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WEIGHT DIAGRAMS

AND THEIR APPLICATION TO THE REDUCTION OF REPRESENTATIONS OF THE GENERAL LINEAR GROUP

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INTRODUCTION

In the application of the SU(3) group to the theory of elementary particles a central position is held by weight diagrams. It is in these diagrams that the multiplets, e.g. the 0^- meson octet, the $\frac{3}{2}^+$ baryon decuplet, are ordered according to the values of their third component of isospin and their hypercharge. Moreover in considering the interactions of elementary particles one sometimes has to deal with the direct product of two irreducible representations of the Lie algebra of SU(3). Such products may be reduced by application of the theory of group characters. Since this presupposes a fairly extensive knowledge of group representations and in particular of integral representations of the general linear group, one may seek an alternative method and this is provided by weight diagrams. We shall later explain this alternative method and apply it to specific examples.

It is therefore worthwhile to look into the whole question of the weight diagrams for the semi-simple Lie algebras of rank 2. These diagrams are two-dimensional figures. In Cartan's notation (Cartan 1913) the algebras in question are the following:

1. A_2, which is the Lie algebra of the special linear group in three dimensions over the complex field $SL(3, \mathbb{C})$. This in turn contains the subgroup SU(3).
2. B_2, which is the Lie algebra of the rotation group in five dimensions over the complex field $SO(5, \mathbb{C})$.
3. C_2, which is the Lie algebra of the symplectic group in four dimensions over the complex field $Sp(4, \mathbb{C})$. This algebra is isomorphic to B_2 and we shall include C_2 in our discussion of B_2, when-

ever we do not state to the contrary.

4. G_2, which is the Lie algebra of Cartan's exceptional group of rank 2. This exceptional group is a subgroup of $SO(7, \mathbb{C})$, the rotation group in seven dimensions over the complex field.

The algebra D_2, which is the Lie algebra of $SO(4, \mathbb{C})$, is not semi-simple; it is the direct sum of two A_1 algebras. We have therefore to consider only three essentially different algebras A_2, B_2, G_2.

CHAPTER I

SUMMARY OF RESULTS FOR LIE ALGEBRAS

1a. Commutation Relations and Root Vectors.

We take a vector space of r dimensions with a coordinate basis $(x_1, x_2, \dots x_r)$ and consider a set of transformations

$$x_i' = \phi_i(x_1, x_2, \dots x_r; a_1, a_2, \dots a_s) \qquad (1.1)$$

depending on the s independent, real and continuous parameters $a_1, a_2, \dots a_s$. We choose the ϕ_i to be analytic functions of the a's. For the set of transformations to form a group there must exist the identity transformation and we suppose that it is given by $a_1 = a_2 = \dots = a_s = 0$, so that

$$x_i = \phi_i(x_1, x_2, \dots x_r; 0, \dots 0).$$

There must also exist the inverse transformation of (1.1); that is to say, there must exist $\bar{a}_1, \bar{a}_2, \dots \bar{a}_s$ such that

$$x_i = \phi_i(x_1', x_2', \dots x_r'; \bar{a}_1, \bar{a}_2, \dots \bar{a}_s).$$

Finally on making successive transformations

$$x_i'' = \phi_i\{\phi_1(x_1, x_2 \dots x_r; a_1 \dots a_s), \phi_2(x_1, x_2 \dots x_r; a_1 \dots a_s) \dots ; a_1', a_2' \dots a_s'\}$$

it must be possible to find $a_1'', a_2'', \dots a_s''$ such that

$$x_i'' = \phi_i(x_1, x_2, \ldots x_r;\ a_1'', a_2'', \ldots a_s'') .$$

If each a_t'' is an analytic function of $a_1, a_2, \ldots a_s, a_1', a_2', \ldots a_s'$, we have a <u>Lie group of order s</u>.

We make an infinitesimal transformation of a function f of the x's in the neighbourhood of the identity:

$$\delta f = \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial a_t}\delta a_t = \delta a_t \left.\frac{\partial \phi_i}{\partial a_t}\right|_{a_1=a_2=\ldots=a_s=0} \frac{\partial f}{\partial x_i} ,$$

summed over repeated indices. We write this

$$\delta f = \varepsilon^A L_A f ,$$

where ε^A has been put for δa_t and

$$L_A = \left.\frac{\partial \phi_i}{\partial a_t}\right|_{a_1=a_2=\ldots=a_s=0} \frac{\partial}{\partial x_i} \qquad (A = 1, 2, \ldots s) .$$

It may be shown (Hamermesh 1962, p.299) that on taking the commutator of L_A and L_B we obtain

$$[L_A, L_B] = C_{AB}{}^D L_D . \qquad (1.2)$$

If the L_A's are elements of a vector space and the product of two elements is defined as their commutator, the last equation shows that the product belongs to the vector space. The algebra of the L_A's is determined by the $C_{AB}{}^D$, which are called the <u>structure constants.</u> We shall sometimes think of $C_{AB}{}^D$ as the BD-element of a matrix C_A.

The commutators always satisfy the Jacobi identity

$$[L_A, [L_B, L_C]] + [L_B, [L_C, L_A]] + [L_C, [L_A, L_B]] = 0 .$$

A set of elements L_A that constitute a vector space and that have a product defined by (1.2), the product obeying the Jacobi identity, are by definition elements of a Lie algebra - the Lie algebra of the group in which the L_A's arose. An infinitesimal transformation of the group is given by

$$I + \varepsilon^A L_A ,$$

summed over A. We shall deal only with Lie algebras of groups of linear transformations: that is to say, with Lie algebras of subgroups of the general linear transformations in n dimensions over the complex field $GL(n, \mathbb{C})$. It can then be proved directly, and without difficulty, that the infinitesimal generators L_A of the subgroup of $GL(n, \mathbb{C})$ are elements of a real vector space and that moreover, if we define the product of two elements as their commutator, they constitute a real Lie algebra (Simms 1969, p.11).

The next step is to put the commutation relations (1.2) into manageable form when the algebra is a semi-simple algebra; that is to say, when it contains no abelian invariant sub-algebra. According to Cartan's criterion a necessary and sufficient condition for this to be so is that the matrix g_{AB} defined by

$$g_{AB} = C_{AD}{}^{E} C_{BE}{}^{D}$$

be non-singular. After come calculation it is found possible (cf. McConnell 1965) to express (1.2) as

$$[H_i, H_j] = 0 , \qquad [H_i, E_\alpha] = r_i(\alpha) E_\alpha$$

$$[E_\alpha, E_{-\alpha}] = \sum_i r_i(\alpha) H_i , \qquad [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad . \tag{1.3}$$

In these equations H_i, H_j stand for commuting L_A's, which are brought to the diagonal by a similarity transformation. Working in the <u>regular representation</u> where L_A is identified with $-C_A$ we write H_i for $-C_i$ when diagonalized. We use Latin indices for A to label the commuting elements: if ℓ is the number of them, ℓ is the <u>rank of the algebra</u> and we let $i,j = 1,2,\ldots \ell$. When A is greater than ℓ, we write it as a Greek letter and we put

$$- C_\tau = E_\tau \quad . \qquad (\tau = \ell+1, \ell+2, \ldots s)$$

It is found that

$$C_{ij}^{\ A} = C_{i\alpha}^{\ j} = C_{i\alpha}^{\ \beta} = 0 . \qquad (\beta \neq \alpha) \tag{1.4}$$

We also write the $\alpha\alpha$-element of C_i as $r_i(\alpha)$; that is,

$$C_{i\alpha}^{\ \alpha} = r_i(\alpha) \tag{1.5}$$

and call $(r_1(\alpha), r_2(\alpha), \ldots r_\ell(\alpha))$ the <u>root vector</u> $\underset{\sim}{r}(\alpha)$. To each $\underset{\sim}{r}(\alpha)$ there corresponds an E_α, so $E_{\alpha+\beta}$ in (1.3) corresponds to $\underset{\sim}{r}(\alpha) + \underset{\sim}{r}(\beta)$, and

$$N_{\alpha\beta} = C_{\alpha\beta}^{\ \alpha+\beta} \quad .$$

This $N_{\alpha\beta}$ will vanish, if $\underset{\sim}{r}(\alpha) + \underset{\sim}{r}(\beta)$ is not a root vector. Moreover for every $\underset{\sim}{r}(\alpha)$ there exists an $\underset{\sim}{r}(-\alpha)$ equal to $-\underset{\sim}{r}(\alpha)$

and a corresponding $E_{-\alpha}$. In order to conform with conventions in the literature of elementary particle theoretical physics (cf. Behrends, Dreitlein, Fronsdal and Lee 1962) we normalize the root vectors, as we can, so that

$$\sum_{\alpha} r_i(\alpha)\, r_j(\alpha) = \delta_{ij} , \qquad (1.6)$$

the summation being over all the α's.

To avoid unnecessary complications we shall henceforth deal almost exclusively with semi-simple Lie algebras of rank 2. When we refer the components $(r_1(\alpha), r_2(\alpha))$ to rectangular axes, it is found that the <u>root diagrams</u> that they constitute can only be the following:

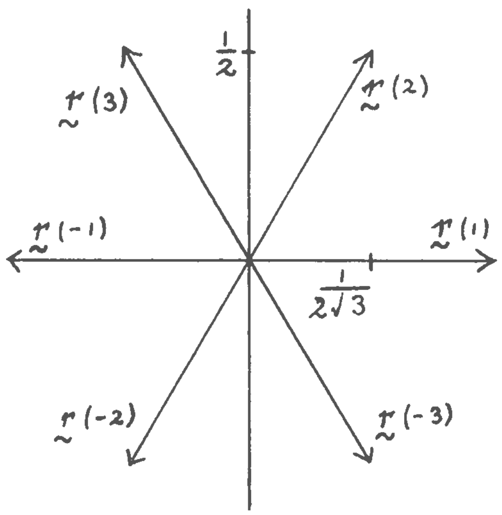


Fig.1.1 - The root diagram for A_2.

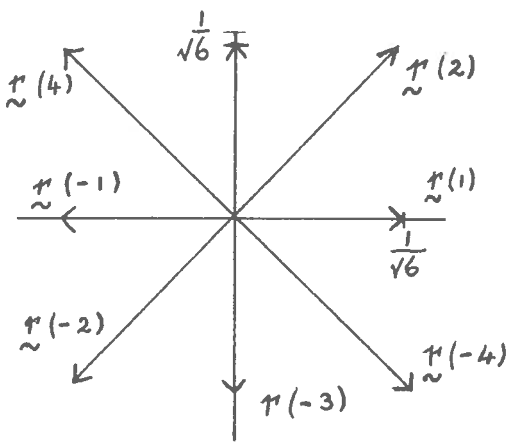


Fig.1.2 - The root diagram for B_2.

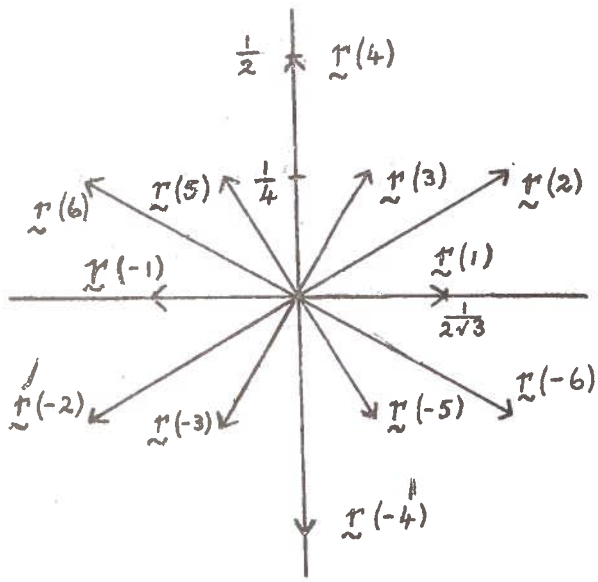


Fig. 1.3 - The root diagram for G_2.

When $r_1(\alpha) > 0$, we say that $\underset{\sim}{r}(\alpha)$ is a <u>positive root vector.</u> For the A_2 algebra $\underset{\sim}{r}(1), \underset{\sim}{r}(2), \underset{\sim}{r}(-3)$ are positive root vectors.

1b. Weights.

From now on we shall deal with representations of the Lie algebras and we shall denote by H_i, E_α not only the elements of the algebras but also the matrices representing them. This will not cause confusion. Since H_1 and H_2 commute, it is possible to find simultaneous eigenvectors of H_1 and H_2 in the representation space. If u is such an eigenvector, so that

$$H_1 u = m_1 u , \qquad H_2 u = m_2 u , \tag{1.7}$$

we say that (m_1,m_2), or more briefly $\underset{\sim}{m}$, is the weight of u. We shall refer to u as a weight vector. If there is only one vector u satisfying (1.7), we say that $\underset{\sim}{m}$ is a simple weight. If there are two, three, ... linearly independent u's, we say that $\underset{\sim}{m}$ is a double weight, triple weight ... , or generally a multiple weight, the weight multiplicity being the number of linearly independent u's that satisfy (1.7) We say that a weight $\underset{\sim}{m}'$ is higher than $\underset{\sim}{m}''$, if $m_1' > m_1''$ or if $m_1' = m_1''$ and $m_2' > m_2''$.

We shall now quote from Racah (1965) some properties of weights and weight vectors of semi-simple Lie algebras:

1. The weights are linear combinations of the root vectors with rational coefficients.
2. Weight vectors corresponding to different weights are linearly independent.
3. The weights are situated in a lattice determined by the root vectors.

This means that two neighbouring weights differ by a root vector. If we take a string of weights like $\underset{\sim}{m}$, $\underset{\sim}{m} + \underset{\sim}{r}(\alpha)$, $\underset{\sim}{m} + 2\underset{\sim}{r}(\alpha)$, ... , there are no gaps in the string, e.g. between $\underset{\sim}{m} + 5\underset{\sim}{r}(\alpha)$ and $\underset{\sim}{m} + 8\underset{\sim}{r}(\alpha)$ we must have $\underset{\sim}{m} + 6\underset{\sim}{r}(\alpha)$, $\underset{\sim}{m} + 7\underset{\sim}{r}(\alpha)$. If E_α acts on the vector u with weight $\underset{\sim}{m}$, then by (1.3)

$$H_i E_\alpha u = E_\alpha H_i u + r_i(\alpha) E_\alpha u = (m_i + r_i(\alpha)) E_\alpha u . \quad (1.8)$$

If the representation is irreducible, either $E_\alpha u$ vanishes or it is a weight vector with weight $\underset{\sim}{m} + \underset{\sim}{r}(\alpha)$. For this reason we may speak of E_α as a displacement operator. If in an irreducible representation one starts with a weight vector and operates successively with the displacement operators, one may obtain all the basis vectors of the representation. The number of these, being the number of linearly independent vectors, is just the total number of weights, multiple weights being counted according to their multiplicity. The total number is therefore the dimension of the irreducible representation. The set of all the weights for a representation, reducible or irreducible, constitute the weight diagram of the representation.

4. If $\underset{\sim}{m}$ is a weight, then

$$\frac{2(\underset{\sim}{m}.\underset{\sim}{r}(\alpha))}{|\underset{\sim}{r}(\alpha)|^2}$$

is zero or an integer.

5. If $\underset{\sim}{m}$ is a weight, its Weyl reflection

$$m - \frac{2(\underset{\sim}{m}.\underset{\sim}{r}(\alpha))}{|\underset{\sim}{r}(\alpha)|^2}\underset{\sim}{r}(\alpha) \qquad (1.9)$$

is also a weight and has the same multiplicity.

We note that (1.9) is obtained from $\underset{\sim}{m}$ by a reflection in the line through the origin that is perpendicular to the root vector $\underset{\sim}{r}(\alpha)$. Two weights connected in such a way are called equivalent weights. We deduce from the last theorem that a weight diagram is invariant under Weyl reflections. In a set of equivalent weights the one that is higher than the others is called the dominant weight. The highest of the dominant weights in a representation is called the highest weight of the representation.

6. If a representation is irreducible, its highest weight is simple.

7. Irreducible representations with the same highest weight are equivalent.

There is therefore a one-to-one correspondence between inequivalent irreducible representations and their highest weights.

8. For an algebra of rank ℓ, there are ℓ fundamental dominant weights such that every dominant weight is a linear combination of these with non-negative integral coefficients.

For an algebra of rank 2 this means that there exist two fundamental dominant weights $\underset{\sim}{w}_1$ and $\underset{\sim}{w}_2$, say, such that every dominant weight is expressible as

$$\lambda \underset{\sim}{w}_1 + \mu \underset{\sim}{w}_2 \tag{1.10}$$

with $\lambda, \mu = 0, 1, 2, \ldots$

9. Every dominant weight (1.10) is the highest weight of an irreducible representation $D(\lambda,\mu)$, say, of the semi-simple Lie algebra that has $\underset{\sim}{w}_1$ and $\underset{\sim}{w}_2$ as fundamental dominant weights.

We note that $\underset{\sim}{w}_1$ is the highest weight of the irreducible representation $D(1,0)$ and that $\underset{\sim}{w}_2$ is the highest weight of $D(0,1)$.

We consider the weights of the regular representation. In the general case of rank ℓ

$$H_i = - C_i \,, \qquad E_\tau = - C_\tau \qquad (i=1,2\ldots \ell;\ \tau = \ell+1,\ldots s)$$

and by (1.4) and (1.5)

$$C_{ij}{}^{A} = 0 \,, \quad C_{i\alpha}{}^{j} = 0 \,, \qquad C_{i\alpha}{}^{\beta} = r_i(\alpha)\, \delta_\alpha{}^{\beta}. \quad (A=1,2\ldots s)$$

The weights in the regular representation are such that m_i is an eigenvalue of H_i, that is, of $- C_i$. Now C_i is a diagonal matrix with the first ℓ diagonal elements equal to zero and the others equal to the ith components of the root vectors. The minus sign is of no consequence, since for every root vector there is an equal and opposite root vector. Hence the weights of the regular representation are the root vectors, plus a zero weight. The only multiple weight is the zero one and its multiplicity is ℓ, the rank of the algebra. Thus, for example, the weights for the regular representation of the algebras A_2, B_2, G_2 are situated at the extremities of the root vectors in Figures 1.1,1.2,1.3 and at the origin, which is a double weight. The dimension of the regular representation is the number of root vectors added to the rank.

1c. The Graphical Method of Calculating Weight Multiplicities.

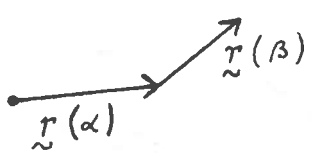
We return to equation (1.8). If we wish to find the multiplicity of a selected weight in the diagram corresponding to an irreducible representation of a semi-simple Lie algebra, we may start with the vector v of a simple weight, e.g. the highest weight, and go from this to the selected weight by a sequence of displacements of weight vectors. This will give a vector

$$E_{\alpha_t} \cdots E_{\alpha_2} E_{\alpha_1} v \,. \tag{1.11}$$

Having done this by all possible sequences we would have to use the commutation relations to see how many linearly independent vectors are so obtained. This will be the required multiplicity.

It is clear that this procedure could become very tedious and so could easily lead to error in calculations. To simplify the approach we shall replace expressions like (1.11) by graphs. When u is a vector of a simple or multiple weight $\underset{\sim}{m}$, we agree to represent $E_\alpha u$ by the directed line from $\underset{\sim}{m}$ to $m + \underset{\sim}{r}(\alpha)$, if the latter is present in the diagram. If it is not present, $E_\alpha u$ is zero. Likewise we represent $E_\beta E_\alpha u$ for $\beta \neq -\alpha$ by

(1.12)



We indicate by placing a heavy dot at $\underset{\sim}{m}$ that we atart with a vector having this weight. It will therefore usually be superfluous to place arrows on the vectors. $E_\alpha E_\beta u$ is represented by

(1.13)

and, as $\underset{\sim}{m} + \underset{\sim}{r}(\alpha) + \underset{\sim}{r}(\beta)$ may be a multiple weight, this may not be equivalent to (1.12). If $\underset{\sim}{r}(\alpha) + \underset{\sim}{r}(\beta)$ is not a root vector, the $N_{\alpha\beta}$ in (1.3) vanishes, E_α and E_β commute and

(1.14)

When $\underset{\sim}{r}(\alpha) + \underset{\sim}{r}(\beta)$ is a root vector, the difference of (1.12) and (1.13) is $N_{\beta\alpha} E_{\alpha+\beta} u$. In considering multiplicities what interests us is the number of linearly independent vectors that are associated with a weight. We are therefore not concerned with non-vanishing numerical multiples of the vector, so that we shall take $N_{\beta\alpha} E_{\alpha+\beta} u$ to be just $E_{\alpha+\beta} u$ and therefore in the present case

(1.15)

If $E_\beta E_\alpha$ vanishes,

 =  .   (1.16)

When $\beta = -\alpha$, we employ the relation

$$[E_\alpha, E_{-\alpha}] = \sum_i r_i(\alpha)H_i ,$$

which gives

$$\begin{aligned} E_\alpha E_{-\alpha}u - E_{-\alpha}E_\alpha u &= \sum_i r_i(\alpha)H_i u \\ &= u \sum_i m_i \; r_i(\alpha) \end{aligned} \qquad (1.17)$$

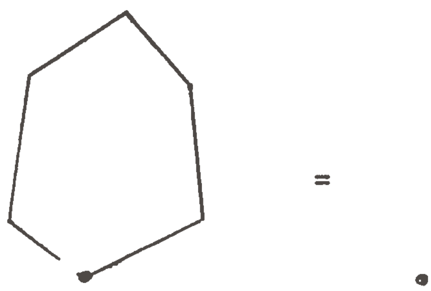
This is depicted as

 −  = •   (1.18)

and for clarity we have included the arrows. The relation (1.18) is true no matter what is the multiplicity of $\underset{\sim}{m}$. If it is a simple weight, there is a unique vector to which it can return and so

 = •   (1.19)

More generally, if we have a simple weight, we can reduce any path that returns to it without leaving the weight diagram to a dot:



(1.20)

To find the multiplicity of a given weight in a diagram for an irreducible representation of a semi-simple Lie algebra of rank 2, one starts from a simple weight, e.g. the highest weight, and examines the paths from this to the given weight. The number of independent paths is taken to be the multiplicity of the weight. The reliability of this method has in fact been checked in specific cases by independent calculations (McConnell 1970).

CHAPTER II

A_2 WEIGHT DIAGRAMS

2a. Irreducible Representations of A_2.

To obtain a weight diagram for a representation of A_2 we take the m_1 axis horizontal and the m_2 axis vertical, and we mark the positions of the points (m_1,m_2) that correspond to the simultaneous eigenvectors of H_1,H_2 that constitute a basis of the representation. In order to classify the representations, we calculate the fundamental dominant weights for A_2.

Let us take a dominant weight $\underset{\sim}{M}$, where

$$\underset{\sim}{M} = M_1\underset{\sim}{e}_1 + M_2\underset{\sim}{e}_2 , \tag{2.1}$$

$\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ being unit vectors in the directions of the m_1,m_2-axes. We find from Fig.1.1 that

$$\frac{2(\underset{\sim}{M}.\underset{\sim}{r}(1))}{|\underset{\sim}{r}(1)|^2} = 2\sqrt{3}M_1 , \quad \frac{2(\underset{\sim}{M}.\underset{\sim}{r}(2))}{|\underset{\sim}{r}(2)|^2} = \sqrt{3}M_1 + 3M_2$$

$$\frac{2(\underset{\sim}{M}.\underset{\sim}{r}(3))}{|\underset{\sim}{r}(3)|^2} = -\sqrt{3}M_1 + 3M_2 \tag{2.2}$$

and therefore by theorem 4 of section 1b

$$2\sqrt{3}M_1 = \tau_1 , \quad \sqrt{3}M_1 + 3M_2 = \tau_2 , \quad -\sqrt{3}M_1 + 3M_2 = \tau_3 , \tag{2.3}$$

where $\tau_1, \tau_2, \tau_3 = 0, \pm 1, \pm 2, \ldots$. The same equations come from $\underset{\sim}{r}(-1), \underset{\sim}{r}(-2), \underset{\sim}{r}(-3)$. On making Weyl reflections of $\underset{\sim}{M}$ with respect to $\underset{\sim}{r}(1), \underset{\sim}{r}(2), \underset{\sim}{r}(3)$ we obtain from (1.9) and (2.2)

$$- M_1 \underset{\sim}{e}_1 + M_2 \underset{\sim}{e}_2 \, , \qquad (\tfrac{1}{2}M_1 - \tfrac{\sqrt{3}}{2} M_2)\underset{\sim}{e}_1 - (\tfrac{1}{2}M_2 + \tfrac{\sqrt{3}}{2} M_2)\underset{\sim}{e}_2$$

$$(\tfrac{1}{2}M_1 + \tfrac{\sqrt{3}}{2} M_2)\underset{\sim}{e}_1 + (\tfrac{\sqrt{3}}{2} M_1 - \tfrac{1}{2} M_2)\underset{\sim}{e}_2 \, .$$

On comparing this with (2.1) we deduce that, if $\underset{\sim}{M}$ is to be a dominant weight, then

i) $M_1 \geqslant - M_1$, so that M_1 is positive or zero

ii) $M_1 \geqslant \tfrac{1}{2}M_1 - \tfrac{\sqrt{3}}{2} M_2$, i.e. $M_1 \geqslant -\sqrt{3}\, M_2$

iii) $M_1 \geqslant \tfrac{1}{2}M_1 + \tfrac{\sqrt{3}}{2} M_2$, i.e. $M_1 \geqslant \sqrt{3}\, M_2$.

We conclude that

$$M_1 \geqslant 0 \, , \qquad M_1 \geqslant \sqrt{3}\, |M_2| \, . \tag{2.4}$$

Returning to (2.3) we see that

$$\tau_1 \geqslant 0 \, , \qquad \tau_2 \geqslant 0 \, , \qquad \tau_3 \leqslant 0$$

$$M_1 = \frac{\tau_1}{2\sqrt{3}} \, , \qquad M_2 = \frac{2\tau_2 - \tau_1}{6}$$

and in order to satisfy (2.4)

$$\tau_1 \geqslant |2\tau_2 - \tau_1| \, ,$$

so that $\tau_1 \geqslant \tau_2$. Let us therefore write

$$\tau_2 = \lambda \, , \qquad \tau_1 - \tau_2 = \mu \, ,$$

where λ and μ are positive integers or zero. Then

$$\underset{\sim}{M} = \frac{\lambda+\mu}{2\sqrt{3}}(1,0) + \frac{\lambda-\mu}{6}(0,1)$$

$$= \lambda\left(\frac{1}{2\sqrt{3}}, \frac{1}{6}\right) + \mu\left(\frac{1}{2\sqrt{3}}, -\frac{1}{6}\right).$$

According to (1.10) the fundamental dominant weights are

$$\left(\frac{1}{2\sqrt{3}}, \frac{1}{6}\right), \qquad \left(\frac{1}{2\sqrt{3}}, -\frac{1}{6}\right).$$

Then theorem 9 of section 1b shows that

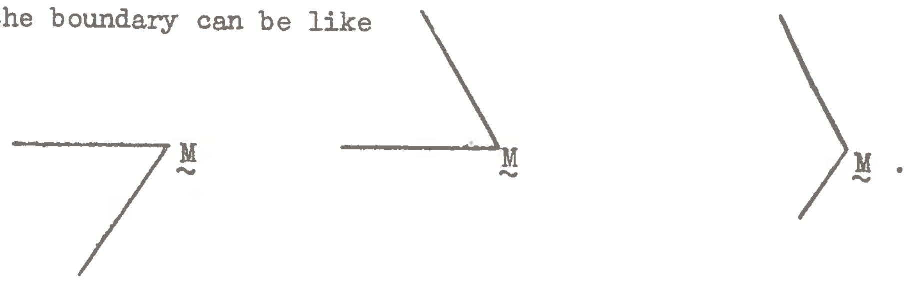
$$\left(\frac{\lambda+\mu}{2\sqrt{3}}, \quad \frac{\lambda-\mu}{\sqrt{6}}\right) \tag{2.5}$$

is the highest weight of the irreducible representation $D(\lambda,\mu)$ of the algebra A_2.

2b. Boundaries of A_2 Weight Diagrams.

We saw in section 1b that weight diagrams are invariant under reflections in lines through the origin perpendicular to the root vectors. On referring to Fig.1.1 we deduce that the weight diagrams for A_2 are invariant for a reflection in the m_2 axis, and for reflections in the lines $m_1 = \pm\sqrt{3}m_2$. Now a reflection in $m_1 = \sqrt{3}m_2$ followed by a reflection in the m_2-axis is equivalent to a rotation about the origin through an angle of $\frac{2\pi}{3}$. The weight diagrams are therefore invariant under such a rotation.

To discuss the shape of the boundary we start from the highest weight $\underset{\sim}{M}$. Since the root vectors $r(1), r(2), r(-3)$ are positive, the boundary lines issuing from $\underset{\sim}{M}$ cannot go in the directions of these vectors. The lines may go in the directions of $\underset{\sim}{r}(-1), \underset{\sim}{r}(-2), \underset{\sim}{r}(3)$ and by taking combinations of two of them at a time we see that in the neighbourhood of the highest weight the boundary can be like



Since the weight diagrams are invariant under the rotations through angle $\frac{2\pi}{3}$, the boundaries are, respectively, an equilateral triangle with vertex downwards, an equilateral triangle with vertex upwards and a hexagon with alternate sides equal in length, all boundaries being symmetrically situated with respect to the m_2 axis.

Let us relate the boundaries to irreducible representations of A_2 . We do not bother about the trivial case of $D(0,0)$, whose weight diagram consists of one weight at the origin. Since $\underset{\sim}{M}$ is simple for an irreducible representation, the corresponding weight vector v is unique apart from a scalar multiple. For the triangle with vertex downwards

$$E_3 v = 0 , \qquad E_{-3} v = 0 ,$$

so

$$0 = [E_3,E_{-3}]v = \sum_i r_i(3) H_i v = (-\frac{1}{2\sqrt{3}} M_1 + \tfrac{1}{2} M_2)v ,$$

which gives

$$M_1 = \sqrt{3}\, M_2 .$$

On comparing with (2.5) we see that the representation is a $D(\lambda,0)$. The highest weight is $\lambda(\frac{1}{2\sqrt{3}} , \frac{1}{6})$ and the reflection of this in the m_2 axis is $\lambda(-\frac{1}{2\sqrt{3}} , \frac{1}{6})$. The difference of the m_1 coordinates is $\frac{\lambda}{\sqrt{3}}$ and, since $|\underset{\sim}{r}(1)| = \frac{1}{\sqrt{3}}$, there are λ units and $\lambda+1$ weights in the horizontal side. To sum up, the boundary of the weight diagram of the irreducible representation $D(\lambda,0)$ is an equilateral triangle with vertex downwards, symmetrically placed with respect to the m_2 axis and having in each side λ units, each being the length of a root vector. The boundary of the weight diagram of the irreducible representation $D(0,\mu)$ is likewise an equilateral triangle with vertex downwards, there being μ units in each side.

The hexagons are therefore the boundaries of the weight diagrams for the irreducible representation $D(\lambda,\mu)$ with λ,μ non-vanishing. According to (2.5) the highest weight is

$$\left(\frac{\lambda + \mu}{2\sqrt{3}} , \frac{\lambda - \mu}{6} \right) .$$

If we reflect this in the line through the origin perpendicular to $\underset{\sim}{r}(3)$, we obtain

$$\left(\frac{\lambda}{2\sqrt{3}} , \frac{\lambda + 2\mu}{6} \right) . \qquad (2.6)$$

This is the extremity of the side that goes from the highest weight in the direction of $\underset{\sim}{r}(3)$. The difference in m_1 coordinates is $\mu/2\sqrt{3}$ and, since the length of the projection of $\underset{\sim}{r}(3)$ on the m_1 axis is $1/2\sqrt{3}$, we deduce that there are μ units in this side. The difference in m_1 coordinates between (2.6) and its reflection in the m_2 axis is $\lambda/\sqrt{3}$, so there are λ units in the horizontal line joining them. Hence the boundary for an irreducible $D(\lambda,\mu)$ representation is a hexagon symmetrically situated with respect to the m_2 axis with alternate sides equal in length and with two horizontal sides, the upper one having λ units of length of a root vector and the lower one having μ units. When $\mu = \lambda$, the hexagon is regular, it is symmetrically placed with respect to both axes and its highest weight lies on the m_1 axis. An example of a $D(\lambda,\lambda)$ representation is the regular representation. We saw at the end of section 1b that the weights in the diagram of the regular representation are situated at the extremities of the six root vectors and at the origin. On referring to (2.5) we see that $\lambda = 1$ and, as the dimension is $6 + 2$, the regular representation of A_2 is $D^{(8)}(1,1)$.

