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The Eta Function in Chern-Simons Field Theory

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Abstract

We discuss a Schwinger expansion technique for computing the η-function of a first order operator in the pure Chern-Simons quantum field theory. When evaluated at zero, the η-function of this operator gives essentially the one-loop correction to the partition function. We illustrate this technique by explicitly computing the one-loop 2-point function in this theory on a flat spacetime background.

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1 Introduction

Chern-Simons field theory has proven to be a useful framework for understanding and generalizing knot and link invariants [1]. One can also show how certain 2 dimensional lattice models arise naturally together with the notion of quantum groups [2]. This gives a unifying 3 dimensional viewpoint on all of these interesting systems. One important technical aspect of Chern-Simons theory is that it gives topological invariants of framed 3-manifolds with a prescription for the behavior under a change in framing. Under a frame change, the partition function picks up a non-trivial phase factor. Witten has shown that the origin of this phase can be traced to a one-loop background field calculation where a certain determinant is observed to have this phase factor. As such, one can see that this is a perturbative effect, the result being that the effective action at one-loop is a shifting of the classical Chern-Simons action [1]. Further confirmation of this result was given in [3] using a Pauli-Villars (higher derivative) regularization scheme.

In this paper, we use - and extend - a regularization technique of McKeon and Sherry [4] (called operator regularization) to explicitly evaluate this phase. The extended method involves an evaluation of the η-function associated to the relevant first order operator in this theory. We find that at the one-loop order, the 2-point $< A\ A >$ function is non-zero in agreement with [1, 3]. This corrects a point made in [5, 6] where a calculation yielded no one-loop correction to this Green's function. A further discussion of this issue can be found in Section 4.

The outline of this paper is as follows. In the next section, we review the relevant aspects of the pure Chern-Simons theory at one-loop. Section 3 contains our calculation of the η-function, while Section 4 relates this work to a supersymmetry anomaly in this theory [5, 7]. We also discuss this SUSY in the context of the Chern-Simons theory dimensionally reduced to 2 dimensions. We close with conclusions together with avenues for future work.

2 Chern-Simons Field Theory at One-Loop

The pure Chern-Simons field theory [1, 8] is generated by the classical action

$$S_c(\mathcal{A}) = \int_M Tr\{\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\}\ , \tag{1}$$

where $\mathcal{A} = \mathcal{A}^a T^a$ is a connection form on some fibre bundle over a smooth manifold M and T^a is a representation of the generators of the bundle group G. The field configurations which are extrema of this action are easily seen to be the flat connections; $\mathcal{F}(\mathcal{A}) = 0$. One can construct a quantum field theory based on this classical theory in the usual way by considering the partition function

$$Z = \int [\mathcal{D}\mathcal{A}]\ \ exp\{\frac{ik}{4\pi}S_c(\mathcal{A})\}\ , \tag{2}$$

where the functional integral is over gauge equivalence classes of connections. One subtle point regarding this particular theory is in the local gauge invariance of the action S_c. Although this action has the usual invariance under infinitesimal gauge transformations, $\delta\mathcal{A} = (d+\mathcal{A})\omega$, it is not invariant under large gauge transformations not connected to the identity. Under such transformations, the action can change by a constant times the winding number of the gauge transformation [1, 9]. The quantum theory nevertheless has the full gauge symmetry when the coupling k is quantized to take integer values, and when the scale in the trace Tr is chosen appropriately.

A perturbative analysis of this theory begins by making a background field expansion of the connection

$$\mathcal{A}_\alpha = A_\alpha + B_\alpha \tag{3}$$

into a classical background part A and a quantum field B. A BRST quantization proceeds exactly as in any Yang-Mills theory, and it is convenient to quantize in the background field gauge $D_\alpha B^\alpha = 0$ [1], where $D_\alpha = \partial_\alpha\delta^{ab} + A^c_\alpha f^{acb}$ is the covariant derivative with respect to the background field A. The partition function is now a functional of A

$$Z[A] = \int [\mathcal{D}B][\mathcal{D}\phi][\mathcal{D}b][\mathcal{D}c]\ \ exp\{iS_q\}\ ,$$
$$S_q = \int d^3x\ Tr[\epsilon^{\alpha\gamma\beta}B_\alpha D_\gamma B_\beta - 2\phi D_\alpha B^\alpha + bD^2 c]\ , \tag{4}$$

where we have rescaled the quantum fields to obtain a more convenient normalization and kept only terms which were bilinear in the quantum fields. The ϕ field is the Lagrange multiplier which enforces the gauge condition and b and c are the usual ghosts needed to produce the correct functional measure. We will be using the conventions where the structure constants of the semisimple algebra are real and completely antisymmetric with $[T^a, T^b] = f^{abc}T^c$. For the fundamental representation of $SU(n)$, the matrices T^a are skew-hermitian and we take $Tr[T^aT^b] = -\frac{1}{2}\delta^{ab}$, while the quadratic Casimir is defined by $\sum_{c,d} f^{acd}f^{bcd} = c_v\delta^{ab}$.

