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Conformal Reduction of WZNW Theories and W-Algebras *

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Abstract: It is shown that Toda field theories can be regarded as reduced WZNW theories and that the reduction generalizes to yield families of conformal and non-conformal integrable field theories. The advantages of regarding the conformal theories as reduced WZNW theories are outlined, and include the natural appearance of two-dimensional gravity, the easy derivation of the general solutions from the standard WZNW solution, and, for the Toda theories, an intuitive understanding and relatively simple construction of the W-algebras.

1 Introduction.

Two of the most celebrated classes of conformally-invariant two-dimensional field theories are the Wess-Zumino-Novikov-Witten (WZNW) theories[1] and the Toda field theories[2]. Recently it has been shown that these theories are not independent and that, in fact, the Toda theories can be obtained from the WZNW theories by placing conformally invariant constraints on the currents[3]. More recently still, it has been shown that a similiar reduction leads to a whole series of conformal integrable field theories, which interpolate between the WZNW and Toda theories [4], and that there even exists a non-conformal version of the constraints that leads to non-conformal integrable systems, in particular to the affine Toda systems.

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In the present talk I should like to describe these reductions of WZNW theory and to outline the various features and advantages that emerge by regarding the reduced conformal theories as constrained WZNW theories. Perhaps the most remarkable feature that emerges is the appearance in all cases of a two-dimensional gravitational field, in non-trivial interaction with itself and with the other fields. The emergence of this gravitational field is is in some sense the converse of the Polyakov's embedding [5] of the string-induced two-dimensional gravity in the WZNW group $SU(1,1)$, but it is present for all WZNW groups. One of the greatest practical advantages that accrues from regarding the conformal theories as constrained WZNW theories is that their general solution can be obtained in a rather simple manner from the general WZNW solution, which is well-known to be quite trivial. For reasons of space the derivation will not be given in this talk but the general method will be indicated (with references for details) and the end-result, which is quite simple, will be presented.

One of the remarkable features of the Toda theories in particular is that they realize [6] the polynomial algebras (so-called W-algebras) defined [7] abstractly by Zamolochikov. Within the confines of Toda theories it is not immediately obvious why these algebras should exist, and one of the great advantages of regarding Toda theory as a reduced WZNW theory is that in the broader WZNW context their existence becomes quite natural and understandable. Indeed in the reduced WZNW context the W-algebras have a simple intuitive interpretation as the algebras of gauge-invariant polynomials of the constrained currents (and their derivatives), the gauge group in question being that one generated by the constraints. This identification not only provides an intuitive understanding of W-algebras, but also provides a relatively simple algorithm for their computation. This is because of the existence of a gauge in which the gauge-invariant polynomials reduce to the currents themselves. In this gauge the W-algebras manifest themselves as the Dirac star algebras of the gauge-fixed constrained currents and, because all constraints are linear in the currents, the W-algebras can then be computed relatively easily from the Kac-Moody algebras of the associated WZNW theories.

For those not completely familiar with two-dimensional conformal field theory we begin by recalling the features of theory which are relevant to our discussion, in particular the WZNW and Toda theories and the Zamolochikov W-algebras. With these aspects of the theory in hand the reduction will be seen to be quite staightforward.

2 Recall of Conformal Field Theory and W-Algebras.

We begin by recalling the situation for conformal invariance in more than two dimensions ($D > 2$). Let $L(\phi(x))$ be the Lagrangian density for any set of tensor fields $\phi(x)$ and $T_{\mu\nu}(\phi(x))$ the corresponding energy-momentum tensor density. If $L(\phi(x))$ is conformally invariant then according to Noether's theorem the generators $L_{\mu\nu}, P_\mu, S_\mu, D$ of the conformal group are moments of $T_{\mu\nu}$. For example, for the dilation D one has $D = \int x^\nu T_{\nu o} d^{D-1}x$. In all the $D > 2$ cases the conformal group is finite-dimensional ($(D+1)(D+2)/2$-dimensional actually), and thus involves only a finite number of moments of $T_{\mu\nu}$. It is also semi-simple and thus admits no central extensions.

