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THE COVERING PROPERTY IN A CAUSAL LOGIC

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Abstract.

We construct a complete orthomodular lattice with an orthogonality relation generated by a causal structure on Minkowski space-time. The properties of this lattice are examined and it is shown that the covering law is not satisfied.

1.Introduction.

There are many difficulties with causality in quantum mechanics. The problem can be seen also in algebraic quantum theory [4,9,10,11] . Motivated by these difficulties,we have investigated the structure of the causal logic of space-time [5,6] .
In the quantum logic approach to relativistic quantum mechanics one is interested in covariant representations of the proper orthomodular lattice in the lattice of projections in Hilbert space.
In this paper we use the causal structure to construct an orthogonality space. The methods and language we have found most appropriate are those of the empirical logic of Foulis and Randall [8] .
We deduce the orthogonality relation from the causal structure of Minkowski space-time. Then we construct a family of double orthocloced sets which form a complete orthomodular lattice.
The main result concernig the representation of this lattice in Hilbert space is negative. This lattice is atomic, with trivial center and does not satisfy the covering law, therefore cannot be represented as the lattice of projections of a von Neumann algebra.

2.Causal structure and orthogonality relation.

We shall start with some facts about orthogonality spaces. An <u>orthogonality space</u> is a pair $(X, \perp)$ where X is a nonempty set and $\perp$ is an orthogonality relation on X which is symmetric and irreflexive. $D \subset X$ is called an <u>orthogonal set</u> if and only if, for all $x, y \in D$ $x \neq y$ implies $x \perp y$.

For $A \subset X$ define $A^{\perp} := \{x \in X ; x \perp a \text{ for all } a \in A\}$ and $A^{\perp\perp} := (A^{\perp})^{\perp}$.

The mapping $\perp : 2^X \to 2^X$ has the following properties [8] : $A \subset A^{\perp\perp}$, $A \cap A^{\perp} = \phi$, $A^{\perp} = A^{\perp\perp\perp}$, if $A \subset B$ then $B^{\perp} \subset A^{\perp}$, $(\bigcup A_i)^{\perp} = \bigcap A_i^{\perp}$.
It is well known [3] that the family $\mathcal{C}(X, \perp) := \{A \subset X ; A = A^{\perp\perp}\}$ forms a complete ortholattice when partially ordered by set theoretic inclusion and equipped with the orthocomplementation $A \longrightarrow A^{\perp}$.
The l.u.b. and g.l.b. are given respectively by

$$\bigvee A_i = (\bigcup A_i)^{\perp\perp} \qquad \bigwedge A_i = \bigcap A_i \qquad A_i \in \mathcal{C}(X, \perp) \quad (2.1)$$

In general $\mathcal{C}(X, \perp)$ need not be orthomodular. This has been discussed in [7] , where conditions equivalent to orthomodularity were given.

We now introduce a causal structure on X which will be used to define the orthogonality space. This structure has a simple physical meaning in the case of Minkowski space-time.

Let $(X, \mathcal{G})$ be a pair where X is a nonempty set and $\mathcal{G}$ is a distinguished covering of X by nonempty subsets. The pair $(X, \mathcal{G})$ will be called a <u>causal space</u>, the family $\mathcal{G}$ a <u>causal structure</u> and an element $f \in \mathcal{G}$ a <u>causal path</u>. Let $x \in X$ we denote by $\beta(x) := \{f \in \mathcal{G};\ x \in f\}$ the set of all causal path containing x.
In the causal space $(X, \mathcal{G})$ one can introduce a natural orthogonality relation:

$$x, y \in X \qquad x \perp y \qquad \text{iff} \quad \beta(x) \cap \beta(y) = \emptyset \tag{2.2}$$

Observe that

$$x \perp y \qquad \text{iff} \quad \forall_{f \in \beta(x)}\ f \cap \{y\} = \emptyset \qquad \text{iff} \quad \forall_{f \in \beta(y)}\ f \cap \{x\} = \emptyset \tag{2.3}$$

and $\qquad A^{\perp} = \{x \in A;\ \forall_{f \in \beta(x)}\ f \cap A = \emptyset\}$

If we understand a causal path as a possible physical signal then $A^{\perp}$ denotes the set of points which are not causally related to any point in the set A.

3.Causal structure and orthomodularity.

