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STOCHASTIC OPTIMAL CONTROL QUANTIZATION OF A FREE RELATIVISTIC PARTICLE

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Abstract. The stochastic variational method of quantization is applied to the case of a free relativistic particle. The WKB approximation of the Klein-Gordon equation is obtained and interpreted in the frame of this method.

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Introduction

The aim of this letter is to show how the method of stochastic variational quantization proposed for nonrelativistic quantum mechanics in our paper [1], may be extended to the case of a free, relativistic, and spinless particle. We use the concept of relativistic Wiener process in augmented space [2,3], and apply it to the stochastic optimal control method [4,5]. We avoid mathematical and "psychological" difficulties with complex valued trajectories, and complex valued action in more general cases, if we interpret our method by "definition by analytic continuation" i.e. in the same way that Feynman's formula is understood in [6]. We omit discussion of boundary problems for partial differential equations which are obtained in the paper.

Stochastic optimal control and Klein-Gordon equation.

Let us augment Minkowski's space-time [2,3] of points

$$(x_1,x_2,x_3,x_4), \qquad x_4=ct, \tag{1}$$

with the metric tensor

$$\eta=\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \tag{2}$$

to become the 5-dimensional space of points

$$(x_1,x_2,x_3,x_4,\tau), \quad \tau \text{ being proper time} \tag{3}$$

where

$$d\tau^2 = -dx_1^2 - dx_2^2 - dx_3^2 + c^2dt^2. \tag{4}$$

Thus we may treat a trajectory $x(\tau) = (x_1(\tau),x_2(\tau),x_3(\tau),x_4(\tau))$ of a particle as a function of a parameter τ, and the condition (4) gives us a 3-dimensional hyperboloid

$$\{(x_1,x_2,x_3,x_4) : \quad x_4 = (x_1^2+x_2^2+x_3^2+1)^{1/2}\} \tag{5}$$

to which the trajectory's velocity $\dot{x}(\tau)=\frac{dx(\tau)}{d\tau}$ always belongs.

The Lagrange function for a relativistic particle is [2,3]

$$L = -m_0 \cdot c \cdot (g_{ik} \cdot \dot{x}_i(\tau)\cdot \dot{x}_k(\tau))^{1/2}, \tag{6}$$

where g_{ik} is a metric tensor; so, for a free particle we have $g_{ik} = \eta_{ik}$, i.e.,

$$L = -m_o \cdot c \cdot \left(\dot{x}_4^2(\tau) - \sum_{i=1}^{3} \dot{x}_i^2(\tau) \right)^{1/2} . \tag{7}$$

Thus the problem of deterministic optimal control for such a particle may be formulated as follows: Find such a function $u(\tau)$ in a set U of admissible control functions with a feed-back property which determines the dynamical equation

$$\dot{x}(\tau) = u(\tau), \quad x(0) = x, \tag{8'}$$

and minimizes the criterion

$$J(u,x) = -m_o \cdot c \cdot \int_0^T \left(u_4^2 - \sum_{i=1}^{3} u_i^2 \right)^{1/2} d\tau . \tag{8''}$$

(8)

Let us notice here that such an optimal control problem has two interesting features. We see from (5) that $\left(\dot{x}_4^2 - \sum_{i=1}^{3} \dot{x}_i^2 \right)^{1/2}$ is equal to 1. Therefore this optimal control problem is equivalent to the search for such an evolution of a particle for which the proper time is maximized. Thus we have, from the physical point of view, a very attractive criterion. Let us stress also that it is an autonomous control problem, in the form given above.

Applying the method of dynamical programming to the autonomous case [4] we obtain for the action

$$S(x) = \inf_{u \in U} \left(-m_o \cdot c \cdot \int_0^T \left(u_4^2 - \sum_{i=1}^{3} u_i^2 \right)^{1/2} d\tau \right) \tag{9}$$

equation

$$\min_{u \in U} \left[\frac{1}{c} \frac{\partial S}{\partial t} \cdot u_4 + \sum_{i=1}^{3} \frac{\partial S}{\partial x_i} \cdot u_i - m_o \cdot c \cdot \left(u_4^2 - \sum_{i=1}^{3} u_i^2 \right)^{1/2} \right] = 0 . \tag{10}$$

