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Markov Processes for Random Fields

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C O N T E N T S

0. INTRODUCTION

The theory of random fields is concerned with the stochastic properties of systems with large numbers of interacting components. Such systems arise naturally in physics, biology, economics and other areas of study.

The basic mathematical structures are as follows. The state of a single component is described by a point of its phase space W. Though the components need not be identical, it simplifies notation to use the one set W for the phase space of each component. The components are labelled by the set S, so that the configuration of the combined systems is described by a point of the system phase space $X = W^S$.

We study the case in which S is a countably infinite set. Systems with a finite number of components arising in applications will be considered in relation to the limiting case of infinitely-many components. Random fields for which the label set S is continuous are not considered here. To simplify questions of measurability we assume that the basic spaces - X and W - are Polish spaces, *i.e.* they are homeomorphic to separable complete metric spaces. Measurability considerations are based on the Borel structure of these spaces.

The equilibrium theory of random fields, founded by Dobrushin (1968), concerns the problem of determining which probability measures on X correspond to a given family of conditional probabilities. This is intimately related to the older problem in statistical mechanics of determining the equilibrium states of a potential. For details of the equilibrium theory see Preston (1974a, 1974b), Georgii (1972, 1974) and Spitzer (1971, 1974a).

In applications most equilibria result from dynamic balance. In the present work we study time development in random field systems. We consider dynamics based on stochastic rather than physical principles. Thus in applications one has the additional problem of determining which, if any, of the stochastic models of time development give a good description of the observed behaviour.

The starting point for Markovian time development is a proposed generator G. We consider generators of the form $G = \Sigma\, G_\Lambda$ with the sum over all nonempty finite subsets of S, where G_Λ is the infinitesimal generator of a Markov jump process in W^Λ which may depend on the configuration of the remaining components. The model proposed by Glauber (1963) for time development of the one-dimensional Ising chain has nonzero generator terms only for $\Lambda = \{k\}$, $k \in S$. The jumping particle models of Spitzer (1970) can be described by a G with nonzero terms for sets $\{j,k\}$, $j,k \in S$, $j \neq k$.

In studying Markov processes associated with a given generator G, we interpret exp(tG) by means of a transition function or *Markov kernel* P(t,x,E). In Sections 1,2 and 3 we review material from probability theory and functional analysis: Markov kernels, strongly continuous linear semigroups of contractions and ergodic theory. In Section 4 we commence the study of interacting Markov processes and give the basic existence, uniqueness and convergence theory for functional semigroups. In Section 5 we introduce certain approximation techniques which allow us to relate the functional semigroups to Markov kernels and also to deduce an "almost product" relationship.

After the proof of the basic existence results the investigation proceeds in several directions. In Section 6 we relate the dynamic to the equilibrium study by considering generators given in terms of conditional probabilities and potentials. The generators given have the property that they are reversible for any equilibrium state of the potential. Heuristically, this means that fluctuations from equilibrium do not distinguish the direction of time. In Section 7 we study reversibility in more detail. With appropriate assumptions we find that G is an essentially selfadjoint operator. We consider relaxation to equilibrium in terms of the spectral properties of G.

In Section 8 we discuss two methods of comparing time development of different systems. The first deals with finite approximations to the infinite case. In the second two processes are compared by combining them into a single process. In Section 9 we consider ordered spaces and generators which preserve the order structure. Order structures occur naturally in birth-death processes and ferromagnetic Ising models.

One difficulty in studying random fields is that the common examples which exhibit phase transitions require rather sophisticated mathematical techniques. In Section 10 we give a simple time dependent model with phase transitions, which is a generalization of the Glauber (1963) model. Nearly all of the previous results are applicable to this model.

Finally, in Section 11, we discuss free energy and the variational principle.

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1. Markov kernels and Jensen's inequality.

The basic space is denoted X and is assumed to have the Borel structure of a separable, complete metric space. The space of bounded, continuous, real valued functions on X is denoted C(X) and is equipped with the supremum norm $\|\cdot\|$. The space of bounded, real valued, Borel functions on X is denoted F(X), and it also uses the supremum norm. We use the term *measure* to mean countably additive, bounded, Borel measure. The total variation norm for measures is denoted $\|\cdot\|_m$. The terms *positive, negative* and *increasing* are not to be taken in the strict sense unless preceded by "strictly".

1.1 Definition. A *measure kernel* $\Phi(x,E)$ is a real valued function defined for each $x \in X$ and Borel $E \subset X$ such that for fixed x, $\Phi(x,\cdot)$ is a measure on X, and for fixed E, $\Phi(x,E) \in F(X)$. The measure kernel Φ is called a *probability kernel* if $\Phi(x,\cdot)$ is a probability measure for each $x \in X$. Φ is called *continuous* if $\Phi(x,\cdot)$ is continuous in the topology of weak convergence of measures.

Given the measure kernel Φ we define its norm by

$$\|\Phi\| = \sup_{x \in X} \|\Phi(x,\cdot)\|_m . \qquad (1.1)$$

When $\|\Phi\|$ is finite and $f \in F(X)$, we construct the new function $\int\Phi(x,dy)f(y)$. If both f and Φ are continuous, then the resulting function is continuous. Similarly, given the measure μ, we construct the new measure $\int\mu(dx)\Phi(x,\cdot)$.

1.2 Definition. A *Markov kernel* $P(t,x,E)$ is a family of measure kernels parametrized by $t \in (0,\infty)$ such that

1^o for fixed t, $P(t,x,E)$ is a probability kernel;

2^o $\int P(t,x,dy)\, P(s,y,E) = P(t+s,x,E)$.

