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SEMINAR NOTES:

Existence and regularity of solutions of gauge field equations

by

J. H. Rawnsley

School of Theoretical Physics

Dublin Institute for Advanced Studies

Dublin 4, Ireland

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In these notes, which are based on a Seminar given by the author, we shall discuss the work of Tyupkin, Fateev and Shvarts [1] on the existence of solutions of the equations of motion for a Yang-Mills-Higgs field. Some new results on the regularity of these solutions at the origin are included. This was not considered by the above authors, and the results we obtain must be regarded as preliminary only. They are quite weak, in comparison with the behaviour of the Prasad-Sommerfield exact solution [2].

In mathematical terms the problem is to find critical points of the functional

$$\mathcal{E}(A,\varphi) = \int_{\mathbb{R}^3} \Big[\frac{1}{4}\sum_{ij} |F_{ij}|^2 + \frac{1}{2}\sum_i |D_i\varphi|^2 + V(\varphi)\Big]\, d^3x,$$

where G is a compact Lie group, $\mathfrak{g}$ its Lie algebra, $A_i : \mathbb{R}^3 \to \mathfrak{g}$, i = 1, 2, 3 the Christoffel symbols of a connection in a principal G-bundle over $\mathbb{R}^3$, $(\cdot,\cdot)_{\mathfrak{g}}$ a G-invariant inner product on $\mathfrak{g}$, $T : G \to \mathrm{End}(E)$ a representation of G on a real inner product space E with G-invariant inner product $(\cdot,\cdot)_E$, $\dot{T} : \mathfrak{g} \to \mathrm{End}(E)$ its derivative, V a G-invariant function on E (the Higgs potential) and

$$D_i\varphi = \partial_i\varphi + \dot{T}(A_i)\varphi, \quad F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

where $\partial_i = \partial/\partial x_i$ and all physical constants are suppressed. F_{ij} is the curvature of the connection and is the gauge field.

Critical points of $\mathcal{E}$ are solutions of $\delta\mathcal{E}_{A,\varphi} = 0$ with $\mathcal{E}(A,\varphi)$ finite. Written out the equations are

$$\sum_i D_i(D_i\varphi) = \partial V/\partial\varphi,$$

$$\sum_i \partial_i F_{ij} + [A_i, F_{ij}] = \varphi \times D_j\varphi, \quad j = 1, 2, 3,$$

where the cross product $\varphi \times \psi \in \mathfrak{g}$ of $\varphi, \psi \in E$ is defined by

$$(A, \varphi \times \psi)_{\mathfrak{g}} = (\dot{T}(A)\varphi, \psi)_E, \quad \forall A \in \mathfrak{g}.$$

Substituting $D_i = \partial_i + \dot{T}(A_i)$ and expanding,

$$\Delta\varphi = \partial V/\partial\varphi - \dot{T}(\operatorname{div} A)\varphi - 2\sum_i \dot{T}(A_i)\partial_i\varphi - \sum_i \dot{T}(A_i)^2\varphi,$$

$$\Delta A_j - \partial_j \operatorname{div} A =$$

$$\varphi \times \partial_j\varphi + \varphi \times \dot{T}(A_j)\varphi - [\operatorname{div} A, A_j] - 2\sum_i [A_i, \partial_i A_j] + \sum_i [A_i, \partial_j A_i] - \sum_i [A_i, [A_i, A_j]].$$

These equations are not elliptic unless we assume we are in the gauge where div A = 0 which we shall now do. Then we have the non-linear self-coupled elliptic system

(1)
$$\Delta\varphi = \partial V/\partial\varphi - 2\sum_i \dot{T}(A_i)\partial_i\varphi - \sum_i \dot{T}(A_i)^2\varphi,$$
$$\Delta A_j = \varphi \times \partial_j\varphi + \varphi \times \dot{T}(A_j)\varphi - 2\sum_i [A_i, \partial_i A_j] + \sum_i [A_i, \partial_j A_i] - \sum_i [A_i, [A_i, A_j]].$$

Expanding, integrating by parts and using div A = 0, $\mathcal{E}$ becomes

$$\mathcal{E}(A,\varphi) = \int_{\mathbb{R}^3} \Big\{ \tfrac{1}{2}\sum_{ij} |\partial_i A_j|^2 + \sum_{ij} (\partial_i A_j, [A_i, A_j])_{\mathfrak{g}} + \tfrac{1}{4}\sum_{ij} |[A_i, A_j]|^2$$
$$+ \tfrac{1}{2}\sum_i |\partial_i\varphi|^2 - \sum_i (\varphi, \dot{T}(A_i)\partial_i\varphi)_E + \tfrac{1}{2}\sum_i |\dot{T}(A_i)\varphi|^2 + V(\varphi) \Big\}\, d^3x.$$

