Brownian Motion on a Submanifold of Euclidean Space

by

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Abstract:

A martingale characterization of Brownian motion on a submanifold of Euclidean space is proved and the implications of the consequent martingale representation are discussed.

§1. Introduction

Price and Williams [1] proved a martingale characterization of Brownian motion on the unit sphere S^2 in $\mathbb{R}^3$. They used the martingale representation which, by Jacod's theorem [2], is implied by martingale characterization, to prove a result on the structure of Brownian motion on S^2. Their martingale characterization was generalized to Brownian motion on a hypersurface in $\mathbb{R}^d$ in [3]: Let f be a real-valued C^2-function defined on an open set in $\mathbb{R}^d$; a process X on $\mathbb{R}^d$ with $f(X_o) = c$ is a Brownian motion on the hypersurface $f^{-1}(c)$ if and only if X is a semimartingale such that

(i) $dX_t - \frac{(d-1)}{2} H(X_t) n(X_t) dt = dM_t$, where M is a continuous local martingale.

(ii) $d\langle XX^T\rangle_t = P(X_t)dt$.

Here H(x) is the mean curvature at x, n(x) is the unit normal at x and P(x) is orthogonal projection of $\mathbb{R}^d$ onto the tangent space at x. In this note we prove a generalization of this result to a submanifold of $\mathbb{R}^d$ of arbitrary co-dimension: a process X on $\mathbb{R}^d$ which starts on a C^2-submanifold V is a Brownian motion on V if and only if X is a semimartingale such that

(i) $dX_t - j(X_t)dt = dM_t$, where M is a continuous local martingale.

(ii) $d\langle XX^T\rangle_t = P(X_t)dt$.

Here j(x) is the trace of one-half times the second fundamental form of the imbedding, and so is normal to the submanifold; the term $- j(X_t)dt$ is the drift which is required to keep the process on the submanifold. This martingale characterization is proved in §3; the proof makes use of Baxendale's equation for Brownian motion on a sub-manifold of $\mathbb{R}^d$ given in [4], and this is described in §2, In §4 we give a result which generalizes the theorem of Price and Williams [1] on the structure of Brownan motion on S^2.

§2 Brownian Motion on a Submanifold

Let V be a C^2-submanifold of $\mathbb{R}^d$; we claim that a process X on $\mathbb{R}^d$ which starts on V and satisfies

$$dX_t - j(X_t)dt = P(X_t)dB_t , \qquad (2.1)$$

is a Brownian motion on V. Here B is a $BM(\mathbb{R}^d)$, a Brownian motion on $\mathbb{R}^d$; $P(x)$ is orthogonal projection of $\mathbb{R}^d$ on the tangent space $T_x(V)$ to V at x, and $j(x)$ is one-half times the trace of the second fundamental form of the imbedding evaluated at x. We have to show that X_t stays on V for all $t>0$ and that X is a diffusion whose generator is a constant multiple of the Laplace-Beltrami operator Δ on V. Now these are all local matters, and it follows from the inverse function theorem that a C^2-submanifold of $\mathbb{R}^d$ is locally a level set $f^{-1}(c)$ of some $\mathbb{R}^d$-valued C^2-function f defined on an open set in $\mathbb{R}^d$, where r is the co-dimension of V, and such that the rank of the derivative $f'(x)$ is equal to r for all in x in $f^{-1}(c)$. It is enough then to establish our claim for a submanifold V which is a level set; in this case it is easy to define $j(x)$ and $P(x)$ on an open neighbourhood W of $f^{-1}(c)$, the open set on which $f'(x)$ has rank r, and (2.1) has meaning as an Itô equation on an open set in $\mathbb{R}^d$. To be precise, let $P(y)$ be the orthogonal projection of $\mathbb{R}^d$ on $E_y = \ker f'(y)$ for all y in W; when x is in V, the subspace E_x coincides with $T_x(V)$, the tangent space to V at x. A vector field v defined on W is said to be a <u>tangent vector field</u> if $v(y)$ belongs to E_y for each y in W; it is said to be a <u>normal vector field</u> if $v(y)$ belongs to $E_y^\perp$ for all y in W. Given a pair v,w of C^1-tangent vector fields, we define a normal vector field $s(v,w)$ by

$$s_y(v,w) = P^\perp(y)(v\cdot \text{grad } w)(y) \tag{2.2}$$

where $P^\perp(y) = I - P(y)$ is the orthogonal projection of $\mathbb{R}^d$ on $E_y^\perp$. Then $v,w \to s_y(v,w)$ is bilinear and symmetric; the restriction of s to V is the second fundamental form of the imbedding of V in $\mathbb{R}^d$. Define the normal vector field j by

$$j(y) = \tfrac{1}{2}\text{trace}_{E_y}(s_y). \tag{2.3}$$

which makes sense since s_y is a quadratic form on E_y.