The one-loop partition function can be represented by a combination of determinants in the usual way [1]

$$Z[A] = exp\{\frac{ik}{4\pi}S_c\}\ \frac{det[-D^2]}{\sqrt{det[H]}}\ , \tag{5}$$

where H is the operator which appears sandwiched between the B and ϕ fields in the action

$$\int d^3x\ \frac{1}{2}\left(\begin{array}{cc} B^a_\alpha & \phi^a \end{array}\right)\left(\begin{array}{cc} -\epsilon^{\alpha\gamma\beta}D_\gamma & -D^\alpha \\ D^\beta & 0 \end{array}\right)^{ab}\left(\begin{array}{c} B^b_\beta \\ \phi^b \end{array}\right)\ . \tag{6}$$

Witten has shown [1] that $det[H]$ picks up a subtle phase factor

$$\frac{1}{\sqrt{det[H]}} = \frac{1}{|\sqrt{det[H]}|}\ exp\{\frac{i\pi}{4}\eta_H(0)\} \tag{7}$$

proportional to the η-function [10] of the operator H evaluated at zero. This function is defined by

$$\eta_H(s) = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty dt\ t^{(s-1)/2}\ Tr[H\ e^{-tH^2}]\ . \tag{8}$$

If we imagine that H has been diagonalized, and the spectrum of eigenvalues is $\{\lambda_n\}$, it is straightforward to show that the above definition is equivalent to

$$\eta_H(s) = \sum_n sign(\lambda_n)\ |\lambda_n|^{-s}\ . \tag{9}$$

In this form, one sees that the η-function measures the spectral asymmetry, or the mismatch between positive and negative eigenvalues. Clearly, if all the eigenvalues appear in pairs $\pm\lambda_n$, then the η-function will vanish identically. In [1], it was shown that the phase factor $\eta_H(0)$ could be obtained through an index theory argument, and that it was essentially proportinal to the Chern-Simons action S_c. Computing the magnitude $|det[H]|$ is straightforward since one can regularize via ζ-function techniques. Here, one takes

$$|det[H]| = \sqrt{det[H^2]}\,, \tag{10}$$

where H^2 is the postive operator

$$H^2 = \begin{pmatrix} -\delta_{\alpha\beta}D^2 - F_{\alpha\beta} & \frac{1}{2}\epsilon_{\alpha\sigma\tau}F^{\sigma\tau} \\ -\frac{1}{2}\epsilon_{\beta\sigma\tau}F^{\sigma\tau} & -D^2 \end{pmatrix} . \tag{11}$$

Remember that we are on a flat spacetime background, so there is no curvature or Christoffel connection to consider here. The determinant of a positive operator M is given by

$$\begin{aligned} det[M] &= exp[-\frac{d\zeta_M}{ds}(0)] \\ \zeta_M(s) &= \frac{1}{\Gamma(s)}\int_0^\infty dt\; t^{s-1}Tr[e^{-Mt}]\,, \end{aligned} \tag{12}$$

and can be evaluated by the techniques in [4] which we will review in the next section. The same applies to the operator D^2. Notice that when the background field A is on-shell ($F_{\alpha\beta}=0$) the magnitude of these determinants cancel automatically, $\frac{det[-D^2]}{|\sqrt{det[H]}|}=1$ since H^2 is then proportional to D^2. The contribution to $\ln det[D^2]$ which is quadratic in the A field turns out to be in general given by

$$\frac{c_v}{32}\int\frac{d^3p}{(2\pi)^3}\,A^a_\alpha(-p)A^a_\beta(p)\,(p^2)^{-1/2}(\delta^{\alpha\beta}p^2 - p^\alpha p^\beta)\,, \tag{13}$$

and this does not cancel the contribution from H^2. We find that

$$\ln\frac{det[-D^2]}{(det[H^2])^{1/4}} = \frac{c_v}{16}\int\frac{d^3p}{(2\pi)^3}\,A^a_\alpha(-p)A^a_\beta(p)\,(p^2)^{-1/2}(\delta^{\alpha\beta}p^2 - p^\alpha p^\beta)\,. \tag{14}$$