2c. Multiplicities on the Boundaries of A_2 Diagrams.

We shall investigate multiplicities in the diagram for a $D(\lambda,\mu)$ representation, and in the course of this we shall cover the cases of $D(\lambda,0)$ and $D(0,\mu)$ representations. We first look at the multiplicities of weights on the boundary.

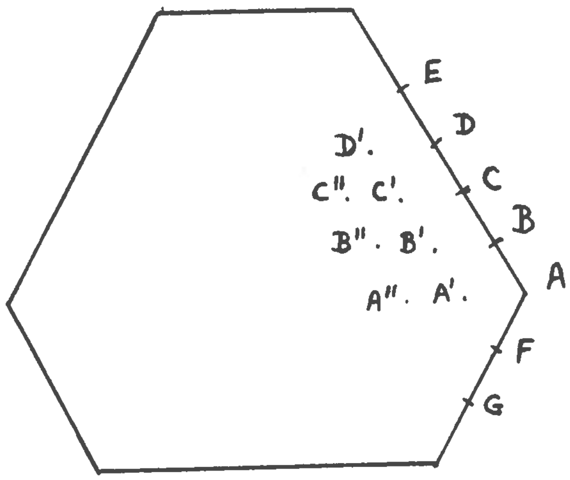


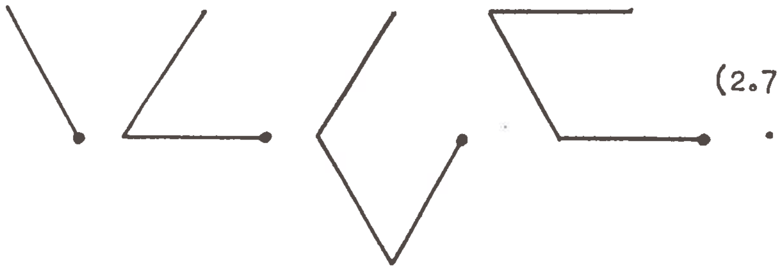
Fig.2.1 - Weights in a $D(\lambda,\mu)$ diagram of A_2 .

The highest weight marked A in Fig.2.1 is simple and starting from it we proceed along various paths to B,C,D,... . The number of independent graphs to the weight in question is the multiplicity.

$\underset{\sim}{B}$ We may go from A to B in a variety of ways, e.g.

AB, AA'B, AFA'B, AA'B'B

with the respective graphs

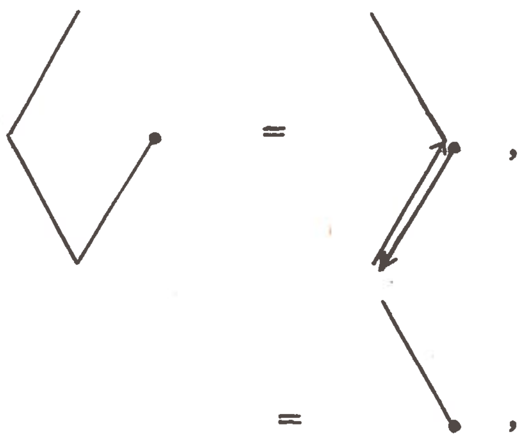


(2.7)

Let us show that the last three are equivalent to the first.

In fact

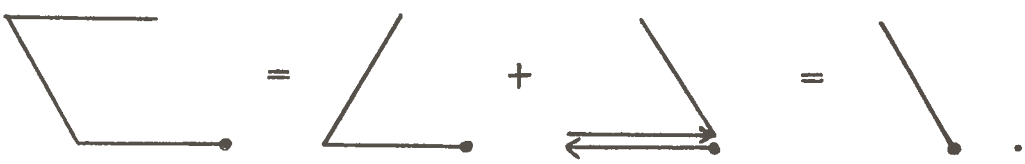
by (1.15), but the last graph vanishes because there is no weight to the right of A. Moreover



by (1.14)

by (1.19),

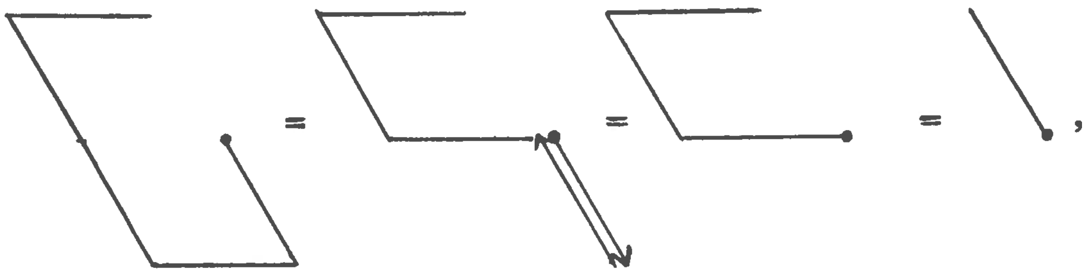
and



(2.8)

We have thus only one independent graph and B is a simple weight.

$\underset{\sim}{C}$ Since B is simple, we can start from it and proceed to C. We have paths obtained by displacing (2.7) by $\underset{\sim}{r}(3)$. In addition we could have a path like BAA'B'C'C and for this



by (2.8).

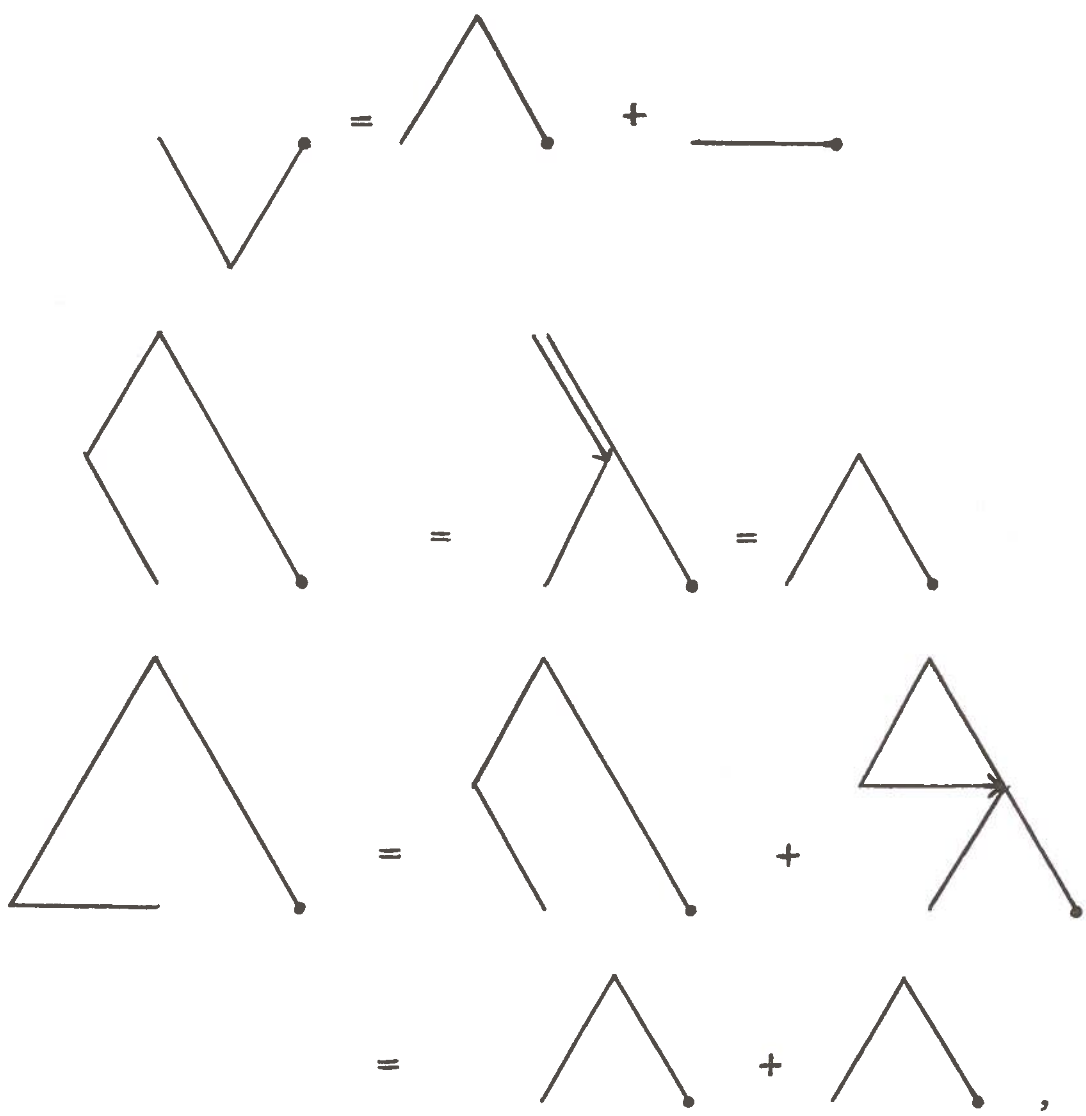
Continuing along D,E, ... we find that all the weights there are simple. Similarly all the weights along the side AFG ... are simple. On rotating twice about the origin through an angle $2\pi/3$ we deduce that all the weights on the boundary are simple.

2d. Multiplicities in Hexagonal Layers.

We shall deal with successive layers of weights assuming that they remain hexagonal. It will clearly suffice to examine layers parallel to the side ABC .... We commence with A',B',C',... .

A' Starting from A we have already two independent graphs

We take several other paths and show that each gives a combination of these two;



by (1.20),

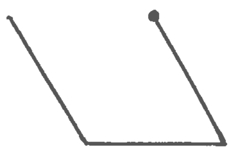
=  .

The multiplicity of A' is therefore 2.

B' We start from B and obtain the independent graphs

 (2.9)

but the latter one exists only if the weight C is present; otherwise B' is on the boundary and we are off the layer. We may as before show that all paths from B to B' give only linear combinations of the graphs (2.9), for example

 =  = .

C' We start from C and obtain multiplicity 2, if D exists; otherwise the multiplicity is 1.

We therefore see that the multiplicity is 2 for all weights along the line A'B'C'... . Arguing as before we conclude that the multiplicity is 2 all along the layer.

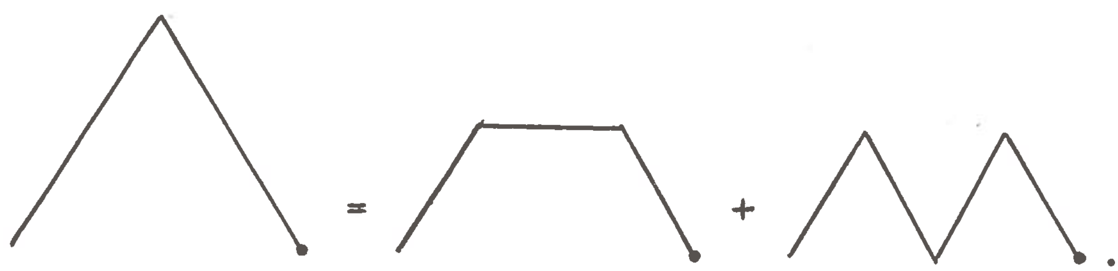
Going to the next layer we consider A",B",C"... .

A" Starting from A we can get two graphs by adding A'A" to those for A', viz.

 (2.10)

If we follow the path ABB'A" , we have

and so get nothing new. If however C exists, we have in addition ABCB'A" and for this



The last graph cannot be reduced to a combination of (2.10), so the multiplicity is then 3. We can bring a classification into these graphs by saying that we can have 2,1,0 horizontal lines in the graphs from A to A", that we have 1 horizontal line only if B is present and that 2 such lines are there only if in addition C is present.

B",C",... The multiplicity of B" is 3 provided that D exists; it is 2 if C exists but D does not. Then it is on the first layer. Continuing, we see that the multiplicity continues to be 3 as long as we keep on the second layer.

There is now a general pattern for the variation of multiplicities. As we go from one layer to the next there is one extra independent graph, since there is one extra horizontal line from the point on the boundary at the same horizontal level. The increase in multiplicity persists as long as we can come in diagonally from the hexagonal boundary. It therefore persists until we reach the triangle obtained by peeling off hexagonal layers of weights. The multiplicity is the same for all weights on this triangular layer.

2e. Multiplicities in Triangular Layers.

Let us first look at the multiplicities on the boundary of a weight diagram, when the boundary is triangular. We take the

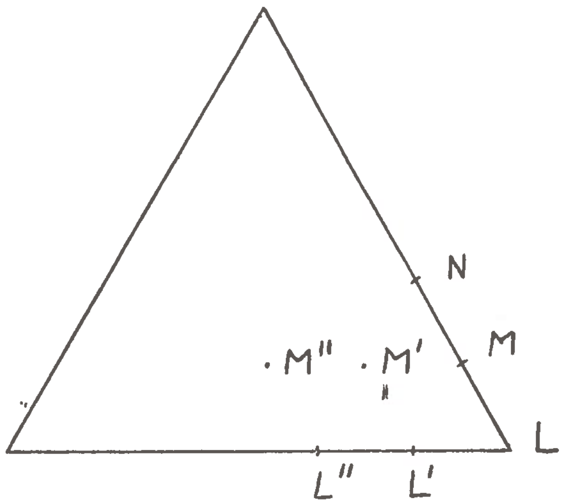
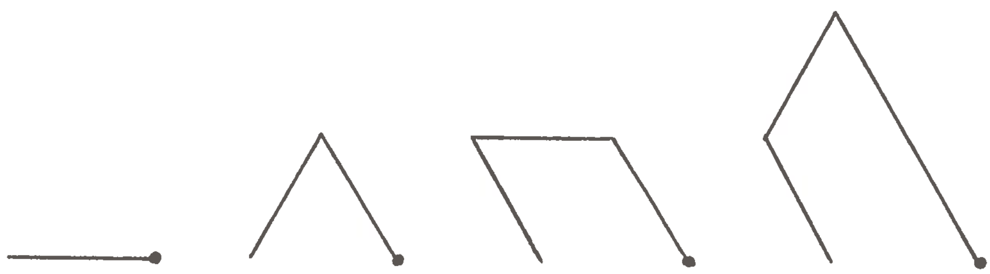


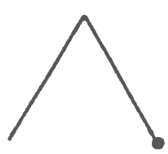
Fig.2.2 - Weight multiplicities for a
$D(0,\mu)$ diagram of A_2 .

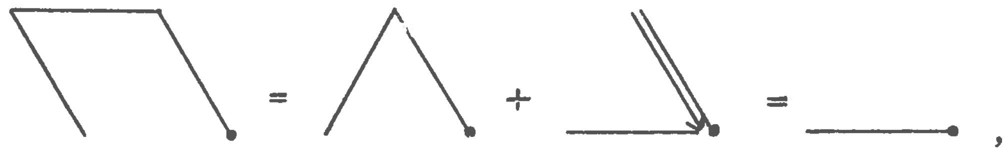
vertex upwards as in Fig.2.2 and calculate the multiplicities of L',L",... .

$\underset{\sim}{L'}$ Proceeding from the highest weight L we have various graphs



etc. . Now

 = , by (1.16),



, by (1.18),

etc. . L' is therefore a simple weight.

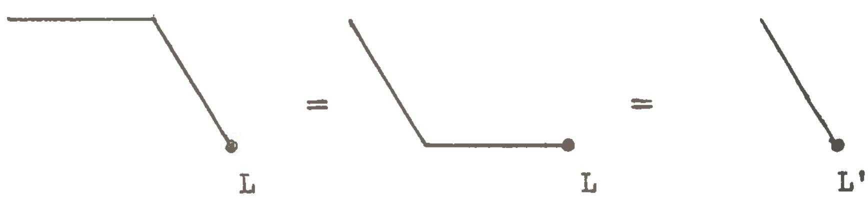
L" Since we have shown that L' is a simple weight, we may go from it to L". We obtain the same type of graphs as in the previous case, so L" is also a simple weight. All the graphs are equivalent to

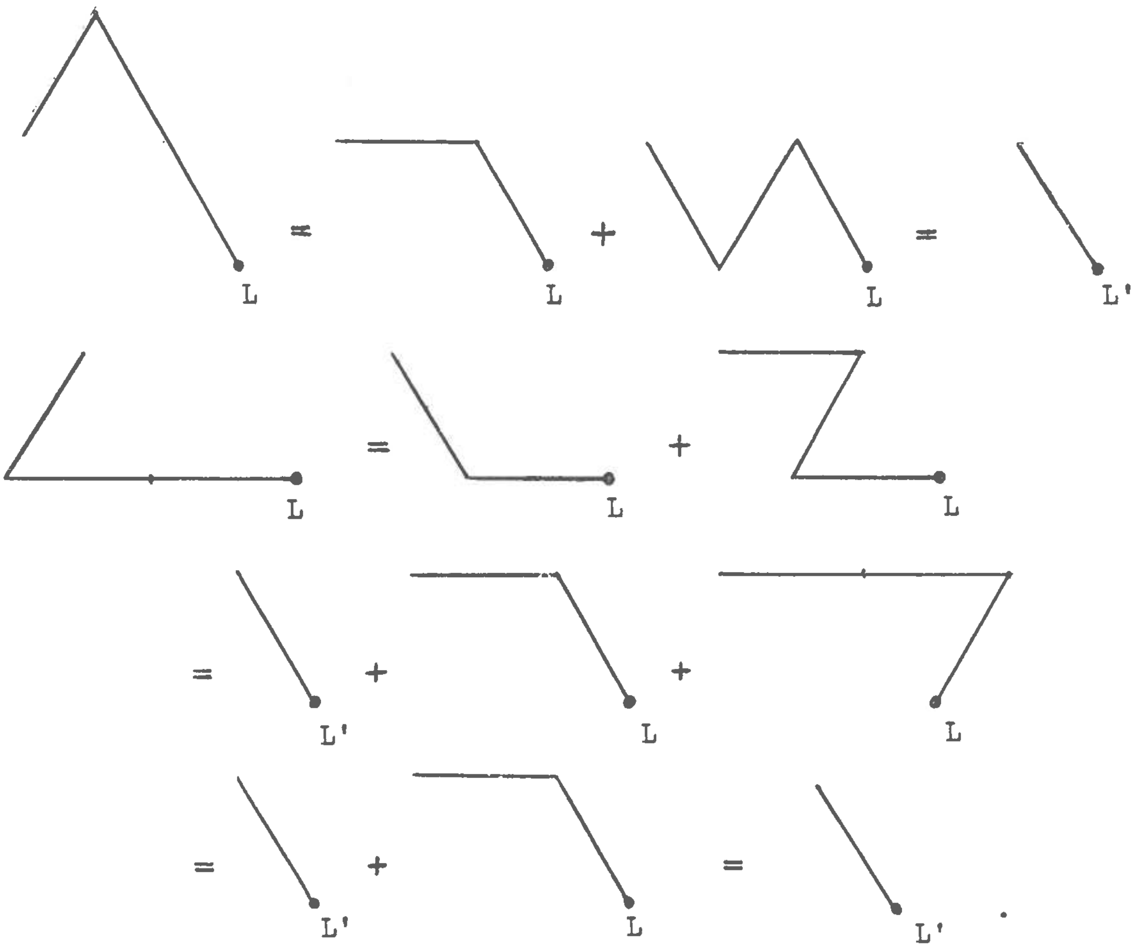
We may argue in this way that the weights along the horizontal base are all simple. By rotating we deduce that all the weights on the boundary are simple.

When we began with a hexagonal boundary and peeled off layer after layer, we arrived at a triangle on which all the weights had the same multiplicity, s say. We can include the case of a triangular boundary in this by putting s = 1 . We now examine what happens when we go to the next triangular layer.

We continue to refer to Fig.2.2. To say that s is the multiplicity of L,L',L",... means that we have s, and only s, independent graphs from the highest weight A of D(λ,μ) to L,L',L",... . To express this in another way: all the independent graphs from A to L',L",... may be obtained by adding LL', LL",... on to the independent graphs from A to L . A similar result holds for graphs from A to M,N ... .

Consider the weight M'. We may take any path from A to M' as passing through L because, if for example it were to come through M, we could replace AM by AL+LM. We therefore turn our attention to the portions of the graphs from L to M' and for clarity we letter the starting points of graphs. Choosing different paths we have, for example,





These show that all independent paths to M' may be reduced to L'M' added to all independent paths to L'. The multiplicity of M' is therefore s and the same is true for every weight in the triangular layer. Going to the next layer we obtain in precisely the same way multiplicity s again.

We conclude that, once we reach the triangle, there is no increase in multiplicity. In particular if the boundary is a triangle, all the weights in the diagram are simple. The rules for multiplicities in A_2 diagrams were first given in Wigner 1937.

2f. A_2 Dimensionality Formula.

We put down weight diagrams for certain irreducible representations of A_2 marking the multiplicity of each weight:

```
  1   1
    1
```
$D^{(3)}(1,0)$

```
    1
  1   1
```
$D^{(3)}(0,1)$

```
    1   1
  1   2   1
    1   1
```
$D^{(8)}(1,1)$

```
1   1   1   1
  1   1   1
    1   1
      1
```
$D^{(10)}(3,0)$

```
      1   1   1
    1   2   2   1
  1   2   3   2   1
    1   2   2   1
      1   1   1
```
$D^{(27)}(2,2)$

```
    1   1   1   1   1
  1   2   2   2   2   1
    1   2   2   2   1
      1   2   2   1
        1   2   1
          1   1
```
$D^{(35)}(4,1)$

```
      1   1   1   1
    1   2   2   2   1
  1   2   3   3   2   1
    1   2   3   2   1
      1   2   2   1
        1   1   1
```
$D^{(42)}(3,2)$.

The weight diagrams are used in elementary particle physics to display particles according to the values of the third component of their isospin and their hypercharge. The $D^{(3)}(1,0)$ weight diagram is used for quarks, the $D^{(3)}(0,1)$ for antiquarks, the $D^{(8)}(1,0)$ for $\frac{1}{2}^+$ baryons, 0^- mesons and 1^- meson resonances, and the $D^{(10)}(3,0)$ diagram for the $3/2^+$ baryon resonances.

The dimension $N(\lambda,\mu)$ of the irreducible $D(\lambda,\mu)$ representation can be calculated by summing the multiplicities in the diagram. Since the boundary of a $D(\lambda,\mu)$ diagram contains alternate sides with λ and μ units, it follows that

$$N(\mu,\lambda) = N(\lambda,\mu) .$$

To calculate $N(\lambda,\mu)$ we shall therefore take $\lambda \geqslant \mu$.

We first find the total number of weights in the hexagonal layers. As we go from one layer to the next, there will be a decrease of one unit of length in each side. Taking account of the increase in multiplicities we have the total number of weights in the hexagons equal to

$$3\{\lambda + \mu + 2[\lambda - 1 + \mu - 1] + \ldots + \mu[\lambda - (\mu - 1) + 1]\}$$

$$= 3 \sum_{r=0}^{\mu-1} (r + 1)(\lambda + \mu - 2r)$$

$$= (\mu + 1)(\tfrac{3}{2}\lambda\mu + 2\mu - \tfrac{1}{2}\mu^2) . \qquad (2.11)$$

When we arrive at the triangle, the multiplicity is $\mu+1$ and the number of units in the side is $\lambda - \mu$. The total number of weights in the triangular layers is therefore equal to

$$(\mu + 1)\{(\lambda - \mu + 1) + (\lambda - \mu) + \dots + 2 + 1\}$$

$$= (\mu + 1)(\tfrac{1}{2}\lambda^2 - \lambda\mu + \tfrac{1}{2}\mu^2 + \tfrac{3}{2}\lambda - \tfrac{3}{2}\mu + 1) .$$

When this is added to (2.11), we obtain altogether

$$(\mu + 1)(\tfrac{1}{2}\lambda^2 + \tfrac{1}{2}\lambda\mu + \tfrac{3}{2}\lambda + \tfrac{1}{2}\mu + 1) ,$$

and therefore

$$N(\lambda,\mu) = \tfrac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2) .$$

This is verified by the specific examples that we took above. We may note in particular for regular hexagonal boundaries that

$$N(\lambda,\lambda) = (\lambda + 1)^3 .$$

CHAPTER III

B_2 WEIGHT DIAGRAMS

3a. Irreducible Representations of B_2.

We first seek the dominant weights for the algebra B_2. Let $\underset{\sim}{M}$ given by

$$\underset{\sim}{M} = M_1\underset{\sim}{e}_1 + M_2\underset{\sim}{e}_2 \qquad (3.1)$$

be any weight. If we take $\underset{\sim}{r}(\alpha)$ to be successively the $\underset{\sim}{r}(1)$, $\underset{\sim}{r}(2)$, ... $\underset{\sim}{r}(-4)$, we find for

$$\frac{2(\underset{\sim}{M}.\underset{\sim}{r}(\alpha))}{|\underset{\sim}{r}(\alpha)|^2}$$

the respective values

$$\begin{array}{cccc} 2\sqrt{6}M_1 , & \sqrt{6}(M_1 + M_2) , & 2\sqrt{6}M_2 , & \sqrt{6}(-M_1 + M_2) \\ -2\sqrt{6}M_1 , & -\sqrt{6}(M_1 + M_2) , & -2\sqrt{6}M_2 , & \sqrt{6}(M_1 - M_2) \end{array} \qquad (3.2)$$

and for the Weyl reflection of $\underset{\sim}{M}$ the respective values

$$-M_1\underset{\sim}{e}_1 + M_2\underset{\sim}{e}_2, \quad -M_2\underset{\sim}{e}_1 - M_1\underset{\sim}{e}_2, \quad M_1\underset{\sim}{e}_1 - M_2\underset{\sim}{e}_2, \quad M_2\underset{\sim}{e}_1 + M_1\underset{\sim}{e}_2 \qquad (3.3)$$

repeated once. From theorem 4 of section 1b and (3.2) we deduce that

$$2\sqrt{6}\, M_1 = \sigma_1 , \qquad 2\sqrt{6}\, M_2 = \sigma_2 , \qquad (3.4)$$

where

$$\sigma_1, \sigma_2, \tfrac{1}{2}(\sigma_1 - \sigma_2) = 0, \pm 1, \pm 2, \ldots .$$

For $\underset{\sim}{M}$ to be a dominant weight we have from (3.1) and (3.3)

$$M_1 \geq -M_1 , \quad \text{so} \quad M_1 \geq 0$$

$$M_1 \geq \pm M_2 , \quad \text{so} \quad M_2 \geq |M_2|$$

$$M_2 \geq -M_2 , \quad \text{so} \quad M_2 \geq 0 ,$$

and therefore

$$M_1 \geq 0 , \qquad M_2 \geq 0 , \qquad M_1 \geq M_2 .$$

Hence in (3.4)

$$\sigma_1 \geq 0 , \quad \sigma_2 \geq 0 , \quad \sigma_1 \geq \sigma_2 ,$$

and

$$\sigma_1, \sigma_2, \tfrac{1}{2}(\sigma_1 - \sigma_2) = 0, 1, 2, \ldots .$$

We therefore write

$$\sigma_1 - \sigma_2 = 2\mu , \quad \sigma_2 = \lambda \qquad (\lambda, \mu = 0,1,2 \ldots)$$

and we have from (3.1) and (3.4) that

$$\underset{\sim}{M} = \frac{1}{2\sqrt{6}} \{ (\lambda + 2\mu)\underset{\sim}{e}_1 + \lambda \underset{\sim}{e}_2 \} .$$

Thus the dominant weight has the general form

$$\lambda\left(\frac{1}{2\sqrt{6}}\ ,\ \frac{1}{2\sqrt{6}}\right) + \mu\left(\frac{1}{\sqrt{6}}\ ,\ 0\right) .$$

The fundamental dominant weights for B_2 are

$$\left(\frac{1}{2\sqrt{6}}\ ,\ \frac{1}{2\sqrt{6}}\right) , \qquad \left(\frac{1}{\sqrt{6}}\ ,\ 0\right) .$$

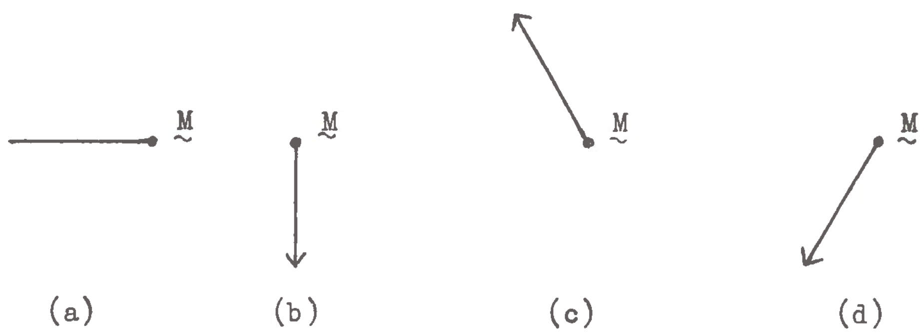
For each zero or positive integral value of λ and of μ we have a $D(\lambda,\mu)$ irreducible representation with highest weight

$$\left(\frac{\lambda+2\mu}{2\sqrt{6}}\ ,\ \frac{\lambda}{2\sqrt{6}} \right) . \qquad (3.5)$$

3b. Boundaries of B_2 Weight Diagrams.

On referring to Fig. 1.2 we see that the perpendiculars through the origin to the root vectors are themselves root vectors, so the weight diagrams are symmetrically placed with respect to the m_1 , m_2 axes and with respect to the lines $m_2 = \pm m_1$. Since a reflection in $\underset{\sim}{r}(1)$ followed by a reflection in $\underset{\sim}{r}(2)$ is equivalent to a rotation about the origin through an angle $\frac{\pi}{2}$, the weight diagrams are invariant under such a rotation. The multiplicities of all the weights in a diagram are determined, when we know those in the sector of the first quadrant between the m_1 axis and the radius vector making an angle $\frac{\pi}{4}$ with it.

We examine the possible shapes of the boundaries of the weight diagrams excluding the trivial case of the $D^{(1)}(0,0)$ representation (McConnell 1966). If $\underset{\sim}{M}$ is the highest weight of an irreducible representation with weight vector v, the lines to neighbouring weights can be only in the directions shown here

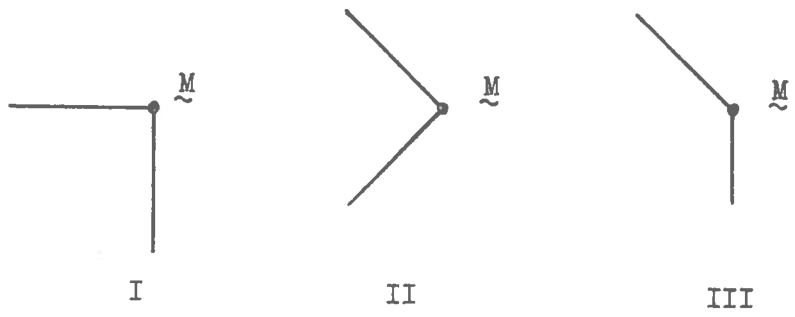


By taking these in pairs we can obtain six possible shapes of the boundary near $\underset{\sim}{M}$. However not all of these are allowed by the commutation relations. If, for example, we have a combination of (a) and (c), then

$$E_3 v = 0 , \qquad E_{-2} v = 0$$

$$N_{3,-2} E_{-1} v = [E_3 , E_{-2}] v = 0 ,$$

which is contrary to the supposition that there is a weight to the left of $\underset{\sim}{M}$ as in (a). We may likewise exclude the combinations (a) and (d), and (b) and (d), so that we are left with three allowed possibilities



Since the diagrams are symmetric with respect to the axes and invariant for the rotation through $\frac{\pi}{2}$, the highest weight $\underset{\sim}{M}$ for case I must have $M_1 = M_2$ and (3.5) shows that it belongs to a $D(\lambda,0)$ representation. The boundary is a square with sides parallel to the axes. In case II the highest weight must lie on the m_1 axis, so by (3.5) the value of λ is zero, the representation is $D(0,\mu)$ and the boundary is a square with vertices on the axes. The boundary for case III is octagonal, the alternate sides being equal in length.

We letter the highest weight A in the $D(\lambda,\mu)$ diagram of

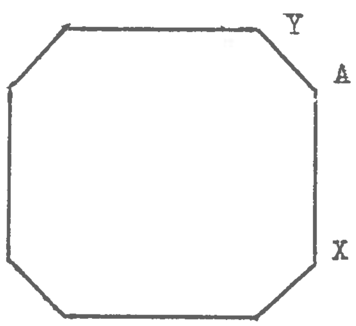


Fig.3.1 - The boundary of a $D(\lambda,\mu)$ diagram of B_2 .