In two dimensions the situation is different. If $x = (x_1, x_2)$ are the usual Cartesian coordinates, then the conformal group consists of all transformations of the form $z \longrightarrow f(z)$ and $w \longrightarrow \bar{f}(w)$, where $z, w = x_1 \pm x_2$, or $z, w = x_1 \pm ix_2$, according as the metric is Minkowskian or Euclidean, and $f(z)$ and $g(w)$ are arbitrary analytic functions. Thus it is an infinite-dimensional group and is a direct product of a left and a right part. Furthermore, it is well-known that each part admits one central extension [1]. For conformally invariant Lagrangians the (three component) energy-momentum tensor density $T_{\mu\nu} = [T_{zw}, T_{zz}, T_{ww}]$ reduces to $[T_{zw} = 0, \quad T_{zz} = L(z) \quad \text{and} \quad T_{ww} = \bar{L}(w)]$, and the Noether generators of the conformal group consist of *all* its moments i.e. consist of the quantities

$$L_n = \oint z^n L(z) dz \quad \text{and} \quad \bar{L}_n = \oint w^n \bar{L}(w). \tag{1}$$

From the structure of the conformal group it follows that the $L(z)$ and $\bar{L}(w)$ commute with each other and that each satisfies a Virasoro algebra i.e. an algebra of the form

$$[L(z), L(z')] = 2L(z)\partial_z \delta(z - z') + \partial_z L(z)\delta(z - z') + \frac{c}{12}\partial_z^3 \delta(z - z'), \tag{2}$$

where the last term is the central extension and c is a constant that depends on the original Lagrangian.

The tensors with respect to the conformal group are called *primary* fields and have the transformation properties

$$\phi(\mathrm{x}) \rightarrow \left(\frac{\partial z}{\partial z'}\right)^j \left(\frac{\partial w}{\partial w'}\right)^{\bar{j}} \phi(\mathrm{x}'), \tag{3}$$

where the quantities j and $\bar{j}$ are called conformal weights and are often integers.

With these properties recalled let us turn to the definition of W-algebras. According to Zamolochikov, who first introduced them [7], a W-algebra is an extension of a Virasoro algebra by primary fields, such that the Poisson bracket (or commutators) of any two primary fields is a polynomial in the fields and their derivatives (both primary and Virasoro), the order of the polynomial being less than the combined order of the two primary fields. In other words a W-algebra consists of the Virasoro algebra, the transformation law (3) (with one of the coordinates (w,say) dormant) and a set of Poisson bracket (or commutation relations) of the form

$$[\phi^{(a)}(z), \phi^{(b)}(z')] = P^{(a,b)}(\phi(z), L(z), \delta(z-z')), \tag{4}$$

where $P^{(a,b)}$ is polynomial of lower order than (a+b) in $L(z)$, $\phi(z)$, $\delta(z-z')$ and their derivatives. In counting the order the delta function and the derivative are each given unit weight.

3 Standard Examples of 2-D conformal Field Theories.

Two standard examples of 2-D conformal field theories are the Wess-Zumino-Novikov-Witten (WZNW) theory and the Toda theory. The WZNW Lagrangian takes the form

$$L_{WZ} = \frac{k}{2}\oint d^2x \mathrm{tr}(g^{-1}(x)\partial g(x))^2 + \frac{2k}{3}\oint d^3x \mathrm{tr}(g^{-1}(x)\partial g(x))^3, \tag{5}$$

where the three-dimensional integral is over a space whose boundary is the two-dimensional one of the first (kinetic) integral. As a result of the addition of the three-dimensional integral, whose variation is purely topological, the field equations of the theory take the form

$$\partial_w J(x) = 0 \quad \text{and} \quad \partial_z \bar{J}(x) = 0, \tag{6}$$

where

$$J(x) = g^{-1}(x)\partial_z g(x) \quad \text{and} \quad \bar{J}(x) = (\partial_w g(x))g^{-1}(x).$$

The field equations mean, of course, that the currents $J(x)$ and $\bar{J}(x)$ are functions only of z and w respectively and from the symmetry of L_{WZ} with respect to (rigid) left and right group multiplication ($g \to hg$ and $g \to gh$), and the Noether theorem, it follows that they satisfy Kac-Moody (KM) algebras with centres k. Thus $J(z)$, for example, satisfies the KM algebra

$$[J_a(z), J_b(z')] = f^c_{ab} J_c(z)\delta(z-z') + k\delta_{ab}\partial_z\delta(z-z'). \tag{7}$$

The Toda Lagrangian, on the other hand, takes the form

$$L_{Toda} = \int d^2x[C_{ij}\partial\phi^i(x)\partial\phi^j(x) + \exp(K_{ij}\phi^i(x)\phi^j(x))], \tag{8}$$

where C and K are the Coxeter and Cartan matrices for any semi-simple simple Lie group. Thus to every Dynkin diagram there is associated a Toda field theory.

