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SKYRME-LIKE MODELS IN GAUGE THEORY

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ABSTRACT

Hierarchies of systems in higher dimensions generalising the Yang-Mills model are reviewed. These involve higher powers of the totally antisymmetrised derivatives of the fields, in common with the Skyrme model. Their solutions as well as their descent to lower dimensions is discussed, and as examples of our procedure Higgs models in 2,4 and 7 dimensions are presented in some detail.

1. Introduction

1.1 *Motivation*

An important role in the study of non perturbative effects in quantum field theory is played by the finite action field configurations which minimise the action. These are the topologically stable instanton solutions of the dynamical equations of the model in question. It is important therefore to find the appropriate instanton solutions of the two models of fundamental interactios, the Strong and the Electroweak. Here the following twin problem arises: The only instantons that the YM-H system in 4 dimensions supports are the well known BPST solutions[1] of the YM field equations, with the Higgs field equal to its constant vacuum value (VEV) everywhere. The first problem is that the trivial Higgs field configuration excludes the possibility of finding non-trivial instanton solutions in the Electroweak model. The second problem is that due to the scale invariance of the YM model in 4 dimensions, the BPST instantons have an arbitrary scale which means that the physically desirable description of a dilute instanton gas is not justified. This problem is also there because of the absence of a Higgs field, which could have provided an absolute scale, the VEV, with respect to which a dilute gas approximation could have been implemented. Thus we see that these twin problems can be tackled if we could construct some extension of the YM-H system which supports instanton solutions in 4 dimensions. This is the main aim of the work reviewed below.

1.2 *Skyrme sigma model*

The task we have set ourselves is similar to, but more complicted than, the resolution of a similar problem, namely the construction of finite action topologically stable solutions in a sigma model in 3 dimensions. This is better known as the Skyrme model[2] which is usually described by the field $U \in SU(2)$. Expressing U in terms of an $O(4)$ valued field ϕ^aas $U = expi(1 \pm \gamma_5)\gamma_a \phi_a / 2$, where $\phi^a \phi^a = 1$. This $O(4)$ sigma model is given by

$$L = \kappa^2 (\partial_\mu \phi^a)^2 + (\partial_{[\mu} \phi^a \partial_{\nu]} \phi^b)^2 \tag{1}$$

where κ is a constant with the dimensions of inverse length..Integrating the trace of the stress tensor over the d-dimensional volume of configuraton space we find the restriction

$$(d-2)\kappa^2 \left\|\partial_\mu \phi^a\right\|^2 + (d-4)\left\|\partial_{[\mu} \phi^a \partial_{\nu]} \phi^b\right\|^2 = 0 \tag{2}$$

from which we see that in $d=3$, it is essential to retain the (quartic) Skyrme term. In essence, a quartic term has been added to the usual (quadratic) sigma model Lagrangian, to render the construction of finite energy solutions in 3 dimensions possible. This is the essence of the procedure described below, where the usual YM system is to be extended by Skyrme-like terms to render the construction of instanton solutions in 4 dimensions possible.

The restriction corresponding to (2) for the usual YM-H system

$$L = tr[\frac{1}{4} F_{\mu\nu}{}^2 + \frac{1}{2} D_\mu \Phi^2 + V(\eta^2, \Phi^2)] \tag{3}$$

in d dimensions is the identity

$$(d-4)\left\|F_{\mu\nu}\right\|^2 + (d-2)\left\|D_\mu \Phi\right\|^2 + (d-0)\left\|V\right\| = 0, \tag{4}$$

from which it is clear that only in $d_{<}4$ can there be 'instanton' solutions with non-trivial Higgs fields and in $d=4$, instanton solutions exist only for trivial configurations of the Higgs field. In a sense, the Higgs dependent terms in (4) could be regarded as Skyrme-like terms in $d<4$.

1.3. *Skyrme Higgs-models*

To find a prescription for constructing suitable Skyrme-terms for the YM sytem in $d \geq 4$, let us first consider the generic YM-H system (3) in $d \leq 4$ as a subsystem of the SU(2) YM system in 4 dimensions. It has been verified in Ref[3] that the SU(2) YM systems on $R_3 x S^1$ and $R_2 x S^2$ respectively, after being subjected to dimensional reduction[4], reduce to the SU(2) YM-H system with $\Phi \in su(2)$, and to the U(1) (Maxwell) Abelian model with a complex valued Higgs field φ. It is well known that both these 3 and 2 dimensional models support finite action topologically stable solutions, namely the monopole[5] and vortex solutions of the respective models.

We see that for the two models exemplified by (3) in 3 and 2 dimensions, the required Skyrme terms are generated by subjecting the 4 dimensional YM model to dimenssional reduction. To employ this prescription for the task at hand, namely to construct instantons in various Higgs-models in dimensions $d \geq 4$, we must first find a suitable higher dimensional generalisation of the usual YM model and subject it to dimensional reduction.

