origin and, after a journey in  $K$ , comes back and discovers that he is younger than Twin 1. The same situation, analysed from  $K'$ , the frame in which Twin 2 is at rest at the origin  $x' = 0$ , might seem to lead to the contradictory result that Twin 1 is now younger than Twin 2 since, in  $K'$ , it is Twin 2 who is at rest and Twin 1 who moves.

The “paradox” is resolved by saying that, if  $K$  is an inertial frame, then  $K'$  cannot be inertial since it is accelerated with respect to  $K$ . This breaks the symmetry of the problem and forbids to draw the (erroneous) conclusion that, as calculated in  $K'$ , Twin 1 is younger than Twin 2.

This explanation is correct and has been analysed by various authors. (See e.g. [15] quoted in [1].) However, to the best of our knowledge, a calculation *restoring the symmetry* of the problem is not available in the literature, and it might therefore not be devoid of interest to present it in this brief note. In other words, it is possible to reformulate the question so that the same treatment applies to both twins, provided one handles appropriately the non-inertial frame  $K'$ .

Given that we have to deal with accelerated frames, it is most convenient to use the formalism of General Relativity applied to Minkowski space. It turns out that this discussion can easily be generalised to the Schwarzschild space-time. We shall thus treat the case of the general-relativistic Twin Paradox in a Schwarzschild background. (This obviously contains the Minkowskian Twin Paradox as the special case  $m = 0$ , where  $m$  is the Schwarzschild mass.)

The development will be conducted in three steps: In Section 2, we shall ob-

tain the ages of the twins as calculated in a frame  $K$  adapted to the Schwarzschild metric. Then, in Section 3, we shall establish the transformation rule linking  $K$  to  $K'$ , the rest frame of Twin 2. Finally, the ages of the two twins will be obtained from an analysis in  $K'$ . In the conclusion, a symmetry argument will then show that the predictions made by the two twins agree with one-another.

## 2. Analysis in the Frame $K$

In this section, we are going to calculate the ages of the twins as predicted by an observer using Schwarzschild coordinates, i.e. in the frame  $K$  defined by (1) below. We shall restrict attention to radial motion in the Schwarzschild geometry  $M$ .

In the tangent space  $TM$  and its dual  $T^*M$ , we shall use respectively the bases

$$[\mathbf{e}] \equiv \left[ \frac{\partial}{\partial t}, \frac{\partial}{\partial r} \right] \quad \text{and} \quad [\theta] \equiv \left[ \frac{dt}{dr} \right], \quad (1)$$

in such a way that the Schwarzschild metric reads, in matrix notation:

$$g = [\theta]^T [g] [\theta] \quad (2)$$

$$[g] \equiv \begin{bmatrix} g_{00} & 0 \\ 0 & g_{11} \end{bmatrix}, \quad g_{00} = -g_{11}^{-1} = 1 - 2m/r. \quad (3)$$

(The coordinates  $t, r$  are the standard Schwarzschild coordinates, and the set of vectors  $[\mathbf{e}]$  in (1) will be referred to as “Frame  $K$ ”.)

The world line of Twin  $i$ ,  $i = 1, 2$ , is represented by the curve  ${}^i\Gamma$  depending on a parameter  ${}^i\lambda$ :

$$\begin{aligned} {}^i\Gamma &\equiv {}^i\mathbf{x} = {}^i\mathbf{x}({}^i\lambda), \quad {}^i\lambda_0 \leq {}^i\lambda \leq {}^i\lambda_1 \\ &\equiv [\mathbf{e}] \begin{bmatrix} {}^iT({}^i\lambda) \\ {}^iR({}^i\lambda) \end{bmatrix}, \end{aligned} \quad (4)$$

where  ${}^i T$  and  ${}^i R$  denote respectively the  $t$  and  $r$  components of the position  ${}^i \mathbf{x}$  of Twin  $i$ . It would be possible to assume, e.g. that  ${}^1 \Gamma$  is a geodesic, i.e. that Twin 1 is freely falling, but this is unnecessary since the treatment below is valid for arbitrary world lines  ${}^i \Gamma$ . We are thus studying a slight generalisation of the Twin Paradox.