From (10) we have

$$u_4^* = \frac{1}{m_o \cdot c^2} \left(\frac{\partial S}{\partial t} \right) \cdot \left(u_4^{*2} - \sum_{i=1}^{3} u_i^{*2} \right)^{1/2} , \tag{11}$$

and

$$u_i^* = -\frac{1}{m_o \cdot c} \left(\frac{\partial S}{\partial x_i} \right) \cdot \left(u_4^{*2} - \sum_{i=1}^{3} u_i^{*2} \right)^{1/2} , \qquad i = 1,2,3, \tag{12}$$

where $u^*(\tau)$ is an optimal control function.

Substituting (11) and (12) in (10) we get

$$\frac{1}{m_0\cdot c^3}\left(\frac{\partial S}{\partial t}\right)^2\cdot\left(u_4^{*2}-\sum_{i=1}^{3}u_i^{*2}\right)^{1/2}-\frac{1}{m_0\cdot c}\sum_{i=1}^{3}\left(\frac{\partial S}{\partial x_i}\right)^2\cdot\left(u_4^{*2}-\sum_{i=1}^{3}u_i^{*2}\right)^{1/2}-m_0\cdot c\cdot\left(u_4^{*2}-\sum_{i=1}^{3}u_i^{*2}\right)^{1/2}=0 \tag{13}$$

which clearly implies

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\sum_{i=1}^{3}\left(\frac{\partial S}{\partial x_i}\right)^2-m_0^2\cdot c^2=0\,. \tag{14}$$

The last equation may be regarded as a Hamilton-Jacobi equation for a free, relativistic particle [2].

Now, following the idea given in [1], we shall obtain a quantum analogue of equation (14) for a free, relativistic particle. We thus extend the deterministic optimal control problem (8) to its simplest stochastic generalization [1]. This means that we should replace the deterministic dynamical equation (8') with

$$\dot{x}(\tau)=\tilde{u}(\tau)+\left(-\frac{i\cdot\hbar}{m_0\cdot c}\right)^{1/2}\cdot\gamma\,,\qquad x(0)=x\,, \tag{15'}$$

and a criterion (8") with

$$J(\tilde{u},x)=E_x\left\{-m_0\cdot c\cdot\int_0^T\left(\tilde{u}_4^2-\sum_{i=1}^{3}\tilde{u}_i^2\right)^{1/2}d\tau\right\} \tag{15"}$$

(15)

where γ is a vector valued white noise, $\tilde{u}(\tau)$ is a feed-back control policy for a stochastic system and E_x denotes the relevant averaging [4,5]. We shall look now for such a policy $\tilde{u}(\tau)$ in a set $\tilde{U}$ of admissible policies [4,5], which minimizes the criterion given in (15").

The method of dynamical programming for stochastic optimal control gives us the equation [4,5],

$$-\frac{i\cdot\hbar}{2\cdot m_0\cdot c}\cdot\frac{1}{c^2}\frac{\partial^2\tilde{S}}{\partial t^2}+\frac{i\cdot\hbar}{2\cdot m_0\cdot c}\cdot\sum_{i=1}^{3}\frac{\partial^2\tilde{S}}{\partial x_i^2}+\min_{\tilde{u}\in\tilde{U}}\left[\frac{1}{c}\cdot\frac{\partial\tilde{S}}{\partial t}\cdot\tilde{u}_4+\sum_{i=1}^{3}\frac{\partial\tilde{S}}{\partial x_i}\cdot\tilde{u}_i-m_0\cdot c\cdot\left(\tilde{u}_4^2-\sum_{i=1}^{3}\tilde{u}_i^2\right)^{1/2}\right]=0\,, \tag{16}$$

for the function

$$\tilde{S}(x)=\inf_{\tilde{u}\in\tilde{U}}E_x\left\{-m_0\cdot c\cdot\int_0^T\left(\tilde{u}_4^2-\sum_{i=1}^{3}\tilde{u}_i^2\right)^{1/2}d\tau\right\}. \tag{17}$$

This function $\tilde{S}(x)$ may be regarded as the action for the system given in (15).