The procedure just proposed however has one further vital element which we now describe. The construction of topologically stable instanton solutions in the above mentioned two examples[3] relies on there being topological lower bounds on the respective actions. These take the form of inequalities that are descended from 4 dimensions, where the lower bound is the integral of residual density descending from the second Chern-Pontryagin (C-P) density $trF \wedge F$ after dimensional reduction. These are respectively the familiar monopole and the vortex charge densities

$$\rho_{monopole} \approx \varepsilon_{ijk} tr D_i \Phi F_{jk} = \partial_i \varepsilon_{ijk} tr \Phi F_{jk} \tag{5a}$$

$$\rho_{vortex} \approx \varepsilon_{\alpha\beta}(4 f_{\alpha\beta} + i D_{[\alpha} \varphi D_{\beta]} \varphi) = \partial_\alpha \varepsilon_{\alpha\beta}(a_\beta + i\varphi^* D_\beta \varphi) \tag{5b}$$

where F_{ij} and f_{ij} are the SU(2) and Abelian field strengths respectively. Both (5a,b) are total divergences like the second C-P density they are descended from. This last property is crucial in guaranteeing that a solution satisfying the appropriate boundary conditions should result in a topologically non-trivial bound. Note that the second term in (5b) does not contribute to the one dimensional (surface) integral because of finite-action asymptotic conditions, leaving just the usual vortex number density.

The total divergence expressions in (5a,b) have their analogues in the case of the sigma model (1) which supports the well known Skyrmion[2] solution, which is also topologically stable. In that case the topological density is the winding number density

$$\rho_{winding} = \varepsilon_{\mu\nu\lambda}\varepsilon^{abcd}\partial_\mu\phi^a\partial_\nu\phi^b\partial_\lambda\phi^c\phi^d \tag{6}$$

whose first variation vanishes as expected.

We conclude therefore that the procedure to be followed below should incorporate first, the generalisation of the YM system to higher dimensions, which we present in Section 2, and second , the dimensional reduction of both the higher dimensional action and the corresponding C-P charge.

2. Generalised YM(GYM) systems

2.1. *Definition*

The higher dimensional YM hierarchy we will present here can be best introduced by generalising the basic topological inequality of the usual YM model in 4 dimensions

$$\int_{M_4} trF(2)^2 \geq \int_{M_4} tr(F(2)\wedge F(2)) \tag{7}$$

because we require that the hierarchy of GYM systems be endowed with similar topological inequalities giving rise to finite action topologically stable solutions. In (7) we have used the notation $F(2) = F_{\mu\nu}$ for the curvature 2-form field strength. We proceed first by generalising this notation to define the $2p$-form field strength $F(2p)$, as the p-fold totally antisymmetrised product of the 2-form curvature $F(2)$. Thus the 4-form field strength $F(4)$ consists of 6 terms involving the 2-fold product of the curvature 2-form, the 6-form strength consists of 70 such terms etc. $F(4)$ in patricular has a very compact expression which we give here since it will occur frequently in the following by way of examples. This is

$$F(4) = F_{\mu\nu\rho\sigma} = \{ F_{\mu[\nu}, F_{\rho\sigma]} \} \tag{8}$$

where the brakets {,} denote anticommutation and [...] cyclic symmetrisation.

It is clear from (7) that the generalisation of the YM system given here by the integrand on the left hand side pertains only to even dimensions since the C-P densities generalising the right hand side are defined only in even dimensions. The most natural way of proceeding here is to note that (7) follows directly from the inequality

$$tr[F(2) - {}^*F(2)]^2 \geq 0 \tag{9}$$

where ${}^*F(2) = ({}^*F(2))(2)$ is the Hodge dual of the curvature 2-form in 4 dimensions, which in this case is also a 2-form. (Note that the saturation of the inequality (9) results in the usual self duality equations.) Using this notation introduced, the generalisation of (9) to $2(p+q)$ dimensions is very natural and straightforward

$$tr[\kappa^{2(q-p)}F(2p) - ({}^*F(2q))(2p)]^2 \geq 0. \tag{10}$$

The dimensions of the constant κ here are those of an inverse length, with $q>p$. Expanding (10) now, we have the generalisation of (8) in which the left hand side *defines* the new generalisation of the YM, and the right hand side is the $(p+q)$-th C-P charge:

$$\int_{M_{2(p+q)}} L_{GYM} \geq 2\kappa^{2(q-p)} \int_{M_{2(p+q)}} trF\wedge F\wedge\ldots\wedge F, \quad (p+q)-times \tag{11a}$$

$$L_{GYM} = tr[\kappa^{4(q-p)}F(2p)^2 + (2q)!(2p)!\,F(2q)^2]. \tag{11b}$$

The hierarchy of generalised YM(GYM) systems in $2(p+q)$ dimensions is defined[7,8] by (11b). For the special cases where $p=q$, the GYM system (11b) in $4p$ dimensions is scale invariant. This interesting subset of the GYM hierarchy will be further discussed in the follwing subsection devoted to a review of known solutions.

The scale-breaking Lagrangian (11b) with $p\neq q$ bears a very close resemblance to the Skyrme sigma Lagrangian (1). In both these Lagrangians, the dimensional constant κ plays the same role. The second terms in both (1) and (11b) can be regarded as Skyrme terms. Recall however that from the scaling viewpoint, we have identified the Higgs dependent terms in the YM-H Lagrangian (3) also as Skyrme-like terms, which incidentally also feature a dimensional constant, η, in the guise of a VEV. Thus in gauge theories, we have two distinct types of Skyrme terms that we can employ, and we will discuss and exploit both of these in the following.