Recently it has been shown that every Toda field theory admits a W-algebra, the W's being coefficients in an equation called the Gelfand Dickey equation[8]. This equation is a linear differential (or pseudo-differential) equation of the same order as the dimension of the defining representation of G, and which is satisfied by certain left-and right-moving functionals of the Toda fields. Its role in our discussion will be to help identify the W-algebras.

4 Reduction of WZNW Theories.

What we want to show is that the Toda theories can be obtained by putting conformally-invariant constraints on the WZNW theories and that by generalizing the constraints one obtains not only the Toda theories but a whole class of theories that interpolate between the WZNW theories and the Toda theories. These theories may be regarded as interacting WZNW theories or as generalizations of the Toda theories in which the individual Toda fields are replaced by WZNW fields,the usual Toda theories being the extreme case in which all the subgroups are abelian. A remarkable feature of the reduction is the emergence of an abelian field that plays the role of two-dimensional gravity.

Some advantages of deriving the Toda theories in this way are:

(i) the emergence of the two-dimensional gravitational theory just discussed

(ii) the emergence of a new set of conformally invariant integrable field theories

(iii) the derivation of the general solutions of these theories from the (trivial) general solutions $g(x) = g(z)\bar{g}(w)$ of the WZNW theories

(iv) the emergence of a simple intuitive explanation of the W-algebras of Toda theory and of a relatively easy algorithm for their computation

(v) the fact that the whole procedure can be generalized so as to obtain a series of non-conformal field theories including the affine Toda field theory.

One also obtains a formula relating the KM and Virasoro centres for the quantized theory [3][4] but this will not be discussed here.

The reduction of the WZNW theories requires setting some of the WZNW currents equal to non-zero constants and since these currents, being space-time vectors, have conformal weights (1,0) or (0,1) with respect to the usual conformal group, the problem is how to set them equal to constants without breaking conformal invariance. By the usual conformal group is meant here the group generated by the Noether currents $L(z)$ and $\bar{L}(w)$ belonging to the energy-momentum tensor of the WZNW theory and the way that is used to circumvent this difficulty is to note that the this conformal group is not unique. In fact, there is a two-parameter family of conformal groups equivalent to it and the procedure will be to choose a member of this family with respect to which some of the currents are no longer vectors but scalars i.e. have conformal weights (0,0). However, to make the appropriate choice of member requires some Lie-algebraic technicalities and these will be discussed in the next section.

5 Lie Algebraic Technicalities.

The simple WZNW groups G which are used for our reduction will be the (maximally non-compact) ones generated by the *real* linear span of the Cartan generators i.e. by the generators $[H_i, E_\alpha]$ in conventional notation. For the A and D algebras, for example, these are the groups SL(n,r) and SO(n,n). Within the Cartan algebra there always exists an element H such that each of the simple roots E_{α_i} is an eigenvector of H with eigenvalue unity or zero.

$$[H, E_{\alpha_i}] = hE_{\alpha_i} \quad \text{where} \quad h = 0, 1, \quad i = 1, 2...l, \tag{8}$$

and l is the rank. (To see this note that H can be written as $\mathbf{w}.\mathbf{H}$, where $\mathbf{w}$ is a sum over any subset of the l fundamental coweights). Then H provides an integer grading of the whole Lie algebra,

$$[H, E_\alpha] = hE_\alpha \quad \text{where} \quad h \in Z. \tag{9}$$

In particular the elements of the algebra of the little group of H, which we shall call B, will have zero grade. It is not difficult to see that the set of little groups B for all possible choices of H are just the non-compact versions of the set of little groups in the adjoint representation of the compact form of G. In particular for $\mathbf{w} = \mathbf{s}$, where $\mathbf{s}$ is the sum over *all* the simple coweights (=half the sum of the positive coroots), the little algebra is the generic one, namely the Cartan algebra itself. (It will be seen later that this case corresponds to the Toda reduction).