The Markov kernel P is called *continuous in* x if, for each $t \in (0,\infty)$, the measure kernel $P(t,\cdot,\cdot)$ is continuous. P is called *strongly continuous in* t if for each $f \in C(X)$

$$\lim_{t \to 0^+} \| f(x) - \int P(t,x,dy)f(y) \| = 0 . \qquad (1.2)$$

Given the Markov kernel P, we associate the family of operators $\{T_t: t \geq 0\}$ acting on $f \in F(X)$ by $T_o f = f$, and for $t > 0$

$$(T_t f)(x) = \int P(t,x,dy) f(y) . \tag{1.3}$$

If P is continuous in x and strongly continuous in t, then T_t acts as a strongly continuous linear semigroup on C(X). In any case, $\{T_t\}$ is a semi-group which maps positive functions to positive functions, constant functions being fixed points of the action of P. We also consider the family of operators $\{T_t' : t \geq 0\}$ acting on the measure μ by $T_o' \mu = \mu$, and for $t > 0$

$$(T_t' \mu)(E) = \int \mu (dx) P(t,x,E). \tag{1.4}$$

$\{T_t'\}$ is a semigroup which maps probability measures to probability measures. T_t and T_t' are related by

$$\int f \, d(T_t' \mu) = \int (T_t f) \, d\mu . \tag{1.5}$$

The proof of the following lemma is straightforward.

1.3 LEMMA. *Let μ be a positive measure and let ν be a measure which is absolutely continuous with respect to μ. Let T_t' be the action of the Markov kernel P on measures given by* (1.4). *Then $T_t' \nu$ is absolutely continuous with respect to $T_t' \mu$.*

Note that $T_t f(x)$ is the expectation value of f with respect to the probability measure $P(t,x, \cdot)$. Then by Jensen's inequality we have the following lemma.

1.4 LEMMA. *Let ψ be a real convex function. Then*

$$\psi(T_t f(x)) \leq T_t(\psi \circ f)(x) \tag{1.6}$$

for each $f \in F(X)$.

1.5 DEFINITION. The probability measure μ is said to be *reversible for the Markov kernel* P if for each $t > 0$ and Borel $E, F \subset X$,

$$\int_F \mu(dx) P(t,x,E) = \int_E \mu(dx) P(t,x,F). \tag{1.7}$$

1.6 DEFINITION. The function $f \in F(X)$ is called *invariant under* the Markov kernel P if $T_t f = f$ for each $t > 0$. The measure μ is called *invariant under* P if $T_t' \mu = \mu$ for each $t > 0$.

We have without difficulty the following.

1.7 LEMMA. *If the probability measure* μ *is reversible for the Markov kernel* P, *then* μ *is invariant under* P.

1.8 THEOREM. *The action* T_t *on* $F(X)$ *of the Markov kernel* P *with invariant probability measure* μ *extends by continuity to a positive, linear contraction semigroup on* $L^p(\mu)$ *for* $1 \leqslant p < \infty$, *the action on* $f \in L^p(\mu)$ *being given by* (1.3).

Proof. For $f \in F(X)$, by Jensen's inequality (1.6),

$$|T_t f|^p \leqslant T_t |f|^p . \tag{1.8}$$

From the invariance of μ

$$\int |T_t f|^p \, d\mu \leqslant \int T_t |f|^p \, d\mu = \int |f|^p \, d\mu . \tag{1.9}$$

Hence T_t acts as a contraction semigroup on the bounded functions in $L^p(\mu)$, which are dense in $L^p(\mu)$. For unbounded $f \in L^p(\mu)$, let $f_n(x)$ equal $f(x)$ if $|f(x)| < n$ and $f_n(x) = 0$ otherwise. From (1.9) we conclude that $\int P(t,x,dy) |f(y)|$ is finite μ-a.e. in x. By the bounded convergence theorem we have

$$\lim_{n \to \infty} T_t f_n(x) = \int P(t,x,dy) f(y) \tag{1.10}$$

whenever the right hand side exists and is finite. Thus the action of T_t in $L^p(\mu)$ is given by (1.3).

The function $s \log s$ is continuous and convex for $0 \leqslant s < \infty$ with the convention $0 \log 0 = 0$. The following is a version of the H-theorem of statistical mechanics. It also results from Jensen's inequality.

1.9 THEOREM. *Let* P *be a Markov kernel with invariant probability measure* μ. *Let* f *be a positive function in* $L^1(\mu)$: *Then*

$$\int (T_t f) \log (T_t f) \, d\mu \leqslant \int f \log f \, d\mu . \tag{1.11}$$

Let P be a Markov kernel with invariant probability measure μ. By Theorem 1.8 we can consider the action T_t of P in $L^1(\mu)$. We define the action of T'_t in $L^1(\mu)$ as follows. For $f \in L^1(\mu)$ by Lemma 1.3 we can write $T'_t(f(x)\ \mu(dx))$ as $g(x)\ \mu(dx)$, with $g \in L^1(\mu)$, so we set $T'_t\ f = g$. On the other hand we have the action in $L^q(\mu)$ which is the adjoint of the action T_t in $L^p(\mu)$ with $1/p + 1/q = 1$. This adjoint turns out to be the same as the action of T'_t just defined. When μ is reversible for P, the actions of T_t and T'_t coincide. In general the two actions are distinct, but the analogs of Theorems 1.8 and 1.9 are valid for the action of T'_t in $L^1(\mu)$.

2. GENERATORS.

In this work we are concerned mainly with Markov kernels which are obtained from generators. The simplest case is that in which the generator is a bounded operator.

2.1 DEFINITION. A *jump generator* $G(x,E)$ is a measure kernel such that

1^0 $\|G\| < \infty$;

2^0 $G(x,E\setminus\{x\}) \geqslant 0$ for all $x \in X$, Borel $E \subset X$;

3^0 $G(x, X) = 0$ for all $x \in X$.

2.2 THEOREM. *Let* G *be a jump generator. Then* $P(t,x,E) = (\exp tG)(x,E)$, $t > 0$, *is a Markov kernel such that for each Borel* $E \subset X$,

$$\lim_{t \to 0^+} \| P(t,x,E) - I_E(x) \| = 0 \tag{2.1}$$

with I_E *the indicator function of* E. *If* G *is continuous as a measure kernel, then* $P(t,x,E)$ *is continuous in* x *and strongly continuous in* t.