Fig.3.1. Let X be the reflection of A in the m_1 axis and Y the reflection of A in the line $m_2 = m_1$. The coordinates of these are

$$A\left(\frac{\lambda+2\mu}{2\sqrt{6}}, \frac{\lambda}{2\sqrt{6}}\right), \qquad X\left(\frac{\lambda+2\mu}{2\sqrt{6}}, -\frac{\lambda}{2\sqrt{6}}\right), \qquad Y\left(\frac{\lambda}{2\sqrt{6}}, \frac{\lambda+2\mu}{2\sqrt{6}}\right).$$

The root vector in the direction XA is $\underset{\sim}{r}(3)$ and its length is $\frac{1}{\sqrt{6}}$. There are λ units of this length in XA. The root vector in the direction AY is $\underset{\sim}{r}(4)$, its length is $\frac{1}{\sqrt{3}}$ and the length of its projection along the m_2 axis is $\frac{1}{\sqrt{6}}$. Hence there are μ units of length $\frac{1}{\sqrt{3}}$ along AY. The boundary for the $D(\lambda,\mu)$ representation has λ units in the sides that are parallel to the axes and μ units in the slant sides. This will include the cases of $D(\lambda,0)$ and $D(0,\mu)$. Since the units of length are unequal, we cannot transfer results from the $D(\lambda,0)$ to the $D(0,\mu)$ diagrams, as we could for the A_2 algebra.

3c. Multiplicities on the Boundaries of B_2 Diagrams.

We shall first look at the multiplicities of weights on the octagonal boundaries. We consider the weights along ABC ... in Fig.3.2.

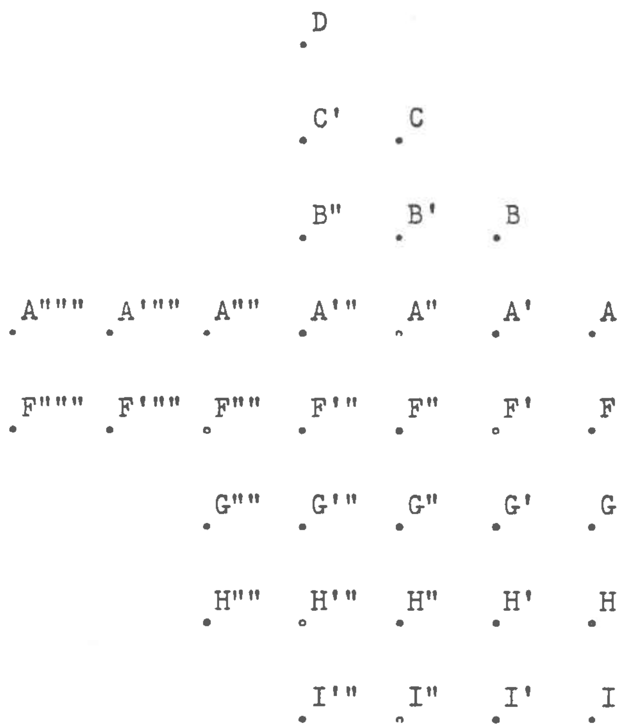
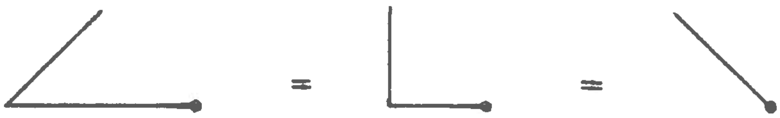
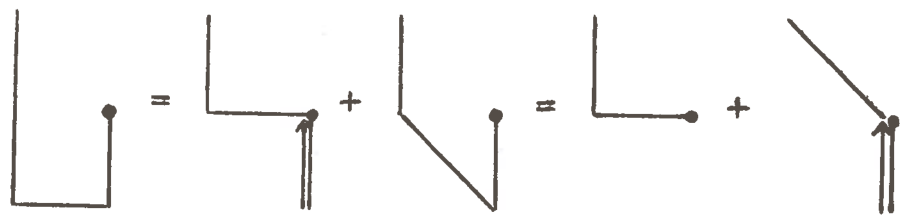


Fig.3.2 - Weights in a $D(\lambda,\mu)$ diagram of B_2 .

$\underset{\sim}{B}$ Starting from A we shall show that the graphs to B are equivalent to AB . Thus

on referring to the root vector diagram of Fig.1.2, but the last graph vanishes because A is the highest weight. Then

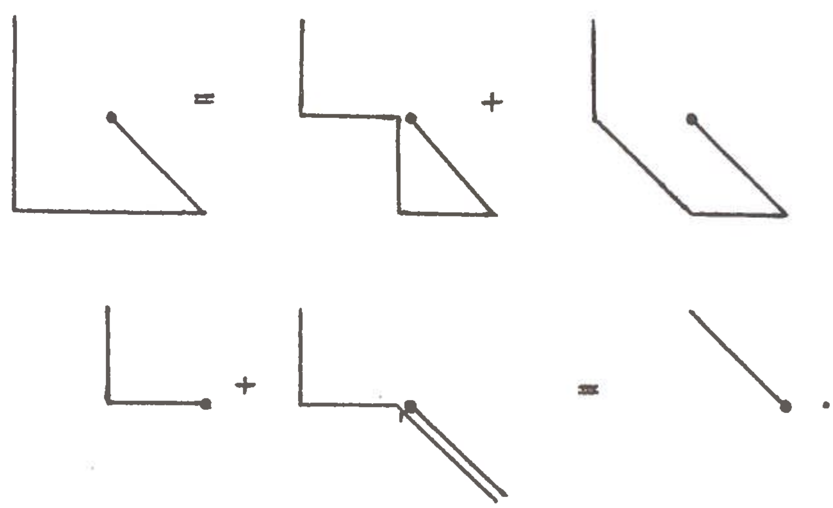




 , etc.,

so we may conclude that B is a simple weight.

C We start from B , which we have shown to be simple, and show that graphs from B to C are equivalent to BC . The only new type of graph that occurs is that which commences from B to A , e.g.



We conclude that C is simple and that similarly D ... are simple. The same argument holds for F, G, ... and we conclude that the weights on the octagonal boundaries are simple. By omitting the slant side and rotating the vertical side we deduce that we have simple weights on the square boundaries of the $D(\lambda,0)$ diagrams, and by omitting the vertical side we see that we have simple weights on the square boundaries of the $D(0,\mu)$ diagrams.

3d. Multiplicities in the $D(\lambda,0)$ Diagrams.

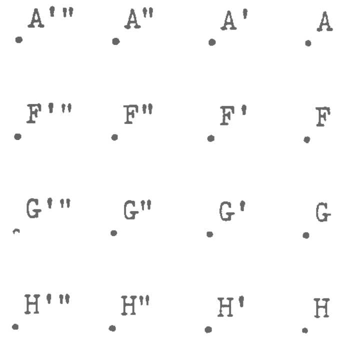


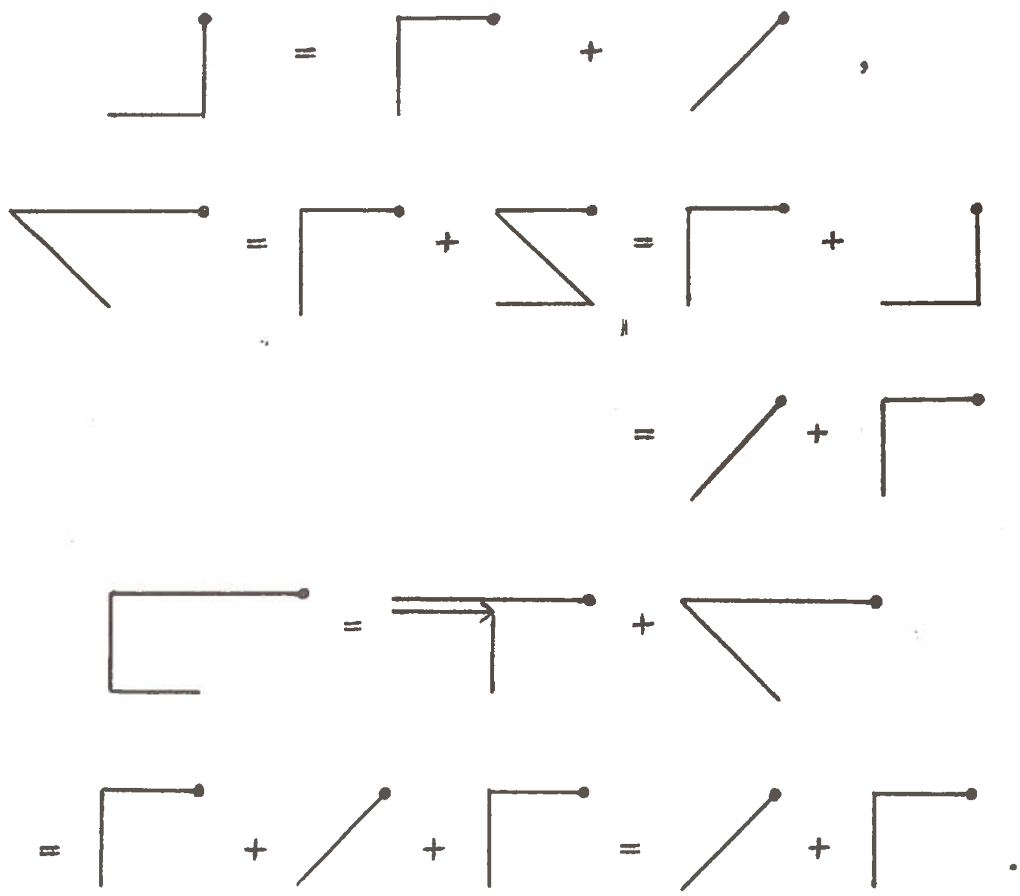
Fig.3.3 - Weights in a $D(\lambda,0)$ diagram of B_2 .

We mark weights near the highest weight in Fig.3.3 and examine multiplicities in the layer just inside the boundary.

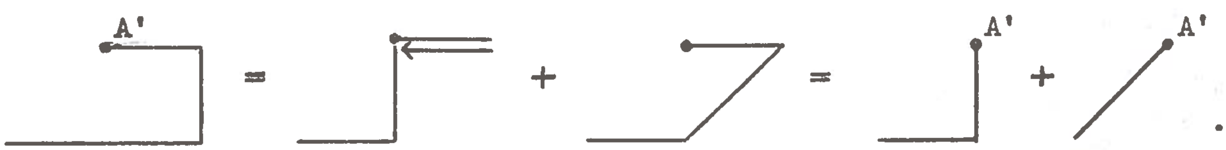
F' There are two linearly independent graphs from A , viz.

(3.6)

All other graphs from A are linear combinations of these, e.g.



F" By shifting one unit towards the left we can use the previous diagrams to go from A' to F" except in the case of paths like A'AFF'F" and in this case



We conclude that F" is a double weight.

We may continue this procedure until we reach the end of this layer. Since all the weights have the same multiplicity, we may say that the linearly independent graphs from A to F", F'",... may be obtained by adding to (3.6) the paths F'F" , F'F'" , ...

There is an alternative way of looking at this. To find the multiplicity of F" we could start from F and go to F" by the independent graphs FF'F" and FAF'F" . This alternative way has the advantage that it can be extended to the calculation of the multiplicities of G", H'", etc. . To deal with G" we start from G and obtain three linearly independent graphs GG'G", GFG'G", GFAF'G" . These may be classified as having 2,1,0 horizontal lines. Having established that the multiplicity of G" is 3, so that there can be three independent graphs from the highest weight A to G" , we can show that the multiplicity of G'" is also 3 by starting from the simple weight A' , and so on for other weights in the layer. Similarly we can classify the linearly independent graphs from H to H'" as those having 3,2,1,0 horizontal lines, viz. HH'H"H'", HGH'H"H'", HGFG'H"H'", HGFAF'G"H'" . We conclude that the multiplicity increases by one as we go from any layer to the next one.

We can now calculate the dimension $N(\lambda,0)$ of the $D(\lambda,0)$ representation of the Lie algebra B_2 . We see from (3.5) that the highest weight is $\left(\frac{\lambda}{2\sqrt{6}}, \frac{\lambda}{2\sqrt{6}}\right)$. As we move in along the diagonal from the highest A to F', each weight is diminished by $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$. This means that the origin will be a weight for λ even but not for λ odd; in this case there is an innermost layer of four weights. To calculate $N(\lambda,0)$ we must treat separately λ odd and λ even.

For λ odd we have on proceeding outwards from the centre of the diagram a succession of layers, the number of their weights being

$$4, \quad 12, \quad 20, \quad \ldots \quad 4(\lambda-2), \quad 4\lambda$$

with the respective multiplicities

$$\frac{\lambda+1}{2}, \quad \frac{\lambda-1}{2}, \quad \ldots \quad 2, \quad 1 .$$

The total number of weights is thus

$$4\{1.\lambda + 2(\lambda-2) + 3(\lambda-4) + \ldots + \tfrac{\lambda+1}{2}.1\}$$

$$= 4\sum_{r=1}^{\frac{\lambda+1}{2}} r(\lambda-2r+2) = 4(\lambda+2)\sum_{r=1}^{\frac{\lambda+1}{2}} r - 8\sum_{r=1}^{\frac{\lambda+1}{2}} r^2$$

$$= \frac{1}{6}(\lambda+1)(\lambda+2)(\lambda+3) .$$

For λ even the number of weights in the layers as we move from the centre are

$$1, \quad 8, \quad 16, \quad \ldots \quad 4(\lambda-2), \quad 4\lambda$$

with the respective multiplicities

$$\frac{\lambda}{2} + 1 \ , \quad \frac{\lambda}{2} , \quad \frac{\lambda}{2} - 1 \ , \quad \ldots \quad 2 \ , \quad 1 \ .$$

The total number of weights is therefore

$$4\lambda \cdot 1 + 4(\lambda-2)2 + \ldots \quad + 4(\lambda-2r)(r+1) + \ldots + 4.2.\frac{\lambda}{2} \quad + 1(\frac{\lambda}{2} + 1)$$

$$= \frac{\lambda}{2} + 1 + 4 \sum_{r=0}^{\frac{\lambda}{2}-1} \{ \lambda + (\lambda-2)r - 2r^2 \}$$

$$= \frac{1}{6} (\lambda+1)(\lambda+2)(\lambda+3)$$

as before. This is the total number of linearly independent vectors in $D(\lambda,\mu)$, and so

$$N(\lambda,0) = \frac{1}{6} (\lambda+1)(\lambda+2)(\lambda+3) \ . \qquad (3.7)$$

3e. Multiplicities in the $D(0,\mu)$ Diagrams.

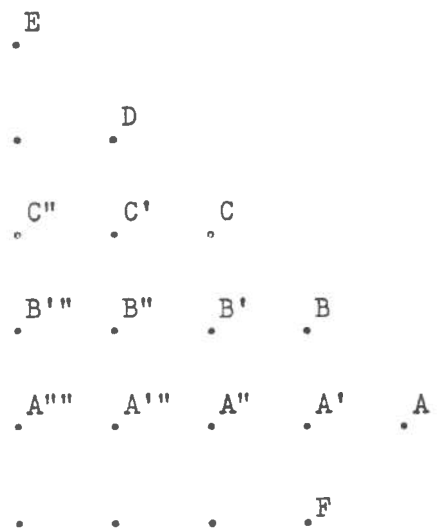


Fig.3.4 - Weights in a $D(0,\mu)$ diagram of B_2 .

We calculate multiplicities in the first quadrant for weights shown in Fig.3.4.

$\underset{\sim}{A}'$ We start from A and see that in fact there is essentially only one graph

(3.8)

Indeed, since there is no weight vertically below A ,

(3.9)

It is easily verified that all other graphs reduce to (3.8). If we start from B, C, ... , we shall find immediately that B', C', ... are simple, so that the first layer consists of simple weights.

$\underset{\sim}{A}''$ We have two linearly independent graphs from A , viz.

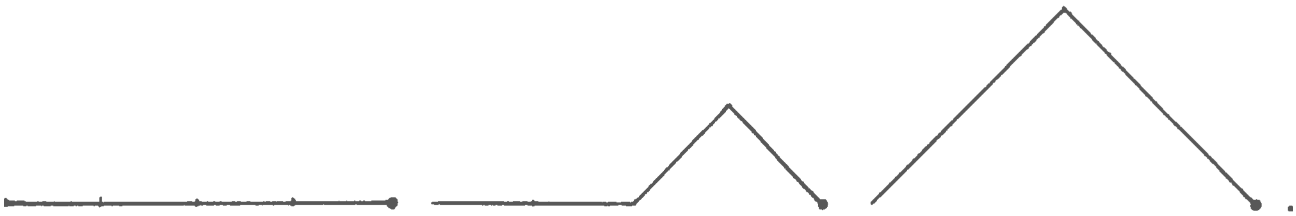
(3.10)

We note that these contain 2 or 0 horizontal lines from A ; the case of one horizontal line is by (3.9) equivalent to the first graph of (3.10). Likewise B", C", ... are all double weights.

A'" The only linearly independent graphs from A are

and these are obtained by adding A"A'" to (3.10). There are only 3 or 1 horizontal lines from A . The multiplicity along the layer A'"B'"C'" ... remains at 2 .

A"" There are three linearly independent graphs from A , viz.



The multiplicity along the layer is 3. The increase by one comes from the possibility of arriving diagonally at A"" from the weight C on the boundary. The next increase arises when we go horizontally two further weights from A .

The law of multiplicity is now clear. The boundary and first layer consists of simple weights, the next two layers consist of double weights, the next two of triple weights, and so on. According to (3.5) the highest weight is $\left(\frac{\mu}{\sqrt{6}}, 0\right)$ and, as we proceed horizontally in units of $\frac{1}{\sqrt{6}}$, we shall have a weight at the origin.

Let us calculate the dimension $N(0,\mu)$ of the $D(0,\mu)$

representation of B_2. The number of units of length in the side is 4μ, so the number of weights on the boundary is 4μ. Working in from the boundary we have for successive layers the number of points where weights occur and their respective multiplicities as follows:

$$\begin{matrix} 4\mu\,, & 4(\mu-1)\,, & 4(\mu-2)\,, & 4(\mu-3)\,, & \ldots & 4.2\,, & 4.1\,, & 1 \\ 1\,, & 1\,, & 2\,, & 2\,, & \ldots & . & & \end{matrix} \qquad (3.11)$$

We now distinguish between μ odd and μ even. For μ odd, (3.11) is

$$\begin{matrix} 4\mu\,, & 4(\mu-1)\,, & 4(\mu-2)\,, & \ldots & 4.2\,, & 4.1\,, & 1 \\ 1\,, & 1\,, & 2\,, & \ldots & 1+\frac{\mu-3}{2}\,, & 1+\frac{\mu-1}{2}\,, & 1+\frac{\mu-1}{2} \end{matrix}$$

and the total number of weights is

$$4(2\mu-1) + 4(2\mu-5)2 + \ldots + 4.5.\frac{\mu-1}{2} + 4.1.\frac{\mu+1}{2} + \frac{\mu+1}{2}$$

$$= \frac{5(\mu+1)}{2} + 4\sum_{s=0}^{\frac{\mu-3}{2}} (2\mu - 1 - 4s)(s+1)$$

$$= \frac{1}{6}(\mu+1)(\mu+2)(2\mu+3)\ . \qquad (3.12)$$

For μ even, (3.11) is

$$\begin{matrix} 4\mu\,, & 4(\mu-1)\,, & 4(\mu-2)\,, & 4(\mu-3)\,, & \ldots & 4.2\,, & 4.1\,, & 1 \\ 1\,, & 1\,, & 2\,, & 2\,, & \ldots & \frac{\mu}{2}\,, & \frac{\mu}{2}\,, & \frac{\mu}{2}+1 \end{matrix}$$

and the total number of weights is

$$4\,\{(2\mu-1)1 + (2\mu-5)2 + \ldots + 3.\frac{\mu}{2}\} + (\frac{\mu}{2} + 1)\ .$$

This has the value (3.12), so

$$N(0,\mu) = \frac{1}{6}(\mu+1)(\mu+2)(2\mu+3) . \qquad (3.13)$$

3f. Multiplicities in the $D(\lambda,\mu)$ Diagrams.

We saw in section 3c that the weights on the boundary are simple, and we now investigate multiplicities of weights within an octagonal boundary. We shall first calculate the multiplicities of A',A",A'",A"" in Fig.3.2 for the cases of different numbers of weights along the bounding sides through the highest weight. From these we shall later deduce the multiplicities of B',B", ... ; C',C", ... ; etc. . The multiplicities of many of the remaining weights can be found by a Weyl reflection through the line $m_2 = m_1$.

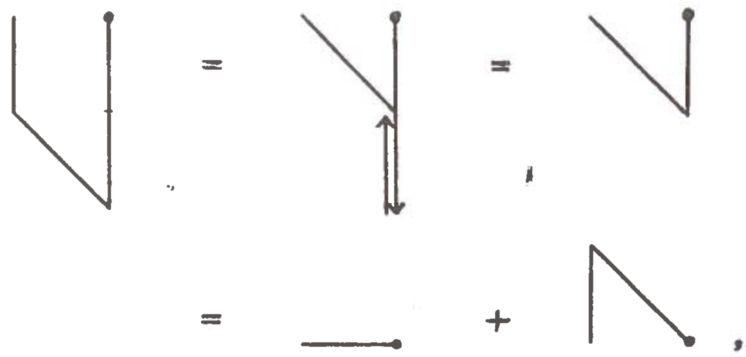
A' (i) B only present

This condition means that C, D, ... do not exist.

Starting from the simple weight A we certainly have 2 independent graphs

 (3.14)

It is immediately verified that all other graphs are linear combinations of these; for example, the path AFGF'A' has its graph

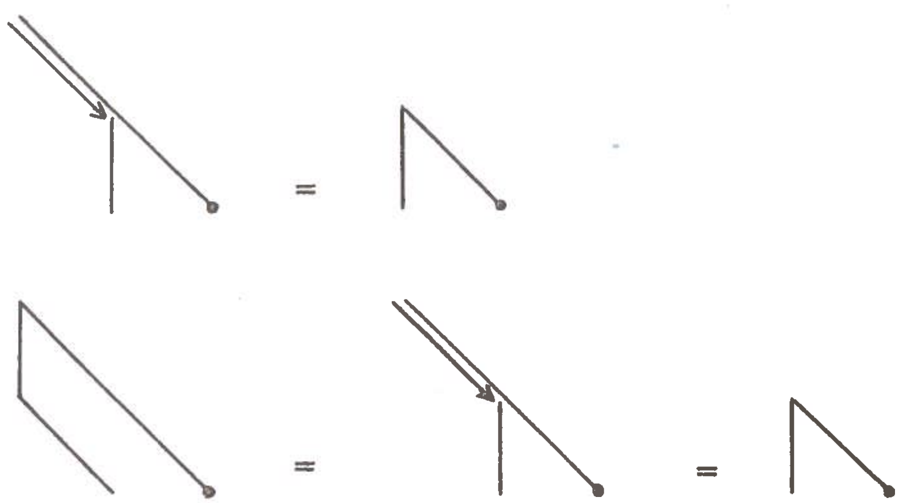


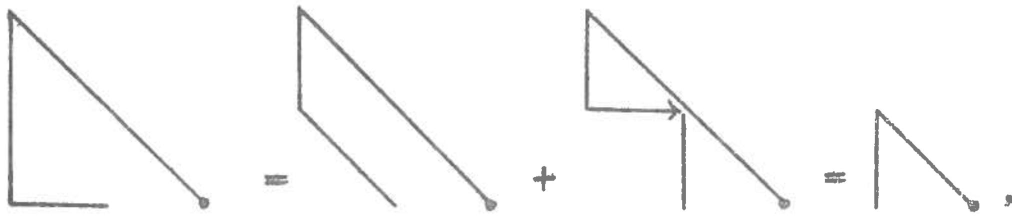
where we have used the information that A and F are simple. The multiplicity of A' is therefore 2 .

(ii) C present

This condition means that C is present and that D, E, ... may also be present.

It will be proved by reducing the new graphs to (3.14) that there is no increase in multiplicity. For only C present,





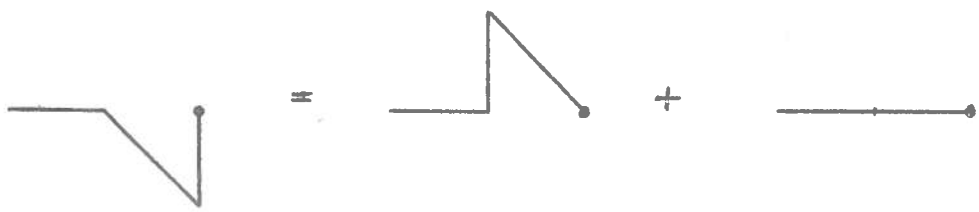
and the same method will be applicable when D, E, ... exist. We conclude that the multiplicity of A' is 2 in all cases.

A" (i) B only present

We can add the path A'A" to (3.14) and we can also travel by the path ABA" , so that we have altogether

(3.15)

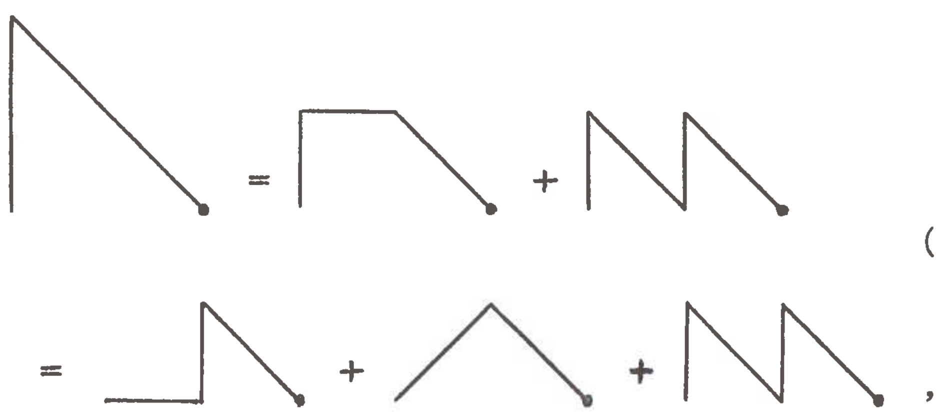
It is clear that these are independent and we shall verify that other graphs are combinations of them. Thus, for example,



and the multiplicity is 3 .

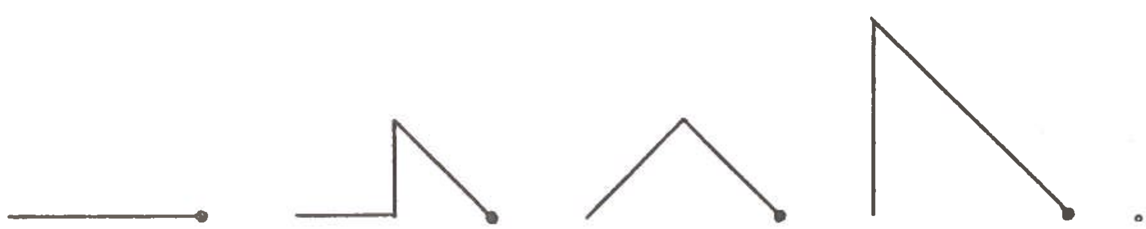
(ii) <u>C present</u>

In addition to (3.15) we have the transition ABCB'A" . Now



(3.16)

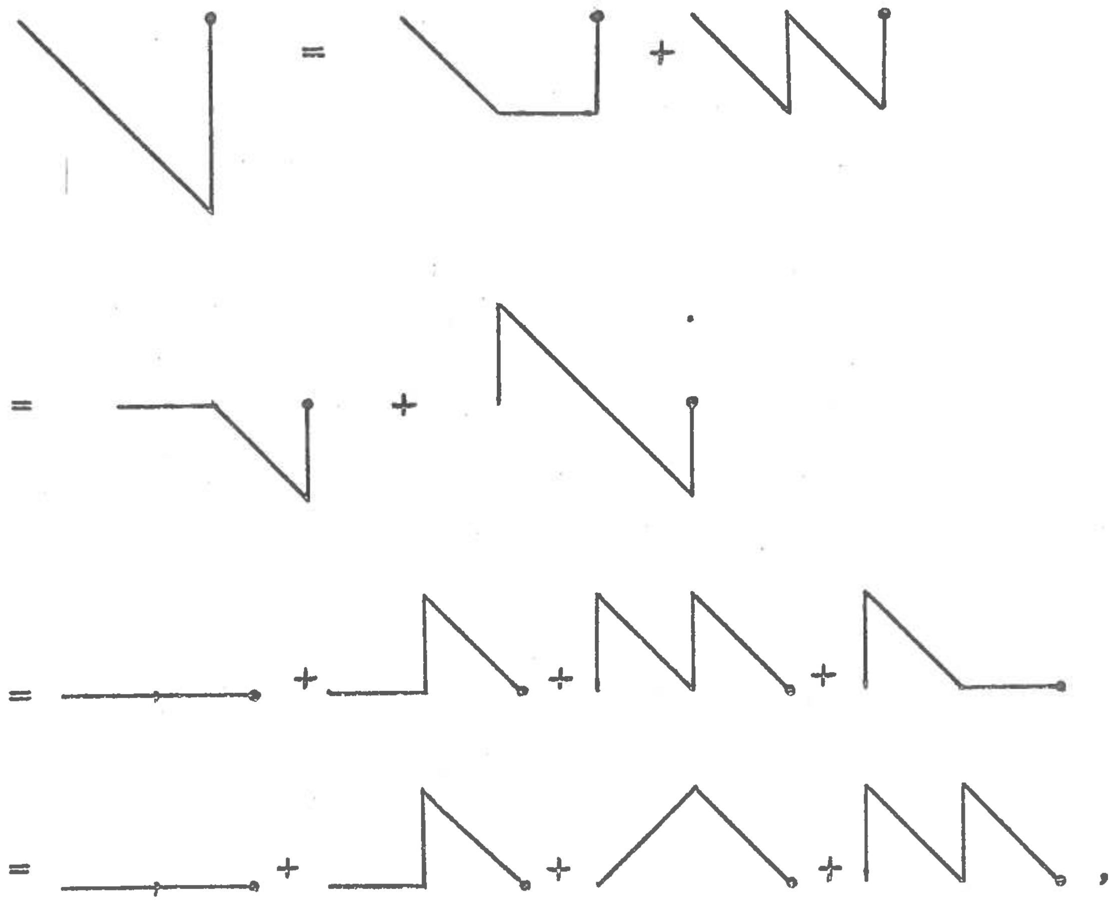
so that the new graph is not included in (3.15). Confining our attention for the moment to graphs that do not go left of C we check that they are linear combinations of



(3.17)

For example,

and, if G exists,



which by (3.16) is a linear combination of (3.17). We therefore have four independent graphs (3.17), if and only if both C and G are present. If G is not present, the linear combination vanishes and there are only three independent graphs in (3.17).

We note that, even when C is absent,

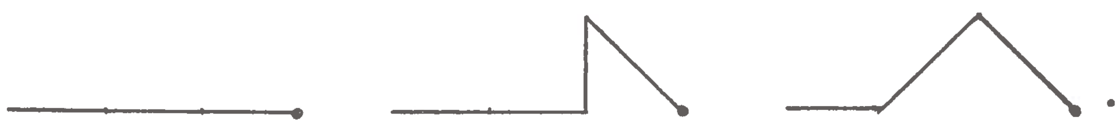
exists but is a linear combination of (3.17).

When we admit graphs that go to the left of C , including those that go to D and E , we may reason as we did for A' that there is no increase in multiplicity. We conclude that, when G is present, the multiplicity of A" is 4 and that it is 3 otherwise.

A'" The graph theory follows the same pattern, so we shall merely state the results and these can be verified without difficulty.

(i) B only present

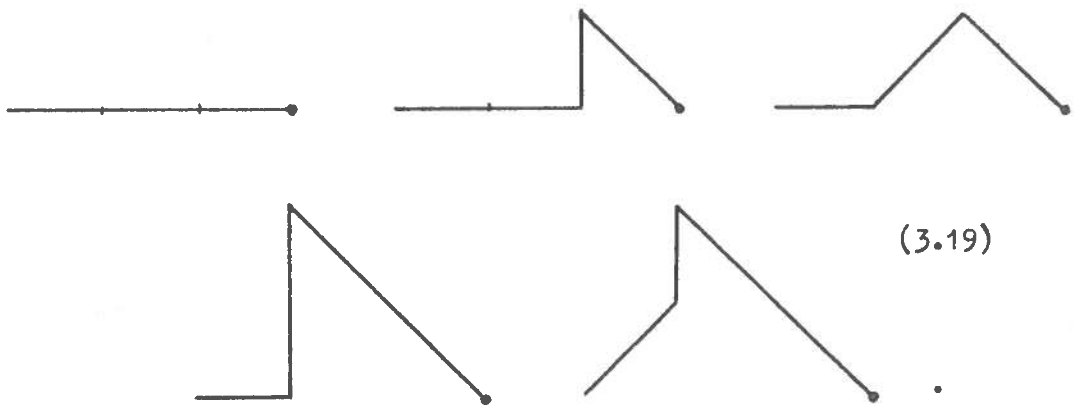
The multiplicity is 3 and the three independent graphs are those obtained by extending (3.15) to the point A'" , viz.