Finally we note that G admits a local Gauss decomposition $G = ABC$, where B is the little group and A and C are the (nilpotent) groups generated by the root vectors E_α with weights which are strictly positive and negative with respect to H. (This decomposition may not be global, but the parameter space may be divided into a finite number of patches on each of which the decomposition is valid up to left- or right-multiplication with a constant group element).

At the KM level we have, correspondingly,

$$[H(z), J^B(z')] = 0 \quad \text{except} \quad [H(z), H(z')] = k\partial_z\delta(z-z')\mathrm{tr}H^2, \tag{10}$$

and

$$[H(z), J_h^\alpha(z')] = hJ_h^\alpha(z)\delta(z-z'). \tag{11}$$

6 Preservation of Conformal Invariance in the Reduction.

We come now to the crucial point. Let $L(z)$ denote the Virasoro operator which is the component $T_{zz}(z)$ of the energy-momentum tensor of the WZNW theory, and with respect to which all the KM currents $J(z)$ are have conformal weights unity, or, more precisely, (1,0). Then we replace $L(z)$ by $\Lambda(z)$, where

$$\Lambda(z) = L(z) + \partial_z H(z), \tag{12}$$

It is to be noted that $\Lambda(z)$ is again a Virasoro operator i.e. satisfies a Virasoro algebra of the form (2). The only difference is that the centre c changes to $c - 12k\mathrm{tr}H^2$. It will turn out that $\Lambda(z)$ is actually the improved (i.e.traceless) energy-momentum tensor of the reduced theory.

Once the crucial change (12) has been made the rest is almost automatic. With respect to the conformal group generated by $\Lambda(z)$ the KM currents $J(z)$ are no longer vectors of conformal weight (1,0) but have the following transformation properties:

(i) Except for $H(z)$ the currents $J^B(z)$ belonging to the little group B are still vectors i.e. have conformal weights (1,0).

(ii) the field $H(z)$ now transforms not as a spin-one vector but as a spin-one *connection.*

(iii) The currents $J^\alpha(z)$ transform as conformal tensors (primary fields) of conformal weight $(1+h)$.

Thus, in particular, *the currents of grade* $h = -1$ *transform as conformal scalars.*

With this information in hand we are ready to impose the constraints, namely,

$$J^{\alpha}_{-1}(z) = J^{\alpha}_{-1}(0) \neq 0, \quad \text{and} \quad J^{\alpha}_{h}(z) = 0, \quad h < -1. \tag{13}$$

Here the set of constraints with non-zero right-hand-side do not break the conformal invariance generated by the new Virasoro operator $\Lambda(z)$ since they are scalars with respect to this operator, and the set of constraints with zero right-hand-side are added so that the complete system of constraints is first class. For the right-handed currents $\bar{J}(w)$ similiar constraints are imposed, but with $h < 0$ replaced by $h > 0$. In order to obtain an intuitive feeling for the meaning of the constraints (13) let us consider the case of G=SL(n,R), in which case the constrained current $J(z)$ takes the form

$$J^{constr.}(z) = \begin{pmatrix} J_{11}(z) & J_{12}(z) & J_{13}(z) & \cdots\cdots & J_{1n}(z) \\ J_{21}(0) & J_{22}(z) & J_{23}(z) & \cdots\cdots & J_{2n}(z) \\ 0 & J_{23}(0) & J_{33}(z) & \cdots\cdots & J_{3n}(z) \\ 0 & 0 & J_{34}(0) & \cdots\cdots & J_{4n}(z) \\ 0 & 0 & 0 & \cdots\cdots & J_{5n}(z) \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ 0 & 0 & 0 & J_{nn-1}(0) & J_{nn}(z) \end{pmatrix},$$

where the $J_{ab}(z)$ denote submatrices of currents which in general are not single entry or even square. Note that the constraints can also be expressed as

$$J^{\alpha}_{neg} = M \quad \text{and} \quad \bar{J}^{\alpha}_{pos} = N, \tag{14}$$