Let ∇ be the differential operator defined on C^1-functions on W by

$$(\nabla g)(y) = P(y)(\text{grad } g)(y), \tag{2.4}$$

and let Δ be the differential operator defined on C^2-functions on W by

$$(\Delta g)(y) = \text{trace } (P(y)(P(y) \text{ grad } g)'(y)). \tag{2.5}$$

It is important to notice that both $(\nabla g)(y)$ and $(\Delta g)(y)$ depend only on the restriction of g to V and that ∇ is the covariant derivative on V, and Δ is the Laplace-Beltrami operator on V. We shall require the following identity:

$$\tfrac{1}{2}(\Delta g)(y) = \tfrac{1}{2} \text{ trace } (P(y)g''(y)) + j(y)\cdot(\text{grad } g)(y). \tag{2.6}$$

A process X in $\mathbb{R}^d$ which satisfies (2.1) is a diffusion since (2.1) is an Itô equation with continuous coefficients; to compute its generator, we apply Itô's formula to the process $g(X)$ with g an arbitrary C^2-function:

$$dg(X_t) = (\text{grad } g)(X_t)\cdot dX_t + \tfrac{1}{2} \text{ trace } (g''(X_t)d\langle XX^T\rangle). \tag{2.7}$$

It follows from (2.1) that the bracket process $\langle XX^T\rangle$ satisfies

$$d\langle XX^T\rangle_t = P(X_t)dt, \tag{2.8}$$

so that (2.7) becomes

$$\begin{aligned} dg(X_t) = {} & (\text{grad } g)(X_t)\cdot P(X_t)dB_t + j(X_t)\cdot(\text{grad } g)(X_t)dt \\ & + \tfrac{1}{2} \text{ trace } (P(X_t)g''(X_t))dt. \end{aligned} \tag{2.9}$$

Using the identity (2.6) we have

$$dg(X_t) - \tfrac{1}{2}(\Delta g)(X_t) = dM_t, \tag{2.10}$$

where M is a continuous local martingale satisfying

$$dM_t = \text{grad } g(X_t)\cdot P(X_t)dB_t; \tag{2.11}$$

it follows from (2.10) that the generator of the diffusion X is $\frac{1}{2}\Delta$. Now let $f_1(y),\dots,f_r(y)$ be the components of $f(y)$ in some orthonormal basis for $\mathbb{R}^r$; then applying (2.10) with $g = f_j$ we have

$$df_j(X_t) = 0, \quad t>0, \quad j=1,\dots,r, \tag{2.12}$$

since $\operatorname{grad} f_j(y)$ is orthogonal to E_y, so that $dM_t = 0$ and $(\Delta f_j)(y) = 0$ for all y in W. Hence X_t stays on V for all $t>0$ almost surely. This establishes the claim.

Now consider a C^1-distribution E of k-dimensional subspaces on $\mathbb{R}^d$. We can define P, s, j and Δ as before; the only difference is that s is symmetric if and only if E is involutive, where the bracket operation $[,]$ on vector fields is defined by

$$[v,w](y) = (v\cdot\operatorname{grad} w)(y) - (w\cdot\operatorname{grad} v)(y). \tag{2.13}$$

Suppose that E is involutive; then, by the classical theorem of Frobenius (see [5], for example), there is a unique maximal integral manifold of E through each point and the above proof establishes the following proposition: Let E be an involutive C^2-distribution on $\mathbb{R}^d$ and let X be a process on $\mathbb{R}^d$ such that $X_0 = x$ and

$$dX_t - j(X_t)dt = P(X_t)dB_t. \tag{2.14}$$

Then X is Brownian motion on the maximal integral manifold through x of E.

§3 Martingale Characterization

The description of Brownian motion on a submanifold V given in §2 suggests the following Martingale Characterization of $BM(V)$:

A process X on $\mathbb{R}^d$ which starts on V is a $BM(V)$ if and only if X is a semimartingale such that

(1) $dX_t - j(X_t)dt = dM_t$, where M is a continuous local martingale.

(2) $d\langle XX^T\rangle_t = P(X_t)dt$.