(3.18)

(ii) C only present

The graphs are now



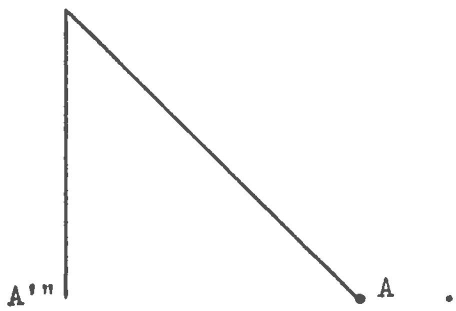
(3.19)

The last graph is linearly independent of the first four and a linear relation exists between these four, if and only if G does not exist. Hence the multiplicity is 5, if G exists, and it is

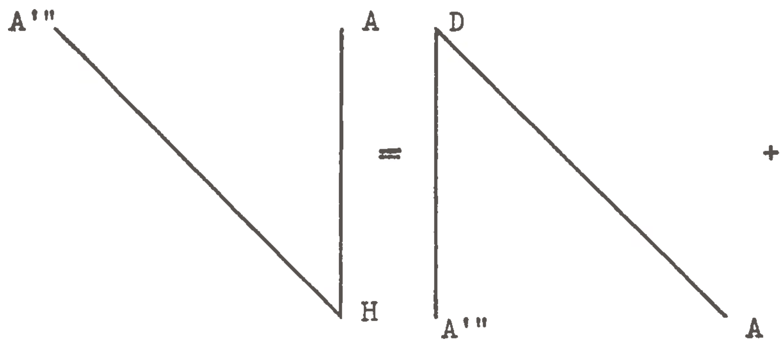
G does not exist.

(iii) D present

There is now an additional graph



It may easily be proved that



+ linear combinations of (3.19) .

The presence of D will give an extra multiplicity 1 , if and only if H exists. The multiplicities of A'" when D is present are as follows:

multiplicity 4, if G is not present,

multiplicity 5, if G is present but H is not,

multiplicity 6, if H is present.

A"" (i) B only present

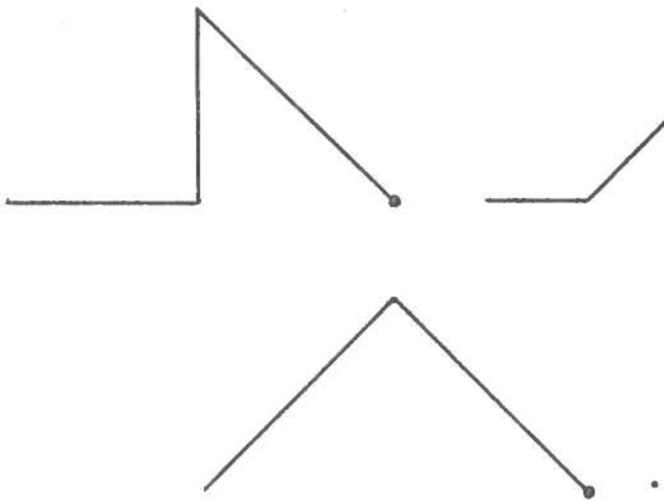
The multiplicity is 3, the three independent graphs being

 .

(3.20)

(ii) C only present

There are six graphs, namely (3.20) and

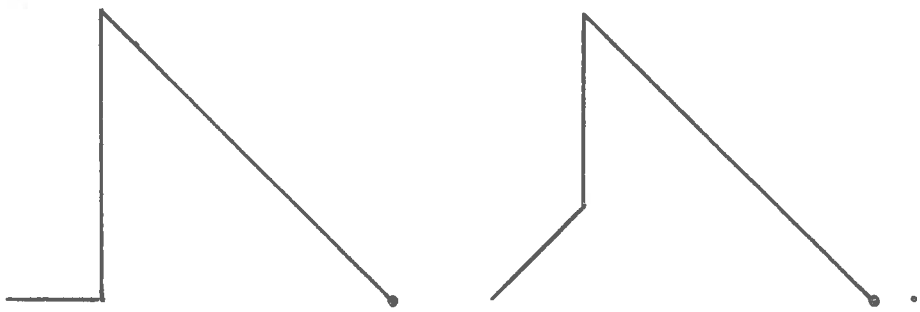


(3.21)

There is one relation between them, when G does not exist. Then the multiplicity is 5; otherwise it is 6 .

(iii) D only present

We can have at most eight independent graphs, viz. (3.20), (3.21) and



If H is not present, the first of these two graphs does not give an extra multiplicity. If moreover G does not exist, the last graph does not give an extra multiplicity. Hence we have

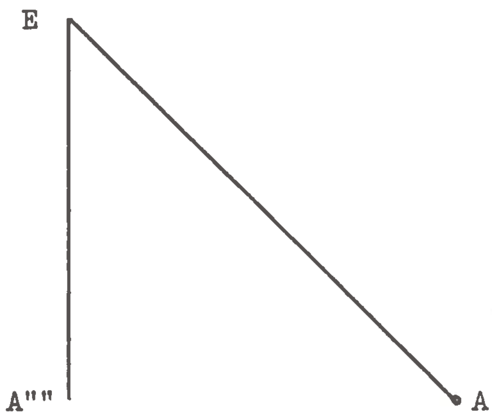
multiplicity 5, if G is not present,

multiplicity 7, if G is present but H is not,

multiplicity 8, if H is present.

(iv) E present

There is a new graph



which gives an extra multiplicity, if and only if I exists. We therefore have

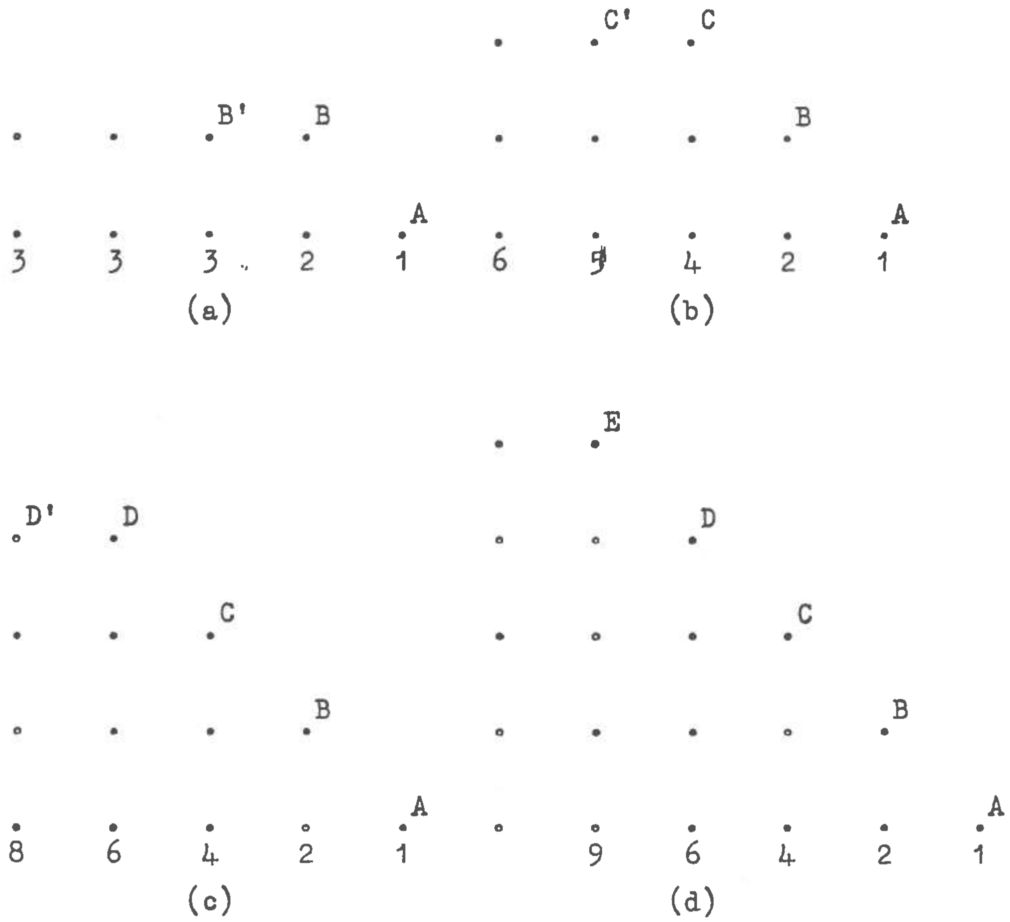
multiplicity 5, if G is not present,

multiplicity 7, if G is present but H is not,

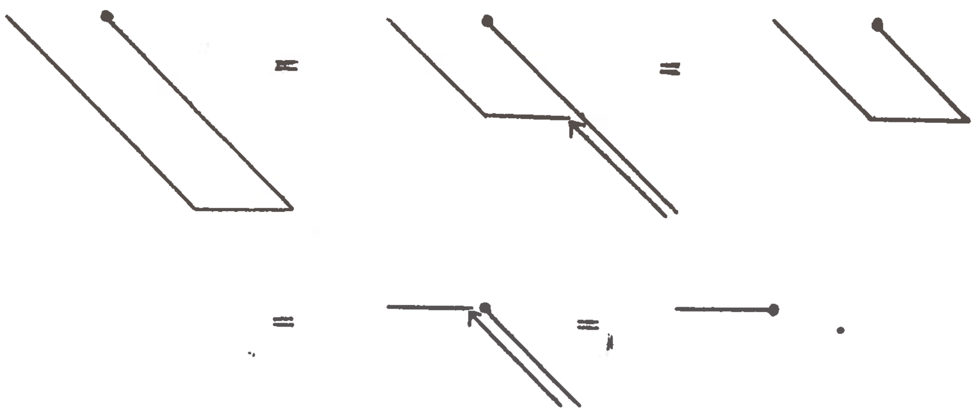
multiplicity 8, if H is present but I is not,

multiplicity 9, if I is present.

We collect the results for the multiplicities of A, A', A", A'", A"" and put them down for different numbers of weights on the slant side of Fig.3.2, assuming also that there are sufficient weights on the vertical side to give the maximum multiplicities.



We enquire whether these results can be taken over for the multiplicities in horizontal strings through B, C, ... The only difference is that now in going, for example, from C to C' we can start in the slant direction CA , which was not possible for the path from A to A' . It is easy to check that nothing new comes from this. Thus, if we start from C ,



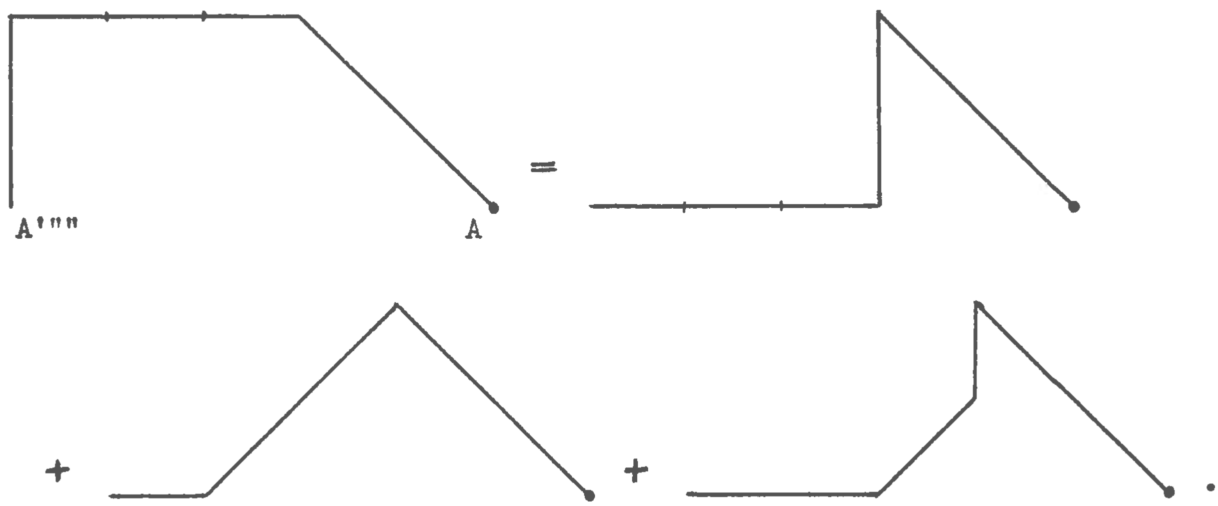
The method employed for (a) above can therefore be applied to the weights on the horizontal string next to the boundary for any other case, and it will give the same multiplicities. Likewise the weights on the next horizontal string are as in (b) above. If we take a weight diagram where the slant side goes as far as E and the vertical side goes at least as far as I , the weight multiplicities for points to the right of the vertical line of symmetry will be as shown in Fig.3.5. If the last weight on the slant side were D , we would omit the bottom row of

			E				
1	1	1	1				
3	3	3	2	1			
	6	5	4	2	1		
		8	6	4	2	1	
			9	6	4	2	1 A

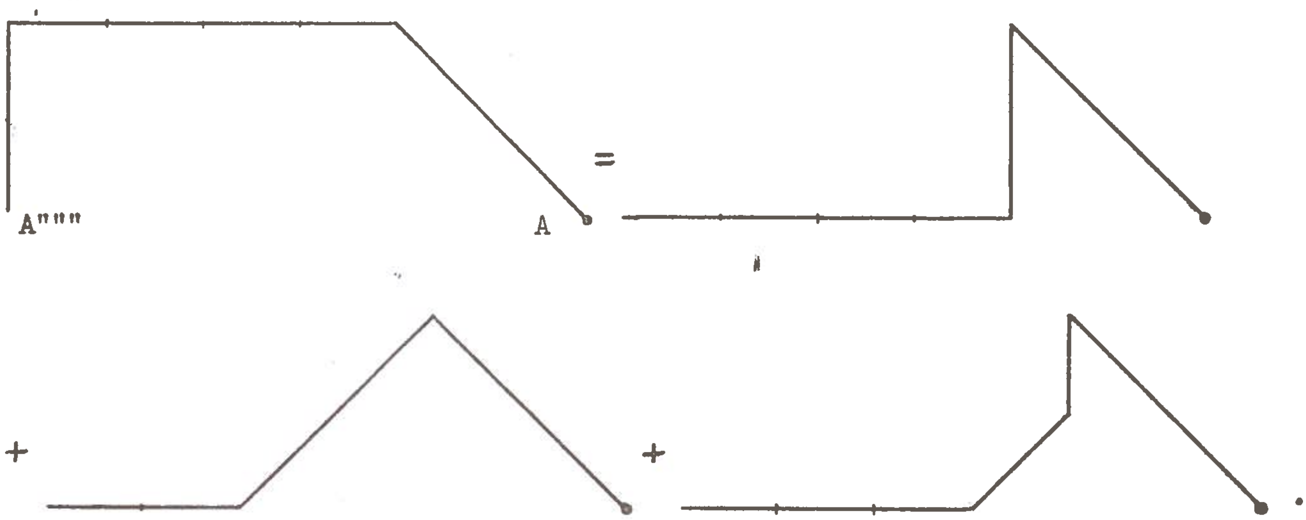
Fig.3.5 - Multiplicities in a weight diagram that has five weights on the slant side and at least five weights on the vertical side.

Fig.3.5. We have the useful results that the multiplicities on the first layer parallel to the horizontal boundary are 1,2, 3,3, ... , that those on the next layer are 1,2,4,5,6, ... , and that the multiplicity is 2 for the layer next to the slant side.

A line through E at right angles to the slant side passes through D",C"",B""",A"""" . According to Fig.3.5 there is no increase of multiplicity to the left of this line for the horizontal strings through E and D . This constancy of multiplicity as one approaches the m_2 axis is in fact a general feature of the weight diagrams. To illustrate this let us go back to the case (b), where C is the last weight on the slant side. The line through C perpendicular to the slant side will pass through A"", which we have seen to be of multiplicity 6. If now we take a graph from A to A'"", it will be reducible to (3.20) and (3.21) extended from A"" to A'"" ; for example



The multiplicity therefore does not increase. Similarly, if we take an example of a graph ending on A"""" ,



We just have (3.20) and (3.21) extended from A""" to A"""" , so again there is no increase in multiplicity.

Many of the multiplicities of weights on horizontal strings through F,F,H etc. may be obtained by a reflection in the line through the origin that makes an angle $\pi/4$ with the positive m_1 m_2 axes. Thus, if as in Fig.3.5 the slant side ends at E , the multiplicity of F' being that of D" is 3, the multiplicity of F" being that of C'" is 5, the multiplicity of F'" is 8. The multiplicities of the remaining weights can be determined directly by the graphical method.

Since the multiplicities in a $D(\lambda,\mu)$ diagram depend so much on the number of weights on the bounding sides, it is not easy to state a simple comprehensive rule for the multiplicity of an individual weight. The graphical method will therefore not provide the multiplicity $N(\lambda,\mu)$ of the representation. However for future reference we note that

$$N(\lambda,\mu) = \frac{1}{6}(1+\lambda)(1+\mu)(\lambda+\mu+2)(\lambda+2\mu+3) . \qquad (3.22)$$

CHAPTER IV

G_2 WEIGHT DIAGRAMS

4a. Irreducible Representations of G_2.

To find the irreducible representations we calculate the dominant weights. If any weight $\underset{\sim}{M}$ is given by

$$\underset{\sim}{M} = M_1 \underset{\sim}{e}_1 + M_2 \underset{\sim}{e}_2 \qquad (4.1)$$

and we take $\underset{\sim}{r}(\alpha)$ to be the $\underset{\sim}{r}(1), \underset{\sim}{r}(2), \ldots \underset{\sim}{r}(-6)$ of Fig.1.3, we find for

$$\frac{2(\underset{\sim}{M} \cdot \underset{\sim}{r}(\alpha))}{|\underset{\sim}{r}(\alpha)|^2}$$

the respective values

$$\begin{aligned} &4\sqrt{3}M_1,\ 2\sqrt{3}M_1 + 2M_2,\ 2\sqrt{3}M_1 + 6M_2,\ 4M_2 \\ &-2\sqrt{3}M_1 + 6M_2,\ -2\sqrt{3}M_1 + 2M_2\ , \end{aligned} \qquad (4.2)$$

and the negatives of these. We find for the Weyl reflections of $\underset{\sim}{M}$ the respective values

$$\begin{aligned} &-M_1\underset{\sim}{e}_1 + M_2\underset{\sim}{e}_2\ ,\quad (-\tfrac{1}{2}M_1 - \tfrac{\sqrt{3}}{2}M_2)\underset{\sim}{e}_1 + (-\tfrac{\sqrt{3}}{2}M_1 + \tfrac{1}{2}M_2)\underset{\sim}{e}_2 \\ &(-\tfrac{1}{2}M_1 - \tfrac{\sqrt{3}}{2}M_2)\underset{\sim}{e}_1 + (-\tfrac{\sqrt{3}}{2}M_1 - \tfrac{1}{2}M_2)\underset{\sim}{e}_2\ ,\quad M_1\underset{\sim}{e}_1 - M_2\underset{\sim}{e}_2 \\ &(\tfrac{1}{2}M_1 + \tfrac{\sqrt{3}}{2}M_2)\underset{\sim}{e}_1 + (\tfrac{\sqrt{3}}{2}M_1 - \tfrac{1}{2}M_2)\underset{\sim}{e}_2 \\ &(-\tfrac{1}{2}M_1 + \tfrac{\sqrt{3}}{2}M_2)\underset{\sim}{e}_1 + (\tfrac{\sqrt{3}}{2}M_1 + \tfrac{1}{2}M_2)\underset{\sim}{e}_2 \end{aligned} \qquad (4.3)$$

repeated once. From theorem 4 of section 1b and (4.2) we deduce that

$$4\sqrt{3}\, M_1 = \tau_1 , \quad 4M_2 = \tau_2 , \qquad (4.4)$$

where

$$\tau_1 , \tau_2 , \tfrac{1}{2}(\tau_1 - \tau_2) = 0, \pm 1, \pm 2, \ldots .$$

For $\underset{\sim}{M}$ to be a dominant weight we have from (4.1) and (4.3) that

$$M_1 \geq - M_1 , \qquad \text{so} \quad M_1 \geq 0 , \quad \tau_1 \geq 0$$

$$M_2 \geq - M_2 , \qquad \text{so} \quad M_2 \geq 0 , \quad \tau_2 \geq 0$$

$$M_1 \geq \sqrt{3}\, M_2 , \qquad \text{so} \quad \tau_1 \geq 3\tau_2$$

and thus

$$\tau_1 - 3\tau_2 = \tau_1 - \tau_2 - 2\tau_2 = 2\lambda . \qquad (\lambda = 0, 1, 2 \ldots)$$

We also write

$$\tau_2 = \mu \qquad (\mu = 0, 1, 2, \ldots)$$

and then by (4.1) and (4.4)

$$\underset{\sim}{M} = \frac{2\lambda + 3\mu}{4\sqrt{3}} \underset{\sim}{e}_1 + \frac{\mu}{4} \underset{\sim}{e}_2$$

$$= \lambda\left(\frac{1}{2\sqrt{3}} , 0\right) + \mu\left(\frac{\sqrt{3}}{4} , \frac{1}{4}\right) .$$

The fundamental dominant weights are therefore

$$\left(\frac{1}{2\sqrt{3}} , 0\right) , \quad \left(\frac{\sqrt{3}}{4} , \frac{1}{4}\right) .$$

For each zero or positive integral value of λ and of μ

we have an irreducible representation $D(\lambda,\mu)$ with highest weight

$$\left(\frac{2\lambda + 3\mu}{4\sqrt{3}}, \frac{\mu}{4}\right) . \qquad (4.5)$$

4b. G_2 Weight Diagrams near the Highest Weight.

The extremities of the root vectors in Fig.1.3 are points in a lattice formed by equilateral triangles. Since neighbouring weights in a diagram are separated from each other by root vectors, the weights also constitute such a lattice. The regular representation is $D^{(14)}(0,1)$, its diagram having weights at the ends of the root vectors and at the origin, which is a double weight. The boundary of the diagram may be viewed in two ways, viz. as a concave twelve-sided polygon determined by the extremities of the twelve root vectors or as the regular hexagon determined by the extremities of $\underset{\sim}{r}(2),\underset{\sim}{r}(4),\underset{\sim}{r}(6),\underset{\sim}{r}(-2),\underset{\sim}{r}(-4),\underset{\sim}{r}(-6)$.

We examine the shape of the weight diagrams for any irreducible representation in the neighbourhood of the highest weight. If $\underset{\sim}{M}$ is the highest weight, the following weights do not exist:

$$\underset{\sim}{M} + \underset{\sim}{r}(-5)\ , \qquad \underset{\sim}{M} + \underset{\sim}{r}(-6)\ , \qquad \underset{\sim}{M} + \underset{\sim}{r}(1)$$
$$\underset{\sim}{M} + \underset{\sim}{r}(2)\ , \qquad \underset{\sim}{M} + \underset{\sim}{r}(3)\ , \qquad \underset{\sim}{M} + \underset{\sim}{r}(4)\ . \qquad (4.6)$$

We draw in Fig.4.1 lines from $\underset{\sim}{M}$ in the directions of $\underset{\sim}{r}(1), \underset{\sim}{r}(2), \ldots \underset{\sim}{r}(-6)$, the lines being full for the

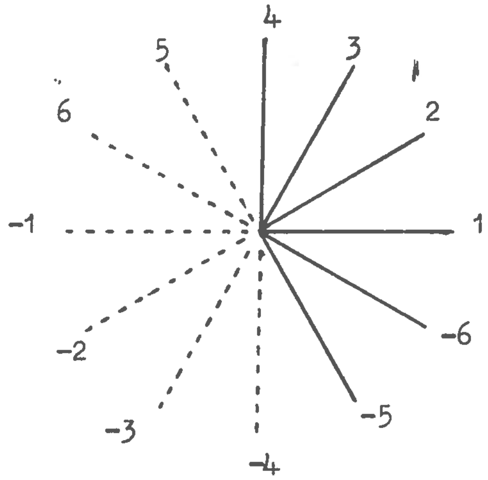


Fig.4.1 - Lines from the highest weight in the directions of the root vectors.

directions corresponding to (4.6) and hatched for the directions in which weights may exist. We indicate by α at the end of a line that it is in the direction of $\underset{\sim}{r}(\alpha)$. We then study separately the possibility of each hatched line being a boundary of the weight diagram.

i) $\underset{\sim}{r}(5)$ <u>a boundary line</u>

The highest weight is simple and, if v is the weight vector,

$$E_\alpha E_\beta v = E_\beta E_\alpha v + N_{\alpha\beta} E_{\alpha+\beta} v\ ,$$

where $N_{\alpha\beta}$ is non-zero, if and only if $\underset{\sim}{r}(\alpha) + \underset{\sim}{r}(\beta)$ is a root vector. When the line 5 is a boundary, $E_4 v = 0$ and hence

$$E_4 E_\beta v = N_{4\beta} E_{4+\beta} v . \qquad (4.7)$$

Fig.1.3 shows that, if we limit our attention to the hatched lines, $N_{4\beta} \neq 0$ for $\beta = -2$ and $\beta = -3$ only, and that in these cases the respective values of $\underset{\sim}{r}(4) + \underset{\sim}{r}(\beta)$ are $\underset{\sim}{r}(6)$ and $\underset{\sim}{r}(5)$. Now, since 5 is a boundary, $E_5 v$ does not vanish and putting $\beta = -3$ in (4.7) we see that $E_{-3}v$ does not vanish and therefore that the line -3 is not outside the diagram. We conclude that the other line through the highest weight that constitutes a boundary is either -3 or -4 .

When the lines through $\underset{\sim}{M}$ on the boundary are 5 and -3 ,

$$E_4 v = E_{-4} v = 0$$

and therefore

$$0 = [E_4, E_{-4}]v = r_1(4) H_1 v + r_2(4) H_2 v = \tfrac{1}{2} M_2 v ,$$

from Fig.1.3. The highest weight lies on the m_1 axis and putting M_2 equal to zero in (4.5) we find that μ is zero and that consequently we are in a $D(\lambda,0)$ representation. When the lines 5 and -4 pass through $\underset{\sim}{M}$, there is no special relation between M_1 and M_2 and the representation is a $D(\lambda\,\mu)$.

ii) $\underset{\sim}{r}(6)$ <u>a boundary line</u>

E_5v now vanishes. We obtain a hatched line by combining $\underset{\sim}{r}(5)$ with $\underset{\sim}{r}(-1)$, $\underset{\sim}{r}(-3)$ and $\underset{\sim}{r}(-4)$ only. Reasoning as above we deduce that the second line through $\underset{\sim}{M}$ is $\underset{\sim}{r}(-4)$ and that the highest weight belongs to a $D(0,\mu)$ representation.

iii) $\underset{\sim}{r}(-1)$ <u>a boundary line</u>

Under this assumption the lines 5 and 6 would be outside and, since -5 and -6 have to be outside anyway, it will be deduced that $\underset{\sim}{M} = 0$. The representation is $D(0,0)$.

iv) $\underset{\sim}{r}(-2)$ <u>a boundary line</u>

It is found that this possibility does not in fact exist.

v) $\underset{\sim}{r}(-3)$ <u>a boundary line</u>

This can occur only in conjunction with 5 as a boundary line, as we discussed above.

4c. Boundaries of G_2 Weight Diagrams

The weight diagrams are invariant for Weyl reflections in lines through the origin perpendicular to the root vectors.

For the G_2 algebra these are just the root vectors themselves. In particular the diagrams are invariant for reflections in the m_1, m_2 axes. Since a reflection in $\underset{\sim}{r}(1)$ followed by a reflection in $\underset{\sim}{r}(2)$ is equivalent to a rotation about the origin through an angle $\frac{\pi}{3}$, the diagrams are invariant under this rotation and it will suffice to examine properties of weight diagrams in the sector enclosed by the lines $\underset{\sim}{r}(1)$ and $\underset{\sim}{r}(3)$.

i) <u>$D(\lambda,0)$ boundary</u>

The highest weight is $(\frac{\lambda}{2\sqrt{3}}, 0)$ on the m_1 axis. The lines through it are $\underset{\sim}{r}(5)$ and $\underset{\sim}{r}(-3)$, which enclose an angle $\frac{2\pi}{3}$. On account of the invariance under the rotation through $\frac{\pi}{3}$, the boundary is a regular hexagon symmetrically placed with respect to the axes and having two horizontal sides.

ii) <u>$D(0,\mu)$ boundary</u>

The highest weight, marked A in Fig.4.2, is $(\frac{\sqrt{3}}{4}\mu, \frac{1}{4}\mu)$. The angle between AB and AF is $\frac{2\pi}{3}$, so an examination of AF will give a description of the whole boundary. There exists a weight at F $(\frac{\sqrt{3}}{4}\mu, -\frac{1}{4}\mu)$ and, since there are no gaps in a string of weights, we have weights all along AF spaced at a distance $\frac{1}{2}$, which is the length of $\underset{\sim}{r}(4)$. The reflection of F in OA will bring us to the point B on the m_2 axis and continuing by rotations through the angle $\frac{\pi}{3}$ we obtain the regular hexagon ABCDEF .

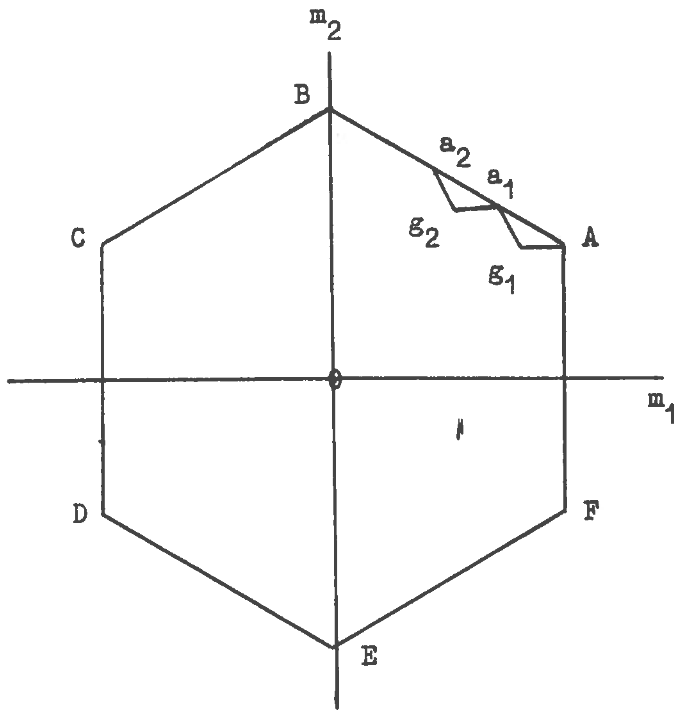


Fig.4.2 - The boundary of the $D(0,\mu)$ diagram of G_2 .

As was mentioned at the beginning of section 4b, there is another way of looking at the shape of the boundary. Since neighbouring weights in the diagram differ only by the root vectors, there will be for A, a_1, a_2, ... B weights g_1, g_2, ... at a distance $\underset{\sim}{r}(-1)$ and $\underset{\sim}{r}(-5)$ away. We could thus regard A g_1 a_1 g_2 a_2 ... as the boundary. This will be the boundary of a concave polygon of 12μ sides.

iii) <u>$D(\lambda , \mu)$ boundary</u>

The highest weight, marked Q in Fig.4.3 is according to (4.5)

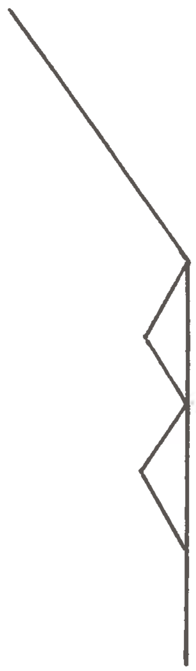


Fig.4.3 - The boundary of the $D(\lambda, \mu)$ diagram of G_2 near the highest weight.

$$\left(\frac{2\lambda + 3\mu}{4\sqrt{3}}, \frac{\mu}{4}\right).$$

There will be a weight at the point P with coordinates

$$\left(\frac{2\lambda + 3\mu}{4\sqrt{3}}, -\frac{\mu}{4}\right).$$

There are μ units of $\frac{1}{2}$ in PQ and each of these gives rise to an indentation as for the $D(0,\mu)$ representation.