The form of $\mathcal{E}$ suggests working in the Sobolev space L_1^2 of square integrable weak first derivatives. In a space of n dimensions $L_k^p \subset C^\ell$ if $\ell < k - n/p$ and then the inclusion is completely continuous (i.e. maps weakly convergent sequences in L_k^p to strongly convergent sequences in C^ℓ). To obtain, as a minimum requirement, that the functions with which we work shall be continuous ($\ell = 0$) it is necessary that $0 < k - n/p$. Since $k = 1$, $p = 2$ this is only possible for $n = 1$. The smoothness of the functions with which one can work with such a functional requires an Ansatz to make the problem 1-dimensional.

If P is an elliptic operator in n-dimensions of order m and Pf = g with $g \in C^k$ then $f \in C^\ell$ with $\ell < k - n/2 + m$. In our case, with $n = 3$, $m = 2$, putting in $g \in C^k$ gives $f \in C^k$. Applying this to equations (1), if we have managed to show A_i and φ are continuous, (1) only gives back that information. Whereas, if we have a one-dimensional Ansatz which preserves ellipticity, then if A_i, φ are C^k, they are C^{k+1} and hence by induction C^∞.

It follows from this discussion that an Ansatz to reduce equations (1) to one independent variable is highly desirable from the technical point of view. This is achieved by specifying the angular dependence of the solution and obtaining equations for the radial dependence. In [3] is discussed the consequences of assuming the radial dependence separates with div A = 0. Then A_i and φ have the following form:

(2)
$$A_i = r^{-2}(1 - K(r))\,\epsilon_{ijk}\, x_j L_k,$$
$$\varphi = r^{-t}\, G(r)\, \varphi_t(x).$$

Our notation differs slightly, and is more convenient for the arguments which follow. The functions φ_t have the following meaning. $\mathfrak{g}$ has a subalgebra isomorphic to $\mathfrak{su}(2)$, so T decomposes as a sum of irreducibles and it is supposed some integer spin t occurs. Then there is a unit vector $\varphi_0 \in E$ fixed under L_3 where L_1, L_2, L_3 are generators of the $\mathfrak{su}(2)$ subalgebra satisfying

$$[L_i, L_j] = \epsilon_{ijk} L_k.$$

Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. For $x \in \mathbb{R}^3$ choose $g \in SO(3)$

with $|x| g e_3 = x$ and set $\varphi_t(x) = |x|^t T(g)\varphi_0$. $\varphi_t(x)$ is independent of the choice of g, and has the following properties:

$$\Delta \varphi_t(x) = 0 \;;$$

$$x\cdot\partial\, \varphi_t(x) = t\,\varphi_t(x) \;;$$

(3)

$$x\cdot \dot{T}(L)\varphi_t(x) = 0 \;;$$

$$\dot{T}(L_i)\varphi_t(x) = \epsilon_{ijk}\, x_j\, \partial_k\, \varphi_t(x) \;;$$

$$(\varphi_t(x), \varphi_t(y))_E = |x|^t |y|^t P_t(\cos\theta)\,, \quad |x||y|\cos\theta = x\cdot y\,.$$

These properties characterize $\varphi_t(x)$ up to a sign and allow one to compute easily with φ defined as in (2). Note $r^t\varphi_t(x)$ depends only on the angular part of x. We call $\varphi_t(x)$ the coherent state labelled by x. It also satisfies

$$T(g)\,\varphi_t(x) = \varphi_t(gx)\;, \quad g\in SO(3).$$

Remark. The calculations below remain valid if one makes the more general Ansatz

$$\varphi(x) = \sum_{\varkappa\in I} r^{-t_\varkappa}\, G_\varkappa(r)\, \varphi_{t_\varkappa}(x)$$

for some subset I of the integer spin subrepresentations of T, this leads only to replacing the G term by a sum over $\varkappa$ of such terms.