Proof. By $(\exp tG)(x,E)$ we mean

$$(\exp tG)(x,E) = I_E(x) + tG(x,E) + \tfrac{1}{2}t^2 \int G(x,dy)\, G(y,E) + \ldots \tag{2.2}$$

From 1^0 above, the limit (2.1) follows, and we can manipulate the absolutely convergent power series to verify that $(\exp tG)(\exp sG) = \exp(t+s)G$. For sufficiently small $t > 0$, $(\exp tG)(x,\cdot)$ is a positive measure, hence for all $t > 0$ by the semigroup property. Because of 3^0, $(\exp tG)(x,\cdot)$ is a probability measure. The proof of the continuity assertion is straightforward.

The above proof is based on the fact that a jump generator acts as a bounded operator on the space of measures. We wish to consider Markov kernels with unbounded generators. For this we need some results from the theory of strongly continuous semigroups.

For the remainder of this section $\mathcal{F}$ denotes a complete normed vector space with norm $\|\cdot\|$ and $\mathcal{D}$ denotes a linear subspace of $\mathcal{F}$. The *closure* $\bar{A}$ of the linear operator A with domain $\mathcal{D}$ and values in $\mathcal{F}$ is defined as follows. The graph of A is $\{(f, Af): f \in \mathcal{D}\}$. $\bar{A}$ is the closure of the graph of A in $\mathcal{F} \times \mathcal{F}$. $\bar{A}$ need not be the graph of a linear operator. When it is, we use the same symbol $\bar{A}$ to denote this operator. A necessary and sufficient condition that $\bar{A}$ be the graph of a linear operator is that for any sequence $\{f_n\} \subset \mathcal{D}$ such that $\lim f_n = 0$ and $\lim A f_n$ exists, then $\lim A f_n = 0$. If $\bar{A}$ coincides with the graph of A, then A is said to be *closed*.

2.3 DEFINITION. The linear operator A with domain $\mathcal{D}$ and values in $\mathcal{F}$ is called *dissipative* if for each $\lambda > 0$ and each $f \in \mathcal{D}$,

$$\| f - \lambda A f \| \geq \| f \| . \tag{2.3}$$

One can verify that a jump generator G acting on measures is dissipative in the norm $\|\cdot\|_m$. Its action on F(X) with norm $\|\cdot\|$ is also dissipative.

2.4 PROPOSITION. *Let A be a dissipative linear operator with domain $\mathcal{D}$ and values in $\mathcal{F}$. Let $\mathcal{D}$ be dense in $\mathcal{F}$. Then the closure $\bar{A}$ is a closed linear dissipative operator.*

Proof. We need only show that $\bar{A}$ is the graph of a linear operator, as linearity and dissipativity are preserved under closure. Let $\{f_n\} \subset \mathcal{D}$, $f_n \to 0$, $A f_n \to g$. By hypothesis we can find $h \in \mathcal{D}$ such that $\| g - h \| \leq \frac{1}{3}\| h\|$. Then $f_n + \lambda h \to \lambda h$, $(1 - \lambda A)(f_n + \lambda h) \to \lambda(h - g) - \lambda^2 Ah$. For sufficiently large n and small λ we get a contradiction to the dissipativity of A unless $g = 0$.

The original proof of the above, using semi-inner products, was given by Lumer (1961).

2.5 DEFINITION. A strongly continuous, linear, semigroup of contractions T_t, $t \geq 0$, is a family of linear operators of $\mathcal{F}$ into itself, such that

1^0 $\| T_t f \| \leq \| f \|$ all $f \in \mathcal{F}$, $t \geq 0$;

2^0 $T_t (T_s f) = T_{s+t} f$ all $f \in \mathcal{F}$, $s,t \geq 0$;

3^0 $\lim_{t \to 0^+} \| T_t f - f \| = 0$ all $f \in \mathcal{F}$.

2.6 DEFINITION. The *infinitesimal generator* A of the strongly continuous, linear, semigroup of contractions T_t is the linear operator

$$A\,f = \lim_{t \to 0^+} (T_t\, f - f)/t \tag{2.4}$$

whose domain is the set of $f \in \mathcal{F}$ for which (2.4) converges.

We give below two basic results from the theory of strongly continuous semigroups. Proofs can be found in Yosida (1965).

2.7 HILLE-YOSIDA THEOREM. *Let* T_t, $t \geq 0$, *be a strongly continuous, linear, semigroup of contractions on* $\mathcal{F}$. *Then the infinitesimal generator* A *of* T_t *is a densely defined, closed, linear, dissipative operator such that the range of* $(1 - \lambda A)$ *is all of* $\mathcal{F}$ *for each* $\lambda > 0$. *Conversely, let* A *be a densely defined, linear, dissipative operator defined on a subspace* $\mathcal{D} \subset \mathcal{F}$ *such that* $\mathcal{D}$ *and* $(1 - \lambda A)\mathcal{D}$ *are dense in* $\mathcal{F}$ *for some* $\lambda > 0$. *Then there is a uniquely defined, strongly continuous, linear, semigroup of contractions whose infinitesimal generator is the closure of A.*

2.8 TROTTER-KATO THEOREM. *Let* $\{T_t^{(n)}\}$ *be a sequence of strongly continuous, linear, semigroups of contractions with infinitesimal generators* $\{A_n\}$. *If there exists* $\lambda > 0$, *and a dense subspace* $\mathcal{D} \subset \mathcal{F}$ *such that*

$$\lim_{n \to \infty} (1 - \lambda A_n)^{-1} f = J\,f \tag{2.5}$$

exists for all $f \in \mathcal{D}$ *and* $J\mathcal{D}$ *is dense in* $\mathcal{F}$, *then there exists a uniquely defined, strongly continuous, linear semigroup of contractions* T_t *whose infinitesimal generator* A *satisfies* $Jf = (1 - \lambda A)^{-1} f$ *for each* $f \in \mathcal{D}$; *and for each* $f \in \mathcal{F}$ *and each* $t_0 > 0$

$$\lim_{n \to \infty} \sup_{0 \leq t \leq t_0} \| T_t\, f - T_t^{(n)} f \| = 0. \tag{2.6}$$

3. Ergodic Theory.

One fundamental problem posed by a Markov kernel P is to determine its invariant probability measures. The set of invariant probability measures is convex. This suggests studying extreme points and the related concept of ergodicity. Roughly speaking, an invariant probability measure μ of P is ergodic if it cannot be divided into smaller invariant parts. There are several precise formulations of this concept, and their interrelationship is subtle.