We complete the definition of GYM systems by discussing the question of the associated Dirac equation. The usual dirac equation in terms of the Dirac symbol $\Gamma_\mu D_\mu$, in terms of the gamma matrices in $2n$ dimensions, would be the simplest generalisation of the the 4 dimensional case. For $n>2$ however this particular generalisation does not capture one of the most attractive features of the Dirac equation in the background of self-dual YM fields in 4 dimensions, namely that for the zero modes, this equation yields the Klein-Gordon equation when it is acted on once more by the symbol. With this in mind, we have suggested a generalised Dirac symbol in $4p$ dimensions, in the background of the (generalised) self dual field which saturates the inequality (11a) for $p=q$. (These generalised self duality equations just introduced will be discussed in detail below, Eq.16, in the context of their solutions.) The new symbol is defined so, that when the zero mode equation is acted on by $\Gamma_\mu D_\mu$, it will yield an elliptic equation generalising the Klein-Gordon equation. This symbol

$$\frac{1}{2}(1\pm\Gamma_{4p+1})\Gamma_{\mu_1\ldots\mu_{2p-1}}\otimes F_{\mu_2\ldots\mu_{2p-1}}D_{\mu_1} \tag{12}$$

was introduced in Ref.9, and the particular Yukawa couplings it gives rise to under dimensional reduction were given in Ref.10. Γ_μ and Γ_{2n+1} are the gamma matrices and the chirality matrix in $2n-4p$ dimensions.

Before proceeding to discuss the solutions these systems support, we briefly allude to and dispose of another hierarchy of higher dimensional models[11,12] which generalises the YM model in 4 dimensions. These models are also defined in even, say $2n$ dimensions by

$$L_n = Tr(F_{\mu\nu}\otimes\Sigma'_{\mu\nu})^n \tag{13}$$

where $\Sigma^{\pm}_{\mu\nu} = (-1/8)(1 \pm \Gamma_{2n+1})[\Gamma_\mu, \Gamma_\nu]$ are the $so_+(2n)$ matrices, and where the trace is taken over both the spinor indices and the gauge group indices. With the exception of the case with $n=3$, for all $n>3$, the systems (12) contain[11] the GYM system (11b) in the appropriate dimensions. In 6 dimensions (12) does not overlap with a GYM system and is given by

$$L_3 = trF_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}. \tag{14}$$

It turns out that for gauge group $SO_+(2n)$, the Euler-Lagrange equations pertaining to (13) have spherically symmetric (see below) solutions. The main difference between the GYM systems (11b) and the alternative hierarchy (13) is that the latter are not endowed with topological inequalities analogous to (11a). This means that any solutions of (13) that may be found are not guaranteed to be topologically stable, and if so, we dispose of the alternative hierarchy (13). We have indeed verified[13] explicitly that the second variation operator in the background of the analytically known spherically symmetric solution of (14) has negative eigenvalues and hence that this solution is unstable. We shall henceforth restrict ourselves to the GYM hierarchy.

Yet another reason for choosing the GYM hierarchy is that the latter involves only the second power of any derivative of the fields by virtue of the built in antisymmetry. This is another property shared with the Skyrme model which the alternative hierarchy (13) does not have.

2.2. *Solutions*

The known finite action topologically stable 'instanton' solutions to the Euler-Lagrange equations of the GYM systems (11b) are either spherically (radially) symmetric or at the least axially symmetric. The most important among these are the self dual solutions which in some cases can be expressed analytically. These occur for the scale invariant cases with $p=q$. There are also non-self-dual instanton solutions for those systems with $p \neq q$. We shall review these two classes of solutions and their properties in Subsections 2.2A and B respectively below.

2.2A *Self dual solutions*

When $p=q$ the GYM system (11b)

$$L_{GYM} = trF(2p)^2 \tag{15}$$

$4p$ Euclidean dimensions, there are p possible axial symmetries as discussed in the Appendix to Ref. 18, if by axial symmetry we understand that imposition of symmetries which results in a two dimensional residual system with $U(1)$ gauge group. The *first* one is the usual (4p-1) dimensional spherical symmetry imposed in the (4p-1) dimensional subspace. The *other* (p-1) axial axial symmetries pertain to the rest of all the odd partitions of $4p$. In our search[18,19] for scale invariant in $4p$ dimensions, and the Euler-Lagrange equations of this system are solved by the self-duality equations

$$F(2p) = {}^*F(2p) \tag{16}$$

which in this case minimise the action ábsolutely, since (16) saturates the inequality (11a) or (10). In the special case of $p=1$, (15) is the usual YM system which is the first member of the GYM hierarchy and (16) is the usual self duality equation. For this reason, we shall often refer to (16) as the generalised self duality equation. Equations (16) in eight dimensions, which is the case $p=2$ here, were independently discovered[14], but not in the context of the GYM dynamics as given by (14).

Before discussing specific solutions of (16), we remark that this is a system of equations involving the algebra valued connection field in $4p$ dimensions and taking into account the gauge freedom, this field has $(4p-1)$ independent components. Counting on the other hand the number of equations (16), we find $(4p)!/2(2p)!$, whose values are not even restricted to the algebra of the gauge group. Clearly we have a potentially overdetermined system here, since the number of equations (16) is greater than the number $(4p-1)$ of independent fields.The exception is the $p=1$ case where these nimbers are equal as expected for the usual YM system in 4 dimensions, where we know that the self duality equations are not overdetermined.

It is therefore not surprising that for $p>1$, equations (16) have non trivial solutions only when the fields are subjected to some stringent symmetries. We have made a detailed study[15] of equations (16) on R_{4p}, imposing spherical symmetry, axial symmetry which involves the imposition of spherical symmetry in a $(4p-1)$ dimensional subspace of R_{4p}, as well as the less stringent symmetries involving the imposition of spherical symmetry in subspaces of dimensions lower than $(4p-1)$. Our conclusion[15] was that the system of equations arising from (16) was greater than the number of independent functions parametrising the connection field subjected to the appropriate symmetry, *except* in the case of axial symmetry, includung spherical symmetry. This investigation[15] was carried out for gauge-group $SO_+(4p)$, which is the gauge-group of the known solutions to be discussed below.