The angle between PQ and QR is $\frac{\pi}{6}$, so the side following QR will be obtained by rotating PQ through the angle $\frac{\pi}{3}$. By rotating the point P we find that the coordinates of R are

$$\left(\frac{\lambda + 3\mu}{4\sqrt{3}}, \frac{\lambda + \mu}{4}\right).$$

The difference in the m_1 coordinates of Q and R is $\frac{\lambda}{4\sqrt{3}}$ and the projection of $\underset{\sim}{r}(5)$ on the m_1 axis is $\frac{1}{4\sqrt{3}}$, so there are λ units in QR. By rotating PQ and QR six times we obtain a dodecagon with alternate sides of lengths $\frac{\lambda}{2\sqrt{3}}$ and $\frac{\mu}{2}$. As in the case of the $D(0,\mu)$ representation, we may say that the side PQ and those obtained from it by rotations are indented. If we take the indentations as part of the boundary, we have a concave polygon with a total of $6 + 12\mu$ sides.

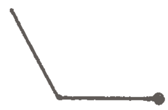
4d. Multiplicities on the Boundaries of G_2 Diagrams

The discussion of the multiplicities of weights on the hexagonal boundary of a $D(\lambda, 0)$ diagram is similar to that carried out in section 2c for the hexagonal boundary of a $D(\lambda, \mu)$ diagram of A_2. The weights on the boundary are therefore simple.

We next examine whether the weights on all boundaries are simple, provided that for the $D(0,\mu)$ and $D(\lambda, \mu)$ representations we take the boundary to be convex. To prove that they are simple it will be sufficient to establish this property for the weights a_1, a_2, ... of the side AB in Fig.4.2; the same method of proof will obtain for the side PQ of Fig. 4.3. The highest weight A is simple. If we proceed from

A to a_1 directly, we have the graph

and, if we go along $A\ g_1\ a_1$, we have



On referring to (1.15) and Fig.1.3 we see that

It may be verified that more roundabout paths may be reduced to $A\ a_1$, e.g.

and a_1 is therefore a simple weight. The same method may be applied to show that the multiplicities of a_2, a_3 etc. are one, and it is applicable also to the weights on sides that are not indented. The weights on the convex boundaries are therefore simple.

We now examine what are the multiplicities at points on a concave boundary where the internal angle is reflex. We return to Fig.4.2 for the $D(0,\mu)$ representation and considering the weight g_1 we see that we may travel to it by

(4.7)

Now

(4.8)

and the last graph is zero when we start from A . The multiplicity of g_1 is therefore one. If we start from a_1 in order to find the multiplicity of g_2 , the last graph of (4.8) no longer vanishes. Then the graphs (4.7) are independent, and the multiplicity of g_2 is 2. This is so for successive g's until the last one, whose weight by symmetry is simple. This method is applicable also to Fig.4.3 and shows that in fact the multiplicity is always 2 for the indentations of the $D(\lambda,\mu)$ boundary.

4e. Multiplicities inside the Boundaries of G_2 Diagrams .

Fig. 4.4 shows weights near the highest weight A of a $D(\lambda,\mu)$ diagram. If F,G,H, are deleted the representation will be $D(\lambda,0)$. We have just found that the multiplicity of both I and H is 2, so we turn our attention to J,K,L .

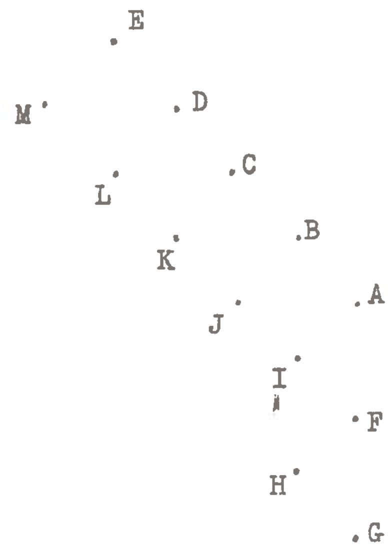


Fig.4.4 - Weights in a $D(\lambda, \mu)$ diagram of G_2 .

On account of the symmetries of weight diagrams it may not be necessary to calculate many multiplicities in a specific weight diagram.

$\underset{\sim}{J}$ Starting from A we have, if C is present, three graphs

The first two are linearly independent because I exists. If H is not in the diagram, there is a linear relation between the three graphs. Hence the multiplicity of J is 2 for $D(\lambda,0)$ and 3 for $D(\lambda, \mu)$.

$\underset{\sim}{K}$ There are four graphs

and for $D(\lambda,0)$ a linear relation exists between the first three. Thus the multiplicity of K is 3 for $D(\lambda,0)$ and 4 for $D(\lambda,\mu)$.

$\underset{\sim}{L}$ It is easily seen that the multiplicity of L is 3 for $D(\lambda,0)$ and 4 for $D(\lambda,\mu)$.

We have assumed throughout that we have not gone so far along the string as to encounter a reduction of multiplicity by a reflection symmetry. Multiplicities for other weights may be readily found by the graphical method. Even for the $D(\lambda,0)$ representation the multiplicities are not constant along the layer $J,K,L, \ldots$. It is therefore not easy to make general statements about weight multiplicities inside the boundary or to use the graphical method to calculate the dimension $N(\lambda,\mu)$ of the $D(\lambda,\mu)$ representation of G_2. For completeness we note that

$$N(\lambda,\mu) = (1+\lambda)(1+\mu)[1+\tfrac{1}{2}(\lambda+\mu)][1+\tfrac{1}{3}(\lambda+2\mu)] \qquad (4.9)$$

$$\times \ [1+\tfrac{1}{4}(\lambda+3\mu)][1+\tfrac{1}{5}(2\lambda+3\mu)] \ .$$

CHAPTER V

REDUCTION OF THE GENERAL LINEAR GROUP UNDER SUBGROUPS WITH LIE ALGEBRAS A_2, B_2, G_2 .

5a. Standard Results for Homogeneous Integral Representations.

There is a well-known procedure for constructing homogeneous integral representations of $GL(n,\mathbb{C})$, the general linear group in n dimensions over the complex field (Boerner 1963, Chapter V). To construct an rth rank tensor one makes a partition $[\lambda_1,\lambda_2, \ldots \lambda_p]$ such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_p = r , \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p .$$

We call a set of empty boxes with successive rows containing $\lambda_1,\lambda_2, \ldots \lambda_p$ boxes a <u>Young diagram</u>. One puts into each box an integer from 1 to n in such a way that in any row the integers are non-decreasing from left to right and in any column they are increasing downwards. We call a Young diagram with integers in its boxes a <u>Young tableau</u> and, if the integers are filled in as just described, we call it a <u>standard Young tableau</u>, or more briefly, a standard tableau. We depict a standard tableau in Fig.5.1. Corresponding to this we write P for the product of permutations of all the elements of each row, viz.

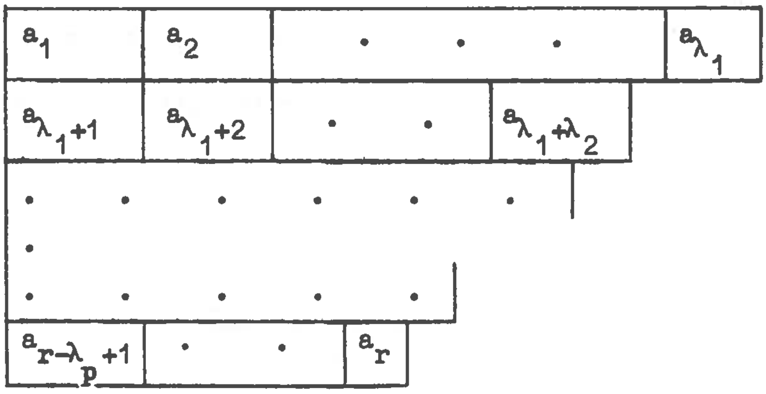


Fig.5.1 - A standard Young tableau.

$$
\begin{aligned}
P = & \{1 + (a_1a_2) + (a_1a_3) + \dots + (a_1a_2a_3) + \dots \\
& + (a_1a_2)(a_3a_4) + \dots + (a_1a_2 \dots a_{\lambda_1})\} \\
& \times \{1 + (a_{\lambda_1+1}a_{\lambda_1+2}) + \dots + (a_{\lambda_1+1} \dots a_{\lambda_1+\lambda_2})\} \\
& \times \dots \times \{1 + \dots + (a_{r-\lambda_p+1} \dots a_r)\} .
\end{aligned} \tag{5.1}
$$

We write Q for the product of permutations of all the elements of each column with a coefficient 1 (-1) for each even (odd) permutation, viz.

$$
\begin{aligned}
Q = & \{1 - (a_1a_{\lambda_1+1}) \dots \pm (a_1a_{\lambda_1+1} \dots a_{r-\lambda_p+1})\} \\
& \times \{1 - (a_2a_{\lambda_1+2}) \dots \pm (a_2a_{\lambda_1+2} \dots)\} \times \dots .
\end{aligned} \tag{5.2}
$$

Starting with a given tensor component $T_{a_1a_2 \dots a_r}$ we construct $QPT_{a_1a_2 \dots a_r}$. It may then be proved that these new tensor components for all different allowed values of $a_1, a_2, \dots a_r$ constitute a basis of an irreducible representation of $GL(n, \mathbb{C})$.

It follows that the degree N of this representation is the number of standard tableaux that can be constructed for the given Young diagram, that is, for the given partition $[\lambda_1, \lambda_2, \ldots \lambda_p]$. The theory of group characters gives (Boerner 1963, p.187)

$$N[\lambda_1, \lambda_2, \ldots \lambda_p] = \frac{v(\lambda_1+n-1, \lambda_2+n-2, \ldots \lambda_{n-1}+1, \lambda_n)}{v(n-1, n-2, \ldots 1, 0)} \qquad (5.3)$$

where

$$v(u_1, u_2, \ldots u_n) = \prod_{i<j} (u_i - u_j)$$

and we identify λ_s with zero for $s > p$. To different Young diagrams there correspond inequivalent irreducible representations.

The tensor component $T_{a_1 a_2 \ldots a_r}$ may be expressible as a product of the basis vectors in the n-dimensional space, i.e.

$$T_{a_1 a_2 \ldots a_r} \equiv x^{(1)}_{a_1} x^{(2)}_{a_2} \ldots x^{(r)}_{a_r} . \qquad (5.4)$$

Then $QPT_{a_1 a_2 \ldots a_r}$ is a homogeneous polynomial of degree r in the x's. We shall write this polynomial as a bracket placed round the entries in the tableau:

$$QPx^{(1)}_{a_1} x^{(2)}_{a_2} \ldots x^{(r)}_{a_r} \equiv \begin{bmatrix} a_1 & a_2 & \ldots & a_{\lambda_1} \\ a_{\lambda_1+1} & a_{\lambda_1+2} & \ldots & a_{\lambda_1+\lambda_2} \\ \cdot & \cdot & \cdot & \\ a_{r-\lambda_p+1} & \ldots & a_r & \end{bmatrix} . \qquad (5.5)$$

As we shall explain later, it may be possible to associate the

$x^{(i)}$'s with the boxes of the Young diagram in different ways.

If the transformations of the x's are

$$x'_j = \sum_{l=1}^{n} s_{jl} x_l , \qquad (5.6)$$

the general rth rank tensor $T_{a_1 a_2 \ldots a_r}$ will have a transformation matrix $s_{j_1 l_1} s_{j_2 l_2} \ldots s_{j_r l_r}$. Likewise in the case of the irreducible representation corresponding to a given Young diagram the elements of the transformation matrix will be homogeneous polynomials of degree r in the $s_{j_i l_i}$. We say that the representation is a <u>homogeneous integral representation</u>; if the polynomials were not homogeneous, the representation would just be an integral representation. If we take a subgroup T(n) of GL(n, $\mathbb{C}$) , the set of s's is restricted and as a result the homogeneous integral representation may reduce. Then by changing the basis, if necessary, a block of the original transformation matrix may now consist of zeros only. If we can prove that a block of zeros arising from T(n) implies that the same block will consist of zeros for GL(n, $\mathbb{C}$) , we may deduce that the irreducible homogeneous integral representations of GL(n, $\mathbb{C}$) will remain irreducible under T(n) .

Let us show that the irreducible homogeneous integral representations of GL(n, $\mathbb{C}$) remain irreducible under SL(n, $\mathbb{C}$) . For the latter group (5.6) is replaced by

$$x'_j = \sum_{l=1}^{n} \sigma_{jl} x_l ,$$

where

$$\sigma_{jl} = \frac{s_{jl}}{\sqrt[n]{\det s}} \quad .$$

A homogeneous polynomial of degree r in s is $(\det s)^{r/n}$ times the same polynomial in σ. Hence, if the latter vanishes, so does the former. Thus a block of zeros for $SL(n, \mathbb{C})$ gives a block of zeros for $GL(n, \mathbb{C})$, and this establishes the theorem. In particular any irreducible homogeneous integral representation of $GL(3, \mathbb{C})$ remains irreducible under $SL(3, \mathbb{C})$, that is, under the group whose Lie algebra is A_2.

Associated with any linear transformation $A(t)$, which belongs to a subgroup of $GL(n, \mathbb{C})$ and depends on a real parameter t in such a way that $A(o)$ is the identity I, there is a Lie algebra with an element O defined by

$$O = \left. \frac{dA(t)}{dt} \right|_{t=0} \quad .$$

An infinitesimal linear transformation $A(\varepsilon)$ is $I + \varepsilon O$. The representation space of the O's is the same as the representation space of the linear transformations A. The transformations A satisfy the condition $\det A \neq 0$, which is not necessarily satisfied by O. This point, however, is not a significant one in the above discussion of irreducible representations and Young tableaux. Thus the tensor components of (5.5) corresponding to standard tableaux are a basis for an irreducible representation of the Lie algebra of $GL(n, \mathbb{C})$, which we shall denote by $gl(n, \mathbb{C})$, the dimension of the representation being given by (5.3).

Suppose that the representation space of the group of linear transformations A is a known direct sum of irreducible subspaces. When A acts in one of these subspaces, O will act in the same subspace. Hence the reduction into irreducible representations of the group leads to the same reduction of the Lie algebra into irreducible representations of the algebra.

We may go from a Lie algebra to a Lie group of linear transformations by making an exponential map. Thus given O we can define A(t) by

$$A(t) = e^{tO} = 1 + tO + \frac{(tO)^2}{2!} + \dots .$$

The representation space of the A's will be the representation space of the O's . On exponentiating in this way the reduction into irreducible representations of the Lie algebra will lead to the reduction into irreducible representations with the same dimensions of the Lie group. Thus to examine the reduction into irreducible representations of a group or its Lie algebra, we may perform the calculations for either and take the results over to the other. The reduction of the group may be performed by using the theory of group characters, the reduction of the algebra by the method of weight diagrams. Since the latter is a more elementary way, we shall employ it exclusively except in Chapter VIII, where the two methods will be combined.

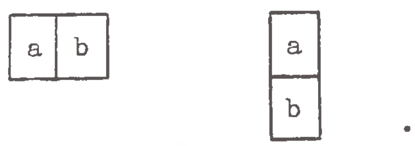
5b. Young Tableaux with Two or Three Entries.

In many applications to physics one is interested in tensors of rank 2 or 3 . Since the theorems quoted in the previous section presuppose quite a good knowledge of representation theory and of the theory of group characters, it is worthwhile to give an alternative more elementary derivation of some of the results. We restrict our investigations to infinitesimal transformations (McConnell 1969).

We shall first deal with bilinear expressions. Let x_a be a basis vector for n-dimensional vector space and let O be an infinitesimal operator in this space; in other words, let $1 + \varepsilon O$ be an infinitesimal transformation. Then we may write

$$Ox_a = \sum_{l} \lambda_{al} x_l \quad . \tag{5.7}$$

We consider the product of two basis vectors x_a and y_b . To the box representing x_a we can add the box representing y_b either on the same horizontal line or in the same vertical line



For the partition [2] when a is equal to 1 , b can have the values 1,2, ... n ; when a is equal to 2 , b can have the values 2,3, ... n , and so on. The dimension of the representation, which is the total number of tensor components, is thus

$$n + (n-1) + \dots + 1 = \frac{n(n+1)}{2} .$$

For the partition $[1^2]$ $b > a$, so the dimension of the representation is

$$n-1 + (n-2) + \dots + 1 = \frac{n(n-1)}{2} .$$

It may be checked that these figures agree with (5.3). The QP-operator for $[2]$ is $1 + (ab)$ and for $[1^2]$ is $1 - (ab)$. The tensor components are therefore, respectively, the symmetric functions

$$x_a y_b + x_b y_a \qquad (a \leqslant b)$$

and the antisymmetric functions

$$x_a y_b - x_b y_a . \qquad (a < b)$$

When $1 + \varepsilon O$ acts on $x_a y_b$, we obtain

$$(1 + \varepsilon O) x_a y_b = (x_a + \varepsilon O x_a)(y_b + \varepsilon O y_b)$$

$$= x_a y_b + \varepsilon O x_a . y_b + x_a . \varepsilon O y_b + \dots .$$

Hence the infinitesimal operator acts in the product space of x_a and y_b according to the rule

$$O(x_a y_b) = (O x_a) y_b + x_a (O y_b) , \qquad (5.8)$$

and so

$$O(x_a y_b \pm x_b y_a) = \sum_l \lambda_{al} x_l y_b + \sum_l x_a \lambda_{bl} y_l$$
$$\pm \sum_l \lambda_{bl} x_l y_a \pm \sum_l x_b \lambda_{al} y_l ,$$

that is

$$O(x_a y_b \pm x_b y_a) = \sum_l \lambda_{al}(x_l y_b \pm x_b y_l) + \sum_l \lambda_{bl}(x_a y_l \pm x_l y_a) . \quad (5.9)$$

Thus when O acts on $x_a y_b \pm x_b y_a$, it produces a linear combination of functions with the same symmetry. In the notation of (5.5)

$$x_a y_b + x_b y_a = [a\ b] , \quad x_a y_b - x_b y_a = \begin{bmatrix} a \\ b \end{bmatrix} ,$$

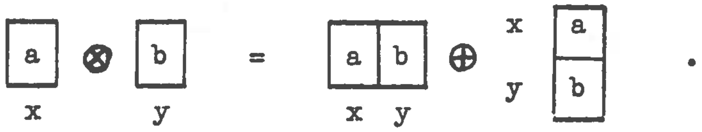
if we identify $x^{(1)}$ with x and $x^{(2)}$ with y . Then we see from (5.9) that

$$O[a\ b] = \sum_l \lambda_{al}[l\ b] + \sum_l \lambda_{bl}[a\ l] \quad (5.10)$$

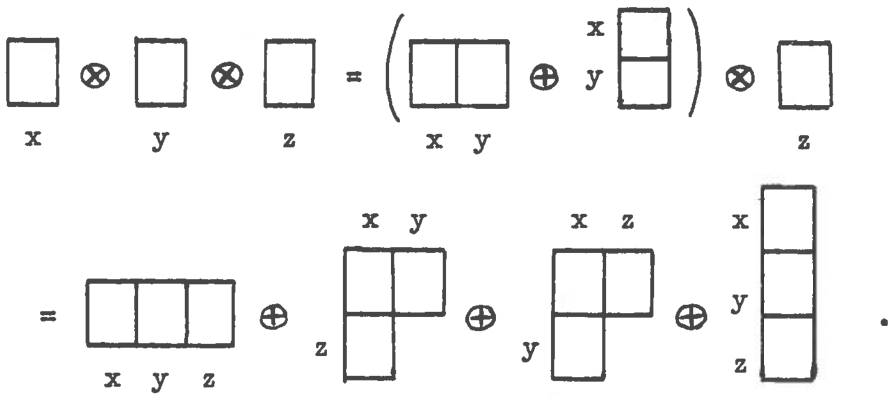
$$O\begin{bmatrix} a \\ b \end{bmatrix} = \sum_l \lambda_{al}\begin{bmatrix} l \\ b \end{bmatrix} + \sum_l \lambda_{bl}\begin{bmatrix} a \\ l \end{bmatrix} . \quad (5.11)$$

It may happen in (5.10) that $l > b$, whereas for the standard tableaux we have agreed to take only $l \leq b$. We can, however, replace $[l\ b]$ by $[b\ l]$. Similarly if $l > b$ in (5.11) , we can write $\begin{bmatrix} l \\ b \end{bmatrix}$ as $-\begin{bmatrix} b \\ l \end{bmatrix}$, so that in fact $O\begin{bmatrix} a \\ b \end{bmatrix}$ is expressible as a linear combination of brackets corresponding to standard tableaux. It will be convenient to retain the definition (5.5) also for sequences $a_1, a_2, \ldots a_r$ that do not give a standard tableau.

The construction of the above sums of products of x_a and y_b may be expressed as

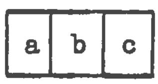


We have put x and y at the sides of the boxes to indicate the variables to which the a, b are suffixes. To extend to the case of trilinear products we write



(5.12)

The functions corresponding to the partition [3] are totally symmetric in the products $x_a\, y_b\, z_c$. The number of functions may be found without recourse to (5.3) as follows: If in



we first put $a = b = 1$, we can have $c = 1, 2, \ldots n$ giving n functions. If we put $a = 1$, $b = 2$ or $a = b = 2$ we can have $c = 2, 3, \ldots n$ and therefore $2(n-1)$ functions. If we put $a = 1$, $b = 3$ or $a = 2$, $b = 3$ or $a = b = 3$, we can have $c = 3, 4, \ldots n$ and therefore $3(n-2)$ functions. Thus the number of totally symmetric functions is $N[3]$, where

$$N[3] = 1.n + 2(n-1) + 3(n-2) + \dots + n.1$$

$$= \sum_{a=1}^{n} a(n+1-a) = \frac{n(n+1)(n+2)}{6} .$$

Similarly the number of totally antisymmetric functions is $N[1^3]$, where

$$N[1^3] = 1 + (1+2) + (1+2+3) + \dots + (1+2+3+ \dots + n-2)$$

$$= \sum_{a=1}^{n-2} \frac{a(a+1)}{2} = \frac{n(n-1)(n-2)}{6} .$$

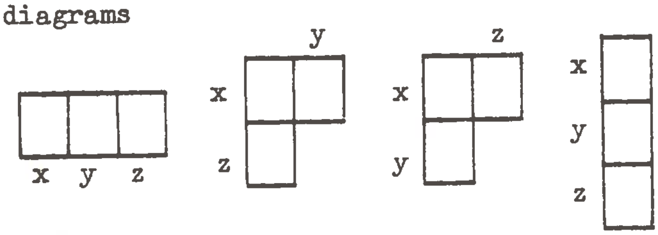
To find the number $N[2, 1]$ of functions of mixed symmetry we note that (5.12) gives

$$n^3 = N[3] + 2N[2,1] + N[1^3] ,$$

from which it follows that

$$N[2,1] = \frac{n^3-n}{3} .$$

We must examine whether each set of functions related to the diagrams



(5.13)

of (5.12) is a basis of an irreducible representation. From (5.8) we deduce that the effect of operating with O on $x_a\, y_b\, z_c$ is given by

$$O(x_a y_b z_c) = (Ox_a)y_b z_c + x_a(Oy_b)z_c + x_a y_b(Oz_c) .$$

Hence for the first diagram of (5.13)

$$OPx_a y_b z_c = O\,\Sigma\, x_a y_b z_c = \Sigma\{(Ox_a)y_b z_c + x_a(Oy_b)z_c + x_a y_b(Oz_c)\}$$

$$= P\,O\,x_a y_b z_c ,$$

so that O and P commute. By writing $Q\,x_a y_b z_c$ as $\Sigma \pm x_a y_b z_c$ we see that the O commutes with Q. These results may be extended to P defined in (5.1), Q defined in (5.2) and T defined in (5.4), since the factors in (5.1) commute among themselves and the factors in (5.2) commute among themselves. Hence for all QPT given by (5.4) and (5.5)

$$O(QPT) = QP(OT) .$$

Now OT is the sum

$$Ox_{a_1}^{(1)}.x_{a_2}^{(2)} \ldots x_{a_r}^{(r)} + x_{a_1}^{(1)}.Ox_{a_2}^{(2)} \ldots x_{a_r}^{(r)} + \ldots + x_{a_1}^{(1)}x_{a_2}^{(2)} \ldots Ox_{a_r}^{(r)}$$

and $O(QPT)$ is therefore the sum of r brackets obtained from QPT by replacing each x successively by Ox. According to (5.7) this is in turn a sum of brackets with the same structure. We conclude that, when an infinitesimal linear transformation is made on a bracket corresponding to a given partition, we obtain a linear combination of brackets corresponding to the same partition. By iterating we can establish the same result for a finite transformation.

The brackets that so arise may not all correspond to standard tableaux and we would next have to show that such brackets are linear combinations of brackets corresponding to standard tableaux. This is obviously true, when the tableau consists of a single row or a single column. We shall now examine whether it is true for a [2, 1] partition.

We can without loss of generality examine the problem by taking the cases of tableaux with entries 1,2,3. The allowed tableaux have the following essentially different entries

$$\begin{matrix} 1 & 1 \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 2 \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 2 \\ 3 & \end{matrix} \qquad \begin{matrix} 1 & 3 \\ 2 & \end{matrix} \tag{5.14}$$

illustrating for $\begin{matrix} a & b \\ c & \end{matrix}$ the general cases

$$a = b < c, \quad a < b = c, \quad a < b < c, \quad a < c < b,$$

respectively. The forbidden tableaux are those with the entries

$$\begin{matrix} 2 & 1 \\ 1 & \end{matrix} \qquad \begin{matrix} 2 & 2 \\ 1 & \end{matrix} \qquad \begin{matrix} 2 & 1 \\ 3 & \end{matrix} \qquad \begin{matrix} 2 & 3 \\ 1 & \end{matrix} \qquad \begin{matrix} 3 & 1 \\ 2 & \end{matrix} \qquad \begin{matrix} 3 & 2 \\ 1 & \end{matrix} \tag{5.15}$$

illustrating respectively, the general cases

$$a > b = c, \quad a = b > c, \quad c > a > b, \quad c < a < b, \quad a > c > b, \quad a > b > c .$$

It will suffice to show that the brackets corresponding to (5.15) are linear combinations of those related to (5.14). For the second diagram of (5.13) we have by definition

$$\begin{bmatrix} a & b \\ c & \end{bmatrix} = (1 - (ac))(1 + (ab))x_a y_b z_c$$

$$= (1 - (ac))(x_a y_b z_c + x_b y_a z_c)$$

and therefore

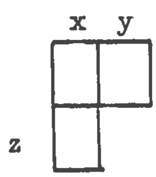
$$\begin{bmatrix} a & b \\ c & \end{bmatrix} = x_a y_b z_c + x_b y_a z_c - x_c y_b z_a - x_b y_c z_a \; . \qquad (5.16)$$

On substituting the values of a,b,c into this we find that

$$\begin{aligned}
\begin{bmatrix} 2 & 1 \\ 1 & \end{bmatrix} &= -\begin{bmatrix} 1 & 1 \\ 2 & \end{bmatrix} , \quad \begin{bmatrix} 2 & 2 \\ 1 & \end{bmatrix} = -\begin{bmatrix} 1 & 2 \\ 2 & \end{bmatrix} \\
\begin{bmatrix} 2 & 1 \\ 3 & \end{bmatrix} &= -\begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \\
\begin{bmatrix} 2 & 3 \\ 1 & \end{bmatrix} &= -\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} \\
\begin{bmatrix} 3 & 1 \\ 2 & \end{bmatrix} &= \begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} \\
\begin{bmatrix} 3 & 2 \\ 1 & \end{bmatrix} &= -\begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} .
\end{aligned} \qquad (5.17)$$

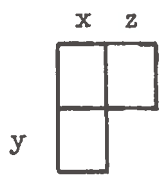
This shows that the brackets with the entries (5.15) are linear combinations of those with the entries (5.14). These latter brackets are linearly independent, as we see from (5.16), so they constitute a basis of a representation of $GL(n, \mathbb{C})$.

What we have established for the partition [2, 1] and x,y,z associated as in



(5.18)

and hold equally well for



(5.19)

What is not evident is whether the brackets for one assignment of x, y, z are linear combinations of those for the other. The bracket $\begin{bmatrix} a & b \\ c & \end{bmatrix}$ for (5.19) has from (5.16) the value

$$x_a z_b y_c + x_b z_a y_c - x_c z_b y_a - x_b z_c y_a$$

$$= x_a y_c z_b + x_b y_c z_a - x_c y_a z_b - x_b y_a z_c \qquad (5.20)$$

and we must examine whether this can be expressed as a linear combination of those for (5.18). On including non-standard tableaux, as we may, there are at most six combinations of a,b,c, viz.,

$$\begin{matrix} a & b \\ c & \end{matrix} \quad \begin{matrix} a & c \\ b & \end{matrix} \quad \begin{matrix} b & c \\ a & \end{matrix} \quad \begin{matrix} b & a \\ c & \end{matrix} \quad \begin{matrix} c & a \\ b & \end{matrix} \quad \begin{matrix} c & b \\ a & \end{matrix} .$$

Some of these will vanish when two of the entries are equal. Inspecting (5.17) we see that for a,b,c all different there are only two linearly independent ones which we shall take to be $\begin{bmatrix} a & b \\ c & \end{bmatrix}$ and $\begin{bmatrix} a & c \\ b & \end{bmatrix}$, namely,

$$x_a y_b z_c + x_b y_a z_c - x_c y_b z_a - x_b y_c z_a$$

and

$$x_a y_c z_b + x_c y_a z_b - x_b y_c z_a - x_c y_b z_a .$$

It is obvious that (5.20) is not a linear combination of these and we can conclude that (5.18) and (5.19) provide two linearly independent bases for representations of $GL(n, \mathbb{C})$. Furthermore we may easily check that we get no other independent basis by putting a y or a z at the first box of the first row. We therefore have two, and only two, independent representations for the [2, 1] partition.

5c. Young Tableaux and Weight Diagrams.

The group $GL(n, \mathbb{C})$ is the set of all non-singular $n \times n$ matrices. If we denote by $I + \sum_A \varepsilon^A O_A$ an infinitesimal transformation of the group, the infinitesimal generators O_A are elements of the Lie algebra denoted in section 5a by $gl(n, \mathbb{C})$. Moreover the infinitesimal generators of any subgroup of $GL(n, \mathbb{C})$ constitute a Lie algebra. In previous sections we considered such algebras for $n = 3,4,5,7$, namely, A_2, C_2, B_2, G_2, respectively.

Suppose that a basis vector with weight $\underset{\sim}{m}(a)$ for an irreducible representation of one of the algebras A_2, B_2, G_2 is denoted by x_a or y_a. Then according to (5.8)

$$\begin{aligned} H_i(x_a y_b) &= (H_i x_a) y_b + x_a (H_i y_b) \\ &= (m_i(a) + m_i(b)) x_a y_b , \end{aligned} \qquad (5.21)$$

so the weight of $x_a y_b$ is the vector sum of the weights of x_a and y_b. More generally, the weight of a continued product

$$x^{(1)}_{a_1}\, x^{(2)}_{a_2} \ldots x^{(r)}_{a_r}$$

is the vector sum of the individual weights of the $x^{(i)}_{a_i}$'s. On performing the QP operation as in (5.5) this sum is unaltered, and the weight of the tensor component represented by the bracket is just the sum $\sum_{t=1}^{r} \underset{\sim}{m}(a_t)$. Thus the weights of all tensor components corresponding to Young tableaux with the same entries in different sequences are equal. This is true for all partitions; thus we have equal weights for

$$\begin{bmatrix} 1 & 3 & 3 & 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 3 \\ 4 & 5 & \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 3 \\ 4 & & \\ 5 & & \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 4 \\ 3 & & \\ 5 & & \end{bmatrix}.$$

We pointed out in section 5a that given a Young diagram with r boxes, or alternatively given any partition of r, we have a corresponding irreducible representation of $GL(n, \mathbb{C})$ and that a basis for this is provided by the bracket symbols of (5.5) that are related to all the standard tableaux that can be constructed for the diagram. Moreover we pointed out that this is equally applicable to $gl(n, \mathbb{C})$. Indeed the only essential difference between $GL(n, \mathbb{C})$ and $gl(n, \mathbb{C})$ consists in the rule of transformation of the product; namely,

$$s(x_a\, y_b) = (sx_a)(sy_b)$$

for $GL(n, \mathbb{C})$ becomes

$$0\,(x_a y_b) = (0\,x_a)y_b + x_a(0 y_b)$$

for $gl(n, \mathbb{C})$. The algebra $gl(n, \mathbb{C})$ is the algebra of all $n \times n$ matrices, singular or not. The dimension of an irreducible representation of the algebra corresponding to the symmetry specified by a Young diagram is given by (5.3).

