Reordering of Non-Lattice Permutations

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1. Reordering with respect to two symbols.

A crucial point in the proof given by Littlewood (Group Characters p. 94) of the Littlewood-Richardson rule for multiplying two Schur functions is the setting-up of a one-to-one correspondence between a non-lattice permutation (nlp) of $\alpha^{\mu_1}\beta^{\mu_2}\gamma^{\mu_3}\ldots$ and a lattice permutation (lp) of some $\alpha^{\mu'_1}\beta^{\mu'_2}\gamma^{\mu'_3}\ldots$. To quote Group Characters p. 95:

> For a given non-lattice permutation of $\alpha^{\mu_1}\beta^{\mu_2}\gamma^{\mu_3}\ldots$, consider first the α's and the β's only. Number the α's and the β's in the order of their appearance.
>
> If β_s precedes α_{t+1} and succeeds α_t, it is said to be of index s-t, and is said to be of positive, zero, or negative index according as s-t is positive, zero or negative.
>
> If the α's and the β's exhibit the lattice property, there is no β of positive index.
>
> Otherwise take the first β of greatest (positive) index and replace it by an α. This step is reversible, an essential part of the argument for the proof depends upon an exact 1 : 1 correspondence. To reverse the step we renumber the α's and the β's, and take the last β of greatest zero or positive index and replace the α immediately following it by a β, unless all the β's are of negative index, in which case we replace the first α in the permutation by a β.

We concentrate our attention on the last paragraph, since a reordering of a nlp to a lp is effected by repeated application of this process. The replacement of the first β of greatest index by an α and the renumbering of the α's and the

β's need no comment. To see how the rule for reversing the step arises we distinguish the cases where the renumbered sequence has a β with positive index, has no β with positive index but has a β with zero index, has only β's with negative index, or finally has no β at all. Let the β that is changed be β_s. Since it is the first β of greatest positive index, there cannot be an α between β_{s-1} and β_s. Moreover there must be at least one α between β_s and β_{s+1}, because otherwise β_{s+1} would be the first β of greatest index. If β_s lies between α_t and α_{t+1}, the sequence may be depicted as

$$\ldots\ \alpha_t \ldots \beta_u \ldots \beta_{s-1} \ldots\ \beta_s \ldots \alpha_{t+1} \ldots \alpha_v \ldots \beta_{s+1} \ldots , \tag{1}$$

where there may be γ's, δ's, etc. at the dots. We have s-t > 0, t ⩾ 0, the case of a symbol with zero subscript being interpreted as the absence of that symbol in the sequence. When β_s is replaced, (1) becomes

$$\ldots\ \alpha_t \ldots \beta_u \ldots \beta_{s-1} \ldots \alpha_{t+1} \ldots \alpha_{t+2} \ldots \alpha_{v+1} \ldots \beta_s \ldots . \tag{2}$$

Let us suppose that t ≠ 0, so that the index of β_{s-1}, namely s-t-1, is positive or zero. In the sequence (2) the index of β_s is at least 2 less than the index of β_s in (1) because of the replacement of the β_s in (1) by α_{t+1} and because there must be at least one α between β_s and β_{s+1} in (1). Hence in (2) the index of β_{s-1} is greater than the index of β_s. Since the sequence of α's and β's after β_{s+1} in (1) is the same as their sequence after β_s in (2) and since the indices of β_{s+1}, β_{s+2} etc. in (1) did not exceed that of β_s, namely s-t, the index of β_{s-1} in (2) is greater than that of all succeeding β's. Thus we return from (2) to (1) when t ⩾ 1 by taking the last β of greatest positive or zero index and replacing the first α following it by a β. That such an α exists follows from our construction of (2), but there may be γ, δ etc. between it and the β.

When t = 0, the sequences (1) and (2) become, respectively,

$$\ldots\ \beta_1 \ldots \beta_{s-1} \ldots \beta_s \ldots \alpha_1 \ldots \alpha_v \ldots \beta_{s+1} \ \ldots \tag{3}$$

and

$$\ldots\ \beta_1 \ldots\ldots\ \beta_{s-1} \ldots\ldots\ \alpha_1 \ldots\ldots\ \alpha_2 \ldots\ldots\ \alpha_{v+1} \ldots\ldots\ \beta_s \ldots\ldots \tag{4}$$

We distinguish the cases of $s > 1$ and $s = 1$. In the former case β_{s-1} is the last β in (4) with greatest positive index, as we argued earlier for $t \geqslant 1$. We may therefore employ the rule as stated for $t \neq 0$ to return from (4) to (3). When $s = 1$, the sequences (3) and (4) become

$$\ldots\ldots\ \beta_1 \ldots\ldots\ \alpha_1 \ldots\ldots\ \alpha_v \ldots\ldots\ \beta_2 \ldots\ldots\ , \tag{5}$$

$$\ldots\ldots\ \alpha_1 \ldots\ldots\ \alpha_2 \ldots\ldots\ \alpha_{v+1} \ldots\ \beta_1 \ldots\ldots\ , \tag{6}$$

respectively. The index of each β_i in (5) cannot exceed that of β_1, so it is less than or equal to +1. The index of each β_i in (6), being 2 less than the index of β_i in (5), is therefore negative. Moreover, since we have dealt with all the other possible cases, the index of every β in (2) is negative only when $t = 0$, $s = 1$. To return from (6) to (5) we replace the first α in (6) by a β, as was stated by Littlewood. When β_1 is the only β in (5), then (6) has no β, so in this case the rule is just to replace the first α in (6) by a β. We have thus considered all possible cases of the renumbered sequence.