3.1 Definition. The Borel set $E \subset X$ is said to be *invariant* under the Markov kernel P if the set of t, for which $P(t,x,E) = 1$ for all $x \in E$, is dense in $(0, \infty)$.

3.2 Definition. The Borel set E is said to be μ-*almost invariant* under the Markov kernel P, with invariant probability measure μ, if for each $t \in (0, \infty)$

$$P(t,x,E) = I_E(x) \quad \mu\text{- a.e.}\,, \tag{3.1}$$

with I_E the indicator function of E.

3.3 Lemma. *Let* E *be* μ*-almost invariant under the Markov kernel* P *with invariant probability measure* μ. *Then there is a Borel* $F \subset E$ *such that* $\mu(F) = \mu(E)$ *and* F *is invariant under* P.

Proof. Select a countable dense set $\tau \subset (0,\infty)$. Define $E_1 = E$ and, for the strictly positive integer k,

$$E_{k+1} = \{x \in E_k:\ P(t,x,E_k) = 1 \quad \text{for all } t \in \tau\} \ . \tag{3.2}$$

We then have $E_{k+1} \subset E_k$, and from (3.1), $\mu(E_2) = \mu(E_1)$.
From the invariance of μ

$$\int P(t,x,E_1 \setminus E_2)\ \mu(dx) = 0; \tag{3.3}$$

so

$$\int_{E_2} P(t,x,E_2)\ \mu(dx) = \int_{E_2} P(t,x,E_1)\ \mu(dx). \tag{3.4}$$

We conclude that E_2, and also E_k for each k, is μ-almost invariant. The set $F = \bigcap E_k$ is invariant under P.

3.4 LEMMA. *Let the probability measure μ be invariant under the Markov kernel P. If the action of P on $L^1(\mu)$ is strongly continuous, and if the Borel set E is invariant under P, then E is μ-almost invariant under P.*

Proof. The assumption of invariance of E amounts to the condition that for t in a dense subset of $(0,\infty)$

$$T_t I_E \geqslant I_E , \tag{3.5}$$

where I_E is the indicator function of E, and T_t is given by (1.3). The invariance of μ yields $T_t I_E = I_E$ in $L^1(\mu)$. This result extends to all $t \in (0,\infty)$ by strong continuity, which yields μ-almost invariance.

3.5 DEFINITION. The probability measure μ invariant under the Markov kernel P is called *ergodic* if each μ-almost invariant set has probability 0 or 1.

3.6 DEFINITION. The probability measure μ invariant under the Markov kernel P is called *extremal* if the only representation $\mu = \frac{1}{2}(\rho + \sigma)$, where ρ and σ are invariant probability measures, is with $\mu = \rho = \sigma$.

3.7 PROPOSITION. *Let μ be a probability measure invariant under the Markov kernel P. Denote by T_t the action of P in $L^1(\mu)$ and by T_t' the action of P on measures (see comments after Theorem 1.9). Let $f \in L^1(\mu)$. Then the following are equivalent.*

1^o $T_t f = f$ *in* $L^1(\mu)$.

2^o $T_t'(f(x)\, \mu(dx)) = f(x)\, \mu(dx)$ *as measures.*

3^o *For each* $c \in R$, $\{x\colon f(x) \geqslant c\}$ *is* *μ-almost invariant.*

Proof. $1^o \Rightarrow 3^o$. Because constants are fixed under the action of T_t, it is sufficient to prove that

$$E = \{x\colon f(x) \geqslant 0\} \tag{3.6}$$

is μ-almost invariant. Let $f^+(x) = f(x)$ on E and 0 otherwise; $f^-(x) = f^+(x) - f(x)$.

$$\int_E f\,d\mu = \int f^+\,d\mu = \int_E \int P(t,x,dy)\, f^+(y)\,\mu(dx) - \int_E \int P(t,x,dy)\, f^-(y)\,\mu(dx). \tag{3.7}$$

The first term on the right hand side of (3.7) is less than or equal to $\int f^+ d\mu$ and the second term is negative. Since f^- is strictly positive on E^c, we conclude that $I_E(x)\, P(t,x,E^c) = 0$ μ-a.e., which implies the μ-almost invariance of E.

$2^o \Rightarrow 3^o$. Let f and E be as above.

$$\int_E f\,d\mu = \int f^+\,d\mu = \int f^+(x)\,\mu(dx)\, P(t,x,E) - \int f^-(x)\,\mu(dx)\, P(t,x,E). \tag{3.8}$$

The first term on the right hand side of (3.8) is less than or equal to $\int f^+ d\mu$, while the second is negative. We conclude that $I_{E^c}(x)\, P(t,x,E) = 0$ μ-a.e., and so E is μ-almost invariant.

$3^o \Rightarrow 1^o$ and 2^o. A finite linear combination of indicator functions of sets of the form of 3^o is invariant in the sense of 1^o and 2^o. We approximate f in $L^1(\mu)$ by such finite linear combinations to obtain the desired conclusion.

In connection with Choquet theory, it is not difficult to show, using the above, that the cone of positive invariant measures is a lattice in its own order.

3.8 THEOREM. *Let μ be a probability measure invariant under the Markov kernel P. Then the following are equivalent.*

1^o *μ is extremal.*

2^o *μ is ergodic.*

3^o *The only fixed points of T_t in $L^1(\mu)$ are the constant functions.*

4^o *Any invariant measure which is absolutely continuous with respect to μ is a constant multiple of μ.*

Proof. $1^o \Rightarrow 2^o$. Let E be a μ-almost invariant Borel set. Then

$$\mu = \tfrac{1}{2}\big((\mu(E^c) + I_E(x))\,\mu(dx) + (\mu(E) + I_{E^c}(x))\,\mu(dx)\big) \tag{3.9}$$

gives a nontrivial representation of μ as a convex combination of invariant probability measures, unless $\mu(E) = 0$ or 1.