The result (5.21) gives a convenient way of reducing the direct product of irreducible representations of A_2, B_2, G_2 into irreducible representations. Consider, for example, the weight diagram for the $D^{(4)}(1, 0)$ representation of B_2 and associate x_1, x_2, x_3, x_4 with the weights in descending order as in Fig.5.2.

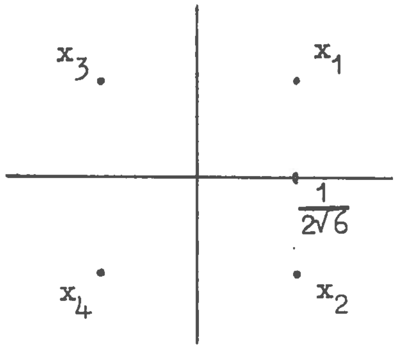


Fig.5.2 - The basis vectors for the $D^{(4)}(1, 0)$ representation of B_2 .

The weights for the product representation $D^{(4)}(1, 0) \otimes D^{(4)}(1, 0)$ will be the sixteen vector sums of all combinations of the weights of x_1, x_2, x_3, x_4, and will therefore appear with the multiplicities shown in Fig.5.3. According to (3.5) and (3.7) the highest weight in this figure can only belong to $D^{(10)}(2, 0)$.

1	2	1
2	4	2
1	2	1

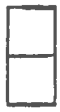
Fig.5.3 - Weight multiplicities for the product representation $D^{(4)}(1,0) \otimes D^{(4)}(1,0)$ of B_2 .

When the weights of this representation are removed from those of Fig.5.3, we are left with the weights of $D^{(5)}(0,1)$ and $D^{(1)}(0,0)$, and thus

$$D^{(4)}(1,0) \otimes D^{(4)}(1,0) = D^{(10)}(2,0) \oplus D^{(5)}(0,1) \oplus D^{(1)}(0,0) . \qquad (5.22)$$

This reduction brings out another point. The highest weight can belong only to the tensor component $x_1 y_1$, which apart from the factor 2 is just $[1\ 1]$. According to the discussion of the last section all the basis vectors of $D^{(10)}(2,0)$ are therefore symmetric bilinear forms. Now we see from (5.3) that the dimension of the irreducible representation of $gl(4,\mathbb{C})$ corresponding to the Young diagram  is 10. Hence this representation of $gl(4,\mathbb{C})$ does not reduce under the subalgebra C_2 .

Since the only other Young diagram with two boxes is  , all the remaining weights must belong to antisymmetric bilinear forms. However we know from equation (3.5) that the highest weight $(\frac{1}{\sqrt{6}}, 0)$ belongs to the irreducible representation $D(0,1)$ of B_2 or C_2 , which by (3.13) is five-dimensional. This means in fact that at the centre of the diagram only one linear combination of $\begin{bmatrix}1\\4\end{bmatrix}$, $\begin{bmatrix}2\\3\end{bmatrix}$ belong to $D^{(5)}(0,1)$.

The other linear combination can belong only to $D^{(1)}(0,0)$. Hence the six-dimensional irreducible representation of $gl(4, \mathbb{C})$ related to the diagram  reduces under C_2 into

$$D^{(5)}(0,1) \oplus D^{(1)}(0,0) .$$

We define the <u>basic representation</u> of A_2,B_2,G_2 to be the irreducible representation of lower order that has a fundamental dominant weight as its highest weight. We shall see immediately that it is the one from which the irreducible representation with the other fundamental dominant weight for highest weight may be obtained in the reduction of the direct product of the basic representation by itself. For A_2 the basic representation may be taken as either $D^{(3)}(1,0)$ or $D^{(2)}(1,0)$, and on using weight diagrams as explained above we see that

$$D^{(3)}(0,1) \otimes D^{(3)}(1,0) = D^{(3)}(0,1) \oplus D^{(6)}(2,0) \qquad (5.23)$$

$$D^{(3)}(0,1) \otimes D^{(3)}(0,1) = D^{(3)}(1,0) \oplus D^{(6)}(0,2) .$$

For B_2 it is $D^{(4)}(1,0)$ and as we see from (5.22) that $D^{(4)}(1,0) \otimes D^{(4)}(1,0)$ contain $D^{(5)}(0,1)$. For G_2 it is $D^{(7)}(1,0)$ and on referring to the weight diagrams of section 4f we see without difficulty that

$$D^{(7)}(1,0) \otimes D^{(7)}(1,0) = D^{(1)}(0,0) \oplus D^{(7)}(1,0) \oplus D^{(14)}(0,1) \oplus D^{(27)}(2,0) .$$

By taking direct products of basic representations we can obtain any representation, including the one-dimensional $D^{(1)}(0,0)$. The latter point has been established for B_2 and G_2, and for A_2 we have on further displacing the weights of (5.23)

$$D^{(3)}(1,0) \otimes D^{(3)}(1,0) \otimes D^{(3)}(1,0)$$
$$= D^{(1)}(0,0) \oplus D^{(8)}(1,1) \oplus D^{(8)}(1,1) \oplus D^{(10)}(3,0) .$$

We note that in a basic representation all weights are simple. If working in such a representation we operate with an E_α on a basis vector x_a with weight $\underset{\sim}{m}(a)$, we obtain

$$E_\alpha x_a = \lambda_a x_b ,$$

where for non-vanishing λ_a the vector x_b corresponds to a simple weight $\underset{\sim}{m}(a) + \underset{\sim}{r}(\alpha)$. We have, of course,

$$H_i x_a = m_i(a) x_a .$$

Hence for any operator O of the algebra

$$O x_a = \lambda x_{a'} ,$$

where λ is a scalar multiple that may be zero, $a' = a$ for $O \equiv H_i$ and the weights $\underset{\sim}{m}(a), \underset{\sim}{m}(a')$ are both simple.

In the following chapters we shall relate in greater detail Young tableaux and weight diagrams for the different semi-simple Lie algebras of rank 2 .

CHAPTER VI

YOUNG TABLEAUX AND WEIGHT DIAGRAMS FOR $gl(5,\mathbb{C})$ AND B_2

6a. Highest Weights and Young Tableaux.

We put down in Fig.6.1 the weight diagram for the $D^{(5)}(0,1)$ representation of B_2. We assign to the weights the basis vectors x_1,x_2,x_3,x_4,x_5 in such a way that their weights are in descending sequence. Referring to Fig.1.2 we may see the

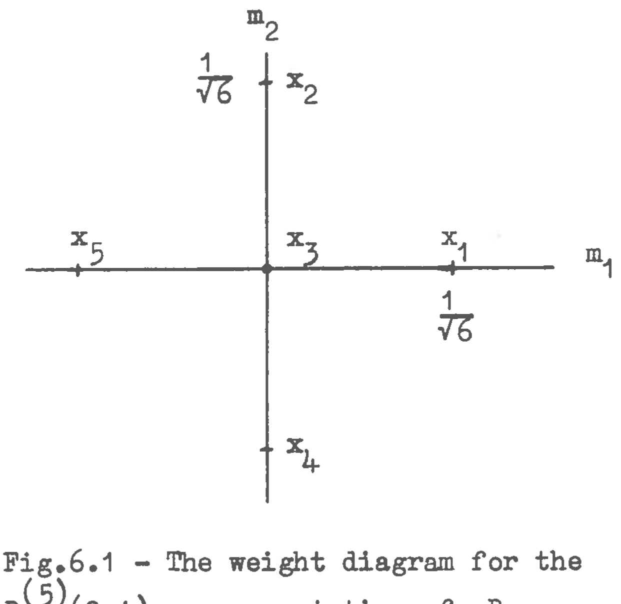


Fig.6.1 - The weight diagram for the $D^{(5)}(0,1)$ representation of B_2.

effect of operating with an E_α on an x_a. Thus E_1,E_2,E_3 operating on x_5 will produce $x_3,x_2,0$, respectively, apart from a multiplying numerical factor.

We may take x_1, x_2, x_3, x_4, x_5 as a basis for $gl(5, \mathbb{C})$. Young diagrams will have at most five rows and (5.3) will become

$$N[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5] = \frac{v(\lambda_1+4, \lambda_2+3, \lambda_3+2, \lambda_4+1, \lambda_5)}{4!\ 3!\ 2!\ 1!} . \tag{6.1}$$

For a given standard tableau we can construct tensor components as in (5.5). The tensor component with the highest weight is that in which there are the maximum number of 1's, then the maximum number of 2's etc., viz.

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 2 & 2 & \cdots & 2 & & \\ 3 & 3 & \cdots & 3 & & \\ 4 & \cdots & 4 & & & \\ 5 & \cdots & 5 & & & \end{bmatrix} .$$

According to section 5c the weight of this is

$$\lambda_1 \underset{\sim}{m}(1) + \lambda_2 \underset{\sim}{m}(2) + \lambda_3 \underset{\sim}{m}(3) + \lambda_4 \underset{\sim}{m}(4) + \lambda_5 \underset{\sim}{m}(5) .$$

Fig.6.1 shows that this equals

$$\frac{1}{\sqrt{6}} (\lambda_1 - \lambda_5, \lambda_2 - \lambda_4) . \tag{6.2}$$

In the reduction of $gl(5, \mathbb{C})$ under its subalgebra B_2 the weight (6.2) will be the highest weight of an irreducible representation of B_2. According to section 3b the boundary of the weight diagram for this representation will be a square with vertices on the axes when $\lambda_2 - \lambda_4 = 0$, which implies

$$\lambda_2 = \lambda_3 = \lambda_4 . \tag{6.3}$$

We shall denote such a boundary by $\diamondsuit$, and we shall denote by $\square$ the square boundary of a weight diagram when the sides are parallel to the axes. Returning again to section 3b we see that (6.2) is the highest weight of an irreducible representation of B_2 with a weight diagram having a $\square$ boundary, if

$$\lambda_1 - \lambda_5 = \lambda_2 - \lambda_4 .$$

This implies

$$0 \leq \lambda_1 - \lambda_2 = -(\lambda_4 - \lambda_5) \leq 0$$

and therefore

$$\lambda_1 = \lambda_2 , \qquad \lambda_4 = \lambda_5 . \tag{6.4}$$

If both (6.3) and (6.4) are satisfied, that is to say, if

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 , \tag{6.5}$$

the representation must be $D^{(1)}(0,0)$. In all other cases the boundary of the irreducible representation of B_2 with (6.2) as highest weight is octagonal. It should be remembered that some of the λ's in the partition may vanish.

When (6.2) is the highest weight of a representation of B_2 with boundary $\square$, it is impossible for the weight of any vector of the representation of $gl(5, \mathbb{C})$ corresponding to the partition $[\lambda_1{}^2,\lambda_3,\lambda_4{}^2]$, that is $[\lambda_1,\lambda_1,\lambda_3,\lambda_4,\lambda_4]$, to be outside the square. Indeed, if a weight were at the point P in Fig. 6.2, another weight would be at Q by re-

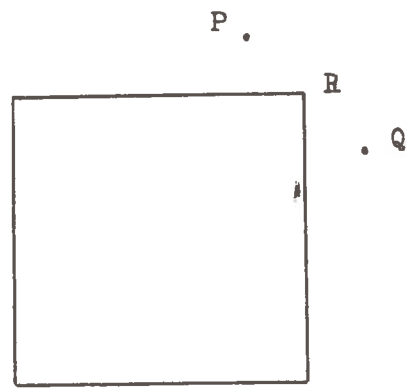


Fig.6.2 - Boundary of a B_2 diagram
with weights outside.

flection. This is excluded, since Q would then be higher than the highest weight R . The impossibility of having a weight outside a ◇ or an octagonal boundary is not evident.

6b. Irreducible Representations of $gl(5,\mathbb{C})$ associated with a $D(0,\mu)$ Boundary.

The Young diagrams that obey (6.3) have highest weight $\frac{1}{\sqrt{6}}(\lambda_1 - \lambda_5,0)$ and this corresponds to the tensor component

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2 & \cdots & 2 & \\ 3 & 3 & \cdots & 3 & \\ 4 & 4 & \cdots & 4 & \\ 5 & 5 \cdots 5 & & & \end{bmatrix} . \qquad (6.6)$$

Since this is the only tensor component for the partition $[\lambda_1,\lambda_2{}^3,\lambda_4]$ that has this weight, the weight is simple and, as we saw above, it is the highest weight of a $D(0,\mu)$ representation of B_2 with a boundary $\diamond$. By referring to Fig.6.1 we see that a displacement vertically upwards is given by the replacements $3 \to 2$ or $4 \to 3$, and that a displacement in the direction of the slant side above the m_1 axis is given by $1 \to 2$ or $4 \to 5$. Of course the transition $1 \to 2$ could be made by the succession $1 \to 3$, $3 \to 2$ but the effect on (6.6) would be just the same as for the direct transition.

We shall proceed along the slant side and consider the multiplicities of the weights. The tensor components at the next weight will be

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 2 \\ 2 & 2 & \cdots & 2 & \\ 3 & 3 & \cdots & 3 & \\ 4 & 4 & \cdots & 4 & \\ 5 & 5 \cdots 5 & & & \end{bmatrix} , \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2 & \cdots & 2 & \\ 3 & 3 & \cdots & 3 & \\ 4 & 4 & \cdots & 5 & \\ 5 & 5 \cdots 5 & & & \end{bmatrix} . \qquad (6.7)$$

The first exists when $\lambda_1 > \lambda_2$; the second exists when $\lambda_2 > \lambda_5$.

We exclude the case (6.5), and then the multiplicity is 1 or 2 according as one or both the inequalities

$$\lambda_1 - \lambda_2 \geqslant 1 , \qquad \lambda_2 - \lambda_5 \geqslant 1$$

are satisfied.

Proceeding from (6.7) we have in the next step $1 \to 2$ in the first bracket if $\lambda_1 - \lambda_2 \geqslant 2$, $1 \to 2$ in the second if $\lambda_1 - \lambda_2 \geqslant 1$; we have $4 \to 5$ in the first bracket if $\lambda_2 - \lambda_5 \geqslant 1$ and $4 \to 5$ in the second when $\lambda_2 - \lambda_5 \geqslant 2$. The multiplicities of the weights are therefore

4 for $\lambda_1 - \lambda_2 \geqslant 2$, $\lambda_2 - \lambda_5 \geqslant 2$

2 for $\lambda_1 - \lambda_2 \geqslant 2$, $\lambda_4 - \lambda_5 = 1$ or $\lambda_1 - \lambda_2 = 1$, $\lambda_2 - \lambda_5 \geqslant 2$

1 for $\lambda_1 - \lambda_2 \geqslant 2$, $\lambda_2 = \lambda_5$ or $\lambda_1 = \lambda_2$, $\lambda_2 - \lambda_5 \geqslant 2$

0 for $\lambda_1 - \lambda_2 = 1$, $\lambda_2 = \lambda_5$ or $\lambda_1 = \lambda_2$, $\lambda_2 - \lambda_5 = 1$.

At each subsequent stage there is a rise in the multiplicity of a factor 2 if both $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_5$ satisfy the next higher inequality, and no change if only one of them obeys this condition. If neither obeys the next higher inequality, we are outside the representation corresponding to the tableau. We conclude that the maximum multiplicity as we go along the slant side is 2^s , where s is the lesser of $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_5$. If $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_5$, $s = 0$ and the weights are therefore all simple for the partitions $[\lambda_1{}^4, \lambda_5]$ and $[\lambda_1, \lambda_2{}^4]$.

The maximum multiplicity occurs s steps from the highest weight. On account of the symmetry of the weight diagrams we must check that this does not bring us more than half-way along the slant side. The total number of steps along the slant side is $\lambda_1 - \lambda_5$, which we write

$$\lambda_1 - \lambda_2 + (\lambda_2 - \lambda_5) \ . \qquad (6.8)$$

When $\lambda_1 - \lambda_2 < \lambda_2 - \lambda_5$, $s = \lambda_1 - \lambda_2$ and (6.8) shows that

$$s < \tfrac{1}{2}(\lambda_1 - \lambda_5) \ ,$$

and the maximum multiplicity is therefore attained before reaching half-way along the slant side. For $\lambda_1 - \lambda_2 > \lambda_2 - \lambda_5$ we obtain the same inequality. When $\lambda_1 - \lambda_2 = \lambda_2 - \lambda_5$,

$$s \ = \ \tfrac{1}{2}(\lambda_1 - \lambda_5)$$

and the maximum multiplicity is reached half-way along.

Returning to (6.6) and (6.7) we note that these, and any of the subsequent tensor components, are annihilated if we put $3 \to 2$ or $4 \to 3$. This means that none of the weights corresponding to the tensor components lie outside the $\Diamond$ boundary.

As we go along the slant side from $\frac{1}{\sqrt{6}}(\lambda_1 - \lambda_5, 0)$, we obtain

a weight at			$\frac{1}{\sqrt{6}}(\lambda_1 - \lambda_5, 0)$	with multiplicity		1
"	"	"	$\frac{1}{\sqrt{6}}(\lambda_1 - \lambda_5 - 1, 1)$	"	"	2
"	"	"	$\frac{1}{\sqrt{6}}(\lambda_1 - \lambda_5 - 2, 2)$	"	"	2^2
"	"	"	. . . .	"	"	...

a weight at $\frac{1}{\sqrt{6}}(\lambda_1 - \lambda_5 - r,r)$ with multiplicity 2^r

" " " . . . . " " ...

" " " $\frac{1}{\sqrt{6}}(\lambda_1 - \lambda_5 - s,s)$ " " 2^s .

Since there are no weights outside the boundary and since all the weights on the boundary of any diagram corresponding to an irreducible representation of B_2 are simple, we deduce that each of the above is the highest weight of an irreducible representation. Moreover, since

$$2^r - 2^{r-1} = 2^{r-1} ,$$

the representation with highest weight $\frac{1}{\sqrt{6}}(\lambda_1 - \lambda_5 - r,r)$, that is $D(2r, \lambda_1 - \lambda_5 - 2r)$ according to section 3a, occurs 2^{r-1} times. The weights on the boundary of the weight diagram corresponding to the partition $[\lambda_1,\lambda_2^3,\lambda_5]$ therefore belong to

$$D(0,\lambda_1 - \lambda_5) \oplus D(2, \lambda_1 - \lambda_5 - 2) \oplus 2D(4, \lambda_1 - \lambda_2 - 4) \oplus \cdots$$
$$\oplus\ 2^{s-1} D(2s, \lambda_1 - \lambda_2 - 2s) .$$

This will not in general be the full reduction under B_2 of the representation of $gl(5,\mathbb{C})$ corresponding to the partition, since there may be weight diagrams of B_2 that do not come as far as the boundary.

6c. Reduction under B_2 of Representations of $gl(5, \mathbb{C})$ with totally Symmetric Bases.

The representation with basis of totally symmetric functions of order r corresponds to the partition $[r]$. This satisfies the conditions (6.3) for a boundary $\diamond$ and, as we saw in the last section, the weights do not go outside the $\diamond$ with highest weight $\frac{1}{\sqrt{6}}(r,0)$. According to (6.1) the dimension of the $gl(5, \mathbb{C})$ representation

$$N[r] = \frac{(r+1)(r+2)(r+3)(r+4)}{24} .$$

We compare this with the dimension $N(0,r)$ of the representation of B_2 with highest weight $\frac{1}{\sqrt{6}}(r,0)$ given by (3.13) viz.

$$N(0,r) = \frac{(r+1)(r+2)(2r+3)}{6} .$$

Taking $r = 0,1,2,3, \ldots$ we have the table:

r	0	1	2	3	4	5	6	7	8
N(0,r)	1	5	14	30	55	91	140	204	285
N[r]	1	5	15	35	70	126	210	330	495

(6.9)

For $r = 0$ and $r = 1$ the representations of $gl(5, \mathbb{C})$ do not reduce under B_2; they are the almost trivial cases of the 1- and 5-dimensional representations. For higher values of r there appears an interesting relation, viz.

$$N[r] = N(0,r) + N(0,r-2) + N(0,r-4) + \ldots$$

ending with $N(0,1)$ for r odd and with $N(0,0)$ for r even. This would suggest that the representation $D[r]$ of $gl(5, \mathbb{C})$, corresponding to the partition $[r]$, reduces under the subalgebra B_2 into irreducible representations as follows:

$$D[r] = D(0,r) \oplus D(0,r-2) \oplus D(0,r-4) \oplus \ldots \oplus \begin{cases} D(0,1) \text{ for } r \text{ odd} \\ D(0,0) \text{ for } r \text{ even.} \end{cases} \quad (6.10)$$

The most direct way of finding whether this is the correct reduction would be by the use of the theory of group characters (Littlewood 1940,p.240). We shall, however, treat it as an exercise in weight multiplicities. It would be conceivable that the same value of $N[r]$ might be obtained from representations with $\square$ or octagonal boundaries, or that it might come from representations with $\diamond$ boundaries, the representations being other than those in (6.10). We shall therefore examine the multiplicities at every point in the first quadrant in the weight diagram associated with the representation of $gl(5, \mathbb{C})$ corresponding to the partition $[r]$.

To do this we first construct the totally symmetric basis vectors associated with weights on the positive m_1 axis, referring to Fig.6.1. For the fundamental 5-dimensional representation there are basis vectors x_3 at the origin and x_1 immediately to the right. We may express these in the notation of (5.5) as [3] and [1]. As we are concerned with only one partition in this section, there should be no danger of confusion with the notation for a partition. For bilinear products we can have at the centre the weights

$$\underset{\sim}{m}(3) + \underset{\sim}{m}(3) , \quad \underset{\sim}{m}(1) + \underset{\sim}{m}(5) , \quad \underset{\sim}{m}(2) + \underset{\sim}{m}(4) ,$$

and these correspond, respectively, to the linearly independent basis vectors

[3 3] , [1 5] , [2 4] .

We note that we can always replace 3 3 by 1 5 or 2 4 without changing the weight. To obtain the vectors for the weight next to the origin we make the substitutions

$$5 \to 3 , \quad 3 \to 1 , \quad 2 \to 0 , \quad 4 \to 0 ,$$

the latter two denoting that we delete terms arising from the displacement of 2 and of 4 . We then obtain for the three points on the m_1 axis

[3 3]
[1 5] , [1 3] , [1 1] , (6.11)
[2 4]

the multiplicities being the number of independent vectors, that is, 3, 1, 1 respectively.

For trilinear products we start with 3 3 3 at the centre, replace 3 3 by 1 5 or 2 4 , and displace, so obtaining

$$\begin{matrix} [3\;3\;3] & [1\;3\;3] & & & \\ [1\;3\;5], & [1\;1\;5], & [1\;1\;3], & [1\;1\;1]. & \\ [2\;3\;4] & [1\;2\;4] & & & \end{matrix} \qquad (6.12)$$

This is equivalent to 3 being added to the entries of the first term of (6.11), and 1 being added to all the terms. The latter produces a shift without change of multiplicity. We note that the multiplicity of the origin is still 3 . For quadrilinear products we may start with [3 3 3 3] and replacing 3 3 by 1 5 and 2 4 obtain 3+2+1 expressions. On displacing (6.12) by addition of a 1 we get altogether

$$\begin{matrix} [3\;3\;3\;3] & & & & \\ [1\;3\;3\;5] & [1\;3\;3\;3] & [1\;1\;3\;3] & & \\ [2\;3\;3\;4], & [1\;1\;3\;5], & [1\;1\;1\;5], & [1\;1\;1\;3], & [1\;1\;1\;1]. \\ [1\;2\;4\;5] & [1\;2\;3\;4] & [1\;1\;2\;4] & & \\ [1\;1\;5\;5] & & & & \\ [2\;2\;4\;4] & & & & \end{matrix} \qquad (6.13)$$

The multiplicity structure is now evident. Coming left from the highest weight we see that the multiplicity of the first two is 1 , the multiplicity of the second two is 1+2 , ... the multiplicity of the pth two is $\frac{1}{2}p(p+1)$. This is also the multiplicity of the weight at the centre for $r = 2p-2$ or $r = 2p-1$. Moreover we can easily write down from (6.13) the r-dimensional basis functions at any point on the positive m_1 axis.

We next examine how the multiplicities of weights vary as we go from a point on the m_1 axis along a layer in the first quadrant parallel to the slant side. Such a displacement is obtained from $1 \to 2$ or $4 \to 5$, and we find that we obtain for the boundary and for the next two layers, respectively,

[11...111], [11...112], [11...122],...,[12...2], [22...2] ;
[11...113], [11...123], [11...223],... [22...23] ;
[11...133], [11...233], [11...2233], ...
[11...115], [11...224], [11...2224], ...
[11...124], [11...125], [11...1225], ...

We see that the first two consist of simple weights, that in the case of the last replacement $4 \to 5$ adds nothing to $1 \to 2$ and that, as far as we have gone, the multiplicity is 3 along this layer. The reason for this constancy of multiplicity is that once 33 has appeared it can be replaced for the same weight only by 15 and 24 . Now on starting from [11...133] and making repeatedly the substitutions $1 \to 2$ the 33 remains and gives multiplicity 3 . The same will be true when we start with [11...1333] , [11...13333], etc. and we conclude that the multiplicity remains constant along each layer. We may therefore study the reductiom of $D[r]$ by merely considering the multiplicities at the points on the positive m_1 axis.

We consider separately the cases of $r = 2p$ and $r = 2p+1$. For the first one we put down the multiplicities for the representation [2p] at each weight on the positive m_1 axis starting at the centre, and subtract the corresponding multiplicity for D(0,2p) .

[2p]	1+2+..+(p+1),	1+2+..+p,	1+2+..p,	...		
	1+2+3,	1+2+3,	1+2,	1+2,	1,	1
D(0,2p)	p+1,	p,	p,	...		
	3,	3,	2,	2,	1,	1
difference	1+2+..+p,	1+2+..+p-1,	1+2+..+p-1,	...		
	1+2,	1+2,	1,	1,	0,	0 .

The difference gives the multiplicity for [2p-2], so we have for each weight the multiplicities as follows:

mult.[2p] = mult.[2p-2] + mult.D(0,2p)

mult.[2p-2] = mult.[2p-4] + mult.D(0,2p-2)

... ...

mult.[4] = mult.[2] + mult.D(0,4)

mult.[2] = mult.[0] + mult.D(0,2) .

Since D[0] ≡ D(0,0) , we deduce that the multiplicity at any point for D[2p] is the sum of the multiplicities of

D(0,2p), D(0,2p-2), ... D(0,2), D(0,0) .

In the case of $r = 2p+1$ we note that D[1] = D(0,1) and find that at any point on the axis the multiplicity for the D[2p+1] representation of $gl(5,\mathbb{C})$ is the sum of the multiplicities for

$$D(0,2p+1), \quad D(0,2p-1), \quad \ldots \quad D(0,3), \quad D(0,1) \quad .$$

We conclude therefore that the reduction (6.10), which we had suggested from inspection of the table (6.9), is in fact correct.

6d. Reduction under B_2 of certain other Irreducible Representations of $gl(5,\mathbb{C})$.

Since irreducible representations of $gl(5,\mathbb{C})$ with totally symmetric basis vectors reduce under B_2 , it may be asked whether this is so also for representations with totally anti-symmetric basis vectors. These will correspond to the partitions $[1^s]$, where s takes the values 1,2,3,4,5. The partition $[1]$ is a special case of (6.3), so its boundary is $\diamond$. By equation (6.1) the value of $N[1]$ is 5 and, since the representation $D(0,1)$ has boundary $\diamond$ and is five-dimensional, we conclude that the representation of $GL(5)$ corresponding to the partition $[1]$ is jus the $D^{(5)}(0,1)$ representation of B_2 . It therefore does not reduce.

The partition $[1^2]$ satisfies (6.4), so its related boundary is $\square$. The highest weight of the representation of $gl(5,\mathbb{C})$ corresponding to this partition has $\begin{bmatrix}1\\2\end{bmatrix}$ for basis vector, and

Fig.6.1 shows that the highest weight is therefore $\frac{1}{\sqrt{6}}(1,1)$. This is the highest weight of the $D^{(10)}(2,0)$ representation of B_2 and, since $N[1^2]$ is also 10, we conclude that the representation of $gl(5,\mathbb{C})$ corresponding to the partition $[1^2]$ is just $D^{(10)}(2,0)$ and is therefore irreducible under B_2 . The representation of $gl(5,\mathbb{C})$ corresponding to $[1^3]$ is also $D^{(10)}(2,0)$. Likewise the representation of $gl(5,\mathbb{C})$ corresponding to the partition $[1^4]$ is $D^{(5)}(1,0)$ and that corresponding to $[1^5]$ is $D^{(1)}(0,0)$, so all the representations of $gl(5,\mathbb{C})$ that have totally antisymmetric functions as bases are irreducible under B_2 .

The boundary of the weight diagram of a representation of $gl(5,\mathbb{C})$ corresponding to the partition $[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5]$ is □, when the conditions

$$\lambda_1 = \lambda_2 , \qquad \lambda_4 = \lambda_5$$

are both satisfied. The tensor component corresponding to the highest weight is

$$\begin{bmatrix} 1\ 1 & & \cdots & & 1\ 1 \\ 2\ 2 & & \cdots & & 2\ 2 \\ 3\ 3 & \cdots & & 3 & \\ 4\ 4 & \cdots\ 4 & & & \\ 5\ 5 & \cdots\ 5 & & & \end{bmatrix} .$$

If we wish to study the multiplicities on the boundary, we can start at the highest weight and work down the vertical side. This is done by the replacements $2 \to 3$, $3 \to 4$. In the first instance $2 \to 3$ will occur only if $\lambda_2 - \lambda_3 \geq 1$ and $3 \to 4$ only if $\lambda_3 - \lambda_4 \geq 1$. However, even if $\lambda_3 = \lambda_4$, we can at a later stage have $3 \to 4$ from a 3 that has already been put into the second row by a $2 \to 3$ replacement. The statement of multiplicity rules would thus involve specifications of $\lambda_2 - \lambda_3$, $\lambda_2 - \lambda_4$, $\lambda_3 - \lambda_4$. We therefore simplify the situation by putting

$$\lambda_3 = \lambda_4 = \lambda_5 = 0 ,$$

and consider only the representations whose bases are the tensors corresponding to the partitions $[2^2]$ and $[3^2]$.

We first examine how the 50-dimensional representation of $gl(5, \mathbb{C})$ related to the partition $[2^2]$ reduces under B_2. The tensor component for the highest weight is

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

and the weight is $\frac{1}{\sqrt{6}}(2,2)$.

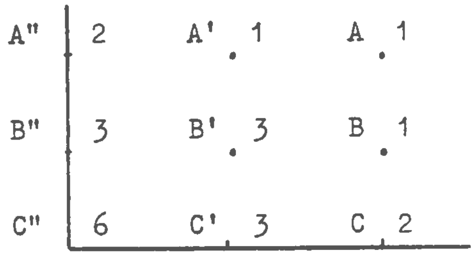


Fig.6.3 - The multiplicities of weights in the diagram for the $D[2^2]$ representation of $gl(5, \mathbb{C})$.

We mark this as A in Fig.6.3, and by writing down the Young tableaux that have the correct weights we assign to other weights in Fig.6.3 the following tensor components:

$$B \quad \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$C \quad \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

$$B' \quad \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$C' \quad \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$C'' \quad \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} .$$

These enable us to affix the multiplicities to every weight of Fig.6.3. The total number of weights for the representation is therefore

$$4\{(1+1+2) + (1+3+3)\} + 6 = 50 ,$$

which is correct. We have all the weights of the diagrams of $D^{(35)}(4,0)$, $D^{(14)}(0,2)$, $D^{(1)}(0,0)$, and the reduction is therefore

$$D[2^2] = D^{(35)}(4,0) \oplus D^{(14)}(0,2) \oplus D^{(1)}(0,0) .$$

It is straightforward but takes rather longer to show that the 175-dimensional representation of $gl(5,\mathbb{C})$ corresponding to the partition $[3^2]$ reduces under B_2 as

$$D[3^2] = D^{(84)}(6,0) \oplus D^{(81)}(2,2) \oplus D^{(10)}(2,0) .$$

As was pointed out at the end of section 5a, the reduction under B_2 of irreducible representations of $gl(5, \mathbb{C})$ may be interpreted as the reduction under the rotation group $SO(5, \mathbb{C})$ of irreducible representations of the general linear group $GL(5, \mathbb{C})$.

CHAPTER VII

YOUNG TABLEAUX AND WEIGHT DIAGRAMS FOR $gl(4,\mathbb{C})$ AND C_2

7a. Boundaries of Weight Diagrams associated with Young Diagrams.