A brief comment on the Ansatz is in order. It must have the feature that when substituted into the functional a critical point of the new functional gives one for the old functional. It is also supposed to be easier to find critical points for the new functional, for example by reducing the numbers of dependent or independent variables. If the Ansatz as in (2) involves restricting A_i to take values in a proper subalgebra of $\mathfrak{g}$ then φ must be chosen so that $\varphi\times D_j\varphi$ lies in this subalgebra. Likewise if φ is in a subrepresentation of E, it is necessary that $\partial V/\partial\varphi$ (for this particular φ) lie also in this subrepresentation. Sometimes it is possible to say a priori this will be so from the break-up of T and $\mathrm{ad}_{\mathfrak{g}}$ when restricted to the subalgebra.

Thus an Ansatz must satisfy two criteria. First that the original equations of motion be satisfied if the Euler-Lagrange equations of the new functional are satisfied; and secondly that the problem is in some way simplified. One of the important features of the Ansatz (2) is that the functional $\mathcal{E}$ becomes

$$\text{(4)}\quad \mathcal{H}(K,G) = 4\pi\int_0^\infty \Big\{K'(r)^2 + \frac{r^2}{2}G'(r)^2 + \frac{r^{-2}}{2}(K(r)^2-1)^2 + \frac{t(t+1)}{2}K(r)^2G(r)^2 + r^2V(G(r))\Big\}\,dr$$

which is a sum of positive terms.

We assume V ≥ 0 everywhere, V(η) = 0 for some η ≠ 0 and look for an absolute minimum of $\mathcal{H}$ over those K, G for which $\mathcal{H}(K,G) < \infty$, $K(r)\to 0$, $G(r)\to\eta$ at ∞, K(o) = 1. In the proof we shall also require the following behaviour of V:

$$\text{(5)}\qquad V'(G) = G.\,F(G)$$

with $|F(G)|$ bounded by a quadratic polynomial in G.

With these assumptions on V, every term in the functional is positive and so for $\mathcal{H}(K,G)$ to be finite all terms must converge separately. Also $\mathcal{H}\geqslant 0$ so has an infimum $m(t)$ taken over the class of functions for which $\mathcal{H}$ converges, and which satisfy

$$K(r) \longrightarrow 0\,,\ r \longrightarrow \infty\ ; \qquad G(r) \longrightarrow \eta\,,\ r \longrightarrow \infty\,;$$

(6) $$K(r) \longrightarrow 1\,,\ r \longrightarrow 0\,.$$

It turns out to be pointless to constrain $G(r) \to 0$ as $r \to 0$ since in the limiting argument used to show $\mathcal{H}$ achieves its infimum it cannot be directly obtained that the limiting function G has this property. We shall however later prove that G vanishes at zero to an order determined by t.

The proof of Fateev-Shvarts-Tyupkin adapts easily to any positive potential, not just those which can be written as a sum of squares. Since we need some of the estimates later we repeat the proof in full. Let K_n, G_n be a sequence of smooth functions satisfying (6) with $\mathcal{H}(K_n, G_n) \to m(t)$. This sequence exists by definition of the infimum. We shall show that, in a suitable Hilbert space $\{(K_n, G_n)\}$ has a limit (K, G), that $\mathcal{H}(K,G)$ exists for this limit and $\mathcal{H}(K,G) = m(t)$.

The Hilbert spaces used are of the Sobolev type. The infinite domain of integration is a source of some of the difficulties encountered here. A second difficulty is the r^2 coefficient of $(G')^2$ which means the Euler-Lagrange are not elliptic at the origin, so that solutions might be singular there. It will require some very careful estimates to show this cannot happen.

For now, let (K,G) be any pairing of smooth functions with $\mathcal{H}(K,G) = M < \infty$, and satisfying (6), then

$$|G(x) - G(y)|^2 = \left|\int_y^x G'(r)\,dr\right|^2 = \left|\int_y^x r^{-1}(rG'(r))\,dr\right|^2$$

$$\leq \int_y^x r^{-2}\,dr \int_y^x r^2 G'(r)^2\,dr$$

$$\leq 2M\left(\frac{1}{y} - \frac{1}{x}\right).$$

Letting $x \to \infty$

(7) $$|\eta - G(y)|^2 \leq \frac{2M}{y}\,, \quad 0 < y < \infty\,.$$

Thus for $0 < a < \infty$,

$$\int_0^\infty r^2 G'(r)^2\,dr + |G(a) - \eta|^2 \leq 2M + \frac{2M}{a}$$

Let $L(r) = G(r) - \eta$; then $L(r) \to 0$ at ∞ and

(8) $$\|L\|_1^2 = \int_0^\infty r^2 L'(r)^2\,dr + |L(a)|^2 \leq 2M\left(1 + \frac{1}{a}\right).$$