The equation (16) is equivalent (in the same way that (10) is equivalent to (13)) to the following:

$$-\frac{i\cdot\hbar}{2\cdot m_0\cdot c}\cdot\frac{1}{c^2}\frac{\partial^2\tilde{S}}{\partial t^2}+\frac{i\cdot\hbar}{2\cdot m_0\cdot c}\cdot\sum_{i=1}^{3}\frac{\partial^2\tilde{S}}{\partial x_i^2}-\frac{1}{m_0 c}\cdot\frac{1}{c^2}\left(\frac{\partial\tilde{S}}{\partial t}\right)^2\cdot\left(\tilde{u}_4^{*2}-\sum_{i=1}^{3}\tilde{u}_i^{*2}\right)^{1/2}+$$

$$+\frac{1}{m_0\cdot c}\cdot\sum_{i=1}^{3}\left(\frac{\partial\tilde{S}}{\partial x_i}\right)^2\cdot\left(\tilde{u}_4^{*2}-\sum_{i=1}^{3}\tilde{u}_i^{*2}\right)^{1/2}+m_0\cdot c\cdot\left(\tilde{u}_4^{*2}-\sum_{i=1}^{3}\tilde{u}_i^{*2}\right)^{1/2}=0 \tag{18}$$

where $\tilde{u}^*(\tau)$ is an optimal control function for a stochastic system.

The equation (18) is thus the exact evolution equation (quantum Hamilton-Jacobi equation) for a free particle, obtained from the stochastic optimal method of quantization. If we imposed on the set of admissible control policies the condition $\tilde{u}_4^2-\sum_{i=1}^{3}\tilde{u}_i^2=1$ then from the equation (18) (which is valid also in this case) we would obtain for the function $\Psi(x,t)=\exp(\frac{i}{\hbar}\tilde{S}(x,t)$ the exact Klein-Gordon equation. Let us consider, however, a case for which we do not assume this condition on control functions.

We shall not try to solve explicitly equation (18), that is, to find an explicit form of the stochastic optimal control $\tilde{u}^*(\tau)$ or an explicit expression for the action $\tilde{S}(x)$, but rather to obtain quasiclassical approximations for $\tilde{u}^*(\tau)$ and $\tilde{S}(x)$ which would be sufficient for a physical interpretation. It is known from the theory of stochastic optimal control [4,5,7,8,9] that, for small $\left(-\frac{i\cdot\hbar}{m_0\cdot c}\right)$, a good approximate solution to the stochastic problem (15) is obtained from the optimal control policy $u^*(\tau)$ for the relevant, deterministic control problem (8). Then the action $\tilde{S}(x)$ of the stochastic system (15) can be approximated by [4,5,9] as

$$\tilde{S}(x)=S(x)+\frac{1}{2}\left(-\frac{i\cdot\hbar}{m_0\cdot c}\right)\cdot S_1(x)+o\left(-\frac{i\cdot\hbar}{m_0\cdot c}\right), \tag{19}$$

where $S(x)$ is the non-quantum relativistic action, defined in (9), which fulfils the equation (14), and $-\frac{1}{2}\cdot\frac{i\cdot\hbar}{m_0\cdot c}\,S_1(x)$ is a first order correction ($\varepsilon^{-1}\cdot o(\varepsilon)\xrightarrow{\varepsilon\to 0}0$). $S_1(x)$ satisfies the equation [4]

$$\frac{1}{c}\cdot\frac{\partial S_1}{\partial t}\cdot u_4^*+\sum_{i=1}^{3}\frac{\partial S_1}{\partial x_i}\cdot u_i^*+\frac{1}{c^2}\cdot\frac{\partial^2 S}{\partial t^2}-\sum_{i=1}^{3}\frac{\partial^2 S}{\partial x_i^2}=0 \tag{20}$$

This last equation may be obtained easily by applying Theorem 9.3, ch. IV of Fleming & Rishel [4].