where M and N are constants matrices of grade minus one and plus one respectively, and *neg* and *pos* refer to the sign of h. The constraints (13) are not invariant with respect to general KM transformations, $J(z) \rightarrow U(z)^{-1}J(z)U(z) + U(z)^{-1}\partial_z U(z)$ but there exists a residual group of KM transformations with respect to which they are invariant. These are the KM transformations for which $U(z)$ lies in the group A of the Gauss decomposition which is generated by the root vectors with negative grade (E^{α}_{h}, for $h < 0$). Thus they are just the transformations that would be generated by the constraints themselves. The idea is to regard these residual KM transformations as gauge transformations and regard only those functions, or functionals, of $J(z)$ which are invariant with respect to this gauge group as physical. Thus finally we have (dimG-dimB)/2 constraints and (dimG-dimB)/2 gauge degrees of freedom, leaving just dimB physical fields altogether. It is possible to choose the gauge (at least locally) so that the physical currents are just the ones $J^{B}(z)$ belonging to the little group B.

7 Field Equations.

It is easy to see that the constraints (13) are consistent with the WZNW field equations (6), indeed are special solutions of some of them, and hence the WZNW field equations can be reduced to field equations for the unconstrained components of the current$J(z)$. After some simple algebra one finds that the reduced field equations take the following form

$$\partial_w J^B(x) = [b(x)Nb^{-1}(x), M], \tag{15}$$

and

$$\begin{aligned} J^A(x) &\equiv a^{-1}(x)\partial_z a(x) = b(x)Mb^{-1}(x), \\ \bar{J}^C(x) &\equiv (\partial_w c(x))c(x)^{-1} = b^{-1}(x)Nb(x), \end{aligned} \tag{16}$$

where M and N denote the constant matrices defined in (14). Note that, in contrast to the WZNW currents, the currents $J^A(x)$, $J^B(x)$, $\bar{J}^C(x)$ and $\bar{J}^B(x)$ are not functions of z and w alone.

The most interesting feature (15) is that the equations for $J^B(z)$ do not involve the fields $J_h(z)$ for $h \neq 0$ and thus are self contained. Furthermore, it is easy to verify that they can be derived from the effective Lagrangian

$$L_{\text{eff}}(b(x)) = L_{WZNW}(b(x)) \quad + \quad \int \text{tr}(b(x)Mb^{-1}(x)N), \quad \text{where} \quad b(x) \in B. \tag{17}$$

This Lagrangian can be interpreted in two ways. First, it can be regarded as the generalization of the WZNW Lagrangian for fields belonging to the group B, but where, because B is reducible, there are interactions between the simple and abelian parts of B. Note, however, that since the constant matrices M and N have grades ± 1 there is a non-zero interaction only between the components of B which differ by one grade (nearest neighbour components). Second, by noting that the Lagrangian (17) reduces to the Toda Lagrangian when B is abelian (i.e. when $b(x) = \exp(H_i\phi_i(x))$ and $M = m_i E_{\alpha_i}$, where the m's are constants and $\text{tr}(H_iH_j) = C_{ij}$), one sees that it can be regarded as a generalization of the Toda Lagrangian to the case in which the nearest-neighbour interacting fields are no longer abelian fields but WZNW fields belonging to the irreducible components of B. Thus (17) may be regarded as describing either interacting WZNW fields or generalized Toda fields.

7 Two-Dimensional Gravity.

As mentioned in the Introduction, the Lagrangian (17) incorporates also a two-dimensional gravitational field. This comes about as follows: Since B is defined as the little group of H it follows that the one-parameter abelian group $R(1)$ generated by H is in the centre of B. Hence, locally at least, B may be written as the direct product $R(1)\otimes B_o$, where B_o denotes the rest of B. If we let $h(x)$ be the WZNW field belonging to $R(1)$ then the Lagrangian (17) can be re-written in the form

$$L_{\text{eff}}(b(x)) = L_{WZNW}(b_o(x)) \quad + \quad \oint (\partial h(x))^2 \quad + \quad \oint h(x) \mathrm{tr}\left(b_o(x) M b_o^{-1}(x) N\right) . \tag{18}$$

