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A Note on Products of Infinite Cyclic Groups.

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Introduction

In his book[2], Fuchs introduces the notion of a subgroup X of a Specker group P being a product and goes on to establish a Lemma [2, Lemma 95.1] which yields a useful characterization of the quotient P/X and enables an easy derivation of Nunkes characterization of epimorphic images of the Specker group [4]. Unfortunately this Lemma is incorrect as we show in section 1. In section 2 by suitably strengthening the hypothesis we regain a characterization of the quotient. Throughout, all groups are additively written Abelian groups and our notation follows the standard works of Fuchs [1],[2].

§1. Suppose $P = \prod_{n=1}^{\infty} \langle e_n \rangle$ is a Specker group, then Fuchs defines a subgroup X of P to be a product $\prod_{n=1}^{\infty} \langle x_n \rangle$ if for every m, the mth co-ordinates of almost all x_n are 0 and X consists of all the formal sums $\sum s_n x_n$. To avoid confusion with the usual meaning of product(i.e. X is a product if it is isomorphic to a cartesian product of infinite cyclic groups) we denote a product(in the sense of Fuchs) by $\sum^*$ and reserve the symbol $\prod$ for the more usual meaning.

Lemma 1 If Y is an endomorphic image of P then Y is a product(in the sense of Fuchs).

Proof. Let $\alpha : P \longrightarrow P$ be an endomorphism with $\operatorname{Im} \alpha = Y$. Let π_n denote the projection of P onto $\langle e_n \rangle$, and set, for each n, $y_n = e_n \alpha$. Since $\langle e_n \rangle$ is slender, the map $\alpha\pi_n : P \longrightarrow \langle e_n \rangle$ maps almost all e_i to zero i.e. $y_i\pi_n = e_i\alpha\pi_n = 0$ for all but a finite number of indices i. Thus for every n, the nth co-ordinates of almost all y_i are zero and so the set of sums $\{\sum s_i y_i\}$ is a product $\sum_{i=1}^{\infty}{}^* \langle y_i \rangle$ in P.

Now define $\beta : P \longrightarrow P$ by $(\ldots\ldots, n_i e_i, \ldots\ldots)\beta = \sum n_i y_i$. For each $n = 1, 2, \ldots\ldots,$ $\alpha\pi_n$ and $\beta\pi_n$ map P into a slender group and agree on $S = \bigoplus_{i=1}^{\infty} \langle e_i \rangle$. Hence $\alpha\pi_n = \beta\pi_n$ for all n. So $\alpha = \beta$ and $Y = \mathrm{Im}\,\alpha = \mathrm{Im}\,\beta = \sum_{i=1}^{\infty}{}^{*} \langle y_i \rangle$, and thus Y is a <u>product</u>(in the sense of Fuchs).

<u>Counter-example.</u> With $P = \prod_{i=1}^{\infty} \langle e_i \rangle$, set $Y = \prod_{i=1}^{\infty} \langle 2^i e_i \rangle$ Then $P/Y \cong \prod_{i=1}^{\infty} \mathbb{Z}(2^i)$ and this is a complete module over the ring J_2 of 2-adic integers. Moreover the torsion submodule of this quotient is not dense in the 2-adic topology. Hence it has a direct summand $H \cong J_2$ and if $\langle x \rangle$ is dense in H then $H/\langle x \rangle$ is divisible. Choose $y \in P$ such that y maps onto x modulo Y and let $X = \langle y, Y \rangle$. Then certainly X is isomorphic to P and hence is an endomorphic image of P. By Lemma 1 X is a <u>product</u>(in the sense of Fuchs). However $P/X \cong \prod_{i=1}^{\infty} \mathbb{Z}(2^i) / \langle x \rangle$ which contains the divisible subgroup $H/\langle x \rangle$. However if the conclusion of Lemma 95.1 in [2] were correct then P/X would be reduced. So X is clearly a counter-example to the quoted Lemma.

Acknowledgement. The above arguments arose from interesting discussions with Peter Neumann and Adolf Mader. The main idea in the counter-example is essentially due to the former.

§2. In this section by introducing an appropriate topological concept we can regain some information about quotients. Let $P = \prod_{i=1}^{\infty} \langle e_i \rangle$ and topologize P with the product topology of the discrete topology on each component. We refer to this topology simply as the product topology on P. The subgroups $P_n = \prod_{i=n}^{\infty} \langle e_n \rangle$ are a basis of neighbourhoods of zero.

Proposition 2. If X is a subgroup of P which is closed in the product topology then

(i) X is a <u>product</u> $\sum_{i=1}^{\infty}{}^{*} \langle x^i \rangle$

(ii) P/X is isomorphic to a cartesian product of cyclic groups.