One can also see that this ratio of determinants does not vanish off-shell by looking at the Schwinger-DeWitt expansion for the heat kernel; the coefficients in this asymptotic expansion have been given by Gilkey [11] for operators of the form $D^2 + X$. In any case, the one-loop partition function $Z[A]$ is simply given by

$$Z[A] = exp\{\frac{ik}{4\pi} S_c + \frac{i\pi}{4}\eta_H(0)\} , \tag{15}$$

at $F_{\alpha\beta} = 0$. Calculation of the effective action away from the classical minimum may involve Vilkovisky-DeWitt type corrections [12, 13].

We would like to define the Chern-Simons quantum field theory through a consistent regularization scheme valid to all orders. A Pauli-Villars regulator was described in [3] which is suitable for this theory. Here, we would like to describe an alternative scheme which extends the operator regularization methods of McKeon and Sherry [4] to deal properly with first order differential operators. In their procedure, determinants of first order operators like the Dirac operator were defined by taking the square root of the determinant of the operator squared. This would not be appropriate here, as such a procedure would miss the subtle phase that arises in $det[H]$. In the next section, we show how their techniques can easily be extended to cover the situation at hand.

3 Calculation of the η-function

In [4], a technique for regulating quantum field theories was described. At the one-loop order, the calculation of Green's functions is reduced to computing the ζ-function of certain operators, and a Schwinger expansion [14] technique was employed. This expansion amounts to writing the exponential of an operator $M = M_0 + M_1$ as a power series in M_1 which can be thought of as a 'perturbation' around the 'free' field operator M_0. The Schwinger expansion is given by

$$\begin{aligned} e^{-Mt} &= e^{-M_0 t} - t\int_0^1 du\; e^{-M_0(1-u)t} M_1 e^{-M_0 ut} \\ &+ t^2 \int_0^1 du \int_0^1 dv\; u\; e^{-M_0(1-u)t} M_1 e^{-M_0(1-v)ut} M_1 e^{-M_0 uvt} + \cdots \end{aligned} \tag{16}$$

where we have explicitly included only terms up to second order in M_1; the general expression can be found in [14]. In our case, the exponential term that arises in the η-function is

$$M = H^2 = (H_0 + H_1)^2 = H_0^2 + (H_1^2 + \{H_0, H_1\}) = M_0 + M_1 \ , \tag{17}$$

where H_0 is simply H with the background field A_α set to zero. We will illustrate the use of this technique by explicitly computing the contribution to $\eta_H(s)$ which is second order in the background field A_α. In this approach, we are computing the one-loop correction to the propagator $< A\ A >$. Other references to the η-function in the physics literature can be found in [15].

We can now apply the Schwinger expansion to the situation at hand where we must compute $Tr[H\ e^{-H^2 t}]$. The terms of order two in the 'perturbation' H_1 are given by

$$Tr[-tH_0 \int_0^1 du\ e^{-H_0^2(1-u)t} H_1^2 e^{-H_0^2 ut} - tH_1 \int_0^1 du\ e^{-H_0^2(1-u)t}\{H_0, H_1\} e^{-H_0^2 ut}$$
$$+t^2 H_0 \int_0^1 du \int_0^1 dv\ u e^{-H_0^2(1-u)t}\{H_0, H_1\} e^{-H_0^2(1-v)ut}\{H_0, H_1\} e^{-H_0^2 uvt}] \ . \tag{18}$$

It is most convenient to perform the trace in momentum space, where for any operator $\mathcal{O}$,

$$\begin{aligned} Tr[\mathcal{O}] &= \int \frac{d^3p}{(2\pi)^3} < p|\mathcal{O}|p > \\ &= \int \frac{d^3p}{(2\pi)^3} d^3x d^3y < p|x >< x|\mathcal{O}|y >< y|p > \ . \end{aligned} \tag{19}$$

