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On Endomorphism Algebras of Mixed Modules

by

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§1. Introduction.

In a remarkable paper [1] some twenty years ago, A.L.S. Corner showed that every countable reduced torsion-free ring is the endomorphism ring of a countable reduced torsion-free abelian group. This has been the starting point for many investigations of the so-called realization problem which may be stated as follows:

Given an algebra A over a commutative ring R, when will A be the endomorphism algebra of an R-module G which belongs to some suitably restricted class $\mathcal{C}$. Complete characterizations of such algrebras A have been obtained in the case where R is a complete discrete valuation ring and $\mathcal{C}$ is the class of torsion-free reduced R-modules ([10]) and also in the case where R = $\mathbb{Z}$ and $\mathcal{C}$ is the class of separable p-groups ([9]) or section 109 in [8]). Such characterizations are, inevitably, much too complicated to lend themselves readily to applications. Consequently Corner [2] tackled the realization problem for primary abelian groups from a different angle. He showed that a suitably large class of rings A could be realized, not as a full endomorphism ring, but rather that the full endomorphism algebra would be the split extension of the given ring A by some ideal whose presence was unavoidable; in the case of primary groups this ideal being precisely the ideal of small endomorphisms ([2]). This idea was subsequently extended to large primary groups in [5] and a similar type of result was produced in [7] for torsion-free modules over a complete discrete valuation ring.

The results of Corner [2], Dugas and Göbel [5] and Dugas, Göbel and Goldsmith [7] are all capable of translation into results on endomorphism algebras in a suitable quotient category. Thus, for example, if $\mathcal{S}$ is the category having primary abelian groups as objects, and morphisms $Hom_{\mathcal{S}}(G,H) = Hom(G,H) / Hom_s(G,H)$, where $Hom_s(G,H)$ consists of the small homomorphisms of G into H, then Corner's result is that if A is a ring whose additive group is the completion of a free p-adic module of at most countable rank, then there exists a primary group G with $E_{\mathcal{S}}(G) = A$.

When dealing with mixed abelian groups (or more generally mixed R-modules), there is a natural category in which to work viz. the category <u>Walk</u> ($_R$<u>Walk</u>). The objects of $_R$<u>Walk</u> are R-modules and its morphisms are given by $Hom_W(G,H) = Hom(G,H) / Hom_t(G,H)$, where $Hom_t(G,H)$ consists of the

R-homomorphisms of G into H with torsion image (see [11]). Recently Dugas [3] has shown that each torsion-free reduced ring A is the Walk-endomorphism ring of a mixed abelian group G. The groups G so realized are all of large infinite rank even when the ring A is of comparatively small cardinality.

Our approach will be to construct a (non-trivial) full embedding of the category of torsion-free reduced R-modules into the category ${}_R\mathrm{Walk}$, where R will be a principal ideal domain. As a consequence of this full embedding we may immediately lift established results from the category of reduced torsion-free R-modules to the category ${}_R\mathrm{Walk}$. A typical, but by no means exhaustive, list of such results is contained in Corollaries 2.4 - 2.6. We note, in particular, that many of the results in the forthcoming paper of Dugas and Göbel [6] can now be established immediately. It is, by now, standard to use such realization results to exhibit a wide range of pathologies and so we desist from such repetition.

We conclude this introduction by noting that all unexplained terms may be found in the standard works of Fuchs [8]; our notation is in accord with [8] with the exception that maps are written on the right.

§2. The embedding theorem.

Throughout let R be a principal ideal domain. We begin with an arbitrary reduced, separable torsion R-module T and T' any pure extension of T by Q/R such that T' is also separable and reduced. Thus we have a pure - exact sequence of R-modules

$$(*) \qquad 0 \longrightarrow T \longrightarrow T' \longrightarrow Q/R \longrightarrow 0$$

which will be fixed for the rest of the section. Note that provided T has no torsion-complete p-component T_p such a sequence exists (see Corollary 68.5 in [8]).

Now, if X is an aribtrary R-module, then (*) yields another pure-exact sequence (see Theorem 60.4 in [8])

$$(*_X) \qquad 0 \longrightarrow T \otimes X \longrightarrow T' \otimes X \longrightarrow Q/R \otimes X \longrightarrow 0.$$

Since $Q/R \otimes X$ is canonically an epimorphic image of $Q \otimes X$ we can form the pullback H(X) of $(*_X)$ with respect to this canonical epimorphism η_X. This yields the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T \otimes X & \longrightarrow & H(X) & \xrightarrow{\pi_X} & Q \otimes X & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \eta_X} & & \\ 0 & \longrightarrow & T \otimes X & \longrightarrow & T' \otimes X & \longrightarrow & Q/R \otimes X & \longrightarrow & 0 \end{array}$$

in which σ_X is epic since η_X is epic. Note that by the construction of a pullback Ker σ_X is mapped isomorphically onto Ker η_X by π'_X. Also Ker η_X is canonically isomorphic to $X/t(X)$. The R-module $H(X)$ has the same torsion-fr rank as X and its torsion submodule is isomorphic to $T \otimes X$. Note, that if X is torsion-free reduced, then $H(X)$ is reduced and hence non-split.

If $U(M) = \bigcap_{0 \neq r \in R} rM$ denotes the first Ulm submodule of an R-module M, then the purity of (*) implies:-

Lemma 2.1. Ker $\sigma_X = U(H(X))$

Proof: Note firstly that it follows from Theorem 61.1. in [8] that $T'_p \otimes X \cong T'_p \otimes$ where B_p is a p-basic submodule of X. Thus $T'_p \otimes X = \bigoplus_p T'_p \otimes B_p$ and since T' is separable it follows readily that $U(T' \otimes X) = 0$. But $U(H(X))\sigma_X \subseteq U(T' \otimes X) =$ and hence $U(H(X)) \subseteq$ Ker σ_X.