We are interested in a causal structure which generates, by (2.2) an orthogonality relation under which the lattice $\mathcal{C}(X, \perp)$ is orthomodular.
The following condition is equivalent to orthomodularity [7] and appropriate for our discussion:

if D is an orthogonal set of X, if $x \notin D^{\perp}$, $x \notin D^{\perp\perp}$ then

$$D^{\perp} \cap (x^{\perp} \cap D^{\perp})^{\perp} \neq \emptyset \tag{3.1}$$

We are now able to formulate the conditions on the family $\mathcal{G}$ for a quite general space $X = \mathbb{R} \times Z$ where $\mathbb{R}$ is a real line and Z is any nonempty set. The family $\mathcal{G}$ consists of graphs of functions (we will identify the function with its graph) $f : S \to Z$; $S \subset \mathbb{R}$ such that

1. $\mathcal{G} \subset \bigcup_{S \subset \mathbb{R}} \{f : S \to Z\ ;\ S$ connected subset of $\mathbb{R}\}$

(3.2) 2. For any $t_1 \leq t_2 \leq t_3$ and for any $z_1, z_2, z_3 \in Z$ if $\beta(t_1, z_1) \cap \beta(t_2, z_2) \neq \emptyset$ and $\beta(t_2, z_2) \cap \beta(t_3, z_3) \neq \emptyset$ then $\beta(t_1, z_1) \cap \beta(t_3, z_3) \neq \emptyset$

3. For any $f \in \mathcal{G}$ and for any $x \in \mathbb{R} \times Z$ the set

$$[f, \{x\}] := \{v \in \mathbb{R};\ (v, f(v)) \notin \{x\}^{\perp}\} \quad \text{is open in } \mathbb{R} \cap \text{domain } f .$$

Theorem 3.1.

Let $(X, \mathcal{G})$ be a causal space where $X = \mathbb{R} \times Z$ and $\mathcal{G}$ satisfies condition (3.2). If A is an orthogonal set for $\mathcal{G}$ and $(t,z) \in \mathbb{R} \times Z$ is such that $(t,z) \notin A^{\perp}$, $(t,z) \notin A^{\perp\perp}$ then $A^{\perp} \cap ((t,z)^{-} \cap A^{\perp})^{\perp} \neq \emptyset$ (3.3)

The proof needs some technical lemmata and can be found with details in [6]. We only point out that from the assumption $(t,z) \notin A^{\perp\perp}$ it follows that there exists $f \in \beta(t,z)$ such that $f \cap A^{\perp} \neq \emptyset$. The proof shows the existence of a point $(\alpha, f(\alpha)) \in A^{\perp} \cap ((t,z) \cup A)^{\perp\perp}$

4.Causal structure in Minkowski space-time.

We shall specify more precisely the causal structure in Minkowski space-time $M = R \times R^3$ with the scalar product $x \cdot y = x_0 y_0 - \underline{x} \cdot \underline{y}$.
Let $\mathcal{G}_\alpha$ be the family of functions $f : R \rightarrow R^3$ such that the following Lipschitz condition is satisfied

$$\| f(x_0) - f(y_0) \| < \alpha \; | x_0 - y_0 | \qquad \alpha > 0 \tag{4.1}$$

We denote the orthogonality relation generated by $\mathcal{G}_\alpha$ as $\perp_\alpha$. The family satisfies conditions (3.2) ; by theorem 3.1 $\mathcal{C}(M, \perp_\alpha)$ is an orthomodular lattice.
Automatically we have that two points $x, y \in M$ are orthogonal if and only if

$$| x_0 - y_0 | \leq \frac{1}{\alpha} \; \| \underline{x} - \underline{y} \| \tag{4.2}$$

This means that x is space- or light-like to y with velocity of light equal to α. One can see that the second assumption for $\mathcal{G}_\alpha$ (3.2) is a causal transitivity condition, the third one is a kind of continuity (signals are propagated inside the light cone only). A special case for the family $\mathcal{G}_\alpha$ given by time-like straight lines has be considered in [5]. The lattice $\mathcal{C}(M, \perp_\alpha)$ is called the <u>causal logic</u> and has connections with the family of closed double cones in Minkowski space. As was shown in [5] the group of automorphisms of this logic consists of Poincaré transformations and dilations.

5.Realization of the causal logic in Hilbert space.

A natural question arises: can we represent $\mathcal{C}(M, \perp_\alpha)$ by projections in a Hilbert space? The general problem of vector-space coordinatization of a lattice was consider by many authors [12,13] ; in particular, the case of a Hilbert space coordinatization [1,14,15]. One of the necessary condition for such a coordinatization is the covering law. In this section we will examine this property in the causal logic $\mathcal{C}(M, \perp_\alpha)$ of Minkowski space-time. We will prove that $\mathcal{C}(M, \perp_\alpha)$ does not satisfy the covering law.

There are many equivalent conditions for the covering property in atomic, orthomodular lattices [2,13,15]. The one most useful in our investigations is connected with the so called Sasaki projection. For an element a in the lattice L define the Sasaki projection $\varphi_a(b) := a \wedge (b \vee a^\perp)$ where $b \in L$

Theorem 5.1.

An atomic orthomodular lattice L has the covering property if and only if for every $a \in L$ and each atom p not orthogonal to a, $\varphi_a(p)$ is an atom of L.

For the proof see [13,15].

Let us examine the properties of $\mathcal{C}(M, \perp_\alpha)$.

Lemma 5.1.

$\mathcal{C}(M, \perp_\alpha)$ is an atomic lattice. The points are the atoms.