$2^o \Rightarrow 3^o$. Let $f \in L^1(\mu)$ be invariant under T_t. Then condition 3^o of Proposition 3.7 and the ergodicity hypothesis imply that f is constant μ- a.e.

$3^o \Rightarrow 4^o$. A measure absolutely continuous with respect to μ can be written $f(x)\ \mu(dx)$ with $f \in L^1(\mu)$. By Proposition 3.7, the invariance of $f(x)\ \mu(dx)$ implies that f is a constant μ- a.e.; hence $f(x)\ \mu(dx)$ is a constant multiple of μ.

$4^o \Rightarrow 1^o$. Let $\mu = \frac{1}{2}(\rho + \sigma)$ with ρ and σ invariant probability measures. Both ρ and σ are absolutely continuous with respect to μ, and so, by 4^o, $\mu = \rho = \sigma$.

3.9 ERGODIC THEOREM. *Let P be a Markov kernel with invariant probability measure μ such that the action T_t of P is a strongly continuous semigroup on $L^1(\mu)$. Then for each $f \in L^1(\mu)$,*

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N (T_t f)(x)\, dt = \tilde{f}(x) \tag{3.10}$$

exists μ-a.e. and in the $L^1(\mu)$-norm. The function $\tilde{f}(x)$ — defined to be zero where (3.10) does not converge — satisfies, for all $t \geq 0$,

$$\int P(t,x,dy)\, \tilde{f}(y) = \tilde{f}(x) \quad \mu\text{-a.e.} \tag{3.11}$$

$$\int f\, d\mu = \int \tilde{f}\, d\mu . \tag{3.12}$$

A proof of the above can be found in Dunford (1958), Ch. VIII. One interpretation of Theorem 3.9 is as follows. If μ is ergodic, then $\tilde{f}$ is constant μ-a.e., and the time average (3.10) equals the space average (3.12).

Not all Markov kernels possess invariant probability measures. Let $X = (0, 1]$. The Markov kernel P with $P(t,x,\{x/(1+tx)\}) = 1$ has no invariant probability measures. One could take X to be compact by identifying 0 and 1. Similarly one has no invariant probability measures for the jump generator G with $G(x,\{x/(1+x)\}) = 1$ and $G(x, E\setminus\{x/(1+x)\}) \leq 0$.

3.10 THEOREM. *Let* X *be compact, and let the Markov kernel* P(t,x,E) *be continuous in* x *and strongly continuous in* t. *Then the set of invariant probability measures of* P *is nonempty, convex and compact in the topology of weak convergence.*

Proof. We have already remarked on convexity. Compactness in the topology of weak convergence follows from the compactness of X and the continuity in x. To show that the set of invariant probability measures is nonempty, start with a probability measure μ and define μ_N by

$$\mu_N(E) = \frac{1}{N} \int_0^N \left(\int \mu(dx)\, P(t,x,E)\right) dt . \qquad (3.13)$$

The sequence $\{\mu_N\}$ has a weak limit point ν, which standard estimates show to be an invariant probability measure.

For the remainder of this section we restrict the discussion to the case in which X is compact and the action T_t of P yields a strongly continuous semigroup on C(X). We define

$$\Phi_N(x,f) = \frac{1}{N} \int_0^N (T_t f)(x)\, dt, \qquad (3.14)$$

$$X' = \{ x \in X: \lim_{N \to \infty} \Phi_N(x,f) \text{ exists for all } f \in C(X)\} . \qquad (3.15)$$

We define the measure kernel $\Phi(x,E)$ such that for $x \notin X'$, $\Phi(x,\cdot) = 0$ and for $x \in X'$

$$\int \Phi(x, dy)\, f(y) = \lim_{N \to \infty} \Phi_N(x,f) \qquad (3.16)$$

for all $f \in C(X)$. The right hand side of (3.16) defines a continuous linear functional on C(X) which corresponds uniquely to a measure on X.

3.11 LEMMA. *For any invariant probability measure* μ *of* P, $\mu(X') = 1$. *If* $f \in L^1(\mu)$, *then in the notation* (3.10),

$$\tilde{f}(x) = \int \Phi(x,dy)\, f(y) \quad \mu\text{- a.e.} \qquad (3.17)$$

Proof. In the definition (3.15) of X' we could have used a countable dense subset, rather than all of C(X), to obtain the same X'. Theorem 3.9 then implies that $\mu(X') = 1$. If $f \in C(X)$, (3.16) coincides with (3.10). Otherwise select a sequence $\{f_n\}$ in $C(X)$ converging to f in $L^1(\mu)$. It follows that $\{\tilde{f}_n\}$ converges to $\tilde{f}$ in $L^1(\mu)$, and (3.17) obtains.

3.12 THEOREM. *Let* X *be compact and let the Markov kernel* P *be continuous in* X *and strongly continuous in* t. *If* μ *is an invariant probability measure of* P, *then for each Borel set* E

$$\mu(E) = \int \mu(dx)\, \Phi(x,E) \tag{3.18}$$

and the measure $\Phi(x,\cdot\,)$ *is an ergodic probability measure* μ-a.e. *in* x.

Proof. To obtain (3.18), apply (3.17) to the indicator function $I_E(x)$, and integrate with respect to μ. Note that for each $x \in X'$, $\Phi(x,\cdot\,)$ is an invariant probability measure. For $f \in C(X)$ we have

$$\int \mu(dx) \int (f(y) - f(x))^2\, \Phi(x,dy) = 0 \tag{3.19}$$

from à straightforward calculation using (3.17). Let

$$X'' = \{x \in X' : \int (f(y) - f(x))^2\, \Phi(x,dy) = 0,\ \text{all } f \in C(X)\}. \tag{3.20}$$

By using a countable dense set rather than all of $C(X)$ in (3.20) we see that $\mu(X'') = 1$. It remains to show for $x \in X''$, $\Phi(x,\cdot\,)$ is ergodic. Let $f \in L^1(\Phi(x,\cdot\,))$. Select a sequence $\{f_n\}$ in $C(X)$ which converges to f in $L^1(\Phi(x,\cdot\,))$. But each $\tilde{f}_n$ is constant $\Phi(x,\cdot\,)$ - a.e. from (3.20). So f is constant $\Phi(x,\cdot\,)$ - a.e.