Spherically symmetric solutions[8]

These solutions are given[8,14] in analytic form on all 4p dimensional[8] flat Euclidean spaces for gauge-group $SO_+(4p)$. In the special case $p=1$, this is chiral $SO(4)$, which coincides with the gauge-group $SU(2)$ of the BPST instantons. Using the same notation as in Eqs.12,13, these solutions all take the same form for all p, namely

$$A^+_\mu = \frac{2a}{r(r^2+a^2)}\Sigma^+_{\mu\nu}x_\nu \tag{17}$$

and hence we shall refer to them as the BPST hierarchy of solutions.

That the analytic form of the solutions (17) is independent of p is not at all surprising, and is a direct consequence of the scale invariance of the GYM systems (15). We have studied[16] the conformal invariance of the BPST hierarchy of instantons in the case of arbitrary p, as had been done[15] for the $p=1$, YM case in 4 dimensions. As a consequence of this scale invariance, it can be shown that the action corresponding to the density (15) on R_{4p} can be expressed on S^{4p}. We shall return to this point below when we discuss other self dual solutions, on compact coset spaces.

Axially symmetric solutions[18,19]

For an $SO_+(4p)$ gauge field in 4p Euclidean dimensions, there are p possible axial symmetries as discussed in the Appendix to Ref.18, if by axial symmetry we understand that imposition of symmetries which results in a two dimensional residual system with $U(1)$ gauge group. The *first* one is the usual $(4p-1)$ dimensional spherical symmetry imposed in the $(4p-1)$ dimensional subspace. The *other* $(p-1)$ axial symmetries pertain to the rest of all the odd partitions of $4p$. In our search[18,19] for axially symmetric solutions to the generalised self duality equations (16), we found that only solutions with axial symmetry of the *first* type just described exist. This is a result of the overdetermined[15] nature of Eqs. 16.

Unlike the spherically symmetric subset of these solutions discussed above, the axially symmetric solutions cannot be found analytically, except in the well known p=1 case[20]. The generalised self duality equations (16) reduce to one complex valued equation for one complex valued function, and one real valued equation for one real valued function. Clearly therefore, imposition of this axial symmetry does not result in Eq. 16 being overdetermined. As in the $p=1$ case given in Ref. 20, the complex valued equation reduces to the Cauchy-Riemann equations and hence is immediately integrated, and the real valued equation then results in the following partial differential equation

$$\Delta\psi = \frac{2p-1}{r}(1-e^{-2\psi}) - \frac{2(p-1)e^{-2\psi}}{(1-e^{-2\psi})}(\vec{\nabla}\psi)^2 \tag{18}$$

where r here is $(x_1^2+x_2^2+...+x_{4p-1}^2)^{1/2}$, x_{4p} is t, and $\vec{\nabla} = (\partial/\partial r, \partial/\partial t)$. For $p=1$, the 2 dimensional partial differential equation (18) can be reduced to the Liouville equation which has well known analytic solutions. For all other $p>1$ however, we have been unable to integrate (18) explicitly. Nevertheless we have been able to show that some non trivial, and not spherically symmetric, solutions of (17) exist[19]. To do this, we have first changed variables according to $r+it = tanh\frac{1}{2}(\rho+i\tau)$, and then converted the partial differential equation (19) to an ordinary differential equation in ρ by suppressing the τ dependence of the function ψ. We have then integrated the ensuing ordinary differential equation numerically, subject to the requisite asymptotic conditions. The resulting solutions depend both on r and on t through their dependence on ρ. We have also verified that these solutions have arbitrary C-P charge, so that they are not restricted to the spherically symmetric case only. We do not expect that we have captured all axially symmetric solutions in this way, since in the $p=1$ case, where the analytic solutions are known[20], the axially symmetric solutions that we have found are only a subset of the complete instanton chain[20], situated on the x_4 axis only at special[21] intervals.

Solutions on G/H

In addition to the solutions on S^{4p} alluded to in the discussion of the spherically symmetric solutions above, the generalised self duality equations can be solved also on other symmetric coset spaces, G/H.

The self dual solutions on $4p$ dimensional spheres, i.e. on the symmetric coset spaces $SO(4p+1)/SO(4p)$, which satisfy the self duality equations (16) pertaining to the scale invariant GYM systems (15), are not all the self dual fields on the spheres. These are the $SO_{\pm}(4p)$ fields for which the action corresponding to the scale invariant Lagrangian (15) on S^{4p} equals the action of this Lagrangian on R_{4p} via a stereographic projection. These solutions, which were mentioned above, are not however all the self dual solutions on the spheres. It is a general feature of gauge fields on symmetric coset spaces that they satisfy the more general self duality equations

$$\kappa^{2(q-p)}F(2p) = (*F(2q))(2p) \tag{19}$$

which saturate the inequality (10) and hence their solutions must satisfy the Euler-Lagrange equations of the scale-breaking GYM Lagrangian (11b). This contrasts with self-dual fields on R_{2n} ,with $n=q+p$, in which case the generalised self duality equation (19) has non trivial instanton solutions only when $p=q$. In this last case, which we shall consider indetail below in Subsection B, the presence of the dimensional constant κ in (18) prevents any solutions which have suitable decay properties at large distances. This problem does not arise when (19) is considered on a *compact* coset space. Technically, what happens in this case is that the curvature field strength on G/H is given up to a factor with the dimensions of an iverse square of a length, and identifying this with the square of κ, Eq.10 becomes dimensionless.