Proof.

Because the order is given by set theoretic inclusion it is enough to prove that each point is an atom. Clearly $a \in \{a\}^{\perp\perp}$, let us take $a = (t_a, x_a)$ and assume that there exists $b = (t_b, x_b) \in \{a\}^{\perp\perp}$ such that $t_a < t_b$ (in the opposite case $t_b < t_a$ the proof proceeds analogously). If $t_a = t_b$ then by 4.2 $a \perp b$. By assumption $b \notin \{a\}^\perp$ so there exists $f \in \beta(b)$ such that $f \cap \{a\} \neq \emptyset$ and by condition 1 for $\mathcal{G}_\alpha$ f is defined for any $t \in (t_a, t_b)$. We shall prove that $(t, f(t)) \in \{a\}^{\perp\perp}$ for any $t \in (t_a, t_b)$.

Let us fix t and assume that $(t, f(t)) \notin \{a\}^{\perp\perp}$ then there exists $h \in \beta(t, f(t))$ such that $h \cap \{a\}^\perp \neq \emptyset$ and using condition 2 for $\mathcal{G}_\alpha$ there exists $g_b \in \mathcal{G}_\alpha$ from $\{a\}^\perp$ to b or $g_a \in \mathcal{G}_\alpha$ from $\{a\}^\perp$ to a which contradicts our assumption that $b \in \{a\}^{\perp\perp}$ or $a \in \{a\}^{\perp\perp}$ respectively. So that if there exists $b \neq a$, $b \in \{a\}^{\perp\perp}$ then there exists $f \in \beta(a) \cap \beta(b)$ such that for any $t \in (t_a, t_b)$, $(t, f(t)) \in \{a\}^{\perp\perp}$.

In Minkowski space-time $\{a\}^\perp = M \setminus (\{a\} \cup V_a)$ where V_a is an open cone with vertex in the point a. But $\{a\}^{\perp\perp\perp} = \{a\}^\perp$ so the time interval (t_a, t_b) should reduce to zero.

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Lemma 5.2.

The center of $\mathcal{C}(M, \perp_\alpha)$ is trivial.

Proof.

Let a, b be atoms in $\mathcal{C}(M, \perp_\alpha)$ such that $a \neq b$, $a \not\perp b$. We shall prove that a and b do not commute. By orthomodularity [2] a and b

commute if and only if $a \wedge b = a \wedge (b \vee a^{\perp})$. So from 2.1 and lemma 5.1 we have

$$a \wedge b = \emptyset \quad , \quad a \wedge (b \vee a^{\perp}) = a$$

Because $\mathcal{G}_{\alpha}$ covers M then for $a \in M$, $a \neq \emptyset$ there exists $f \in \beta(a)$ and no point from this path f is orthogonal to a , therefore the center contains only $\emptyset$ and M . ■

Combaing the Sasaki projection with theorem 3.1 and lemma 5.1 we have the following result.

Lemma 5.3.

$\mathcal{L}(M, \perp_{\alpha})$ does not satisfy the covering law.

Proof.

Observe that by the lattice operation (2.1) the condition (3.3) means that $A^{\perp} \cap (x^{\perp} \cap A^{\perp})^{\perp} = A^{\perp} \wedge (x \vee A)$. Taking as A a point $A = (t_A, x_A)$ by lemma 5.1 we have $A^{\perp\perp} = A$ and we are in the case of the Sasaki projection $\varphi_{A^{\perp}}(p)$ associated with $A^{\perp}$.

Let us take an atom $p = (t_p, x_p)$ such that $p \in A^{\perp}$, $p \notin A^{\perp\perp}$ then by theorem 3. (a point is an orthogonal set) there exists $(a, f(a))$ such that $(a, f(a)) \in \varphi_{A^{\perp}}$ and $t_p < a < t_A$. We shall prove the existance of another atom in $\varphi_{A^{\perp}}$ different from $(a, f(a))$. From $p \notin A^{\perp}$ there exists $g \in \beta(t_p, x_p)$ such that $g \cap A = \emptyset$ and $t_p < a < t_A$. Connectness of g (3.2) gave us that $a \in \mathrm{dom}\, g$ and so $(a, f(a)) \perp (a, g(a))$. But $(a, g(a)) \notin A^{\perp}$ and $(a, g(a)) \notin A^{\perp\perp}$ we can repeat the arguments of the theorem 3.1 starting from the point $(a, g(a))$ and as a result we have $(a_1, h(a_1)) \in \varphi_{A^{\perp}}$ which is different from $(a, f(a))$.

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Corollary.

$\mathcal{L}(M, \perp_{\alpha})$ cannot be represented faithfully by a lattice of projections in the Hilbert space.

Proof.

Assuming such a representation exists and using lemma 5.1 and 5.2 we obtain a contradiction with lemma 5.3 (the lattice of all projections in Hilbert space satisfies the covering law) .

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