4. Generators of interacting Markov processes.

We commence the study of interacting Markov processes given in terms of an infinite sum of jump generators. The system phase space X is assumed to be of the form W^S, where S is a countably infinite set and W is phase space of the individual component. The topology of W is assumed to have a metric in which W is separable and complete; then the product topology for X has this property as well (*i.e.* W and X are Polish spaces). We consider generators of the form $\Sigma\, G_\Lambda$ where the sum is over the nonempty, finite subsets of S and G_Λ is a jump generator which affects the configuration of only those components labelled by Λ. G_Λ can, however, depend on the configuration of the total system. The first basic question is whether there is a Markov kernel on X which corresponds in a reasonable way to $\Sigma\, G_\Lambda$.

For generalized jump processes of this kind two assumptions appear natural. One is that the jumping rate of each individual component should be finite. The second is that the jumping behaviour of any one component should be influenced predominantly by a finite number of its neighbours. Assumptions of this kind appear in the theorems below.

We introduce further notation. The symbol Λ, possibly primed or subscripted, *always denotes a finite subset of* S. The limit

$$\lim \Lambda \to S \qquad (4.1)$$

is to be taken on the net of finite subsets of S ordered by inclusion. For $\Gamma \subset S$, $F(X|\Gamma)$ denotes the set of bounded Borel functions f such that f(x) depends only on the values of x on Γ. $C(X|\Gamma)$ denotes $C(X) \cap F(X|\Gamma)$. $F_f(X)$ is the set of functions which are in $F(X|\Lambda)$ for some Λ. $F_0(X)$ denotes the closure of $F_f(X)$ in the uniform norm $\|\cdot\|$. We define $C_f(X)$ and $C_0(X)$ similarly. If W is discrete, then $C_0(X) = F_0(X)$. If W is compact, then so is X and $C(X) = C_0(X)$.

For $f \in F(X)$ we define $\delta_i f$ by

$$\delta_i f = \sup_{\substack{x,y \,\in\, X \\ x=y \text{ except at } i}} |f(x) - f(y)| \;. \qquad (4.2)$$

We denote by $F^*(X|\Lambda)$ the set of functions $f \in F(X)$ such that $\delta_i f = 0$ unless $i \in \Lambda$. For $F^*(X|\phi)$ we have the special notation $F_\infty(X)$. If $f \in F_\infty(X)$, then $f(x) = f(y)$ whenever the set on which x and y differ is finite. Heuristically speaking, the star spaces allow variation at infinity. $F^*_f(X)$ denotes those f which are in $F^*(X|\Lambda)$ for some Λ, and $F^*_o(X)$ is the closure of $F^*_f(X)$ in the uniform norm.

In working with the product space X we employ the following *left subscript* notation. For $\Gamma \subset S$, which will be specified or understood from context,

$$({}_y x)_j = \begin{cases} y_j & \text{if } j \in \Gamma \\ x_j & \text{if } j \in \Gamma^c \end{cases} . \tag{4.3}$$

The reader should recall the norm $\|\cdot\|$ for parametrized measures (1.1).

4.1 DEFINITION. A *generator* G *on* X is a formal sum $\Sigma\, G_\Lambda$ with a term for each nonempty, finite subset of S such that for each Λ, $G_\Lambda(x,\cdot)$ is a positive measure on W^Λ which is a Borel function of x and satisfies

$$\| G_\Lambda(x,\cdot) \| < \infty . \tag{4.4}$$

The *operator* of G_Λ is the linear transformation of F(X) into itself given by

$$(G_\Lambda f)(x) = \int (f({}_y x) - f(x))\, G_\Lambda(x,dy), \tag{4.5}$$

where the notation (4.3) is used with $\Gamma = \Lambda$. The *operator* of G is defined with domain the set of $f \in F(X)$ for which

$$\Sigma \| G_\Lambda f \| < \infty , \tag{4.6}$$

and, for such f,

$$Gf = \Sigma\; G_\Lambda f. \tag{4.7}$$

G is called *continuous* if each G_Λ is continuous in the topology of weak convergence of measures.

For technical reasons, we do not define G_Λ to be a jump generator.

However, it operates as one, and we shall think of it as such when convenient.

In the theory of continuous-time, countable Markov chains, the generator is called the Q-matrix (see Chung (1967)). If $\{Q_{xy} : x \neq y\}$ is uniformly bounded, then there is a space X and a generator $G = \Sigma\, G_\Lambda$ on X corresponding to Q. Let $W = \{0, 1\}$ and $S = \{1,2, \ldots\}$. The states of the countable Markov chain are identified with the points of $X = W^S$ which have a finite number of 1's. The Q_{xy}, $x \neq y$, term is included in G_Λ, where Λ is the set on which x and y differ. Through this correspondence one can exhibit pathological examples in random field evolutions based on ones in Markov chain theory.

The results below are based upon comparisons with semigroups acting on ℓ^1 spaces. For $f \in F(X)$ we define the sequence δf on S whose terms are $\{\delta_i f\}$. For the sequence $v = \{v_i\}$ on S, the norm $\|\cdot\|_1$ is defined by

$$\| v\|_1 = \Sigma_{i \in S} |v_i| . \tag{4.8}$$

Particularly important are those f for which $\|\delta f\|_1 < \infty$. For two sequences v, w on S we write

$$v \leqslant w \tag{4.9}$$

to mean

$$v_i \leqslant w_i \quad \text{all } i \in S. \tag{4.10}$$

For G_Λ we define

$$\delta_i G_\Lambda = \sup_{\substack{x,\ y \in X \\ x=y \text{ except at } i}} \| G_\Lambda(x,\cdot) - G_\Lambda(y,\cdot)\|_m . \tag{4.11}$$

4.2 DEFINITION. The generator $G = \Sigma\, G_\Lambda$ on X is called *local* if each G_Λ satisfies

$$\lim_{\Lambda' \to S} \sup_{\substack{x,y \in X \\ x = y \text{ on } \Lambda'}} \| G_\Lambda(x,\cdot) - G_\Lambda(y,\cdot) \|_m = 0. \tag{4.12}$$

We can now state an existence and uniqueness theorem.