This completes the proof of the rule for reversing by one step in a well-defined manner the process of constructing a lp monomial function of α and β from a nlp one. It may be noted that there exist permutations which cannot be reversed, for example ,

$$\alpha_1\ \alpha_2\ \alpha_3\ \beta_1\ \beta_2\ \beta_3\ \alpha_4\ \beta_4\ . \tag{7}$$

This may be seen directly by attempting to replace each α of (7) in turn by a β and examining whether this β would be changed back to an α by the reordering rule. It may also be seen by observing that the last β of greatest positive or zero index in (7), namely β_4, has no α following it. We should note that we are not entitled to go back to β_3, whose index is the same as that of β_4, and replace α_4

Having dealt with the first step in going from a nlp to a lp we repeat the process until the sequence has no β with positive index. The reverse process being well defined at every stage, we return from lp to the nlp in a unique way. Hence there is a one-to-one correspondence between a nlp of $\alpha^{\mu_1}\beta^{\mu_2}$ and a lp of $\alpha^{\mu'_1}\beta^{\mu'_2}$, where obviously

$$\mu'_1 + \mu'_2 = \mu_1 + \mu_2$$

$$\mu'_1 \geqslant \mu_1 + 1, \quad \mu'_1 \geqslant \mu_2 .$$

2. Reordering with respect to three or more symbols

To quote again from Group Characters p. 95:

Next the β's and γ's only are considered, and each γ is given an index relative to the β's. If necessary the first γ of greatest positive index is converted into a β .... .

This step may destroy the lattice property of the α's and β's. If so, the first β of index +1, which may or may not be the symbol converted from a γ to a β, is converted into an α, ...

This process is continued consecutively with the γ's, δ's etc., until we arrive at a lattice permutation of $\alpha^{\mu'_1}\beta^{\mu'_2}\gamma^{\mu'_3}\ldots$.

Let us confine our attention for the moment to continued products of α, β, γ only. The central problem is to understand how the one-to-one correspondence may be preserved, when the above rules are applied both to α, β and to β, γ. We should first remark that, if all the β's have been replaced by α's, the reordering is nevertheless performed with respect to the β's and γ's and that this amounts to replacing each γ by a β with the same suffix, unless μ_3 exceeds $\mu_1 + \mu_2$. For the purpose of establishing the Littlewood-Richardson rule it would suffice to take $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \ldots$ and then $\mu_3 < \mu_1 + \mu_2$ anyway.

An an example of a nlp we have

$$\alpha\ \gamma\ \beta\ \alpha\ \gamma\ \alpha\ \beta\ \gamma\ \beta\ \alpha\ . \tag{8}$$

and the partition (μ_1, μ_2, μ_3) is (4, 3, 3). The sequence is already ordered with respect to α and β, and reordering with respect to β and γ changes (8) to

$$\alpha_1 \; \beta_1 \; \beta_2 \; \alpha_2 \; \gamma_1 \; \alpha_3 \; \beta_3 \; \gamma_2 \; \beta_4 \; \alpha_4 \; . \tag{9}$$

This has now to be recorded to respect to α and β:

$$\alpha_1 \; \beta_1 \; \alpha_2 \; \alpha_3 \; \gamma_1 \; \alpha_4 \; \beta_2 \; \gamma_2 \; \beta_3 \; \alpha_5 \; , \tag{10}$$

which is a lattice permutation and corresponds to the partition (5, 3, 2).

While there is a well-defined procedure for going forward from (8) to (10), the reverse process is not well-defined; in other words starting from (10) we have no *a priori* way of knowing whether we reverse first with respect to β and γ or with respect to α and β. If we reverse first with respect to β and γ and then with respect to α and β, we obtain

$$\alpha_1 \; \beta_1 \; \beta_2 \; \alpha_2 \; \gamma_1 \; \alpha_3 \; \beta_3 \; \gamma_2 \; \gamma_3 \; \alpha_4 \; , \tag{11}$$

which differs from (8) though it still corresponds to the (4, 3, 3) partition. On the other hand, if we apply the rules to bring (11) to a lattice permutation, we obtain (10). Hence the two nlp's (8) and (11) corresponding to the same partition are reordered to the same (5, 3, 2) lp.

We can therefore speak of a one-to-one-correspondence between (8) and (10), if and only if we prescribe the order in which the reverse steps are taken. In the general case of monomials in the symbols α, β, γ, δ, etc. we carry out the same type of procedure for bringing a nlp to a lp. By reversing the order of the substitutions of pairsof consecutive symbols we can establish a one-to-one correspondence between the nlp and the lp.