But we have already seen that, unlike the rest of the components of the current which transform like primary fields, the components in the direction H transform like spin-one connections, and it is not difficult to deduce from this that the field $h(x)$ transforms like $\sqrt{g}$ where $g_{\mu\nu}$ is a two-dimensional metric. Accordingly, if one *defines* a metric as $g_{\mu\nu} = h(x)\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is any flat (constant) non-singular metric, introduces general coordinate transformations, and extends the tensor properties of the currents to be the same with respect to general coordinate transformations, one finds that the Lagrangian (18) may be written as

$$L_{\text{eff}}(b) = L_{WZNW}(b_o) \quad + \oint R\triangle^{-1}R \quad + \quad \oint \sqrt{(g)} \mathrm{tr}\left(b_o M b_o^{-1} N\right), \tag{19}$$

where $R(x)$ is the Gauss curvature and $\triangle$ the two-dimensional d'Alembertian operator. It is clear that this Lagrangian describes a theory in which a two-dimensional gravitational field $h(x)$ and the WZNW fields $b_o(x)$ interact with themselves and with each other. The purely gravitational part of the interaction (which is obtained by setting $b_o(x) = 1$) is just the Liouville gravitational interaction which is induced by string theory in less than 26 dimensions [9]. This Liouville theory was embedded in an SU(1,1) Kac-Moody theory by Polyakov [5] in order to facilitate its quantization, so our procedure may be regarded as the converse of Polykov's for SU(1,1), and a generalization of the converse for the other groups.

8 Solutions of the Field equations.

The general solution of the field equations (15)(16) for the fields $b(x)$ are obtained from the general solution for the WZNW equations for the group G, namely, $g(x) = g(z)\bar{g}(w)$, where $g(z) \in G$ and $\bar{g}(w) \in G$ are any arbitrary functions of the coordinates z and w respectively. I do not have time to describe the procedure by which the solution of the reduced system is obtained from this solution, but it is not difficult and is given in [4]. Here we shall simply present the result, which is that the general solution takes the form

$$b(x) = b(z)D(z,w)b_o(w), \tag{20}$$

where $b(z) \in B$ and $\bar{b}(w) \in B$ are again arbitrary functions of z and w respectively, and $D(z,w)$ is the B part in the Gauss decomposition of $c(z)a(w)$, where $a(w)$ and $c(z)$ are the solutions of the remaining equations in (14) and its right-handed counterpart, namely,

$$\partial_z a(z) = a(z)(b(z)Mb^{-1}(z)) \quad \text{and} \quad \partial_w a(w) = a(w)(\bar{b}^{-1}(w)N\bar{b}(w)), \tag{21}$$

with initial values $c(0) = a(0) = 1$. It might be thought, of course, that this solution is not complete because it leaves the differential equations (21) still to be solved. However, because of the nilpotency of the groups A and C these equations can be solved by simple iteration. Indeed if $c(z)$, for example, is decomposed into its H grades $c_h(z)$ then the solution is

$$c_h(z) = \int_o^z dz' c_{h-1}(z')(b(z')Mb^{-1}(z')), \qquad c_o(z) = 1. \tag{22}$$

Note the resemblance between the general solution (20) and the general solution $b(z)\bar{b}(w)$ for non-interacting WZNW fields belonging to the little group B. Indeed (20) reduces to this solution in the non-interacting case, for which $M = N = 0$ and hence, from (21), $D(z,w) = 1$.

9 The W-Algebras of Toda Theory.

In this section we show that the W-algebras that have emerged in the Toda theory become much more understandable and tractable in the reduced WZNW context. First we identify them by means of the equation $\partial_z g(z) = J(z)G(z)$ connecting the WZNW fields $g(z)$ with their currents $J(z)$. These equations can be regarded as first-order differential equations for $g(z)$, given $J(z)$, and, it turns out that, in the constrained case, they can easily be reduced to higher order differential (or pseudo-differential) equations for those components of $g(z)$ that are gauge-invariant with respect to the residual gauge group discussed earlier. Since the coefficients of the powers of ∂_z in these higher-order equations are gauge-invariant with respect to the residual gauge group by construction, and are polynomials in the constrained currents and their derivatives because the group A is nilpotent, we see that they are gauge-invariant polynomials in the constrained currents and their derivatives. The crucial point now is that the higher-order equations obtained in this way are just the Gelfand-Dicke equations. Since the coefficients of the latter equations are just the base elements of the W-algebra of the Toda theory this immediately gives us an identification of the W-algebra as the algebra of local gauge-invariant polynomials in the constrained currents.