4.3 THEOREM. *Let* $G = \Sigma_\Lambda G_\Lambda$ *be a generator on* X *and let* K *be a real number such that for each* $j \in S$,

$$\Sigma_{\Lambda \ni j} \; \| G_\Lambda \| \leqslant K, \tag{4.13}$$

$$\Sigma_{i \in S} \; \Sigma_{\Lambda \ni j} \; \delta_i \; G_\Lambda \leqslant K . \tag{4.14}$$

Then there is a uniquely defined, positive, linear semigroup of contractions T_t, $t \geqslant 0$, *on* $F_o^*(X)$, $\| \cdot \|$, *so that for each* $f \in F_o^*(X)$ *and each real* $t_o > 0$ *we have*

$$\lim_{\Lambda \to S} \; \sup_{0 \leqslant t \leqslant t_o} \; \| T_t f - \exp(t\Sigma_{\Lambda' \subset \Lambda} G_{\Lambda'})f \| = 0 . \tag{4.15}$$

Further, if $\| \delta f \|_1 < \infty$, *then*

$$\delta \, T_t f \leqslant \exp(tC) \; \delta f, \tag{4.16}$$

where C *is the matrix with elements*

$$C_{ij} = \Sigma_{\Lambda \ni j} \; \delta_i \, G_\Lambda . \tag{4.17}$$

If G *is local, then* T_t *maps* $F_o(X)$ *into itself. If* G *is continuous and local, then* T_t *maps* $C_o(X)$ *into itself.*

We omit the proof of the above, for it is along the same lines as that of the more general result below. For $f \in F(X)$, $i \in S$, $c \in W$, we define the operator Δ_i^c by

$$\Delta_i^c \; f(x) = f(x) - f({}_c x) , \tag{4.18}$$

where the notation (4.3) is used with $\Gamma = \{i\}$. For $i \in \Lambda^c$ we define

$$B(i, \Lambda) = \sup_{\substack{x,y \in X \\ x=y \text{ except at } i}} \tfrac{1}{2} \begin{bmatrix} \| G_\Lambda(x,\cdot\,) - G_\Lambda(y,\cdot\,) \|_m \\ - | G_\Lambda(x,\{x_\Lambda\}) - G_\Lambda(y,\{y_\Lambda\}) | \\ + | G_\Lambda(x, W^\Lambda \backslash \{x_\Lambda\}) - G_\Lambda(y, W^\Lambda \backslash \{y_\Lambda\}) | \end{bmatrix} . \tag{4.19}$$

For $i \in \Lambda$ we define $B(i, \Lambda) = \delta_i \, G_\Lambda$, and

$$D(i,\Lambda) = \inf_{\substack{x,\, y \,\varepsilon\, X \\ x=y \text{ except at } i}} \tfrac{1}{2} \left[\begin{array}{l} \| G_\Lambda(x,\cdot) + G_\Lambda(y,\cdot) \|_m \\ -\| G_\Lambda(x,\cdot) - G_\Lambda(y,\cdot) \|_m \\ +2 \max \{0, G_\Lambda(y,\{x_\Lambda\}) - G_\Lambda(x,\{x_\Lambda\})\} \\ +2 \max \{0, G_\Lambda(x,\{y_\Lambda\}) - G_\Lambda(y,\{y_\Lambda\})\} \end{array} \right], \qquad (4.20)$$

where x_Λ, y_Λ represent the points of W^Λ whose coordinates agree with x, y. From (4.5) we get the following:

4.4 LEMMA. *Let* $f \,\varepsilon\, F(X)$, $i \,\varepsilon\, S$. *Let* $x \,\varepsilon\, X$ *and* $c \,\varepsilon\, W$ *be such that*

$$\Delta_i^c f(x) \geq \delta_i f - \varepsilon . \qquad (4.21)$$

Then

$$G_\Lambda(\Delta_i^c f)(x) \leq \varepsilon \| G_\Lambda \| . \qquad (4.22)$$

4.5 LEMMA. *Let* f, i, x, c *be as above and satisfy* (4.21). *Then for any* $\Lambda \ni i$

$$- \Delta_i^c (G_\Lambda f)(x) \geq D(i,\Lambda)\, \delta_i f - \varepsilon \| G_\Lambda \| - B(i,\Lambda) \textstyle\sum_{j \,\varepsilon\, \Lambda \setminus \{i\}} \delta_j f . \qquad (4.23)$$

Proof. Let (4.21) be satisfied.

$$- \Delta_i^c (G_\Lambda f)(x) = G_\Lambda(x, W^\Lambda) f(x) - G_\Lambda({}_c x, W^\Lambda) f({}_c x) - \int f({}_y x)(G_\Lambda(x,dy) - G_\Lambda({}_c x,dy)) . \qquad (4.24)$$

There is for fixed x a constant J such that as a function of a

$$|f({}_a x) + J| \leq \tfrac{1}{2} \delta_i f , \qquad (4.25)$$

with the a, c modification of x at the i-position. Then as a function of $y \,\varepsilon\, W^\Lambda$,

$$|f({}_y x) + J| \leq \tfrac{1}{2} \delta_i f + \textstyle\sum_{j \,\varepsilon\, \Lambda \setminus\{i\}} \delta_j f . \qquad (4.26)$$

The addition of J to f does not change (4.24) so

$$-\Delta_i^c(G_\Lambda f)(x) \geq \tfrac{1}{2}\delta_i f(\| G_\Lambda(x,\cdot) + G_\Lambda({}_c x,\cdot)\|_m - \| G_\Lambda(x,\cdot) - G_\Lambda({}_c x,\cdot)\|_m)$$
$$- \varepsilon \| G_\Lambda \| - \delta_i G_\Lambda \Sigma_{j \in \Lambda \setminus \{i\}} \delta_j f \ . \tag{4.27}$$

The desired result (4.23) follows by noting that changing $G_\Lambda(x,\{x_\Lambda\})$ does not change (4.24).