From

$$\frac{t(t+1)}{2}\int_0^\infty G^2 K^2\,dr \leq M$$

we get

$$\int_0^\infty |L(r) + \eta|^2 K(r)^2\,dr \leq \frac{2M}{t(t+1)} \leq M \qquad \text{for } t \geq 1\,.$$

Also, since $|L(r)|^2 \leq M/r$ by (7), if we choose $c > M/\eta^2$,

$$|L(r) + \eta|^2 \geq \eta^2 - L(r)^2 \geq \eta^2 - M/r \geq \eta^2 - M/c > 0$$

if $r \geq c$. Thus

$$1 \leq \frac{|L(r) + \eta|^2}{\eta^2 - M/c}\,, \quad r \geq c$$

so that

$$\int_c^\infty K(r)^2\,dr \leq \int_c^\infty \frac{K(r)^2\,|L(r) + \eta|^2}{\eta^2 - M/c}\,dr \leq \frac{M}{\eta^2 - M/c}\,.$$

Thus

$$|K(x)^2 - K(y)^2|^2 = \Big|\int_y^x 2K(r)K'(r)\,dr\Big|^2 \le 4\int_y^x K(r)^2\,dr \int_y^x K'(r)^2\,dr$$

$$\le 4M\int_y^x K(r)^2\,dr \le \frac{4M^2}{\eta^2 - M/c}$$

for $x \ge y \ge c$. Letting $x \to \infty$

(9) $$|K(y)|^4 \le \frac{4M^2}{\eta^2 - M/c}, \quad y \ge c.$$

For any $x \ge y$

$$|K(y)| = |K(x) - K(y) - K(x)| \le |K(x)| + \Big|\int_y^x K'(r)\,dr\Big|$$

$$\le |K(x)| + (x-y)^{1/2} M^{1/2}$$

Taking $x = c$, for $y \le c$ we have

(10) $$|K(y)| \le |K(c)| + (c-y)^{1/2} M^{1/2} \le |K(c)| + c^{1/2} M^{1/2}.$$

Thus for any $0 \le a < \infty$ (using (9) if $a \ge c$ and (5) if $a \le c$) $|K(a)|^2 \le AM$ for some A determined by (9) and (10). Thus

(11) $$\|K\|_2^2 = \int_0^\infty K'(r)^2\,dr + |K(a)|^2 \le (A+1)M.$$

Now consider the sequence K_n, G_n with $\mathcal{H}(K_n, G_n) \to m(t)$. Since $\mathcal{H}(K_n, G_n)$ is a bounded sequence so are $\|L_n\|_1$ and $\|K_n\|_2$ by (8) and (11). Since the closed balls in Hilbert space are weakly compact there are weakly convergent subsequences which we relabel L_n, K_n, and let L, K denote the weak limit. Moreover passing to further subsequences we can suppose $\|L_n\|_1$ and

$\|K_n\|_2$ are convergent (numerical) sequences (by compactness of closed intervals in $\mathbb{R}$).

Let $0 < a < b < \infty$ and consider

$$\int_a^b r^2 L'(r)^2\,dr + L(a)^2, \quad \int_a^b K'(r)^2\,dr + K(a)^2$$

which are Sobolev norms. The inclusion maps of the Sobolev spaces $W^1[a,b]$ into $C[a,b]$ are completely continuous (that is weakly convergent sequences become strongly convergent). Hence $L_n \to L$ and $K_n \to K$ strongly in $C[a,b]$ for each $0 < a < b$. In particular $L_n(a) \to L(a)$ and $L_n(b) \to L(b)$. Thus $\int_0^\infty r^2 L_n'(r)^2\,dr$ and $\int_0^\infty K_n'(r)^2\,dr$ are convergent sequences. Since $\mathcal{H}(K_n, G_n)$ is a convergent sequence, so is

$$\int_0^\infty \frac{r^{-2}}{2}(1-K_n^2)^2 + \frac{t(t+1)}{2} G_n^2 K_n^2 + r^2 V(G_n)\,dr.$$