(Note that we take $< p|x >= e^{-ip\cdot x}$ and $A_\alpha(p) = \int d^3x\ e^{-ip\cdot x} A_\alpha(x)$.) In computing the η-function, we must evaluate the individual terms in (18), and each of these involves both momentum integrals (which come from the Fourier transform of the fields and operators) as well as u-v parameter integrals. There is also the t integral in the definition of $\eta_H(s)$ that must be carried out. In practice, it is most convenient to do the momentum and t integrals first, leaving the u-v parameter integrals for last, but it is also possible (at least at this order) to begin with the u-v integrals. We have checked our calculation by performing these integrals both ways. The most difficult part in the calculation is in resolving the final u-v parameter integrals and we have found the following three integrals useful:

$$\int_0^1 du\ u^{\alpha-1}(1-u)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \ ,$$

$$\int_0^1 du \int_0^1 dv\; u(uv - u^2v^2)^\alpha = \frac{\Gamma(1+\alpha)^2}{2\,\Gamma(2+2\alpha)} ,$$
$$\int_0^1 du \int_0^1 dv\; u^2v(uv - u^2v^2)^{\alpha-1} = \frac{\Gamma(1+\alpha)^2}{\Gamma(2+2\alpha)} . \tag{20}$$

Without further ado, we can now list the results of this calculation. The first term in (18) involving H_1^2 is identically zero, while the second term yields

$$\frac{ic_v}{2\pi^{3/2}} \frac{\Gamma(1-\frac{s}{2})^2\Gamma(\frac{s}{2})}{\Gamma(2-s)\Gamma(\frac{s+1}{2})} \int \frac{d^3p}{(2\pi)^3}\, (p^2)^{-s/2} \epsilon^{\alpha\beta\gamma} A^a_\alpha(-p)\, p_\beta\, A^a_\gamma(p) . \tag{21}$$

Notice that this contribution is singular at $s = 0$, being proportional to $\Gamma(s/2)$. It is a theorem [10] that the η-function is regular at $s = 0$, so we know that the remaining term in (18) must come to the rescue. Indeed, this piece gives

$$-\frac{ic_v}{2\pi^{3/2}} \frac{\Gamma(1-\frac{s}{2})^2}{\Gamma(2-s)\Gamma(\frac{s+1}{2})} [\Gamma(\frac{s}{2}) + \Gamma(1+\frac{s}{2})] \times$$
$$\int \frac{d^3p}{(2\pi)^3}\, (p^2)^{-s/2} \epsilon^{\alpha\beta\gamma} A^a_\alpha(-p)\, p_\beta\, A^a_\gamma(p) . \tag{22}$$

Combining the above terms gives

$$\eta_H(s) = -\frac{ic_v}{2\pi^{3/2}} \frac{\Gamma(1-\frac{s}{2})^2\Gamma(1+\frac{s}{2})}{\Gamma(2-s)\Gamma(\frac{1+s}{2})} \int \frac{d^3p}{(2\pi)^3} (p^2)^{-s/2} \epsilon^{\alpha\beta\gamma} A^a_\alpha(-p)\, p_\beta\, A^a_\gamma(p) + \cdots \tag{23}$$

for the $\mathcal{O}(A^2)$ contribution to the η-function. This result is of course regular at $s = 0$, yielding

$$\eta_H(0) = -\frac{ic_v}{2\pi^2} \int \frac{d^3p}{(2\pi)^3}\; \epsilon^{\alpha\beta\gamma}\; A^a_\alpha(-p)\, p_\beta\, A^a_\gamma(p) + \cdots . \tag{24}$$

This is simply the first term in the Chern-Simons action written in momentum space. In x-space, this is equal to

$$\frac{c_v}{\pi^2} \int d^3x\; Tr[\epsilon^{\alpha\beta\gamma}\, A_\alpha\, \partial_\beta\, A_\gamma] , \tag{25}$$

and is in agreement with [1, 3].

In principle, there is no obstruction to computing the contributions to $\eta_H(s)$ due to terms of higher order in A_α with this technique. In [1] it was shown that there is only one additional piece to $\eta_H(0)$ which is proportional to $\int d^3x\ Tr[\epsilon^{\alpha\beta\gamma}\ A_\alpha A_\beta A_\gamma]$. One proceeds in the same way starting with a Schwinger expansion valid to order A^3. We have carried out the calculation through the stage where one does the momentum and t integrals. Unfortunately, one is left with the integrals over the parameters in the Schwinger expansion and these have proved to be difficult - though probably not impossible - to resolve. The result of the calculation before carrying out these final integrals is too lengthy to include here.