Proof. Part (i) is a well-known result due to Nunke [3]. He shows that there are elements x^n in X with (a) $x^n_i = 0$ for $i < n$ (b) $x^n_n = 0$ if and only if $x^n = 0$ (c) x^n_n divides u_n for all u in $X \cap P_n$.
(Subscripts denote components in the product P). Moreover if X is closed, $X = \Sigma^* \langle x^n \rangle$.

In establishing (ii) we let $d_n = x^n_n$ in order to simplify notation. Notice that it follows easily from the properties (a) (b) (c) that if $X = \Sigma^* \langle y^n \rangle$ also, then $y^n_n = d_n$ and $x^n_{n+1} - y^n_{n+1}$ is a multiple of d_{n+1}.

Suppose $a \in P$ is given by a = (a_1, a_2, ...) then we may write

$$a_1 = r_1 d_1 + s_1 \quad \text{where} \quad 0 \leq s_1 < d_1 .$$

Also
$$a_2 - r_1 x^1_2 = r_2 d_2 + s_2 \quad \text{where} \quad 0 \leq s_2 < d_2$$
$$a_3 - r_1 x^1_3 - r_2 x^2_3 = r_3 d_3 + s_3 \quad \text{where} \quad 0 \leq s_3 < d_3 \quad \text{etc.}$$

Define a map ϕ from P onto the cartesian product of the cyclic groups of order d_i by $\phi(a) = (s_1, s_2, \ldots, s_n, \ldots)$. We must verify that ϕ is a well-defined homomorphism. Suppose $X = \Sigma^* \langle y^n \rangle$ then since $y^1_1 = x^1_1 = d_1$ we get that r_1 and s_1 are uniquely defined.

Now $a_2 - r_1 y^1_2 = a_2 - r_1(x^1_2 + k d_2) = (r_2 + r_1 k) d_2 + s_2$ (some k∈ and so s_2 is defined as before. Note that $x^1 - y^1 - k x^2 \in X$ and so by property (c) $x^1_3 - y^1_3 - k x^2_3$ is a multiple of d_3. Making this substitution one easily obtains that $a_3 - r_1 y^1_3 - (r_2 + k) y^2_3 \equiv s_3 \bmod d_3$ and so s_3 is defined as before. Repeating this type of argument easily gives that ϕ is well defined. Moreover ϕ is easily seen to be a homomorphism.

Finally $\operatorname{Ker}\phi = \{a \in P \mid s_1 = s_2 = \ldots\ldots = 0\}$ i.e. if $a \in \operatorname{Ker}\phi$ then

$$a_1 = r_1 d_1$$
$$a_2 = r_2 d_2 + r_1 x_2^1$$
$$a_3 = r_3 d_3 + r_2 x_3^2 + r_1 x_3^1 \text{ etc.}$$

i.e. $a = \sum_{i=1}^{\infty} r_i x^i$ and so $\operatorname{Ker}\phi = X$.

Hence $P/X \cong \prod \mathbb{Z}(d_i)$ where $\mathbb{Z}(d_i)$ is to be interpreted as $\mathbb{Z}$ if $d_i = 0$.

Given Proposition 2 one can easily recover the characterization of homomorphic images of P (Nunke [4] or Fuchs [2, Prop. 95.2]).

Corollary 3. Every epimorphic image of P is the direct sum of a cotorsion group and a direct product of infinite cyclic groups.

Proof. Let K be a subgroup of P and let $\bar{K}$ be the closure of K in the product topology. From Proposition 2, $\bar{K}$ is a product, say $\bar{K} = \sum_{i=1}^{\infty}{}^{*} \langle x_i \rangle$ and $P/\bar{K}$ is a product of cyclic groups. Let $P = P_1 \oplus P_2$ where P_1, P_2 are the products of the $\langle e_n \rangle$ with $d_n \neq 0$ and $d_n = 0$ respectively. Then $\bar{K} \leq P_1$ and $P_1/\bar{K}$ is algebraically compact since it is a product of finite cyclic groups. Since $\bigoplus_{i=1}^{\infty} \langle x_i \rangle$ is contained in K, the quotient $\bar{K}/K$ is cotorsion and this combined with $P_1/\bar{K}$ being cotorsion implies P_1/K is also cotorsion [1, 54(D)].

References.

[1] L. Fuchs, Infinite Abelian Groups Vol I, Academic Press, New York and London 1970.

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[4] R.J. Nunke, Slender groups, Acta Sci. Math. Szeged, 23 (1962) pp 67-73.