DIAS-TP-77-07

A Non-unitary Pairing of
Two Polarizations of the Kepler Manifold

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March 1977.

Abstract

A real and a Kaehler polarization of the Kepler manifold are paired, giving a formal operator between the two spaces of polarized sections. The operator is shown to exist and give a non-unitary isomorphism of these spaces.

Introduction.

In [5] J.-M. Souriau showed that the regularized Kepler problem had a phase space which could be identified with $T_0^*S^n$, the complement of the zero section in the cotangent space of an n-sphere. He also pointed out the existence of a complex structure for this space which we observed in [2] was actually a positive polarization [1].

The tangents to the projection $T_0^*S^n \longrightarrow S^n$ give a real polarization whose leaves are the cotangent spaces. We use the method described in [3] to calculate the Blattner-Kostant- Sternberg pairing [1] of these two polarizations.

In order to show that the pairing defines a bicontinuous isomorphism of the Hilbert spaces of polarized sections we use the reproducing kernels of the spaces of spherical harmonics on S^n. A sketch of the proof is given here. Details will be found in [4].

The length function on $T_0^*S^n$ taken as a Hamiltonian preserves the complex polarization, so we may quantize it there. This quantization is then transported to S^n by the pairing and shown to be

$$\left[-\Delta + \tfrac{1}{4}(n-1)^2\right]^{1/2}$$

on a suitable dense domain, where Δ is the usual Laplacian. This operator has spectrum $k + \frac{1}{2}(n-1)$, $k = 0, 1, \ldots,$ which coincides with the semi-classical spectrum obtained by Weinstein [6], but with different multiplicities.

For simplicity we only consider $n \geqslant 3$ since $T_0^*S^n$ is simply-connected in this case.

The two quantizations.

If we identify S^n with the set of unit vectors in $\mathbb{R}^{n+1}$ then we may further identify $T_0^*S^n$ with

$$\{(e,x) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid e\cdot e = 1,\ e\cdot x = 0,\ x \neq 0\}$$

where the fundamental 1-form θ becomes

$$\theta = x \cdot de .$$

$\omega = d\theta$ is the usual symplectic structure. The complex structure of [2,5] is obtained by identifying (e,x) with $z \in \mathbb{C}^{n+1}$ where

$$z = |x| e + ix \ , \quad z\cdot z = 0 \ , \ z \neq 0 .$$

Then

$$\theta = i\partial|x| - i\bar\partial|x| \ , \qquad \omega = 2i\bar\partial\partial|x|$$

and hence the space F of tangents of type (0,1) form a positive polarization.

The integral curves of the Hamiltonian h defined by

$$h(e,x) = 2\pi\,|x|$$

are

$$z(t) = \sigma_t(z) = (\exp -2\pi i t) z .$$

Clearly this flow preserves F.

Let G denote the tangent spaces to the projection

$$\pi : T_0^*S^n \longrightarrow S^n \ , \quad (e,x) \longmapsto e ,$$

then G is a real polarization, and $F \cap \bar{G} = 0$. G is not invariant under the flow σ_t.

Let K^F, K^G denote the canonical bundles of F and G, see [3]. Then K^G is the pull-back to $T_0^*S^n$ of $\Lambda^n(S^n)$ and since S^n is orientable K^G (and hence K^F) is trivial. Thus the first Chern class of the symplectic structure vanishes and $(T_0^*S^n, \omega)$ admits a metaplectic structure, unique for $n \geqslant 3$. Let (Q^F, i^F), (Q^G, i^G) be the half-form bundles for this metaplectic structure.

Let ρ denote the Riemannian volume on S^n, then

$$\pi^*\rho = \sum_{j=0}^{n} (-1)^j e_j \, de_0 \wedge de_1 \wedge \cdots \wedge \widehat{de_j} \wedge \cdots \wedge de_n$$

is a closed, nowhere-vanishing section of K^G. It follows, since $T_0^*S^n$ is simply-connected, that there is a covariant constant, nowhere-vanishing section φ_0 of Q^G with $i^G(\varphi_0 \otimes \varphi_0) = \pi^*\rho$. We would like to construct such a section for Q^F. Q^F is trivial since Q^G is, the problem is to see that it can be trivialized by a covariant constant section.