G/H= SO(2n+1)/SO(2n)

In Ref.22, we have verified that the $SO_{\pm}(2n)$ field strength, which at the north pole of the sphere S^{2n} is given by $\Sigma^{\pm}_{\mu\nu}$, satisfies the generalised self, and respectively anti-self, duality equation (19) for arbitrary p and q, with $n=p+q$. In other words, the $SO(2n)$ field strength which splits up as $SO_{+}(2n)\oplus SO_{-}(2n)$, stisfies (19) simply.

Another interesting result from the work of Ref.22 is, that the spin connection identification[23] of the $SO_{\pm}(4p)$ gauge connection

$$A^{\pm}_{\mu} = -\frac{1}{2}\omega^{mn}_{\mu}\Sigma^{\pm}_{mn} \tag{20}$$

in terms of the Einstein-Cartan spin connection ω^{mn}_{μ} with m and n the frame indices, which already satisfies the generalised self-duality condition (16), implies the following double-self-duality for the $2p$-form Riemann curvature

$$R(2p) = *R(2p)* \tag{21}$$

in which one of the Hodge duals is taken over the coordinate indices $\mu, \nu,...$, and the other one is taken over the frame indices $m,n,...$. For $p=1$, the condition (21) which is equivalent to condition (16), was discussed before in Ref.23. $R(2p)$ in (21) is the totally antisymmetrised p-fold product of the 2-form Riemann curvature. It is interesting to note that the double self duality condition (20) actually solves the Euler-Lagrange equations of the generalised Einsten-Hilbert system

$$L_{GEC} = \varepsilon^{\mu_1\mu_2...\nu_1...\nu_{2p}} e^{n_1}_{\nu_1}..e^{n_{2p}}_{\nu_{2p}} \varepsilon_{n_1...n_{2p}m_1...m_{2p}} R^{m_1m_2}_{\mu_1\mu_2}..R^{m_{2p-1}m_{2p}}_{\mu_{2p-1}\mu_{2p}} \tag{22}$$

Note that for $p>1$, the dynamics of (22) is not Torsion free and hence we should more properly denote is as a generalised Einstein-Cartan system.

G/H=SU(n+1)/SU(n)xU(1)

In Ref.24, we have verified that the $SU(n)xU(1)$ gauge fields on CP^n satisfy the generalised self duality equations (19) for arbitrary p and q, with $n=p+q$. In terms of the specific subset of the $SU(n+1)$ Gell-Mann matrices λ_{μ}, labelled by $\mu = n,...,(n^2+2n-1)$, this curvature 2-form is expressed[25], at a fixed point in the appropriate patch simply as $[\lambda_{\mu},\lambda_{\nu}]$. In terms of the $SU(n)$ and $U(1)$ valued components of the curvature $2p$-forms on CP^n, the generalised self duality equation (19) reads

$$({}^{*}F(2q)_{SU_n})(2p) = (-)^{p-1}[(p-1)!(q-1)!]F(2p)_{SU_n} \qquad (23a)$$

$$({}^{*}F(2q)_{U_1})(2p) = (-)^{p}(p!/q!)F(2p)_{U_1} \qquad (23b)$$

In the particular case of p=1, Eqs.23 were obtained before in Ref.26, but not in the context of the GYM dynamics of the Lagrangian (11a). Similar self duality equations for gauge fields on HP^n, again only for $p=1$, were verified in Ref.26.

In the background of the above partial results stating that the H-algebra valued field $2p$-form field strength is self dual in the sense of (19) on the $2n$ dimensional coset space G/H with $n=p+q$ and for arbitrary q, for the cases where $G/H = SO(2n+1)/SO(2n)$ and $SU(n+1)/SU(n)xU(1)$, we speculate that this statement remains valid for any symmetric coset space.

We finish this Subsection with a final remark. It is a general feature of the above instanton solutions, that the C-P charge of the dierect sum fields, e.g. $SO_{\pm}(2n)\oplus SO_{-}(2n)$ and $SU(n)\oplus U(1)$, vanishes since this is the sum of the C-P charges of the self and the anti-self dual field strengths.

2.2B. *Non self dual solutions*

Instantons in 2n dimensions

It was stated in the previous Subsection that the generalised self duality equation (19) does not support instanton solutions in R_{2n}, except when $n=2p$. This does not mean however that there are no instanton solutions at all. There is in fact no reason to doubt the existence of such solutions, since the inequality (11a) supplies the action with a non trivial topological lower bound whether or not this inequality is saturated. The only technical disadvantage of not being able to saturate (10) or (11a) is, that instead of solving first order self-duality equations it is necessary to solve the second order Euler-Lagrange equations. This is precisely the case that obtains also for the Skyrme sigma model (1), where the Skyrmion is evaluated numerically.

Unfortunately, these second order Euler lagrange equations cannot be integrated to give analytic solutions in closed form but have to be integrated numerically. This is so even in the special case of the spherically symmetric fields when these second order equations are ordinary differential equations. In Ref.27, we have shown using numerical methods that there is a spherically symmetric $SO_{\pm}(2n)$ gauge field in R_{2n} which satisfies the requisite asymptotic conditions that result in finite action and topological stability. These solutions have unit C-P charge as expected.