We associate the basis vectors x_1, x_2, x_3, x_4 of the representation space of the algebra $gl(4,\mathbb{C})$ with the weights of

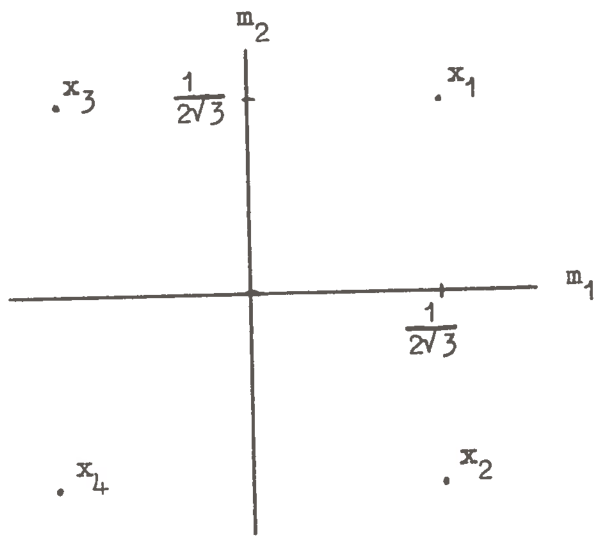


Fig.7.1 - Basis vectors associated with the weights of the $D^{(4)}(1,0)$ representation of C_2.

the $D^{(4)}(1,0)$ representation of C_2 in descending order as shown in Fig.7.1. For the irreducible representation $D[\lambda_1,\lambda_2,\lambda_3,\lambda_4]$ of $gl(4,\mathbb{C})$ associated with the partition $[\lambda_1,\lambda_2,\lambda_3,\lambda_4]$ the dimension $N[\lambda_1,\lambda_2,\lambda_3,\lambda_4]$, as deduced from (5.3) is given by

$$N[\lambda_1,\lambda_2,\lambda_3,\lambda_4] = \frac{1}{12} v\ (\lambda_1+3,\ \lambda_2+2,\ \lambda_3+1,\ \lambda_4)\ . \qquad (7.1)$$

The tensor component associated with the highest weight is

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 & \\ 3 & \cdots & 3 & & \\ 4 & \cdots & 4 & & \end{bmatrix} \qquad (7.2)$$

and the weight is $\lambda_1\underset{\sim}{m}(1) + \lambda_2\underset{\sim}{m}(2) + \lambda_3\underset{\sim}{m}(3) + \lambda_4\underset{\sim}{m}(4)$, where $\underset{\sim}{m}(1)$, $\underset{\sim}{m}(2)$, $\underset{\sim}{m}(3)$, $\underset{\sim}{m}(4)$ are the weights of x_1,x_2,x_3,x_4, respectively. Thus the highest weight is

$$\frac{1}{2\sqrt{6}}\,(\lambda_1 - \lambda_4 + \lambda_2 - \lambda_3,\ \lambda_1 - \lambda_4 - \lambda_2 + \lambda_3) \qquad (7.3)$$

and it is simple because (7.2) is the only bracket with λ_1 1's, λ_2 2's, λ_3 3's, λ_4 4's .

We see from (7.1) that for any positive integer τ

$$N[\lambda_1+\tau,\ \lambda_2+\tau,\ \lambda_3+\tau,\ \lambda_4+\tau] = N[\lambda_1,\lambda_2,\lambda_3,\lambda_4]\ .$$

The reason for this is that we go from a tensor component for $D[\lambda_1+\tau,\ \lambda_2+\tau,\ \lambda_3+\tau,\ \lambda_4+\tau]$ to the corresponding tensor component for $D[\lambda_1+\tau,\ \lambda_2+\tau,\ \lambda_3+\tau,\ \lambda_4+\tau]$ by adding to the bracket τ columns with entries 1,2,3,4, and this can be done in only one way. Since

$$\underset{\sim}{m}(1) + \underset{\sim}{m}(2) + \underset{\sim}{m}(3) + \underset{\sim}{m}(4) = 0\ ,$$

the positions of the weights are unaltered by these additions. Thus $D[\lambda_1{}^4]$ is one-dimensional with the tensor component

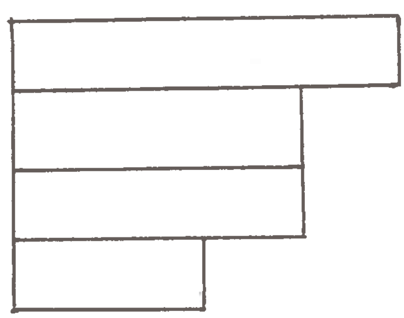
$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2 & \cdots & 2 & 2 \\ 3 & 3 & \cdots & 3 & 3 \\ 4 & 4 & \cdots & 4 & 4 \end{bmatrix}$$

having zero weight, and the weight diagram for the $D[\lambda_1+\tau, \lambda_2+\tau, \lambda_3+\tau, \lambda_4+\tau]$ representation of $gl(4, \mathbb{C})$ is the same as that for the $D[\lambda_1,\lambda_2,\lambda_3,\lambda_4]$ representation. To express this differently, the weight diagram for $D[\lambda_1,\lambda_2,\lambda_3,\lambda_4]$ is the same as the weight diagram for $D[\lambda_1-\lambda_4,\lambda_2-\lambda_4, \lambda_3-\lambda_4]$. This result will often simplify calculations.

Returning to (7.3) we study the conditions under which it is the highest weight of a $D(\lambda,0)$ or a $D(0,\mu)$ representation of C_2. On comparing it with (3.5) we see that it is the highest weight of a $D(\lambda,0)$ representation, if

$$\lambda_1-\lambda_4+\lambda_2-\lambda_3 = \lambda_1-\lambda_4-(\lambda_2-\lambda_3) ,$$

that is, if $\lambda_2=\lambda_3$. The partition is then $[\lambda_1, \lambda_2{}^2, \lambda_4]$ and the Young diagram has the shape



The highest weight is $\frac{1}{2\sqrt{6}}(\lambda_1-\lambda_4,\ \lambda_1-\lambda_4)$ and this is the highest weight of the $D(\lambda_1-\lambda_4,0)$ representation of C_2.
Now equations (7.1) and (3.7) give

$$N[\lambda_1,\lambda_2^2,\lambda_4] = \tfrac{1}{12}(\lambda_1-\lambda_2+1)(\lambda_1-\lambda_2+2)(\lambda_1-\lambda_4+3)(\lambda_2-\lambda_4+2)(\lambda_2-\lambda_4+1)$$

$$N(\lambda_1-\lambda_4,0) = \tfrac{1}{6}(\lambda_1-\lambda_4+1)(\lambda_1-\lambda_4+2)(\lambda_1-\lambda_4+3) .$$

By studying the variation of the quotient of the two expressions on the right hand side one finds without difficulty that

$$N[\lambda_1,\lambda_2^2,\lambda_4] \geq N(\lambda_1-\lambda_4,0) ,$$

the equality occurring only when $\lambda_2=\lambda_1$ or $\lambda_4=\lambda_2$. This means that all the weights of $D[\lambda_1^3,\lambda_4]$ and of $D[\lambda_1,\lambda_2^3]$ are accounted for by those of $D(\lambda_1-\lambda_4,0)$ and of $D(\lambda_1-\lambda_2,0)$ respectively, and therefore that these representations of $gl(4,\mathbb{C})$ do not reduce under C_2. In particular the one-dimensional $D[\lambda_1^4]$ and the representations $D[\lambda_1]$ and $D[\lambda_1^3]$ do not reduce.

We see from (7.3) that the highest weight is the highest weight of a $D(0,\mu)$ representation, if

$$\lambda_1-\lambda_4-\lambda_2+\lambda_3 = 0 ,$$

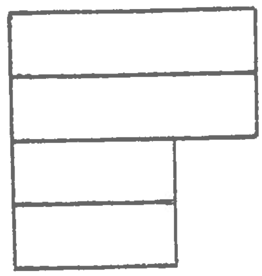
so that

$$\lambda_1-\lambda_2-(\lambda_3-\lambda_4) = 0$$

and therefore

$$\lambda_1 = \lambda_2 , \qquad \lambda_3 = \lambda_4 \tag{7.4}$$

giving a Young diagram



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This will include $D[\lambda_1{}^4]$ and also $D[\lambda_1{}^2]$. For (7.4) the highest weight (7.3) becomes $\frac{1}{\sqrt{6}}(\lambda_1 - \lambda_3, 0)$, which is the highest weight of the $D(0, \lambda_1 - \lambda_3)$ representation of C_2. From (7.1) and (3.13)

$$N[\lambda_1{}^2, \lambda_3{}^2] = \frac{1}{12}(\lambda_1-\lambda_3+1)(\lambda_1-\lambda_3+2)^2(\lambda_1-\lambda_3+3)$$

$$N(0, \lambda_1-\lambda_3) = \frac{1}{6}(\lambda_1-\lambda_3+1)(\lambda_1-\lambda_3+2)(2\lambda_1-2\lambda_3+3)$$

and so $N[\lambda_1{}^2,\lambda_3{}^2] > N(0,\lambda_1-\lambda_3)$ for $\lambda_1 > \lambda_3$. Apart from the one-dimensional $D[\lambda_1{}^4]$, all the representations $D[\lambda_1{}^2,\lambda_3{}^2]$ of $gl(4,\mathbb{C})$ reduce under C_2. We shall now examine how they reduce.

7b. Reduction of the $D[\lambda_1{}^2,\lambda_3{}^2]$ Representation of $gl(4,\mathbb{C})$ under C_2.

Let us put $\lambda_1 - \lambda_3 = r$ and, as the weight diagrams are the same for $D[\lambda_1{}^2,\lambda_3{}^2]$ as for $D[r^2]$, we shall consider the reduction problem for the latter. The highest weight for this is $\frac{1}{\sqrt{6}}(r,0)$ and corresponding tensor component is

$$\begin{bmatrix} 1\ 1\ 1 & \cdots & 1\ 1 \\ 2\ 2\ 2 & \cdots & 2\ 2 \end{bmatrix}. \tag{7.5}$$

If we move along the m_1 axis to the next weight $\frac{1}{\sqrt{6}}(r-1,0)$, then according to Fig.7.1 either a 1 is changed to a 3 or a 2 is changed to a 4. We may therefore say that when we move from any weight on the m_1 axis to the next one on the left, we replace a 2 by a 4 and change each 24 pair into a 13 pair in such a way that we obtain a bracket corresponding to a standard tableau, which implies among other things that the 1's are as far left as possible on the first row and the 4's as far right as possible on the second row. Starting with (7.5) and proceeding to the left we obtain

$$\begin{array}{ccccc}
\begin{bmatrix} \cdots 1111 \\ \cdots 4444 \end{bmatrix} & \begin{bmatrix} \cdots 111 \\ \cdots 444 \end{bmatrix} & \begin{bmatrix} \cdots 11 \\ \cdots 44 \end{bmatrix} & \begin{bmatrix} \cdots 11 \\ \cdots 24 \end{bmatrix} & \begin{bmatrix} \cdots 11 \\ \cdots 22 \end{bmatrix} \\
\left.\begin{matrix} \begin{bmatrix} \cdots 1112 \\ \cdots 3444 \end{bmatrix} \\ \begin{bmatrix} \cdots 1113 \\ \cdots 2444 \end{bmatrix} \end{matrix}\right\} & \left.\begin{matrix} \begin{bmatrix} \cdots 112 \\ \cdots 344 \end{bmatrix} \\ \begin{bmatrix} \cdots 113 \\ \cdots 244 \end{bmatrix} \end{matrix}\right\} & \left.\begin{matrix} \begin{bmatrix} \cdots 12 \\ \cdots 34 \end{bmatrix} \\ \begin{bmatrix} \cdots 13 \\ \cdots 24 \end{bmatrix} \end{matrix}\right\} & \begin{bmatrix} \cdots 12 \\ \cdots 23 \end{bmatrix} &
\end{array} \tag{7.6}$$

$$
\begin{array}{lll}
\left.\begin{array}{l}
\begin{bmatrix} \cdots & 1123 \\ \cdots & 2344 \end{bmatrix} \\
\begin{bmatrix} \cdots & 1133 \\ \cdots & 2244 \end{bmatrix} \\
\begin{bmatrix} \cdots & 1122 \\ \cdots & 3344 \end{bmatrix}
\end{array}\right\}
&
\left.\begin{array}{l}
\begin{bmatrix} \cdots & 123 \\ \cdots & 234 \end{bmatrix} \\
\begin{bmatrix} \cdots & 122 \\ \cdots & 334 \end{bmatrix}
\end{array}\right\}
&
\begin{bmatrix} \cdots & 22 \\ \cdots & 33 \end{bmatrix}
\\
&
\begin{bmatrix} \cdots & 222 \\ \cdots & 333 \end{bmatrix}
&
\\
\left.\begin{array}{l}
\begin{bmatrix} \cdots & 1222 \\ \cdots & 3334 \end{bmatrix} \\
\begin{bmatrix} \cdots & 1223 \\ \cdots & 2334 \end{bmatrix}
\end{array}\right\}
& &
\\
\begin{bmatrix} \cdots & 2222 \\ \cdots & 3333 \end{bmatrix} . & &
\end{array}
\qquad \text{(7.6) ctd.}
$$

The omitted entries are 1 on the first row and 2 on the second row. We have bracketed together tensor components with the same entries which therefore differ only in the distribution of the 2's and 3's among the rows. As we move left the multiplicities of the weights are

$$1, \quad 1+1, \quad 1+2+1, \quad 1+2+2+1, \quad 1+2+3+2+1, \quad \ldots .$$

It is not clear from what we have done so far whether the multiplicities will continue to increase. Let us confine our attention to brackets that all refer to one weight. There are two considerations to be borne in mind: the number of 4's in the second row and the number of 23's that are replacing the 14's . For no 4 in the second row there is no replacement and just one bracket. For the replacement of one 4 there are two entries $\begin{smallmatrix}2\\4\end{smallmatrix}$ and $\begin{smallmatrix}3\\4\end{smallmatrix}$ in the last column, provided that there was one 4 left in the second row; that is provided that we had at least two 4's to start with: otherwise we have multiplicity 2 . For the replacement of two 4's and with at least four 4's present originally we can end the first row with 22, 23, 33 . We may look on this as 2 placed before 2,3 for one replacement and 33 added on. If we had originally only three 4's , we now end the first row with 22 or 23 , so there are two brackets. When we make replacements of three 4's , we can end the first row with 2 placed before 22, 23, 33 and, if there were originally at least six 4's , also with 333 .

Hence for any even number 2t of 4's initially present the number of brackets increases by unity for the first t replacements and the number of brackets then is t+1 . When we make t+1 replacements, we have only (t-1) 4's left on the second row and the multiplicity is the same as for t-1 replacements. The multiplicity will decrease steadily by one until all the 4's have disappeared. Now we see from (7.6) that we have 2t 4's after 2t displacements from the highest weight, so the multiplicity here is

$$1 + 2 + 3 + \dots + t + (t+1) + t + (t-1) + \dots + 1$$
$$= t(t+1) + (t+1) = (t+1)^2 .$$

For $t = 2$ the multiplicity is 9 , which agrees with (7.6).

For an odd number $2t+1$ of 4's , t replacements give multiplicity $t+1$. When $t+1$ replacements have been made, there are left only t 4's in the second row and this gives again multiplicity $t+1$. When we make a further replacement, we reduce the multiplicity by one. The total multiplicity of the weight obtained by $2t+1$ displacements from the highest weight is therefore obtained by adding $t+1$ to $(t+1)^2$ and so is $(t+1)(t+2)$, which again agrees with (7.6) .

We combine these results saying that after s displacements along the m_1 axis from the highest weight the multiplicity for the $D[r^2]$ representation is $\frac{1}{4}(s+2)^2$ for s even and $\frac{1}{4}(s+1)(s+3)$ for s odd. Thus the multiplicity of the origin will be obtained by putting $s = r$. Our investigations were based on the fact that one tensor component at the weight in question is

$$\begin{bmatrix} 11 \ldots 11 \ldots 11 \\ \underbrace{22 \ldots 2}_{r-s}\underbrace{4 \ldots 44}_{s} \end{bmatrix} \qquad (7.7)$$

and our discussion centred entirely on the last s columns. According to Fig.7.1 we can find a tensor component at any point along the string of weights in the first quadrant going backwards at an angle $\frac{\pi}{4}$ to the m_1 axis, if we replace 2 by 3 a prescribed number of times in (7.7). In obtaining the tensor components we are precluded from replacing 23 on the second row by 14 because there are no vacancies in the first row for a 1 .

The multiplicities will therefore come entirely from replacing 14 in the last s rows by 23 . The multiplicity is therefore exactly the same as that derived from (7.7); in other words, the multiplicity is constant along the string of weights.

The tensor components for the weights on the string through the highest weight, obtained by putting $s = 0$ in (7.7) and making the replacement $2 \to 3$ repeatedly, are

$$\begin{bmatrix} 11 & \dots & 11 \\ 22 & \dots & 22 \end{bmatrix}, \quad \begin{bmatrix} 11 & \dots & 11 \\ 22 & \dots & 23 \end{bmatrix} \quad \dots \quad \begin{bmatrix} 11 & \dots & 11 \\ 33 & \dots & 33 \end{bmatrix} . \qquad (7.8)$$

If we make a vertical upwards displacement from one of the weights of the string, the tensor component is obtained by putting $2 \to 1$ in one of the brackets (7.8). The tensor component vanishes, so there are no weights in the first quadrant vertically above the string. It may be proved similarly that there are no weights of the $D[r^2]$ diagram outside the boundary of the $D(0,r)$ representation of C_2 in any quadrant. Thus the boundary of the $D(0,r)$ diagram is the boundary of the $D[r^2]$ diagram.

Since the multiplicities are constant for $D[r^2]$ along layers parallel to the boundary and since the same is true for $D(0,r)$, we write down from section 3e the multiplicities for $D(0,r)$, $D(0,r-1)$, ... $D(0,r-s)$ for weights on the m_2 axis obtained by 0,1, ... s left displacements from the highest weight:

s	2t+1	2t	...	5	4	3	2	1	0
D(0,r)	1 + t	1+t		3	3	2	2	1	1
D(0,r-1)	1 + t	t		3	2	2	1	1	
D(0,r-2)	t	t		2	2	1	1		
⋮									
D(0,r-2t)	1	1							
D(0,r-2t-1)	1								

For $s = 2t$ the total number of weights is

$$1 + 1 + 2 + 2 + \ldots + t + t + (t+1) ,$$

which is just the $\frac{1}{4}(s+2)^2$ found for $D[r^2]$. For $s = 2t+1$ we have an additional $t+1$, as we had for $D[r^2]$. The examination of the weight multiplicities has therefore given the reduction under C_2 of the $D[r^2]$ irreducible representation of $GL(4, \mathbb{C})$ as

$$D[r^2] = D(0,0) \oplus D(0,1) \oplus D(0,2) \oplus \ldots \oplus D(0,r) .$$

The reduction of $D[\lambda_1{}^2,\lambda_3{}^2]$ is therefore

$$D[\lambda_1{}^2,\lambda_3{}^2] = D(0,0) \oplus D(0,1) \oplus D(0,2) \oplus \ldots \oplus D(0,\lambda_1-\lambda_3) \tag{7.9}$$

7c. Reduction of the $D[\lambda_1,\lambda_2^2,\lambda_4]$ Representation of $gl(4,\mathbb{C})$ under C_2.

We saw in section 7a that the weight diagram of $D[\lambda_1,\lambda_2^2,\lambda_4]$ is the same as that of $D[\lambda_1-\lambda_4,(\lambda_2-\lambda_4)^4]$ and that its highest weight is the highest weight of the $D(\lambda_1-\lambda_4,0)$ representation of C_2. We shall write

$$\lambda_2 - \lambda_4 = a\ , \qquad \lambda_1 - \lambda_2 = b\ , \tag{7.10}$$

so that

$$\lambda_1 - \lambda_4 = a + b\ ,$$

and shall examine the reduction under C_2 of the irreducible representation $D[a+b,a^2]$ of $gl(4,\mathbb{C})$.

The tensor component for the highest weight of the $D[a+b,a^2]$ representation is

$$\begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ 2 & \cdots & 2 & & & \\ \underbrace{3 \ \cdots \ 3}_{a} & & & \underbrace{\qquad\qquad}_{b} & & \end{bmatrix} .$$

We would obtain the same weight by the replacement $23 \rightarrow 14$ but, as we have already the maximum number of 1's, we cannot construct a second tensor component and the weight is simple. We now proceed step-by-step to the left on the horizontal line through the highest weight. For each step we replace a 1 by a 3 and then replace 23 by 14 in all possible ways. We thus obtain in turn

$$\begin{bmatrix} 1 & \cdots & 1 & \cdots & 13 \\ 2 & \cdots & 2 & & \\ 3 & \cdots & 3 & & \end{bmatrix}, \qquad \begin{bmatrix} 1 & \cdots & 1\,1 & \cdots & 1\,1 \\ 2 & \cdots & 2\,3 & & \\ 3 & \cdots & 3\,4 & & \end{bmatrix}$$

giving multiplicity 2 for $a \geq 1$

$$\underbrace{\begin{bmatrix} 1 & \cdots & 1 & \cdots & 33 \\ 2 & \cdots & 2 & & \\ 3 & \cdots & 3 & & \end{bmatrix}, \begin{bmatrix} 1 & \cdots & 1 & \cdots & 13 \\ 2 & \cdots & 3 & & \\ 3 & \cdots & 4 & & \end{bmatrix}}_{a \geq 1}; \underbrace{\begin{bmatrix} 1 & \cdots & 1\,1 & \cdots & 11 \\ 2 & \cdots & 3\,3 & & \\ 3 & \cdots & 4\,4 & & \end{bmatrix}}_{a \geq 2}$$

giving multiplicity 2 for $a = 1$ and 3 for $a \geq 2$

$$\underbrace{\begin{bmatrix} 1 & .. & 1 & .. & 333 \\ 2 & .. & 2 & & \\ 3 & .. & 3 & & \end{bmatrix}, \begin{bmatrix} 1 & .. & 11 & .. & 133 \\ 2 & .. & 23 & & \\ 3 & .. & 34 & & \end{bmatrix}}_{a \geq 1}; \underbrace{\begin{bmatrix} 1 & .. & 11 & .. & 113 \\ 2 & .. & 33 & & \\ 3 & .. & 44 & & \end{bmatrix}}_{a \geq 2}; \underbrace{\begin{bmatrix} 1 & .. & 111 & .. & 111 \\ 2 & .. & 333 & & \\ 3 & .. & 444 & & \end{bmatrix}}_{a \geq 3}$$

giving multiplicity 2 for $a = 1$, 3 for $a = 2$, 4 for $a \geq 3$

$$\begin{bmatrix} 1 & \cdots & 1\,3\,3 & \cdots & 3 \\ 2 & \cdots & 2 & & \\ 3 & \cdots & 3 & & \end{bmatrix}, \quad \cdots \quad \begin{bmatrix} 1 & \cdots & 1 .. 1 & \cdots & 1 \\ 2 & \cdots & 3 .. 3 & & \\ 3 & \cdots & 4 .. 4 & & \end{bmatrix}$$

giving multiplicity 2 for $a = 1$, 3 for $a = 2$, ... b for

$a = b-1, \quad b+1 \quad$ for $\quad a \geq b$.

Since we can no longer add 1's , there can be no further increase in multiplicity for increasing values of a .

When we make another shift, we obtain

$$\underset{a \geq 1}{\begin{bmatrix} 1 \,..\, 113 \,..\, 3 \\ 2 \,..\, 23 \\ 3 \,..\, 34 \end{bmatrix}} ; \underset{a \geq 2}{\begin{bmatrix} 1 \,..\, 113 \,..\, 3 \\ 2 \,..\, 3 \\ 3 \,..\, 4 \end{bmatrix}} , \ldots \begin{bmatrix} 1 \,..\, 1 \,..\, 11 \,..\, 1 \\ 2 \,..\, 3 \,..\, 3 \\ 3 \,..\, \underbrace{4 \,..\, 4}_{b+1} \end{bmatrix}$$

giving multiplicity 1 for $a = 1$, 2 for $a = 2$, ... $b+1$ for $a \geq b+1$.

The effect of the additional 3 is therefore to reduce the previous multiplicity by one. When we have made $b+u$ shifts, we require at least u columns to get

$$\begin{bmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 & 3 & \ldots & 3 \\ 2 & \ldots & 2 & 3 & \ldots & 3 \\ 3 & \ldots & 3 & \multicolumn{3}{l}{\underbrace{4 \;\ldots\; 4}_{u}} \end{bmatrix}$$

and multiplicity 1 . If $a < u$, the tensor component vanishes and we are outside the weight diagram. Altogether we have multiplicity 0 for $a < u$, 1 for $a = u$, 2 for $a = u+1$, ...

$b+1$ for $a \geqslant u+b$. At this stage all the places in the first row have been filled with 1's and there is no further increase in multiplicity. We conclude that the multiplicity increases by one as we shift left to a maximum of $b+1$, but that this maximum is attained only if $a \geqslant b$.

We next consider the case of $a < b$. As we go left from the highest weight, the multiplicity increases steadily up to $a+1$ and no more. As more 3's are introduced, the multiplicity will drop by one for each additional 3 .

There are two observations that we may make about the tensor components associated with the weights on the horizontal line through the highest weight. We notice that they all vanish, if we make any of the substitutions

$$2 \to 1 , \quad 2 \to 3 , \quad 4 \to 1 , \quad 4 \to 3 .$$

This implies that we have no weight of the $D[a+b,a^2]$ representation in the sector of the upper half-plane bounded by the horizontal line and $m_2 = \pm m_1$. We may likewise establish a similar theorem for the four boundary lines of the diagram for the $D(a+b\,,0)$ representation of C_2 , and we deduce that these also constitute the boundary of the diagram of the $D[a+b,a^2]$ representation of $gl(4, \mathbb{C})$. The second point is that for $D[a+b,a^2]$ the multiplicities in the vertical line down from the highest weight are just those that we have obtained for the horizontal line through the highest weight.

We have now sufficient information to obtain the reduction under C_2 of the $D[a+b,a^2]$ representation of $gl(4, \mathbb{C})$. As we come down from the highest weight, we have for $a \geq b$

multiplicity 1 at $\frac{1}{2\sqrt{6}}(a+b,a+b)$

" 2 at $\frac{1}{2\sqrt{6}}(a+b,a+b-2)$

" 3 at $\frac{1}{2\sqrt{6}}(a+b,a+b-4)$

. . . .

" b+1 at $\frac{1}{2\sqrt{6}}(a+b,a-b)$, $\frac{1}{2\sqrt{6}}(a+b,a-b-2)$, etc. .

The points at which the multiplicity is increasing are the highest weights of the representations of C_2 :

$$D(a+b,0),\ D(a+b-2,1),\ D(a+b-4,2),\ \ldots\ D(a-b,b)\ .$$

Moreover the multiplicities on this side, and therefore on the whole boundary, are accounted for by these representations. Hence it would seem that for $a \geq b$ the reduction might be

$$D[a+b,a^2] = D(a+b,0) \oplus D(a+b-2,1) \oplus D(a+b-4,2) \oplus \ldots \oplus D(a-b,b)\ . \qquad (7.11)$$

Similarly, when $a < b$, it would seem that the reduction might be

$$D[a+b,a^2] = D(a+b,0) \oplus D(a+b-2,1) \oplus D(a+b-4,2) \oplus \ldots \oplus D(b-a,a)\ . \qquad (7.12)$$

To establish (7.11) and (7.12) it would be sufficient to show that the sum of the weights, due account being taken of their multiplicities, in the representations on the right-hand sides of (7.11) and (7.12) is equal to $N[a+b,a^2]$. According to (7.1)

$$N[a+b,a^2] = \frac{1}{12}(a+1)(a+2)(b+1)(b+2)(a+b+3) , \qquad (7.13)$$

and we note that this expression is symmetrical in a and b . The sum of the weights for the representations on the right-hand side of (7.11) is by (3.22)

$$\sum_{r=0}^{b} N(a+b-2r,r) = \frac{1}{6}(a+b+3)\sum_{r=0}^{b}(r+1)(a+b+1-2r)(a+b+2-r)$$

$$= \frac{1}{6}(a+b+3)\sum_{r=0}^{b}\{(a+b+1)(a+b+2) + r(a^2+2ab+b^2-3) - 3r^2(a+b+1) + 2r^3\}$$

$$= \frac{1}{12}(a+b+3)(a+1)(a+2)(b+1)(b+2)$$

$$= N[a+b,a^2] ,$$

by (7.13). We can go from the right-hand side of (7.11) to the right-hand side of (7.12) by interchanging a and b and, since the expression for the sums of the weights just calculated is symmetric in a and b , we have now

$$\sum_{r=0}^{a} N[a+b-2r,r] = N[a+b,a^2] .$$

The two sums are identical for $a = b$. The reductions (7.11) and (7.12) have therefore been established. The reductions are possible because all the weight diagrams for the C_2 algebra actually reach the boundary of the weight diagram for the $D[a+b, a^2]$ representation of $gl(4, \mathbb{C})$.

We return to the more general notation $D[\lambda_1, \lambda_2{}^2, \lambda_4]$ using (7.10). For $\lambda_2 - \lambda_4 \geq \lambda_1 - \lambda_2$

$$D[\lambda_1, \lambda_2{}^2, \lambda_4] = D(\lambda_1 - \lambda_4, 0) \oplus D(\lambda_1 - \lambda_4 - 2, 1) \oplus D(\lambda_1 - \lambda_4 - 4, 2) \oplus \cdots \oplus D(2\lambda_2 - \lambda_1 - \lambda_4, \lambda_1 - \lambda_2) \tag{7.14}$$

and for $\lambda_2 - \lambda_4 \leq \lambda_1 - \lambda_2$

$$D[\lambda_1, \lambda_2{}^2, \lambda_4] = D(\lambda_1 - \lambda_4, 0) \oplus D(\lambda_1 - \lambda_4 - 2, 1) \oplus D(\lambda_1 - \lambda_4 - 4, 2) \oplus \cdots \oplus D(\lambda_1 + \lambda_4 - 2\lambda_2, \lambda_2 - \lambda_4) . \tag{7.15}$$

When $\lambda_2 - \lambda_4 = \lambda_1 - \lambda_2$, the last representation in the reduction is $D(0, \lambda_2 - \lambda_4)$. Then the multiplicities on the boundary increase until we come to the middle point of a bounding side.

7d. Reduction of the $D[\lambda_1{}^2,\lambda_3,\lambda_4]$ Representation of $gl(4,\mathbb{C})$ under C_2.

The weight diagram of $D[\lambda_1{}^2,\lambda_3,\lambda_4]$ is the same as that of $D[(\lambda_1-\lambda_4)^2,\lambda_3-\lambda_4]$. We therefore study the reduction of the $D[p^2,r]$ representation of $gl(4,\mathbb{C})$, where

$$\lambda_1 - \lambda_4 = p , \qquad \lambda_3 - \lambda_4 = r . \tag{7.16}$$

According to (7.3) the highest weight is $\frac{1}{2\sqrt{6}}(2p-r,r)$, which is also the highest weight of the $D(r,p-r)$ representation of C_2 according to (3.5). Since the boundary is octagonal, we shall use the letters in Fig.3.2 to designate the weights.

The tensor component corresponding to the highest weight A is

$$\left[\begin{matrix} \overbrace{1\ 1\ \ldots\ 1\ \ldots\ 1\ 1}^{p} \\ 2\ 2\ \ldots\ 2\ \ldots\ 2\ 2 \\ \underbrace{3\ 3\ \ldots\ 3}_{r} \end{matrix}\right] . \tag{7.17}$$

Since we cannot replace 23 by 14 in this, the highest weight is simple. According to Fig.7.1 we go vertically downwards from the highest weight by the replacement $3 \to 4$. This will give a unique tensor component after each replacement, so the weights here are simple. Since any of the replacements

$$4 \to 2,\ 4 \to 1,\ 4 \to 3,\ 3 \to 1,\ 3 \to 2 \tag{7.18}$$

will annihilate these tensor components, there are no weights for $D[p^2,r]$ to the right of the vertical line down from the highest weight. We go from (7.17) to the tensor components for the weights along the slant side ABC ... by repeated replacements $2 \to 3$. The weights will be simple and the tensor components will be annihilated by (7.18). We conclude that the boundary of the $D(r,p-r)$ diagram is the boundary of the $D[p^2,r]$ diagram and that the weights on the boundary of this are simple.