We have to show that if (1) and (2) hold, then there exists B, a $BM(\mathbb{R}^d)$ such that

$$dM_t = P(X_t)dB_t. \tag{3.1}$$

Let $\tilde{B}$ be a $BM(\mathbb{R}^d)$ independent of X, so that

$$d\langle X\tilde{B}^T\rangle_t = 0, \quad d\langle\tilde{B}\tilde{B}^T\rangle_t = I dt, \tag{3.2}$$

and B be the process on $\mathbb{R}^d$ with $B_0 = 0$ and

$$dB_t = P(X_t)dX_t + P^{\perp}(X_t)d\tilde{B}_t. \tag{3.3}$$

Then it follows from (1), (2) and (3.2) that B is a continuous local martingale on $\mathbb{R}^d$ and

$$d\langle BB^T\rangle_t = I dt, \tag{3.4}$$

so that B is a $BM(\mathbb{R}^d)$. Moreover, from (3.2) we have

$$P(X_t)dM_t = P(X_t)dB_t; \tag{3.5}$$

it remains to show that $dM_t = P(X_t)dM_t$. Let $\tilde{M}$ be the process on $\mathbb{R}^d$ given by $\tilde{M}_0 = 0$ and

$$d\tilde{M}_t = P(X_t)dM_t; \tag{3.6}$$

then

$$d\langle \tilde{M}\tilde{M}^T\rangle_t = P^{\perp}(X_t)\,d\langle MM^T\rangle_t P^{\perp}(X_t) = P^{\perp}(X_t)P(X_t)P^{\perp}(X_t)\,dt = 0, \tag{3.7}$$

using (3.6) and (2), so that $\tilde{M}$ is a continuous local martingale whose bracket process vanishes. It follows that

$$d\tilde{M}_t = 0 \tag{3.8}$$

so that

$$dM_t = P(X_t)\,dM_t \tag{3.9}$$

and the proof is complete.

§4 Martingale Representation

Let X be a Brownian motion starting at x on a level set V of $\mathbb{R}^d$, and let Y satisfy $Y_0 = 0$ and

$$dY_t = P(X_t)\,dX_t, \tag{4.1}$$

so that dY_t is the tangential component of dX_t; by (2.1) we have

$$dY_t = P(X_t)\,dB_t \tag{4.2}$$

so that Y is a continuous local martingale. Suppose now that $\tilde{X}$ is a $BM(V)$, starting at x, which is adapted to the filtration of X; let $\tilde{Y}$ satisfy $Y_0 = 0$ and

$$d\tilde{Y}_t = P(\tilde{X}_t)\,d\tilde{X}_t. \tag{4.3}$$

Then we have the following

Martingale Representation: The processes Y and $\tilde{Y}$ are related by the Itô equation

$$d\tilde{Y}_t = C_t\,dY_t, \tag{4.4}$$

where

(i) for each t, C_t is an orthogonal transformation of $\mathbb{R}^d$ such that

$$C_t P(X_t) C_t^T = P(\tilde{X}_t). \tag{4.5}$$

(ii) the process C is X - predictable.

Let $\{n_1, \ldots, n_r\}$ be an orthonormal set of normal vector fields on V; let $\{b^1, \ldots, b^r\}$ be a set of independent $BM(\mathbb{R}^1)$-processes independent of both X and $\tilde{X}$. Then, by the argument in §3, the processes B and $\tilde{B}$ such that $B_0 = \tilde{B}_0 = 0$ and

$$dB_t = dY_t + \sum_{j=1}^{r} n_j(X_t)\,db^j, \quad d\tilde{B}_t = d\tilde{Y}_t + \sum_{j=1}^{r} n_j(\tilde{X}_t)\,db^j \tag{4.6}$$

are both $BM(\mathbb{R}^d)$ and X and $\tilde{X}$ satisfy

$$dX_t - j(X_t)\,dt = P(X_t)\,dB_t, \quad d\tilde{X}_t - j(\tilde{X}_t)\,dt = P(\tilde{X}_t)\,d\tilde{B}_t. \tag{4.7}$$

Moreover, $\tilde{B}$ is B- predictable so that, by the martingale representation theorem for $BM(\mathbb{R}^d)$, there exists a B - predictable process C of orthogonal transformation on $\mathbb{R}^d$ such that

$$d\tilde{B}_t = C_t\,dB_t. \tag{4.8}$$

Hence, from (4.6), we have

$$C_t\,dY_t + \sum_{j=1}^{r} C_t n_j(X_t)\,db^j = d\tilde{Y}_t + \sum_{j=1}^{r} n_j(\tilde{X}_t)\,db^j; \tag{4.9}$$

forming the bracket process of both sides with b^k we have

$$C_t n_k(X_t)\,dt = n_k(\tilde{X}_t)\,dt, \quad k=1,\ldots,j, \tag{4.10}$$

which establishes (4.5). By subtraction we have $d\tilde{Y}_t = C_t\,dY_t$, establishing (4.4). It follows from (4.5) that C is X - predictable.

Remark:

For a hypersurface we have $r=1$ and then $x \to n(x)$ is the Gauss map of the hypersurface. When the hypersurface is the unit sphere S^2 in $\mathbb{R}^3$, the normal subspace $P^{\perp}(X_t)\mathbb{R}^3$ is spanned by the vector X_t and we recover the theorem of Price and Williams [1] as a special case.

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