Consider

$$\beta_0 = |x|^{-2} \sum_{j=0}^{n} (-1)^j \bar{z}_j \, dz_0 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n$$

which is certainly a section of K^F. Let U_j be the subset of $T_0^* S^n$ where $e_j \neq 0$. Then $z_j \neq 0$ on U_j and one can compute

$$\beta_0 | U_j = 2(-1)^j z_j^{-1} dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

showing β_0 does not vanish on U_j and is closed. Since the U_j, $j = 0, \ldots, n$ cover $T_0^* S^n$ this shows β_0 is covariant constant and vanishes nowhere as required. Let Ψ_0 be a section of Q^F with $i^F(\Psi_0 \otimes \Psi_0) = \beta_0$ then Ψ_0 is a nowhere vanishing covariant constant section of Q^F.

Since ω is exact it is integral. Let π: $L \to T_0^* S^n$ be any Hermitian line bundle with connection α having curvature ω. Since $\omega = d\theta = d(2i\partial|x|)$, there are sections s_0, t_0 of L which vanish nowhere such that

$$s_0^* \alpha = \theta , \qquad t_0^* \alpha = 2i\partial|x| .$$

But θ vanishes on G, and $2i\partial|x|$ on F so that s_0 and t_0 are G and F-polarized respectively. Any F-covariant constant section of $L \otimes Q^F$ thus has the form $f t_0 \otimes \Psi_0$ with f holomorphic, and any G-covariant constant section of $L \otimes Q^G$ has the form $g s_0 \otimes \Phi_0$ with g constant along the leaves of G. That is: g is a function on S^n.

Now θ is real so $|s_0|^2 \equiv 1$ whilst

$$d \log |t_0|^2 = 2\pi i (2i\partial|x| + 2i\bar{\partial}|x|) = -4\pi \, d|x|$$

or

$$|t_0|^2 = e^{-4\pi|x|} .$$

Also

$$(s_0, t_0) = e^{-2\pi|x|} .$$

Further

$$\beta_0 \wedge \overline{\beta_0} = 2^{n+2} |x|^{n-2} \lambda$$

where λ is the Liouville volume on $T_0^* S^n$. Thus

$$\langle \Psi_0, \Psi_0 \rangle_0 = 2^{n/2+1} |x|^{n/2-1} .$$

The Hilbert space $\mathcal{H}_F$ corresponding with F can be identified with the holomorphic functions f on $T_0^* S^n$ with norm

$$\| f \|_F^2 = \int_{T_0^* S^n} |f(z)|^2 e^{-4\pi|x|} \, 2^{n/2+1} |x|^{n/2-1} \lambda$$

and the Hilbert space $\mathcal{H}_G$ corresponding with G with $L^2(S^n)$ with norm

$$\| g \|_G^2 = \int_{S^n} |g(e)|^2 \, d\rho(e) .$$

Evidently

$$\sigma_{-t}^* \beta_0 = e^{2\pi i (n-1) t} \beta_0$$

so that

$$\sigma_{-t}^* \Psi_0 = e^{\pi i (n-1) t} \Psi_0 .$$

Also σ_t lifts to L to give an action on sections

$$(\sigma_t^L s)(z) = \sigma_t \cdot s(\sigma_{-t} z)$$

This gives rise to a unitary one-parameter group U_t on $\mathcal{H}_F$

$$(U_t f)(z) = e^{\pi i (n-1) t} f(e^{2\pi i t} z) .$$

<u>The pairing: the formal expression.</u>

On U_j we have

$$\pi^* \rho = (-1)^j e_j^{-1} de_0 \wedge \cdots \wedge \widehat{de_j} \wedge \cdots \wedge de_n , \quad \beta_0 = 2(-1)^j z_j^{-1} dz_0 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n$$

so that

$$\beta_0 \wedge \overline{\pi^* \rho} = 2 i^n |x|^{-1} \lambda .$$

From [3] we have

$$\langle \beta_0, \pi^* \rho \rangle_0 \lambda = (-1)^n \beta_0 \wedge \overline{\pi^* \rho} ,$$

so that

$$\langle \beta_0, \pi^* \rho \rangle_0 = 2(-i)^n |x|^{-1}$$

Changing the pairing by a constant of modulus one will not materially affect the later considerations, so we may assume

$$\langle \Psi_0, \Phi_0 \rangle_0 = 2^{1/2} |x|^{-1/2} .$$

Then

$$\langle f, g\rangle = \langle f t_0\otimes\psi_0, g s_0\otimes\varphi_0\rangle = \int_{T_0^*S^n} f(z)\overline{g(e)}\, e^{-2\pi|x|}\, 2^{1/2}|x|^{-1/2}\lambda .$$

The pairing thus defines a map $T : \mathcal{H}_F \longrightarrow \mathcal{H}_G$ given formally by

$$(Tf)(e) = 2^{1/2}\int_{x\cdot e=0} f(|x|e+ix)\,e^{-2\pi|x|}|x|^{-1/2}\,dx .$$

This expression has also been obtained by R. Blattner [private communication].