Write $f_n(r)$ for the integrand, and $f(r)$ for the same integrand with K_n and G_n replaced by K and G. Then $\lim_{n\to\infty} \int_0^\infty f_n(r)\,dr$ exists, $f_n(r) \to f(r)$ pointwise, $f_n(r) \ge 0$ everywhere. Thus

$$\int_a^b f(r)\,dr = \int_a^b f(r) - f_n(r)\,dr + \int_a^b f_n(r)\,dr$$

and the first term of the right hand side tends to zero as $n \to \infty$. Thus

$$0 \le \int_a^b f(r)\,dr \le \int_a^b f(r) - f_n(r)\,dr + \int_0^\infty f_n(r)\,dr$$

and letting $n \to \infty$,

$$0 \le \int_a^b f(r)\,dr \le \lim_{n\to\infty} \int_0^\infty f_n(r)\,dr.$$

Since this is true for all a and b, $\int_0^\infty f(r)\,dr$ exists and

$$\int_0^\infty f(r)\,dr \leq \lim_{n\to\infty} \int_0^\infty f_n(r)\,dr.$$

But

$$\|L\|_1 \leq \lim_{n\to\infty} \|L_n\|_1 , \quad \|K\|_2 \leq \lim_{n\to\infty} \|K_n\|_2$$

since in any Hilbert space the norm of a weak limit is less than or equal to the limit of the norms if the latter exists. But

$$\mathcal{H}(G_n,K_n) = \tfrac{1}{2}(\|L_n\|_1^2 - L_n(a)^2) + \|K_n\|_2^2 - K_n(a)^2 + \int_0^\infty f_n(r)\,dr$$

so that $\mathcal{H}(G,K)$ exists, and

$$\mathcal{H}(G,K) \leq \lim_{n\to\infty} \mathcal{H}(G_n,K_n) = m(t)$$

By definition of the infimum we have $\mathcal{H}(G,K) \geq m(t)$ and hence $\mathcal{H}(G,K) = m(t)$. Thus the functional $\mathcal{H}$ achieves its infimum on the functions satisfying (6) and which are in the Sobolev space with norms (8) and (11). The functions G, K are then continuous on $(0, \infty)$ and satisfy the Euler-Lagrange equations weakly. Since these are elliptic, G, K are C^∞ and satisfy the Euler-Lagrange equations strongly.

It remains to see what happens around r = 0. Since there is no r^2 term in (11), the Sobolev convergence of K_n implies $K_n(0) \to K(0)$ or $K(0) = 1$ and K is continuous at zero, so $K(r) \to 1$ as $r \to 0$. Choose $\delta > 0$ so $\frac{1}{2} \leq K(r) \leq 2$ for $0 \leq r \leq \delta$.

The Euler-Lagrange equations are

(12a) $$K''(r) = r^{-2}(K(r)^2 - 1)K(r) + \frac{t(t+1)}{2} G(r)^2 K(r) ;$$

(12b) $$(r^2 G'(r))' = t(t+1) G(r) K(r)^2 + r^2 V'(G(r)) .$$

Assuming $r^2V'(G)$ is small when $r \to 0$, since $K^2 \to 1$ (12b) is approximately

$$(r^2 G'(r))' = t(t+1)\, G(r)$$

for $r \sim 0$. This has solutions $G = r^t$ and $G = r^{-t-1}$. The functional converges only for $G \sim r^t$ at 0 so this indicates that solutions of (12) with finite have G vanishing to some order at r = 0. This we now show. In fact we re-write (12b) as

(13) $$(r^2 G'(r))' - t(t+1) G(r) = t(t+1) G(r)(K(r)^2 - 1) + r^2 F(G(r)) G(r)$$

using (5). r^{1-t} and r^t are integrating factors for the LHS of (13). If we use r^{1-t} we have no control of the RHS, so we multiply by r^t, and writing $E(r) = t(t+1)(K(r)^2-1)/r + r\,F(G(r))$ we have

(14) $$(r^{t+2} G'(r) - t\, r^{t+1} G(r))' = r^{t+1} E(r)\, G(r) .$$

Moreover $r^{-1}(K(r)^2 - 1) \in L^2[0,\delta]$, $|r\,F(G(r))|^2 \leq B r^2 G(r)^2 \leq Cr$ for some C and r sufficiently small. So E is L^2 near the origin.