By virtue of (4), the tangent vector  ${}^i \mathbf{t}$  to  ${}^i \Gamma$  is given by<sup>†</sup>

$${}^i \mathbf{t} = [\mathbf{e}] \begin{bmatrix} {}^i \dot{T}({}^i \lambda) \\ {}^i \dot{R}({}^i \lambda) \end{bmatrix} , \quad (5)$$

where a dot over a function denotes the derivative of this function with respect to its argument. As a result, the proper-time lapse  $\Delta {}^i \tau$  along  ${}^i \Gamma$  reads, by (2), (3), (5):

$$\Delta {}^i \tau = \int_{{}^i \lambda_0}^{{}^i \lambda_1} (g({}^i \mathbf{t}, {}^i \mathbf{t}))^{1/2} d {}^i \lambda \quad (6)$$

$$= \int_{{}^i \lambda_0}^{{}^i \lambda_1} (g_{ab} {}^i t^a {}^i t^b)^{1/2} d {}^i \lambda \quad (7)$$

$$= \int_{{}^i \lambda_0}^{{}^i \lambda_1} (g_{00} {}^i \dot{T}^2 + g_{11} {}^i \dot{R}^2)^{1/2} d {}^i \lambda \quad (8)$$

$$= \int_{{}^i \lambda_0}^{{}^i \lambda_1} {}^i \gamma^{-1} {}^i \dot{T} d {}^i \lambda \quad (9)$$

$${}^i \gamma \equiv (g_{00} + g_{11} {}^i \beta^2)^{-1/2} , \quad {}^i \beta \equiv {}^i \dot{R}/{}^i \dot{T} . \quad (10)$$

We are now going to prove that, when calculated in  $K'$ , the rest frame of Twin 2, the expressions for  $\Delta {}^i \tau$ ,  $i = 1, 2$ , are identical with (9). Geometrically, this is obvious since, by (6),  $\Delta {}^i \tau$  involves a tensor contraction of the metric and the tangent vector to the trajectory. Such a contraction is invariant under any frame change. In particular, the result of the integration (6) must have the same numerical value when calculated in the Schwarzschild frame  $K$  and the rest frame

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<sup>†</sup> We assume, without loss of generality, that the parameter  ${}^i \lambda$  is chosen so that  ${}^i \dot{T} > 0$ .

$K'$  of Twin 2. However, it is enlightening to do the calculation in  $K'$  in some detail since it shows the origin of the asymmetry between the twins in the standard treatment, as explained in the conclusion.

### 3. Analysis in the Frame $K'$

The frame  $K'$  is the frame in which Twin 2 is at rest. This means that the time vector  $\mathbf{e}'_{(0)}$  of  $K'$  is the proper velocity  $\mathbf{u}$  of Twin 2. The spatial vector  $\mathbf{e}'_{(1)}$  is then most conveniently taken orthogonal to  $\mathbf{e}'_{(0)}$  and of square norm  $-1$ .

The calculation of  $\mathbf{u}$  in  $K$  is elementary:

$$\begin{aligned}\mathbf{u} &\equiv \frac{d^2 \mathbf{x}}{d^2 \tau} \\ &= \frac{d^2 \mathbf{x}}{d^2 \lambda} \left( \frac{d^2 \tau}{d^2 \lambda} \right)^{-1} \\ &= [\mathbf{e}] \begin{bmatrix} {}^2 \dot{T} \\ {}^2 \dot{R} \end{bmatrix} \left( \frac{d^2 \tau}{d^2 \lambda} \right)^{-1},\end{aligned}\tag{11}$$

where  ${}^2 \tau$  is the proper time of Twin 2, and use has been made of (4). The derivative term in (11) is evaluated from (9) as

$$d^2 \tau = {}^2 \gamma^{-1} {}^2 \dot{T} d^2 \lambda, \tag{12}$$

in such a way that the final form of (11) reads:

$$\mathbf{u} = [\mathbf{e}] \begin{bmatrix} 1 \\ {}^2 \beta \end{bmatrix} {}^2 \gamma \tag{13}$$

Furthermore, there is no difficulty in finding a vector  $\mathbf{w}$  orthogonal to  $\mathbf{u}$  and of square norm  $-1$  as

$$\mathbf{w} = [\mathbf{e}] \begin{bmatrix} {}^2 \beta \chi \\ \chi^{-1} \end{bmatrix} {}^2 \gamma \tag{14}$$

$$\chi \equiv (|g_{11}|/g_{00})^{1/2}. \tag{15}$$

Consequently,  $K'$  is obtained from  $K$  by the transformation

$$[e'] \equiv [e'_{(0)}, e'_{(1)}] = [u, w] = [e]B \quad (16)$$

$$B \equiv {}^2\gamma \begin{bmatrix} 1 & {}^2\beta\chi \\ {}^2\beta & \chi^{-1} \end{bmatrix} . \quad (17)$$

(In the special case of the Minkowski metric,  $g_{00} = -g_{11} = 1$ , and  $B$  is then clearly a Lorentz transformation.)