Proof of existence.

Let $\mathcal{H}_k$ denote the space of polynomials in $z_0,\dots,z_n$ homogeneous of degree k, regarded as functions on $T_0^*S^n$, and $\mathcal{S}_k$ denote the space of spherical harmonics of degree k on S^n. Then $\mathcal{H}_k\subset\mathcal{H}_F$, $\mathcal{S}_k\subset\mathcal{H}_G$ for each k = 0,1,... and in [4] we show T maps $\mathcal{H}_k$ one-one onto $\mathcal{S}_k$ for each k. There are real numbers $c_k > 0$ with

$$\|Tf\|_G^2 = c_k\|f\|_F^2$$

for all $f\in\mathcal{H}_k$, so that T restricted to $\mathcal{H}_k$ is a multiple of a unitary operator. We also show

$$c_k^2 = \frac{\Gamma(k+(n-1)/2)\,\Gamma(k+n-1/2)^2}{\Gamma(k+3n/4)\,\Gamma(k+3n/4-1/2)\,\Gamma(k+n-1)} .$$

Now

$$c_{k+1}^2/c_k^2 = 1 - \frac{(n^2+2n-4)k + (n-1)(n^2+2n-2)}{16(k+n-1)(k+3n/4)(k+3n/4-1/2)}$$

is strictly less than one, so that c_k is a monotone decreasing sequence. Hence $\|T\| = c_0$, $\|T^{-1}\| = \lim_{k\to\infty} c_k^{-1}$. Thus T is bounded, and for constants $a_1,\dots,a_N,b_1,\dots,b_N$,

$$\lim_{k\to\infty}\frac{\Gamma(k+a_1)\dots\Gamma(k+a_N)}{\Gamma(k+b_1)\dots\Gamma(k+b_N)}$$

is ∞, 1 or 0 according as $\sum_{i=1}^N a_i$ is greater than, equal to or less than $\sum_{i=1}^N b_i$. In our case $a_1+a_2+a_3 = 5n/2 - 3/2 = b_1+b_2+b_3$, so that

$$\lim_{k\to\infty} c_k = 1.$$

Thus $\|T^{-1}\| = 1$ so that T is a continuous isomorphism of $\mathcal{H}_F$ onto

$\mathcal{H}_G$ with a continuous inverse, but is not unitary.

Finally we observe that for f in $\mathcal{H}_k$,

$$U_t f = e^{2\pi i(k+(n-1)/2)t} f$$

since f is homogeneous of degree k. The spectrum of $-\Delta$ on $\mathcal{S}_k$ is $k(k+n-1) = (k+(n-1)/2)^2 - (n-1)^2/4$, so that TU_tT^{-1} has generator

$$2\pi[-\Delta+(n-1)^2/4]^{1/2}.$$

Acknowledgement.

The author wishes to thank R. Blattner and D. J. Simms for their help and interest in this work.

References.

1. Blattner, R. J. The metalinear geometry of non-real polarizations, in Differential Geometric Methods in Mathematical Physics, Bonn July, 1975.
2. Rawnsley, J. H. Coherent states and Kaehler manifolds. Preprint Dublin, 1976. DIAS-TP-76-33.
3. Rawnsley, J. H. On the pairing of polarizations. Preprint, Dublin, 1977. DIAS-TP-76-34.
4. Rawnsley, J. H. Existence of the pairing of two polarizations of the Kepler manifold. Preprint, Dublin, 1977. DIAS-TP-77-08.
5. Souriau, J.-M. Sur la variété de Kepler. Symposia Mathematica XIV, Academic Press, London, 1974.
6. Weinstein, A. Quasi-classical mechanics on spheres. Symposia Mathematica XIV, Academic Press, London, 1974.