Integrate from y to x:

(15) $$x^{t+2}G'(x) - t\,x^{t+1}G(x) - [y^{t+2}G'(y) - t\,y^{t+1}G(y)] = \int_y^x r^{t+1} E(r)\, G(r)\,dr .$$

Now by (7), $y^{t+1}G(y) \to 0$ as $y \to 0$, and $r^{t+1}G(r)$ is certainly L^2 near 0. We need to show $y^{t+2}G'(y) \to 0$ as $y \to 0$. But

$$(r^2 G'(r))' = t(t+1)G(r)K(r)^2 + r^2 F(G(r))\, G(r)$$

and for $0 \le r \le \delta$, $K^2 \le 4$ so

$$\delta^2 G'(\delta) - y^2 G'(y) = t(t+1)\int_y^\delta G(r)K(r)^2\,dr + \int_y^\delta r^2 F(G(r))G(r)\,dr .$$

Now $F(G)G$ is bounded by a cubic and G by $M^{\frac{1}{2}}r^{-\frac{1}{2}}$ so $r^2F(G)G \sim r^{\frac{1}{2}}$. Thus

$$|\delta^2 G'(\delta) - y^2 G'(y)| \leq t(t+1)M^{1/2}4\int_y^\delta r^{-1/2}\,dr + D(\delta^{3/2} - y^{3/2})$$

$$= t(t+1)M^{1/2}8(\delta^{1/2} - y^{1/2}) + D(\delta^{3/2} - y^{3/2})$$

which remains bounded as $y \to 0$ so $y^2G'(y)$ is bounded and hence $y^{t+2}G'(y) \to 0$ as $y \to 0$. Letting $y \to 0$ in (15) we get

(16) $$x^{t+2}G'(x) - t\,x^{t+1}G(x) = \int_0^x r^{t+1}E(r)\,G(r)\,dr .$$

Put $D(r) = r^{-t}G(r)$ so that (16) becomes

(17) $$x^{2t+2}D'(x) = \int_0^x r^{2t+1}E(r)\,D(r)\,dr$$

and

(18) $$r^{t+1}D(r) \to 0 \quad \text{as} \quad r \to 0 .$$

Applying Cauchy-Schwartz to (17)

$$x^{4t+4}\,D'(x)^2 \leqslant \int_0^\delta E(r)^2\,dr \int_0^x r^{4t+2}D(r)^2\,dr$$

for $0 \le x \le \delta$. $\int_0^\delta E(r)^2\,dr$ is some fixed constant.

Suppose we have shown

$$r^k\,D(r) \longrightarrow 0 \quad \text{as} \quad r \longrightarrow 0$$

with k real, $t+1 \ge k > \frac{1}{2}$, then $|r^kD(r)| < \varepsilon$ for r sufficiently small so

$$x^{4t+4}D'(x)^2 \leqslant \varepsilon^2 \int_0^\delta E(r)^2\,dr \int_0^x r^{4t+2-2k}\,dr$$

$$= \varepsilon^2 \int_0^\delta E(r)^2\,dr \; \frac{x^{4t+3-2k}}{4t+3-2k} .$$

Thus

$$D'(x) \leqslant \varepsilon\left(\frac{\int_0^\delta E(r)^2\,dr}{4t+3-2k}\right)^{1/2} x^{-k-1/2}$$

Thus

$$D(x) - D(y) \leqslant \varepsilon\left(\frac{\int_0^\delta E(r)^2\,dr}{4t+3-2k}\right)^{1/2} \frac{y^{-k+1/2} - x^{-k+1/2}}{k-1/2}$$

Multiplying by $y^{k-\frac{1}{2}}$ shows

$$y^{k-1/2}\,D(y) \leqslant y^{k-\frac{1}{2}}D(x) + (\quad)\,\varepsilon$$

Letting $y \to o$, since ε was arbitrary, shows

$$y^{k-\frac{1}{2}} D(y) \longrightarrow o \quad , \quad y \longrightarrow o .$$

Thus so long as $k > \frac{1}{2}$ we can improve by $r^{\frac{1}{2}}$ each time. Since $r^{t+1}D(r) \to o$ by (13), by induction we have

$$r^{\frac{1}{2}} D(r) \longrightarrow o \quad , \quad r \longrightarrow o .$$

Trying one more step brings in a logarithmic term.

Using (12a) in a similar fashion one can show $1-K$ vanishes as $r^{3/2}$. Feeding this and $G \sim r^{t-1/2}$ at $r = 0$ back gives the improved estimate that $E(r)/r^{1/2}$ is continuous on $[0, \delta]$. This gives an improvement in (17) which allows the argument to be carried out one more step and shows $G \sim r^t$ at $r = 0$. Likewise $1-K \sim r^2$.

Thus φ vanishes to order t and A to first order at $r = 0$. It is probable that φ and A are C^{∞} or even analytic at $r = 0$, but the crude estimates made above do not yield this.

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