Merons in 2n dimensions

Since the existence of the meron[27] solution to the Euler-Lagrange equations of the YM model in 4 dimensions is a consequence of conformal invariance, we would expect that the conformal invariant GYM systems (15) in $4p$ dimensions should also support meron solutions. Indeed, even the alternative models (13) in $2n$ dimensions which we have discarded on grounds that they cannot support topologically stable solutions, should also support meron solutions since they are manifestly conformal invariant in 2n dimensions. In fact, that the Euler-Lagrange equations of (13) cannot support topologically stable solutions is irrelevant in this context, since merons do not exhibit any such stability. In Ref.28, we have verified that for spherically symmetric $SO_{\pm}(2n)$ gauge fields, the Euler-Lagrange equations of the systems (13) and of course also (15), as well as any linear combinations of these, support meron solutions.

3. Descent to Higgs Models

Having given a natural GYM hierarchy of models in all even dimensions, we are now in a position to generate various Abelian and non-Abelian Higgs models by dimensional reduction. Given a gauge connection on a product space $M_{2n} = R_d \times K_{2n-d}$, where K_{2n-d} is compact and most typically a symmetric coset space, it is well known that the components of this connection on the subspace K_{2n-d} give rise to a Higgs field on R_d after dimensional reduction. The gauge group of the residual model on R_d as well as the Higgs multiplet, depend on both the gauge group in the 2n dimensional theory and the structure of the compact subspace. The calculus of dimensional reduction used here is our adaptation[30,31] of the formalism of Ref.4, which is specific to descents by 2 and three dimensions, to the case of descents by arbitrary dimensions. We shall take the results of Refs.4,30,31 as given and will use them in the following, to implement the programme described under Subsection 1.3.

Technically the most important items in Subsection 1.3 were the topological charge densities (5a,b) descended from the second C-P density. Both of these, the monopole and vortex charges, can be evaluated as surface, resp. line, integrals by virtue of the fact that both densities (5a,b) are total divergences. This property is crucial in enabling the construction of topologically stable solutions, since the surface integrals can be controlled by requiring suitable asymptotic properties from the solutions. While in these two familiar cases this property was well known, it remains to be verified that the situation holds also for other Higgs models arrived at by the dimensional descent of GYM systems.

In Refs.30,31,32, we have verified that under dimensional reduction from $M_{2n} = R_d \times K_{2n-d}$ to R_d, the n-th C-P charge reduces to a surface integral in d dimensions:

$$\int_{M_{2n}} trF\wedge\ldots\wedge F = \int_{R_d} \nabla.\Omega = \int_{\Sigma_{d-1}} \Omega \qquad (24)$$

where Σ_{d-1} is the (d-1) dimensional surface in R_d, for arbitrary $K_{2n-d} = S^{2n-d}$. In Ref.25, we have verified the same result for $K_{2n-d} = CP^{2n-d}$. Having obtained the result (24) both for spheres and for complex projective spaces, we would expect that it remains true for the case of arbitrary symmetric coset spaces. In the following however, we shall employ exclusively spheres since these are the simplest and we are interested only in the qualitative aspects of the models thus derived.

The procedure for deriving the Higgs models below starts from the inequality (11a), with the integrations taken instead over the product space $M_{2n} = R_d \times K_{2n-d}$. Here of course we have in mind that $n>2$, otherwise the ensuing models will simply be the usual Abelian Higgs and the YMH models on d=2 and 3 respectively, which we discussed previously in Subsection1.3. Indeed, without sacrificing any generality, we can restrict to the cases where $p=q$. After the proper imposition of symmetries, in our cases those of the spheres employed in the dimensional reduction, the coordinates on these spheres are integrated out to yield a residual inequality

$$\int_{R_d} L_{residual}[A_\mu,\Phi] \geq \int_{\Sigma_{d-1}} \Omega[A_\mu,\Phi] \qquad (25)$$

where we have used the result (24). Provided that the asymptotic conditions of the fields solving the Euler-Lagrange equations of the residual Lagrangian permit the surface integral on the right hand side of (24) not to vanish, this inequality acts as a topolgical lower bound and we can expect non trivial instanton solutions. These boundary conditions are qualitatively the same as those for the well known monopole[5] and vortex[6] solutions.

There is a particular family of models satisfying the inequality (25) which is canonical in the sense that the $d=2$ and $d=3$ members of that hierarchy generalise the usual Abelian Higgs and the YMH models in a very natural way which we discuss presently.

The mode of descent[33] from $2n$ down to d dimensions is so organised in this case that the spherically symmetric fields of the residual models in question have gauge groups $SO(d)$, and the corresponding Higgs fields take their values in the *vector* representations. These spherically symmetric gauge and Higgs field configurations are

$$A_\mu = \frac{1}{r^2}(1+f(r))\Gamma_{\mu\nu}x_\nu \tag{26a}$$

$$\Phi = \eta r^{-1}h(r)i\Gamma_{\mu}x_\mu \tag{26b}$$

in the notation of Eqs.12,13. The required instanton asymptotic behaviours are

$$-1 \xleftarrow[0\leftarrow r]{} f(r) \xrightarrow[r\to\infty]{} 0 \tag{27a}$$

$$0 \xleftarrow[0\leftarrow r]{} h(r) \xrightarrow[r\to\infty]{} 1. \tag{27b}$$

Here we note a very interesting common feature of all these fields (26) in the infinite asymptotic region, which is familiar in 3 dimensional case and namely that in the Dirac string gauge the Higgs field is a constant and the $SO(3)$ gauge field breaks down to an $SO(2)$ (Abelian) field with a line singularity along the z-axis. Using a suitable[33] $SO(d)$ gauge transformation to gauge the Higgs field (26b) at infinity away to a contant, the $SO(d)$ gauge field (26a) in this region breaks down to an $SO(d\text{-}1)$ singular 1-form