Although the identification of the W-algebra of Toda theory as the algebra of gauge-invariant polynomials of the constrained WZNW theory is very natural and intuitive it is not of great help for practical computations in arbitrary gauges. However, there exists a set of gauges in which it is very useful and practical, and in which we obtain an alternative interpretation of the W-algebras as Dirac star algebras. These are the (DS) gauges introduced [10] by Drinfeld and Sokolow. In these gauges the local gauge-invariant polynomials in the constrained currents reduce to the currents themselves,

$$P(J_{(i)}(z), \partial_z^n J_{(k)}(z)) = J_{(i)}^{DS}(z), \tag{23}$$

where the $J_{(i)}(z)$, of which there are l, form a basis for the W-algebra. The gauge-fixing is complete in these gauges and the system of constraints obtained by combining the original constraints and the gauge fixing form a second class system of constraints. Hence their Poisson-bracket algebra (which, from (23), is just the W-Poisson-bracket algebra) is not their normal Kac-Moody algebra but the corresponding Dirac star algebra,

$$[P_{(i)}, P_{(k)}] = [J_{(i)}^{DS}, J_{(k)}^{DS}]^* = [J_{(i)}^{DS}, J_{(k)}^{DS}] - [J_{(i)}^{DS}, C_\alpha][C_\alpha, C_\beta]^{-1}[C_\beta, J_{(k)}^{DS}]. \tag{24}$$

We thus obtain an alternative identification of the W-algebra as the Dirac star algebra of the constrained currents in the DS gauge. This identification is very useful for practical purposes because in this gauge the gauge-fixing as well as the original constraints impose *linear* conditions on the currents. This means that the the constraints C_α in (24) can be replaced by components J_α of the currents themselves, in which case the right-hand-side of (24) can be expressed completely in terms of KM commutators. Furthermore, because of the nilpotency of the gauge group it turns out that the inverse constraint matrix $[J_\alpha^{DS}, J_\beta^{DS}]^{-1}$ is easy to compute and is a polynomial in the currents. Again we shall not give the details of the computation here but refer to the literature [3] in which, as examples, the W-algebras for the groups $G = A_2, B_2$ and G_2 are computed. It is well-known that that the W-algebra for G_2, which involves the Poisson bracket of two sixth-order polynomials, is very difficult to compute by direct methods. Indeed, as far as we know it has not yet been computed this way.

10 Reduction to Affine Toda Theory.

The reduction described up to now has been conformally invariant, but there exists a natural non-conformally-invariant generalization which leads, inter alia, to the affine Toda theories. The generalization is obtained by noting that in the equation (15) for the reduced field equations no use was made of the fact that the group B was the little group of H. Thus, in principle, one could use *any* subgroup B and any two cosets A and C in the Gauss decomposition (so long as they were complete in the sense that every group element g could be written as $g = abc$) and then impose the constraints (14). The constraints would still be special solutions of the original WZNW field equations. The only difference would that there would be no reason for the constraints to be conformally invariant, or expressible linearly and/or locally in terms of the original currents $J(z)$, and, in general, they would not be so. However, this would not in itself make them uninteresting and to illustrate the kind of theory that one would obtain we show now indicate how the affine Toda theories can be obtained by such a reduction. The reduction consists of simply replacing the conditions

$$M = \sum_{\alpha_i} \mu_{\alpha_i} E^{\alpha_i} \quad \text{and} \quad N = \sum_{\alpha_i} \nu_{\alpha_i} E^{\alpha_i} \tag{25}$$

where the α_i denote the l simple roots, by a similiar sum in which i denotes not only the simple roots but also the most negative root α_o, say. Thus $i = 0, 1, 2, ...l$ instead of just $1, 2, ...l$. It is not difficult to see that in this case the effective Lagrangian (18) reduces to the affine Toda one. In particular, for SL(2,R), the Lagrangian (17) reduces to the sinh-Gordon Lagrangian.

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