Under a change (16) of frame, the components of an arbitrary (contravariant) vector  $V$  and of the metric  $g$  transform as

$$V = [e][V] = [e'] [V'] , \quad [V'] = B^{-1}[V] \quad (18)$$

$$[g'] = B^T [g] B . \quad (19)$$

This implies, for the matrix  $B$  given by (17), the tangent vectors  ${}^i t$  of (5) and the metric (2), (3), that the proper-time lapses  $\Delta {}^i \tau$  as calculated in  $K'$ , the rest frame of Twin 2, read:

$$\Delta {}^i \tau = \int_{i\lambda_0}^{i\lambda_1} (g'_{ab} {}^i t'^a {}^i t'^b)^{1/2} d {}^i \lambda \quad (20)$$

$$= \int_{i\lambda_0}^{i\lambda_1} ([{}^i t']^T [g'] [{}^i t'])^{1/2} d {}^i \lambda \quad (21)$$

$$= \int_{i\lambda_0}^{i\lambda_1} \{ (B^{-1} [{}^i t])^T (B^T [g] B) (B^{-1} [{}^i t]) \}^{1/2} d {}^i \lambda \quad (22)$$

$$= \int_{i\lambda_0}^{i\lambda_1} (g_{ab} {}^i t^a {}^i t^b)^{1/2} d {}^i \lambda , \quad (23)$$

in agreement with the prediction (7) made in the Schwarzschild frame  $K$ . We shall now conclude with the analysis of the results.

#### 4. Conclusion

In the light of the considerations of Sections 2 and 3, the argument of the absence of paradox in the twin problem can be reformulated as follows: We proved that the time lapses obtained in the rest frame of Twin 2 agree with those obtained in the Schwarzschild frame. By interchanging the roles of the twins, this means that the results in the rest frame of Twin 1 *also* agree with those in the Schwarzschild frame, and consequently that the predictions of the twins agree with one-another. It is important to realise that, *in our framework*, the two twins are equivalent since their world lines  ${}^1\Gamma$  and  ${}^2\Gamma$  were treated as completely arbitrary curves. It was, for instance, *not* assumed that one of them was a geodesic. In other words, our method restores the symmetry of the problem.

In the standard Special-Relativistic treatment, one usually restricts attention to global inertial frames. This means that one considers only holonomic frames in which the metric  $g$  equals the Minkowski metric  $\eta$ . Then, with the same notation as in (7), the proper-time lapses  $\Delta^i\tau$  of the twins are written as

$$\Delta^i\tau = \int_{i\lambda_0}^{i\lambda_1} (\eta_{ab} {}^it^a {}^it^b)^{1/2} d^i\lambda \quad . \quad (24)$$

In a non-inertial frame, the metric cannot be assumed globally Minkowskian, and the transformation law (19) *must* be taken into account<sup>†</sup>. Such a framework is, therefore, necessarily asymmetric since (24) cannot hold in both rest frames of the twins.

The fact that the rest frame of at least one twin is non-inertial appears very clearly in our framework since, by (17), the frame  $[e']$  is, in general, non-holonomic and the connection is non-vanishing (even in the Minkowskian case  $g_{00} = -g_{11} =$

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<sup>†</sup> It is indeed obvious from (21), (22) that the outcomes of the calculations would *not* be the same in  $K$  and  $K'$  if the *transformed* metric  $B^T[g]B$  were not used, but rather the Minkowskian value  $[\eta]$ .



1) when  $B$  depends on the position. Moreover, this must be so for at least one twin if the two twins are to meet again after completing their journey through space.

Finally, it should also be noted that we never had to exploit the explicit values of the metric functions  $g_{00}$  and  $g_{11}$ . The conclusions thus hold for all the metrics of the form (2), (3), irrespective of  $g_{00}$  and  $g_{11}$ .

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