$$'A_i = \frac{1}{r^2(1+x_d/r)}\Gamma_{ij}x_j, \qquad 'A_d = 0; \quad \mu = i,d \tag{28}$$

in the Dirac string gauge. The field strength components corresponding to (28) were computed in Ref.34

$$'F_{ij} = -\frac{1}{r^2}\left[\Gamma_{ij} + \frac{1}{r^2(1+x_d/r)}x_{[i}\Gamma_{j]k}x_k\right] \tag{29a}$$

$$'F_{i,d} = \frac{1}{r^3}\Gamma_{ij}x_j. \tag{29b}$$

which agree with the familiar $d=2$ and $d=3$ results. The asymptotic field strengths (29) in this constant-Higgs gauge are very helpful in certain of the computations involved in constructing a dilute gas of instantons.

To illustrate the procedure of constructing Higgs models given in this Section, we give the residual Lagrangians and the corresponding topological charge densities featuring in the inequality (25) for three examples. One of these is the SO(7) Higgs model descended from the scale invariant GYM system on $R_7 \times S^1$ which is interesting in its own right, and the others are physically relevant cases, namely the generalised Abelian Higgs model[35] on R_2 descended from the same GYM system on $R_2 \times S^6$, and, the $SU(2)xSU(2)xU(1)$ Higgs model[33] descended from $R_4 \times S^4$.

Generalised monopole on R_7

This is the simplest possible example of a non trivial generalised Higgs model, and was first discussed in Ref.36. The Lagrangian and topological densities in 7 dimensions are given by

$$L_7 = tr(F_{ijkl}^2 + [F_{ij}, D_k\Phi]^2) \tag{30a}$$

$$\rho_7 \approx \partial_u \varepsilon_{ijklmnu} tr\Phi F_{ij}F_{kl}F_{mn} \tag{30b}$$

and the corresponding self duality equations are

$$F_{ijkl} = \frac{1}{3!}\varepsilon_{ijklmnu}[F_{mn}, D_u\Phi]. \tag{31}$$

It was shown in Refs.37, that the spherically symmetric fields (26), solve the self duality equation (31). The solution however has not been given in closed form as was in the case of the BPS monopole[38] in 3 dimensions.

It is interesting to note that in this case self dual solutions exist, in spite of the fact that the self duality equations (16) from which (31) are descended are is often overdetermined[15]. This is because this 'monopole' in 7 dimensions bears a similar relation to the axially symmetric instantons in 8 dimensions which we have shown exist[18,19], as the 3 dimensional BPS monopole[38] bears to the axially symmetric instanton chain in 4 dimensions[20].

Generalised vortices on R_2

Expressing the Abelian gauge field as $f_{\alpha\beta}$, and the O(2) Higgs field as a complex valued quantity φ as in (5b), the residual Lagrangian and topological densities for this model are

$$L_2 = 12[(\eta^2-|\varphi|^2)f_{\alpha\beta} - iD_{[\alpha}\varphi D_{\beta]}\varphi^*]^2 + (\eta^2-|\varphi|^2)^2|D_\alpha\varphi|^2 + \lambda(\eta^2-|\varphi|^2)^4 \tag{32a}$$

$$\rho_2 \approx \partial_\alpha\varepsilon_{\alpha\beta}[\eta^6 a_\beta - 3i(\eta^4 - \eta^2|\varphi|^2 + \frac{1}{3}|\varphi|^4)\varphi D_\beta\varphi^*]. \tag{32b}$$

The dimensionless coupling parameter λ can take any positive value without invalidating[6] the topological inequality (25).

It is interesting to note that only the first term in (32b), namely the first Chern-Simons density in 2 dimensions, can contribute to the line integral on the large circle since all the other terms decay faster according to finite action conditions. This situation obtains also for the usual vortex number density (5b), as pointed out there. When the dimensionless coupling strength parameter λ is equal to one, we have the following self duality equations corresponding to (32)

$$[(\eta^2-|\varphi|^2)f_{\alpha\beta} - iD_{[\alpha}\varphi D_{\beta]}\varphi^*] = 3\varepsilon_{\alpha\beta}(\eta^2-|\varphi|^2)^2 \tag{33a}$$

$$D_\alpha\varphi = i\varepsilon_{\alpha\beta}D_\beta\varphi. \tag{33b}$$

The descent mechanism in fact results in (32a) with the particular value $\lambda = 1$. It turns out here as in the case of the usual Abelian Higgs vortices[6], that the self duality equations (33) cannot be solved in closed form even though solutions exist. We have verified in Ref.35 using numerical methods, that radially symmetric solutions of arbitrary winding number to (33) exist.

The most prominent feature of these generalised self dual vortices is their quantitative difference from the usual vortices: For the same value of the dimensional parameter η, the profile of the generalised vortex is sharper than that of the usual vortex. This feature is demonstrated graphically in our numerical studies in Ref.35.

These generalised self dual Abelian Higgs vortices were also exploited recently in constructing a generalisation[39] of the self dual Chern-Simons vortices in 2+1 dimensions introduced in Refs.40. The generalised Abelian Higgs vortices thus played in Ref.39, the same role that the usual Abelian Higgs played in the work of Ref.40. As in the case of the latter work[40], the self dual vortex solutions of Ref.39 are not expressed analytically but are found using numerical methods. Here too there are fairly striking differences in the quantitative aspects of the two types of vortices, illustrated graphically in Ref.39.