Conversely let m be an arbitrary element of Ker σ_X and let r be an arbitrary non-zero element of R. Then there is an element $y \in H(X)$ with $m - ry = t \in t(H(X))$. But then $t = t\sigma_X = m\sigma_X - ry\sigma_X = -ry\sigma_X \in r(T' \otimes X) \cap T \otimes X = r(T \otimes X)$ by the purity of the sequence $(*_X)$. Hence $m \in rH(X)$ follows. Since r was arbitrary non-zero, we ha $m \in U(H(X))$ and so Ker $\sigma_X \subseteq U(H(X))$.

We remark that the construction of $H(X)$ is functorial: every $f \in \mathrm{Hom}(X,Y)$ yields homomorphisms $Q \otimes X \to Q \otimes Y$ and $T' \otimes X \to T' \otimes Y$ which in turn give rise to a unique homomorphism $H(f): H(X) \to H(Y)$ by the universal property of the pullback. We denote this functor by H. In order to place our construction in a functorial setting let U be the subfunctor of the identity defined by $U(X) = \bigcap_{0 \neq r \in R} rX$ and $U(f) = f|U(X)$, the restriction of f to $U(X)$; let F be the functor defined by $F(X) = X/t(X)$ and $F(f) = \bar{f}$ where $\bar{f}$ is the mapping induced by f on the quotient.

Proposition 2.2. The functors UH and F are naturally equivalent.

Proof: By lemma 2.1. $UH(X) =$ Ker σ_X and since π'_X maps Ker σ_X isomorphically onto the kernel of η_X the assertion follows from the observation that Ker $\eta_X \cong X/t(X)$.

In the following let $_R\mathcal{C}$ denote the category of torsion-free reduced R-mod

Theorem 2.3. Let R be a principal ideal domain, T be a separable reduced torsion R-module and T' be a pure extension of T by Q/R such that T' is separat and reduced. Then there is a full embedding $\bar{H}: {_R\mathcal{C}} \longrightarrow {_R\mathrm{Walk}}$ such that

(i) $\bar{H}(X)$ is reduced, non-split and of the same torsion-free rank as X.

(ii) $t(\bar{H}(X)) \cong T \otimes X$.

(iii) $\bar{H}(X)/t(\bar{H}(X))$

(iv) $U\bar{H}(X) = X$ and $\bar{H}(X)/U\bar{H}(X) = T' \otimes X$.

Proof: For $X \in R^{\omega}$ let $\bar{H}(X) = H(X)$ and for $f: X \to Y$ let $\bar{H}(f) = H(f) + \mathrm{Hom}_t(H(X), H(Y))$. The only assertion still to be verified is that $\bar{H}$ is a full embedding. By Proposition 2.2 UH is naturally equivalent to F which is the identity functor on R^{ω}. Therefore we may identify X and UH(X). Consider the homomorphisms $h : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(H(X), H(Y))$ and $u: \mathrm{Hom}(H(X), H(Y)) \to \mathrm{Hom}(X,Y)$ induced by H and U respectively. Then h u is the identity on Hom(X,Y), thus h is monic and u is epic. Furthermore $\mathrm{Ker}\, u = \mathrm{Hom}_t(H(X), H(Y))$ since $g|UH(X) = 0$ implies that Im g is torsion as an epimorphic image of the torsion module $H(X)/UH(X) \cong T' \otimes X$. On the other hand, if Im g is torsion, then $g(UH(X) = 0$ because $UH(Y) \cap t(H(Y)) = 0$. Thus we conclude that the map $f \mapsto \bar{H}(f)$ is an isomorphism and $\bar{H}$ is a full embedding.

Remarks: (a) An alternative way to construct the functor H is the following: Let $M = H(R)$, a mixed module of torsion-free rank one. Then it is readily seen that the functors H and $M \otimes -$ are naturally equivalent. (b) As indicated in in the above proof $E(H(X))$ is the split extension of E(X) by $\mathrm{Hom}_t(H(X), H(X))$, i.e. there are ring homomorphisms $E(X) \overset{h}{\to} E(H(X)) \overset{u}{\to} E(X)$ such that $hu = id_{E(X)}$ and $\ker u = \mathrm{Hom}_t(H(X), H(X))$.

Corollary 2.4. Let R be a principal ideal domain. If A is a countable reduced torsion-free R-algebra then there are $2^{\aleph_0}$ countable mixed R-modules M_i with $M_i/t(M_i)$ divisible, $E_W(M_i) \cong A$ and $\mathrm{Hom}_W(M_i, M_j) = 0$ for $i \neq j$.

Proof: By an unpublished extension of a well-known theorem of Corner [1] there exist countable reduced torsion-free modules with $E(X_i) \cong A$ and $\mathrm{Hom}(X_i, X_j) = 0$ for $i \neq j$. Now Theorem 2.3 yields the assertion by choosing an appropriate torsion module T, for example an unbounded countable direct sum of cyclics.

In the finite rank case Corner's result gives

Corollary 2.5. Let R be a principal ideal domain and A a countable reduced torsion-free algebra of finite rank n. Then there exists a reduced mixed module M of torsion-free rank 2n such that $M/t(M)$ is divisible and $E_W(M) = A$.

Corollary 2.6. If R is a principal ideal domain and not a complete discrete valuation ring and A is any cotorsion-free R-algebra, then there exists a reduced mixed R-module M with M/t(M) divisible and $E_W(M) = A$.

Proof: This is a consequence of Corollary 5.4 in [4], which ensures the existenc of a cotorsion-free R-module X with E(X) = A.

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