As a preparation for the reduction of $D[p^2,r]$ under C_2 we examine the multiplicities along the string A,A',A",A'", ... The tensor components for A' will be obtained by replacing 2 by 4 in (7.17) and possibly later replacing 14 by 23. Now when 2 is replaced by 4, the 4 will have to be in the third row and the displaced 3 will have to go to the end of the second row, or alternatively the 4 will replace the 2 at the end of the second row. Altogether we shall have

$$A' \quad \begin{bmatrix} 1 \,..\, 1 \,..\, 11 \\ 2 \,..\, 2 \,..\, 23 \\ 3 \,..\, 4 \end{bmatrix} ; \begin{bmatrix} 1 \,..\, 1 \,..\, 11 \\ 2 \,..\, 2 \,..\, 24 \\ 3 \,..\, 3 \end{bmatrix} , \begin{bmatrix} 1 \,..\, 1 \,..\, 12 \\ 2 \,..\, 2 \,..\, 23 \\ 3 \,..\, 3 \end{bmatrix} . \qquad (7.19)$$

If we refer to (7.6), we note that for A the last two entries of the first two rows are the same as those in the last bracket of (7.6) and that the last two entries of the first two rows in the last two brackets of (7.19) are the same as the last two in the two brackets just to the left of the last bracket of (7.6).

In fact these entries arise merely from the $14 \to 23$ replacement. Going along the string to the next weight we have

$$A''\begin{bmatrix}1..11..11\\ 2..22..23\\ 3..44\end{bmatrix};\begin{bmatrix}1..1..11\\ 2..2..34\\ 3..4\end{bmatrix},\begin{bmatrix}1..1..12\\ 2..2..33\\ 3..4\end{bmatrix};\begin{bmatrix}1..1..11\\ 2..2..44\\ 3..3\end{bmatrix},$$

$$\begin{bmatrix}1..1..12\\ 2..2..34\\ 3..3\end{bmatrix},\begin{bmatrix}1..1..13\\ 2..2..24\\ 3..3\end{bmatrix},\begin{bmatrix}1..1..22\\ 2..2..33\\ 3..3\end{bmatrix}.$$

Again we may compare the last four brackets with the four brackets in the middle of (7.6). The multiplicities are therefore

A'''	A''	A'	A
$1+(1+1)+(1+2+1)+(1+2+2+1)$	$1+(1+1)+(1+2+1)$	$1+(1+1)$	1
$1+2+4+6$	$1+2+4$	$1+2$	1 .

(7.20)

The next point to note is that, when we proceed from one of these weights along a layer parallel to the slant side, the multiplicity is constant. The line of reasoning runs much the same as in section 7b, and the result may be checked by constructing explicitly the tensor components.

We refer to Fig.3.5 of section 3d and write down successively the multiplicities at $A, A', A'', A''', \ldots$ for representations of C_2 whose highest weights are $A, A', A'', A''', \ldots$

under the assumption that we have at least five weights on every side of the octagon:

A""	A'"	A"	A'	A
9	6	4	2	1
6	4	2	1	
4	2	1		
2	1			
1 .				

It thus appears from (7.20) that the multiplicities for $D[p^2,r]$ are found by superimposing the multiplicities of the C_2 representations

$$D(r,p-r), \quad D(r,p-r-1), \quad D(r,p-r-2), \quad D(r,p-r-3), \quad \ldots \; . \qquad (7.21)$$

These will also give the correct multiplicities in the layers parallel to the slant side. We have so far not paid attention to the layers parallel to the vertical side. If, however, the representations (7.21) exhaust all the weights of $D[p^2,r]$, then automatically the layers parallel to the vertical side are taken care of.

We therefore propose that the irreducible representation $D[p^2,r]$ of $gl(4,\mathbb{C})$ reduces under C_2 into irreducible representations of C_2 as follows:

$$D[p^2,r] = D(r,p-r) \oplus D(r,p-r-1) \oplus D(r,p-r-2) \oplus \ldots \oplus D(r,0) . \tag{7.22}$$

To establish this it will be necessary and sufficient to prove that

$$N(r,p-r) + N(r,p-r-1) + \ldots + N(r,0) = N[p^2,r] .$$

According to (3.22)

$$\sum_{t=0}^{p-r} N(r,p-r-t) = \frac{1}{6}\sum_{t=0}^{p-r} (1+r)(1+p-r-t)(2+p-t)(3+2p-r-2t)$$

$$= \frac{1+r}{6}\sum_{t=0}^{p-r} \{6+13p+9p^2+2p^3-8r-10pr+2r^2-3p^2r+pr^2$$

$$-t[13+18p+6p^2-10r-6pr+4r^2] + 3t^2[3+2p-r] - 2t^3\}$$

$$= \frac{1}{12} (1+r)(p-r+1)(p+2)(p+3)(p-r+2)$$

$$= N[p^2,r] ,$$

by (7.1). This establishes the result (7.22). On returning to (7.16) we deduce the reduction

$$D[\lambda_1^2,\lambda_3,\lambda_4] = D(\lambda_3-\lambda_4,\lambda_1-\lambda_3) \oplus D(\lambda_3-\lambda_4,\lambda_1-\lambda_3-1) \oplus D(\lambda_3-\lambda_4,\lambda_1-\lambda_3-2) \oplus \ldots + D(\lambda_3-\lambda_4,0) , \tag{7.23}$$

which we had set out to perform.

7e. The General Problem of the Reduction of the $D[\lambda_1,\lambda_2,\lambda_3,\lambda_4]$ Representation of $gl(4,\mathbb{C})$ under C_2.

The discussion of section 7a and equations (7.9),(7.14),(7.15),(7.23) show how the $D[\lambda_1,\lambda_2,\lambda_3,\lambda_4]$ representation of $gl(4,\mathbb{C})$ reduces under C_2 when any two λ's are equal. When they are all unequal, we write

$$\lambda_1 - \lambda_4 = p\ , \quad \lambda_2 - \lambda_4 = q\ , \quad \lambda_3 - \lambda_4 = r$$

and consider $D[p,q,r]$. The tensor component for the highest weight, which is simple, is

$$\left[\begin{array}{l} \overbrace{1\ 1\ \ \cdots\cdots\ \ 1\ 1}^{p} \\ 2\ \cdots\ 2 \\ \underbrace{3\ \cdots\ 3}_{r} \end{array}\right] .$$
$$\underbrace{\qquad\qquad\qquad}_{q}$$

The highest weight is $\frac{1}{2\sqrt{6}}(p+q-r,p-q+r)$ and this by (3.5) is the highest weight of the $D(p-q+r,q-r)$ representation of C_2.

It may be proved, by the method employed in the previous section, that the boundary of the weight diagram of the $D(p-q+r,q-r)$ representation of C_2 is also the boundary of the weight diagram of the $D[p,q,r]$ representation of $gl(4,\mathbb{C})$. Clearly it is also the boundary of the diagram for all $D[p,q+\tau,r+\tau]$ representations but, since $N[p,q+\tau,r+\tau] \neq N[p,q,r]$ for $\tau \neq 0$, the weight diagrams are different.

Information about the reduction of $D[p,q,r]$ may be obtained by peeling off the weights belonging to $D(p-q+r,\ q-r)$ and other representations of C_2. The weights on the boundary for $D[p,q,r]$ are not simple and the reductiom will not be an easy matter. Indeed, on account of the difficulty, pointed out in section 3f, of specifying multiplicities for the $D(\lambda,\mu)$ representations of C_2, it will not be possible to deduce from a study of weight diagrams alone a closed formula for the reduction. By constructing the tensor components for weights on and inside the boundary one can obtain the partial reduction

$$D[p,q,r] = D(p-q+r,q-r) \oplus D(p-q+r,q-r-1) \oplus D(p-q+r,q-r-2) \oplus \dots \oplus D(p-q+r-2,q-r+1) \oplus D(p-q+r-2,q-r) \oplus D(p-q+r-2,q-r-1) \oplus \dots \quad (7.24)$$

This may be useful as a guide to guessing the reduction in a specific case. One would have to check the dimensions and also to verify that the multiplicities at each point are correct. If, for example, we wish to reduce $D[6,5,3]$, we may be guided by (7.24) to write

$$D[6,5,3] = D(4,2) \oplus D(4,1) \oplus D(4,0) \oplus D(2,3) \oplus D(2,2) \oplus D(2,1) . \quad (7.25)$$

The total number of weights on each side is 630. To check the multiplicity at the point $\frac{1}{\sqrt{6}}(1,1)$ we construct explicitly the tensor components for the $D[6,5,3]$ representation of $gl(4, \mathbb{C})$ and find that there are 23 of them. By employing the rules given in sections 3d and 3f and using the graphical method whenever the rules are not stated we obtain for the representations on the right-hand side of (7.25) the multiplicities at $\frac{1}{\sqrt{6}}(1,1)$ to be, respectively,

$$6, 4, 2, 5, 4, 2 .$$

This adds up correctly to the multiplicity 23 and so verifies (7.25) .

The reduction under C_2 of the irreducible representations of $gl(4, \mathbb{C})$ may, as pointed out in section 5a, be interpreted in terms of the reduction under the symplectic group $Sp(4, \mathbb{C})$ of irreducible representations of $GL(4, \mathbb{C})$.

CHAPTER VIII

YOUNG TABLEAUX AND WEIGHT DIAGRAMS FOR $gl(7,\mathbb{C})$, $o(7,\mathbb{C})$ AND G_2

8a. Relations between $gl(7,\mathbb{C})$, $o(7,\mathbb{C})$ and G_2 .

We associate the basis vectors $x_1,x_2,x_3,x_4,x_5,x_6,x_7$ of the representation space of the algebra $gl(7,\mathbb{C})$ with the weights of the $D^{(7)}(1,0)$ representation of G_2 in descending

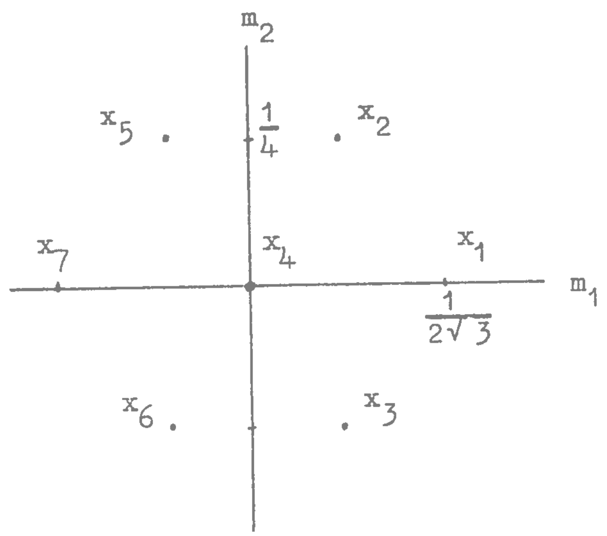


Fig.8.1 - Basic vectors associated with the weights of the $D^{(7)}(1,0)$ representation of G_2 .

order as shown in Fig.8.1. The root diagram for G_2 is that of Fig.1.3 and the elements of this Lie algebra are consequently

$$H_1,H_2,E_1,E_2,E_3,E_4,E_5,E_6,E_{-1},E_{-2},E_{-3},E_{-4},E_{-5},E_{-6} \ . \qquad (8.1)$$

The matrix representing H_1 is diagonal and the diagonal elements are the first components of the weights in the representation of G_2 in question. We saw in section 4c that the weight diagrams are symmetric about the m_1 m_2 axes, so the matrix representing H_1 is traceless and similarly the matrix representing H_2 is traceless. Moreover the non-vanishing elements in the matrix representative of any displacement operator E_α are off the diagonal. Hence all the matrices corresponding to (8.1) are traceless. This shows that G_2 is a subalgebra of $sl(7, \mathbb{C})$ and by exponentiating, as explained in section 5a, it may be deduced that Cartan's exceptional semi-simple group of rank 2, which we shall denote by G, is a subgroup of $SL(7, \mathbb{C})$.

It may be verified that G is also a subgroup of $SO(7, \mathbb{C})$, the rotation group in seven dimensions over the complex field. It was found (McConnell 1968) that G has an invariant bilinear form which on assigning basis vectors as in Fig.8.1 becomes

$$- x_1y_7 - x_2y_6 + x_3y_5 + x_4y_4 + x_5y_3 - x_6y_2 - x_7y_1 .$$

In particular

$$- 2\, x_1x_7 - 2\, x_2x_6 + 2\, x_3x_5 + x_4^{\,2} \qquad (8.2)$$

is invariant. If now we put

$$\frac{x_1+x_7}{\sqrt{2}} = i\, z_1 , \qquad \frac{-x_1+x_7}{\sqrt{2}} = z_7$$

$$\frac{x_2+x_6}{\sqrt{2}} = i\, z_2 , \qquad \frac{-x_2+x_6}{\sqrt{2}} = z_6$$

$$\frac{x_3+x_5}{\sqrt{2}} = z_3 , \qquad \frac{x_3-x_5}{\sqrt{2}} = i\, z_5$$

$$x_4 = z_4 ,$$

the invariant (8.2) becomes

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 + z_7^2 .$$

G is therefore a subgroup of the orthogonal group $O(7, \mathbb{C})$ and, as it is also a subgroup of $SL(7, \mathbb{C})$, it is a subgroup of $SO(7, \mathbb{C})$. The order of the Lie algebra $so(7, \mathbb{C})$, or B_3 in Cartan's notation, is 21, while the order of G_2 , being the number of elements (8.1), is 14. Thus G_2 is not isomorphic to $so(7, \mathbb{C})$ and G is not isomorphic to $SO(7, \mathbb{C})$.

We might proceed, as in previous chapters, to examine the reduction of representations of $gl(7, \mathbb{C})$ under G_2 using the method of weight diagrams. Since however G_2 is a subalgebra of $so(7, \mathbb{C})$, it will be of greater interest to study first of all the reduction of representations of $gl(7, \mathbb{C})$ under $so(7, \mathbb{C})$. This we shall do employing the theory of group characters for the respective groups.

8b. Reducibility of Representations of $GL(7, \mathbb{C})$ with totally Symmetric Bases.

If $\{\lambda\}$ is the Schur S-function for the general complex linear group in a finite number of dimensions corresponding to a partition $[\lambda_1, \lambda_2, \ldots \lambda_t]$, say, and (λ) is the S-function for the orthogonal group, then (Littlewood 1940, p.240)

$$\{\lambda\} = (\lambda) + \Sigma g_{\delta\eta\lambda}(\eta) , \qquad (8.3)$$

summed for the S-functions $\{\delta\}$ of the set δ such that $\{\lambda\}$ appears in a product $\{\delta\}\{\eta\}$, $g_{\delta\eta\lambda}$ being the coefficient of $\{\lambda\}$ in the product. The set δ is that of partitions into even parts only:

$$[2], [4], [2^2], [6], [4,2], \ldots . \qquad (8.4)$$

For odd n the dimension of the representation of $O(n, \mathbb{C})$ corresponding to the partition $[\lambda_1, \lambda_2, \ldots \lambda_t]$ is (Littlewood 1940, p.236)

$$\frac{\prod\limits_{p<q} [(a_p - a_q)(a_p + a_q + n - 2)] \prod (2a_2 + n - 2)}{(n - 2)! \, (n - 4)! \, \ldots \, 1!} ,$$

where

$$a_s = \lambda_s - s + 1 ; \qquad s = 1, 2, \ldots \frac{n-1}{2} . \qquad (8.5)$$

The dimension for $O(7, \mathbb{C})$ is therefore

$$\frac{1}{3!5!}\{(a_1-a_2)(a_1-a_3)(a_2-a_3)(a_1+a_2+5)(a_1+a_3+5)(a_2+a_3+5) \times$$

$$(2a_1+5)(2a_2+5)(2a_3+5) \quad . \qquad (8.6)$$

We apply this to a representation with a basis of totally symmetric functions corresponding to the partition $[r]$. Equation (8.3) becomes

$$\{r\} = (r) + \Sigma g_{\delta\eta r} (\eta) \, .$$

We take different members of (8.4) for δ and find the corresponding η's by application of nodes (Littlewood 1940,p.94). For $\delta = [2]$ we have $[\eta] = [r-2]$ only and this occurs only once, so that $g_{\delta\eta r} = 1$. For $\delta = [4]$ we have $\eta = [r-4]$ once only and similarly for $\delta = [6]$, $[8]$, etc. . For $\delta = [2^2]$ it is not possible to find an $[\eta]$ that will give $[r]$ according to the rules of application of nodes. Similarly we get no contribution for $[4,2]$, $[6,2]$, $[4,2^2]$, ... Thus

$$\{r\} = (r) + (r-2) + (r-4) + \ldots + \begin{cases} (1) & \text{for } r \text{ odd} \\ (0) & \text{for } r \text{ even} \end{cases}$$

and under $O(7,\mathbb{C})$ we have a reduction of the representation of $GL(7,\mathbb{C})$ into the direct sum of representations of $O(7,\mathbb{C})$ corresponding to the partitions $[r]$, $[r-2]$ $[r-4]$ etc. . We write the result

$$D[r] = d(r) \oplus d(r-2) \oplus d(r-4) \oplus \ldots \oplus \begin{cases} d(1) & \text{for } r \text{ odd} \\ d(0) & \text{for } r \text{ even} . \end{cases} \quad (8.7)$$

If we denote by m any of the $r, r-2, r-4, \ldots$, we deduce from (8.5) that

$$a_1 = m \ , \quad a_2 = -1 \ , \quad a_3 = -2$$

and from (8.6) that the dimension $N(m)$ of the $d(m)$ representation of $O(7, \mathbb{C})$ is given by

$$N(m) = (1+m)(1+\tfrac{m}{2})(1+\tfrac{m}{3})(1+\tfrac{m}{4})(1+\tfrac{2m}{5}) \ . \qquad (8.8)$$

Moreover since self-associate Young diagrams do not exist for an odd dimensionality, e.g. for $GL(7, \mathbb{C})$, the irreducible representations of $O(7, \mathbb{C})$ do not further reduce under $SO(7, \mathbb{C})$ (Weyl 1946,pp.155 and 164).

We next examine whether these representations of $SO(7, \mathbb{C})$ reduce under G . We work in terms of the Lie algebras considering the possible reduction of $so(7, \mathbb{C})$ under G_2 . The basis vector of the highest weight in the representation of $so(7, \mathbb{C})$ corresponding to the partition $[m]$ is

$$x_1^{(1)} x_1^{(2)} \ldots x_1^{(m)} \quad .$$

This is also the basis vector of the highest weight of the

$D(m,0)$ representation of G_2. From equation (4.9), viz.

$$N(\lambda,\mu) = (1+\lambda)(1+\mu)[1+\tfrac{1}{2}(\lambda+\mu)][1+\tfrac{1}{3}(\lambda+2\mu)][1+\tfrac{1}{4}(\lambda+3\mu)]$$

$$\times\ [1+\tfrac{1}{5}(2\lambda+3\mu)] \qquad (8.9)$$

for the dimension $N(\lambda,\mu)$ of the $D(\lambda,\mu)$ representation of G_2 we deduce that

$$N(m,0) = N(m) ,$$

as given by (8.8) . This shows that the representations of $so(7,\mathbb{C})$ with bases consisting of totally symmetric tensor components do not further reduce under G_2 , and it will follow that the representations of $SO(7,\mathbb{C})$ with such bases do not further reduce under the exceptional group G .

We may therefore deduce from (8.7) that

$$D[r] = D(r,0) \oplus D(r-2,0) \oplus D(r-4,0) \oplus \dots \oplus \begin{cases} D(1,0) & \text{for } r \text{ odd} \\ D(0,0) & \text{for } r \text{ even} \end{cases} ,$$

where $D[r]$ may be taken as referring to either $GL(7,\mathbb{C})$ or $gl(7,\mathbb{C})$ and $D(s,0)$ is then taken to refer, respectively, to G or G_2 . The reduction for the algebras could also, in principle, be obtained by constructing the weight diagram for the representation $D[r]$ and removing from it successively the weights of the diagrams for $D(r,0)$, $D(r-2,0)$ etc. . However, since as we saw in section 4e, it is not easy to state general multiplicity rules for G_2 diagrams, it would be difficult to obtain by this method the reduction for r greater than 3, say.

To sum up, the irreducible representations of the general linear group in seven dimensions, which have as a basis totally symmetric tensor components, reduce under the rotation group in seven dimensions but do not further reduce under the exceptional group.

8c. Reducibility of Representations of $GL(7, \mathbb{C})$ with totally Antisymmetric Bases.

We consider the representations corresponding to the partitions $[1^r]$, where $r = 1,2,\dots 7$. Then (8.3) becomes

$$\{1^r\} = (1^r) + \Sigma\, g_{\delta\eta 1^r}(\eta) \quad .$$

For $\delta = [2]$ we have to construct one column of elements by adding two nodes with the same label. This is not permitted by the rule of application of nodes which forbids the placing of two nodes with the same label in one column. Hence the g in the above equation vanishes for $\delta = [2]$ and it likewise vanishes for every other δ of (8.4). Thus

$$\{1^r\} = (1^r),$$

and the representation does not reduce under $O(7,\mathbb{C})$. Since it is possible to place two nodes with the same label in any Young diagram that contains more than one column, we see that the only representations of $GL(7,\mathbb{C})$ that do not reduce are those corresponding to the partition $[1^r]$.

We next examine the possible reduction as we go from $gl(7,\mathbb{C})$ to G_2. For $gl(7,\mathbb{C})$ the dimension $N[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7]$ of the representation corresponding to the partition $[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7]$ is, according to (5.3), given by

$$N[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7] = \frac{v(6+\lambda_1,5+\lambda_2,4+\lambda_3,3+\lambda_4,2+\lambda_5,1+\lambda_6,\lambda_7)}{1!\ 2!\ 3!\ 4!\ 5!\ 6!} \tag{8.10}$$

with

$$v(z_1,z_2,z_3,z_4,z_5,z_6,z_7) = \prod_{i<j}(z_i-z_j),$$

and so

$$N[1] = 7 = N[1^6]$$

$$N[1^2] = 21 = N[1^5]$$

$$N[1^3] = 35 = N[1^4]$$

$$N[1^7] = 1.$$

The representation $D[1^7]$ of $gl(7, \mathbb{C})$ is just the $D^{(1)}(0,0)$ of G_2 and the $D[1]$ or $D[1^6]$ representation of $gl(7, \mathbb{C})$ is just the $D^{(7)}(1,0)$ of G_2. Thus for the partitions $[1]$, $[1^6]$, $[1^7]$ there is no reduction as we go from $gl(7, \mathbb{C})$ to G_2, or from $GL(7, \mathbb{C})$ to G. Furthermore it is easy to prove (cf. McConnell 1968) that $D[1^2]$ or $D[1^5]$ of $gl(7, \mathbb{C})$ reduces under G_2 to

$$D^{(7)}(1,0) \oplus D^{(14)}(0,1) .$$

Let us therefore pay attention to $D[1^3]$. For this the basis vectors are totally antisymmetric trilinear forms. Since Fig.8.1 and indeed all G_2 weight diagrams are invariant for a rotation about the origin through an angle $\frac{\pi}{3}$, we chiefly consider basis vectors that have weights in the first quadrant enclosed between the m_1 axis and a radius vector making with it an angle $\frac{\pi}{3}$. We write pqr with $p \leq q \leq r$ at a point

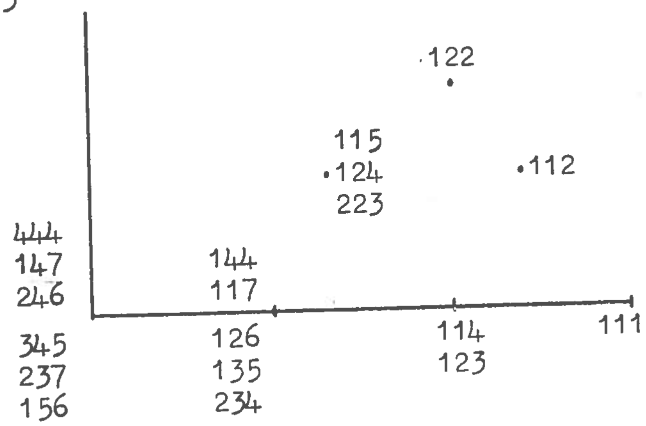


Fig.8.2 - Weights for trilinear forms in the basis vectors of $D^{(7)}(1,0)$.

where the weight is obtained by taking the vectorial sum of the weights related to x_p, x_q, x_r in Fig.8.1, and thus obtain Fig.8.2. The basis vectors for $D[1^3]$ must have three different entries, so from this figure we can immediately put down in Fig.8.3 the multiplicities for this representation.

```
      1   1   1
    1   3   3   1
  1   3   5   3   1
    1   3   3   1
      1   1   1
```

Fig.8.3 - The weight multiplicities for the $D[1^3]$ representation of $gl(7, \mathbb{C})$.

```
      1   1   1
    1   2   2   1
  1   2   3   2   1
    1   2   2   1
      1   1   1
```

Fig.8.4 - The multiplicities for the $D^{(27)}(2,0)$ representation of G_2 .

It is found from section 4e that the multiplicities for the $D^{(27)}(2,0)$ representation of G_2 are as in Fig.8.4. On subtracting these from those of Fig.8.3 we obtain the reduction under G_2 of the $D[1^3]$ representation of $gl(7, \mathbb{C})$:

$$D^{(35)}[1^3] = D^{(27)}(2,0) \oplus D^{(7)}(1,0) \oplus D^{(1)}(0,0) \; .$$

Since there is no reduction in going from $gl(7,\mathbb{C})$ to $so(7,\mathbb{C})$, we may interpret this equation as expressing the reduction of the $d(1^3)$ representation of $so(7,\mathbb{C})$ under G_2.

We conclude that irreducible representations with totally antisymmetric bases, and these alone, do not reduce under $SO(7,\mathbb{C})$ and that, apart from the almost trivial cases of the one- and the seven-dimensional representations, they do reduce under the exceptional group.

8d. Reducibility of Representations of $GL(7,\mathbb{C})$ with Bases having Mixed Symmetry.

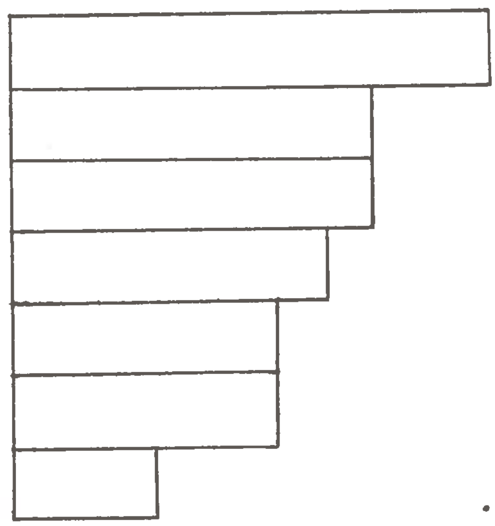
The tensor component for the highest weight of the irreducible representation $D[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7]$ of $GL(7,\mathbb{C})$ that corresponds to the partition $[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7]$ is

$$\begin{bmatrix} 1\;1\;1 \;\ldots\ldots\ldots\ldots\; 1 \\ 2\;2 \;\ldots\ldots\ldots\; 2 \\ 3\;3 \;\ldots\ldots\; 3 \\ 4\;4 \;\ldots\; 4 \\ 5 \;\ldots\; 5 \\ 6 \;\ldots\; 6 \\ 7 \;\ldots\; 7 \end{bmatrix} .$$

It is clearly simple. On referring to Fig.8.1 we see that the highest weight is

$$\left(\frac{2(\lambda_1-\lambda_7) + (\lambda_2-\lambda_6) + (\lambda_3-\lambda_5)}{4\sqrt{3}} , \frac{\lambda_2-\lambda_3 + \lambda_5-\lambda_6}{4} \right). \qquad (8.11)$$

If this is the highest weight for a $D(\lambda,0)$ representation of the exceptional group G, then according to (4.5) the m_2 will vanish. This requires that $\lambda_2 = \lambda_3$ and $\lambda_5 = \lambda_6$, so the Young diagram will look like



(8.12)

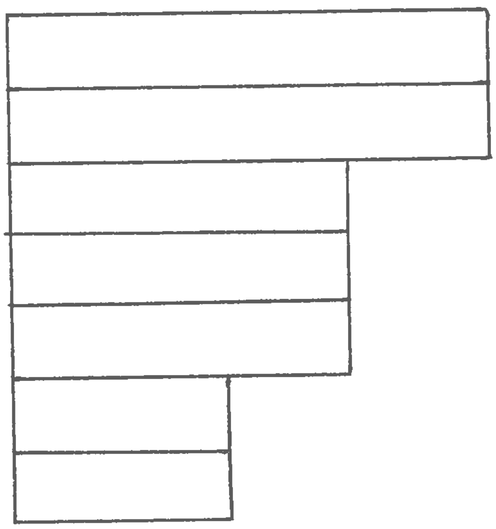
This will include the cases of the partitions

$[r]$, $[r^3]$, $[r^6]$, $[r^7]$

and in particular

$[1]$, $[1^3]$, $[1^6]$, $[1^7]$.

If (8.11) is the highest weight of a $D(0,\mu)$ representation of G , equation (4.5) shows that $m_1 = \sqrt{3}m_2$ and this is found to imply that $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4 = \lambda_5$ and $\lambda_6 = \lambda_7$. The Young diagram has the shape



(8.13)

This will include the Young diagrams for the partitions

$$[r^2] , \quad [r^5] , \quad [r^7] ,$$

and in particular

$$[1^2] , \quad [1^5] , \quad [1^7] .$$

We shall examine the simplest case of a representation whose basis consists of tensor components that are neither totally symmetric nor totally antisymmetric, namely, the representation that corresponds to the partition [2,1] . Its Young diagram belongs neither to (8.12) nor to (8.13). When the representation of $GL(7, \mathbb{C})$ is reduced under $SO(7, \mathbb{C})$, equation (8.3) gives

$$\{2,1\} = (2,1) + (1)$$

and therefore

$$D[2,1] = d(2,1) \oplus d(1) \,. \qquad (8.14)$$

Now from equation (8.10)

$$N[2,1] = 112$$

and, since the dimension of $d(1)$ is 7, the dimension of $d(2,1)$ must be 105, as we can verify by referring to equation (8.6). We express (8.14) more precisely by

$$D^{(112)}[2,1) = d^{(105)}(2,1) \oplus d^{(7)}(1) \,. \qquad (8.15)$$

Let us see how the irreducible $D[2,1]$ representation of $gl(7, \mathbb{C})$ reduces under G_2. Returning to Fig.8.2 we can readily calculate the multiplicities for this representation; for example, the point with the entries 115, 124, 223 has as basis vectors the tensor components corresponding to the Young tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \,.$$

The weight multiplicity is therefore 4 . We then plot all the multiplicities in Fig.8.5.

```
            1     1
      1     3     4     3     1
   1     4     8     8     4     1
      3     8    10     8     3
   1     4     8     8     4     1
      1     3     4     3     1
            1     1
```

Fig.8.5 - The weight multiplicities for the $D^{(112)}[2,1]$ representation of $gl(7, \mathbb{C})$.

We see from (4.5) that the highest weight in Fig.8.5 is the highest weight of $D(1,1)$, which according to (8.9) is 64-dimensional. By using the graphical method of Chapter IV it may be shown that the multiplicities of the $D^{(64)}(1,1)$ weight diagram are as in Fig.8.6. When the weights of this

```
            1     1
      1     2     2     2     1
   1     2     4     4     2     1
      2     4     4     4     2
   1     2     4     4     2     1
      1     2     2     2     1
            1     1
```

Fig.8.6 - The weight multiplicities for the $D^{(64)}(1,1)$ representation of G_2 .

figure are subtracted from those of Fig.8.5, we are left with those of $D^{(7)}(1,0)$ as in Fig.8.1, those of $D^{(27)}(2,0)$ as in Fig.8.4 and those of $D^{(14)}(0,1)$, since this being the regular representation has a diagram with two weights at the origin and one at the extremity of every root vector of Fig.1.3. Hence the $D[2,1]$ representation of $gl(7,\mathbb{C})$ reduces under G_2 as

$$D^{(112)}[2,1] = D^{(64)}(1,1) \oplus D^{(27)}(2,0) \oplus D^{(14)}(0,1) \oplus D^{(7)}(1,0) .$$

On comparing with (8.15), adopted for Lie algebras, we deduce that the $d^{(105)}(2,1)$ representation of $so(7,\mathbb{C})$ reduces under G_2 as

$$d^{(105)}(2,1) = D^{(64)}(1,1) \oplus D^{(27)}(2,0) \oplus D^{(14)}(0,1) .$$

We must remember that the (2,1) here refers to a partition, while the (1,1), (2,0), (0,1) refer to (λ,μ) values.

We conclude that, when a basis for an irreducible representation of $GL(7,\mathbb{C})$ consists of tensor components with mixed symmetry, there is in general a reduction of the representation as we pass to the subgroup $O(7,\mathbb{C})$ and a further reduction as we go to its subgroup G. We obtained for the above specific example the first reduction from the theory of group characters and the second by combining this with an inspection of weight diagrams. While it is not easy to state in precise terms general theorems for such successive reductions, the reduction for simpler Young diagrams can often be carried out without too much difficulty.

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