The substitution of (11) and (12) into (20) gives us a more convenient form of (20), namely

$$\frac{1}{m_0 \cdot c} \cdot \frac{1}{c^2} \left(\frac{\partial S_1}{\partial t}\right) \cdot \left(\frac{\partial S}{\partial t}\right) - \frac{1}{m_0 \cdot c} \sum_{i=1}^{3} \left(\frac{\partial S_1}{\partial x_i}\right)\left(\frac{\partial S}{\partial x_i}\right) + \frac{1}{c^2}\left(\frac{\partial^2 S}{\partial t^2}\right) - \sum_{i=1}^{3}\left(\frac{\partial^2 S}{\partial x_i^2}\right) = 0 \tag{21}$$

To obtain (21) we should notice, of course, that

$$\left(u_4^{*2} - \sum_{i=1}^{3} u_i^{*2}\right)^{1/2} = 1$$

Let us consider, now, a Klein-Gordon equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} + \frac{m_0^2 \cdot c^2}{\hbar^2}\right]\Psi = 0 \tag{22}$$

for a function $\Psi(x)=\exp\left(\frac{i}{\hbar}\tilde{S}(x)\right)$ where $\tilde{S}(x)$ is the action for the system given in (15). Thanks to the deterministic policy approximation, $\Psi(x)$ may be written as

$$\Psi(x) = \exp\left(\tfrac{i}{\hbar}\tilde{S}(x)\right) = \exp\left(\tfrac{i}{\hbar} S(x) + \frac{1}{2 m_0 c} S_1(x) + O\left(-\frac{i\hbar}{m_0 c}\right)\right), \tag{23}$$

where $\varepsilon^{-1} \cdot O(\varepsilon) \xrightarrow{\varepsilon \to 0} c_1$, c_1 being constant value.

Equating coefficients of $\hbar^{-2}$ and $\hbar^{-1}$ to zero we find that the function $\Psi(x)$ given in (23) fulfils (22) up to $O(1)$ if classical action $S(x)$ and "correction" $S_1(x)$ fulfil the set of equations

$$\left.\begin{aligned} &\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \sum_{i=1}^{3}\left(\frac{\partial S}{\partial x_i}\right)^2 - m_0^2 \cdot c^2 = 0 \\ &\frac{1}{m_0 \cdot c} \cdot \frac{1}{c^2}\left(\frac{\partial S_1}{\partial t}\right)\left(\frac{\partial S}{\partial t}\right) - \frac{1}{m_0 c}\sum_{i=1}^{3}\left(\frac{\partial S_1}{\partial x_i}\right)\left(\frac{\partial S}{\partial x_i}\right) + \frac{1}{c^2}\frac{\partial^2 S}{\partial t^2} - \sum_{i=1}^{3}\frac{\partial^2 S}{\partial x_i^2} = 0 \end{aligned}\right\} \tag{24}$$

But this set of equations is exactly the same as the set (14), (21). This means that the WKB approximation of the Klein-Gordon equation is obtained here as a result of the first order deterministic optimal policy approximation for a problem of stochastic optimal control for a relativistic, spinless particle. In this sense we may regard the Klein-Gordon equation (at least in its WKB approximation) as a consequence of the stochastic variational method of quantization.

Discussion

Let us notice that the set of equations (14), (21), which determines approximately the quantum action $\tilde{S}(x)$ and quantum state $\psi(x)=\exp(\frac{i}{\hbar}\tilde{S}(x))$, is identical to Maslov's set of canonical equations for a Klein-Gordon equation [12]. It is enough to identify Maslov's phase function with classical action, and Maslov's amplitude $\varphi_0(x,t)$ with $\exp\left(\frac{1}{2m_0 c}\cdot S_1(x)\right)$. Thus, the procedure of stochastic variational quantization, combined with a method of approximation through classical (deterministic) optimal control policy, gives us the interpretation of Maslov's supposition about the form of solution for the Klein-Gordon equation [12]. It explains why the the first of Maslov's canonical equations is the Hamilton-Jacobi equation, and gives the meaning of the second canonical equation [12].

Our considerations suggest that the method of stochastic variational quantization need not be restricted to a free particle in relativistic physics. Especially, it seems that the same idea, without any new basic assumptions, may be useful for the quantization of a relativistic particle in a given gravitational field. For instance, if we consider a Wiener process on a fixed Riemannian manifold [10, 11], with a Lagrange function (6) given on it, we may obtain, in quite similar way, the quantum Hamilton-Jacobi equation for the action defined for a particle in this manifold.

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