It is interesting to note again, that Eqs.33 have non trivial solutions in spite of the fact that they are descended from (16) which are in general overdetermined[15]. This is for the same reason as in the above example, and namely because this generalised Abelian Higgs model bears the same relationship to the axially symmetric[18,19] subsystem of the GYM model (15) in 8 dimensions, as the usual Abelian Higgs model[6] bears to the axially symmetric[20] subsysem of the usual YM model.

Localised instantons on R_4

This is a prototype model for the Electroweak and Srtong interactions. Its main feature is that the instanton solutions it supports exhibit a non trivial Higgs field configuration. As a consequence the instanton fields are localised to an absolute scale, namely the VEV of the Higgs field η, with respect to which the localisation is not a power but an exponential.

Here we shall restrict ourselves to the minimal, or simplest and most symmetric, model given in Ref.33. The gauge group is $SU(2)\times SU(2)\times U(1)$, and the Higgs multiplet is the following 4×4 field

$$\Phi = \begin{bmatrix} 0 & \varphi \\ -\varphi^{\dagger} & 0 \end{bmatrix} \tag{34}$$

where φ is a 2×2 complex valued field. The Lagrangian and topological charge densities are, respectively,

$$L_4 = tr[\, F_{\mu\nu\rho\sigma}^2 + 4\lambda_1 \{ F_{[\rho\sigma}, D_{\mu]}\Phi \}^2 - 18\lambda_2 (\{ (S. F_{\mu\nu} \} + D_{[\mu}\Phi D_{\nu]}\Phi)^2 -$$
$$- 54\lambda_3 \{ S. D_\nu \Phi \}^2 + 54\lambda_4 S^4 \,] \tag{35a}$$

$$\rho_4 = \partial_\mu \varepsilon_{\mu\nu\rho\sigma} tr\beta [-\eta^4 A_\nu (F_{\rho\sigma} - \frac{2}{3} A_\rho A_\sigma) + \frac{1}{6}\eta^2 \Phi \{ F_{[\rho\sigma}, D_{\mu]}\Phi \} +$$
$$+ \frac{i}{6}\Phi (\{ S. F_{\rho\sigma} \} + D_{[\rho}\Phi D_{\sigma]}\Phi) D_\nu \Phi \,] \tag{35b}$$

where $S = -(\eta^2 + \Phi^2)$, and $\beta = \gamma_5$, and note that (35a) is a positive definite quantity taking account of the fact that the gauge and Higgs fields here are taken to be antihermitian. The dimensionless coupling strengths $\lambda_1, \dots, \lambda_4$ can take any positive values without invalidating the topological inequality (25), without which topological stability would be lost. The descent mechanism in fact results in (35a) with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$. As was the case for the vortex number density (32b), only the first term in (35b), namely the second Chern-Simons density in 4 dimensions, contributes to the surface integral of the topological charge. It is likewise a consequence of the finite action conditions.

In contrast to the previous two examples, the self duality equations here, corresponding to the case $\lambda_1 = \dots = \lambda_4 = 1$, are overdetermined and have no non trivial solutions. For this reason we do not list them here. We are therefore restricted to seek only non self dual solutions, to the second order Euler-Lagrange equations of (35a). As expected, we have not found a result that can be expressed analytically, but it is possible to find the asymptotic solutions of these in both regions $r>>1$ and $r<<1$. Work is at present in progress to integrate the second order Euler-Lagrange equations numerically[41].

Since we regard the model defined by (35a) as a prototype for one describing the fundamental interactions, it is reasonable to seek sphaleron[42] solutions in the static limit. It is clear that the static limit of (35a) remains formally the same except that the first term now on R_3 vanishes due to the antisymmetry, and the R_4 indices $\mu, \nu, \dots$ must now be replaced by $i,j,..=1,2,3$. The following Ansatz made in Ref.43, which unlike the *usual* spaleron[42] Ansatz is genuinely spherically symmetric hence suppressing the U(1) field rigorously,

$$A_i = -\frac{1}{4} f(r) [\Phi^\infty, \partial_i \Phi^\infty], \quad \Phi = h(r)\Phi^\infty \tag{36a, b}$$

where r is the radial variable in 3 dimensions, the matrix $\Phi^\infty = i\gamma_\mu q_\mu$, and $q_\mu = (\sin\mu \sin\theta \cos\phi, \sin\mu \sin\theta \sin\phi, \sin\mu \cos\theta, \cos\mu)$ in terms of the polar and asymuthal angles (θ, ϕ), and the parameter μ, $0 \le \mu \le \pi$. The sphaleron solution is the field configuration corresponding to the value of this parameter $\mu = \pi/2$.. It is shown in Ref.43 that the energy integral corresponding to the fields (36a,b) takes its maximum value for the sphaleron solutions with $\mu = \pi/2$, and is unstable[43] agaist fluctuations in this parameter. The sphaleron solution itself can be found numerically, and this part of the work is now in progress[41].

For completeness, since we regard this model as a prototype for fundamental interactions, we should seek the vortex like Cosmic String solutions that it might support. This remains for the future, but the construction of a Semilocal String[44] promises to be a straightforward task.

Finally we mention that this 4 dimensional model can be extended as

$$L = \kappa^4 L_{YM} + L_4 \tag{37}$$

by adding the usual YM system without invalidating the inequalities (25). The system (37), as also the GYM system (11b), bears an obvious resemblance to the Skyrme model (1). It promises to supply a non trivial dynamics for a *dilute gas* of instantons[45] in 4 dimensions, which would be the single most significant outcome of the present programme.

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