4 Chern-Simons SUSY Anomaly

In [5], an unusual symmetry was discovered in the Chern-Simons theory when quantized in the Landau gauge. If we take the quantum action to be

$$\int d^3x\ Tr[\ \epsilon^{\alpha\beta\gamma}(A_\alpha\partial_\beta A_\gamma + \frac{1}{3}A_\alpha[A_\beta, A_\gamma]) - 2\phi\partial\cdot A + b\partial\cdot Dc]\ , \qquad (26)$$

where D_α is now the covariant derivative with respect to the connection form A, then it is straightforward to verify that the following transformations are symmetries of this action:

$$\begin{aligned} \delta A_\alpha &= \epsilon^\beta\epsilon_{\alpha\beta\gamma}\partial^\gamma c \\ \delta b &= -2\epsilon^\alpha A_\alpha \\ \delta c &= 0 \\ \delta\phi &= \epsilon^\alpha D_\alpha c\,. \end{aligned} \qquad (27)$$

Here, ϵ^α is a constant Grassmann odd vector parameter. It was later shown [7] how a superspace could encode both this 'supersymmetry' together with the usual BRST symmetry on the same footing. Ward identities (non-anomalous) were also derived in [5] for this symmetry, and an attempt was made at a one-loop calculation using dimensional regularization. The problem is in extending the $\epsilon^{\alpha\beta\gamma}$ symbol to d dimensions, and the conjectured extension does not exist. If one tries to use a more conventional definition for this tensor [16, 17] - a method which successfully yields the chiral anomaly in 4 dimensions - one cannot invert the A field propagator in the dimensionally extended space. A Pauli-Villars regulator was used in [3], and it was shown that one of the non-anomalous Ward identities was violated; the key point being that one does not have the freedom here to renormalize an integer coupling parameter.

One expects that the phase in the anomaly is essentially $\eta_H(0)$ and it would be interesting to recover this by an analysis of the functional measure *ala* Fujikawa [18]. In the Landau gauge, the operators occuring in the action (26) do not seem well suited for this task, as the fields must first be decomposed into eigenfunctions. Although this symmetry is peculiar to the Landau gauge, the following transformations of the quantum fields in the action (18) :

$$\begin{aligned} \delta B_\alpha &= \epsilon^\beta \epsilon_{\alpha\beta\gamma} D^\gamma c \\ \delta b &= -2\,\epsilon^\alpha B_\alpha \\ \delta c &= 0 \\ \delta\phi &= \epsilon^\alpha D_\alpha c\,, \end{aligned} \tag{28}$$

where the covariant derivatives are with respect to the background field, are indeed symmetries when the background field is on-shell, $F_{\alpha\beta} = 0$. The Hermitian operators H and D^2 in this action are much nicer to deal with. One must check to see whether the regularized functional measure in (4) is invariant under the above transformations (28). Work on this approach is in progress.

One last point regarding this symmetry deserves mention. If one dimensionally reduces the Chern-Simons theory in (26) to two dimensions

$$S_q \rightarrow \int Tr[B\epsilon^{ij}F_{ij} + \phi\partial_i A^i + b\partial\cdot Dc]\,, \tag{29}$$

the symmetry above also reduces correspondingly. Here, $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ is the curvature of the connection and the scalar field B is the descendent of the higher dimensional field $A_\alpha = (A_i, B)$. The new feature is that the 'third' component of ϵ^α becomes a scalar in 2 dimensions and the symmetry

$$\delta A_i = \tilde{\epsilon}\,\epsilon_{ij}\partial^j c$$

$$\begin{aligned}
\delta B &= 0 \\
\delta b &= 2\tilde{\epsilon} B \\
\delta c &= 0 \\
\delta \phi &= 2\tilde{\epsilon}[B, c]
\end{aligned} \tag{30}$$

is potentially valid on any Riemann surface (since $\tilde{\epsilon}$ is a scalar parameter); not just on flat manifolds. Recall that the covariant generalization of a constant vector parameter ($\partial_\alpha \epsilon_\beta = 0$) on R^n is the constraint $\nabla_\alpha \epsilon_\beta = 0$ and this leads to the integrability condition that the Riemann tensor vanish. The issue of an anomaly needs to be readdressed in this case.

5 Conclusion

We have exhibited a Schwinger expansion technique for computing the η-function of an operator and applied it to the case of the Chern-Simons theory. This gives essentially the one-loop correction to the partition function. Unfortunately, terms of higher order in the background field become increasingly difficult to compute in this approach. We have also shown how this is likely to be related to the anomalous supersymmetry in this theory, and that there is a well defined set of transformations (28) to consider in a Fujikawa [18] analysis of the anomaly. In 2 dimensions, there is a potentially non-anomalous descendent of this symmetry that could be valid on any Riemann surface, and this problem warrants future consideration.

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