For two operators G, H we define the commutator bracket [G, H] by

$$[G, H] = GH - HG \ . \tag{4.28}$$

4.6 Lemma. *For* $f \in F(X)$, $i \in \Lambda^c$

$$| [\Delta_i^c , G_\Lambda] f(x) | \leq B(i,\Lambda) \Sigma_{j \in \Lambda} \delta_j f \ . \tag{4.29}$$

Proof. We have

$$[\Delta_i^c, G_\Lambda] f(x) = \int (f({}_{yc}x) - f({}_c x))(G_\Lambda(x,dy) - G_\Lambda({}_c x,dy)), \tag{4.30}$$

with the c modification of x at the i-position. For fixed ${}_c x$ we can select a constant J so that

$$|J| \leq \tfrac{1}{2} \Sigma_{j \in \Lambda} \delta_j f \ , \tag{4.31}$$

and, as a function of $y \in W^\Lambda$,

$$|f({}_{yc}x) - f({}_c x) + J | \leq \tfrac{1}{2} \Sigma_{j \in \Lambda} \delta_j f \ . \tag{4.32}$$

The addition of J to $f({}_{yc}x) - f({}_c x)$ changes (4.30) by

$$J(G_\Lambda(x,W^\Lambda) - G_\Lambda({}_c x,W^\Lambda)). \tag{4.33}$$

Then (4.30) follows from reasoning like that in Lemma 4.5.

4.7 Definition. The comparison matrix $B = (B_{ij})$ of the generator $G = \Sigma_\Lambda G_\Lambda$ on X is defined

$$-B_{ii} = \Sigma_{\Lambda \ni i} D(i,\Lambda), \tag{4.34}$$

$$B_{ij} = \Sigma_{\Lambda \ni j} B(i,\Lambda), \; i \neq j, \tag{4.35}$$

with $B(i,\Lambda)$ and $D(i,\Lambda)$ as given between (4.19) and (4.20).

4.8 THEOREM. *Let* $G = \Sigma_\Lambda G_\Lambda$ *be a generator on* X *with comparison matrix* B. *Assume that there are real numbers* α, K *such that for all* $j \in S$

1^0 $\quad \Sigma_{i \in S} B_{ij} \leqslant \alpha;$

2^0 $\quad \Sigma_{\Lambda \ni j} \| G_\Lambda \| \leqslant K.$

Then the domain of the operator of G *contains* $F_f^*(X)$ *and the closure of* G *as defined on* $F_f^*(X)$ *is the infinitesimal generator of a uniquely defined, strongly continuous, positive, linear semigroup of contractions* T_t, $t \geqslant 0$, *on* $F_0^*(X)$ *with norm* $\|\cdot\|$. *For each* $f \in F_0^*(X)$ *and each real* $t_0 > 0$

$$\lim_{\Lambda \to S} \; \sup_{0 \leqslant t \leqslant t_0} \| T_t f - \exp(t\Sigma_{\Lambda' \subset \Lambda} G_{\Lambda'}) f \| = 0. \tag{4.36}$$

If, in addition, one has

3^0 $\quad \alpha < 0,$

for each $f \in F_0^*(X)$ *with* $\| \delta f \|_1 < \infty$ *there is a function* $f_\infty \in F_\infty(X)$ *such that*

$$e^{-\alpha t} \| T_t f - f_\infty \| \text{ is bounded in } t \geqslant 0, \tag{4.37}$$

and for any $g \in F_0^*(X)$ *there is a* $g_\infty \in F_\infty(X)$ *such that*

$$\lim_{t \to \infty} \| T_t g - g_\infty \| = 0. \tag{4.38}$$

If, in addition, G *is local, then* $F_0(X)$ *is invariant under* T_t *and for* $f \in F_0(X)$ *the limit function* f_∞ *is a constant.*

Proof. We assume 1^0 and 2^0 are satisfied with specified α and K. For $f \in F_0^*(X)$ with $\|\delta f\|_1 < \infty$ we have

$$\begin{aligned} \Sigma_\Lambda \| G_\Lambda f \| &\leqslant \Sigma_\Lambda \| G_\Lambda \| \; \Sigma_{j \in \Lambda} \delta_j f \\ &= \Sigma_{j \in S} \delta_j f \; \Sigma_{\Lambda \ni j} \| G_\Lambda \| \leqslant K \| \delta f \|_1 , \end{aligned} \tag{4.39}$$

so that f is in the domain of $\mathbb{G}$. For $\lambda > 0$ let the function g be defined by

$$(1 - \lambda G)f = g \quad . \tag{4.40}$$

We proceed to show that

$$(1 - \lambda B)\, \delta f \leqslant \delta g \quad . \tag{4.41}$$

To prove this we have for $i \in S$, $c \in W$,

$$\Delta_i^c \,(1 - \lambda \Sigma_\Lambda\, G_\Lambda)\, f = \Delta_i^c\, g \quad , \tag{4.42}$$

$$\begin{aligned}(1 - \lambda \Sigma_{\Lambda \not\ni i}\, G_\Lambda)\, \Delta_i^c\, f - \lambda \Sigma_{\Lambda \ni i}\, \Delta_i^c\, G_\Lambda\, f \\ = \Delta_i^c\, g + \lambda \Sigma_{\Lambda \not\ni i}\, [\Delta_i^c\, , G_\Lambda]\, f \quad .\end{aligned} \tag{4.43}$$

Because of (4.39) we may approximate the left hand side of (4.43) by finite sums, so for any $\varepsilon > 0$ one obtains from Lemmas 4.4, 4.5 and 4.6,

$$\begin{aligned}\delta_i f + \lambda \Sigma_{\Lambda \ni i}\, (D(i, \Lambda)\delta_i f - B(i,\Lambda)\Sigma_{j \in \Lambda \setminus \{i\}} \delta_j f) \\ \leqslant \delta_i g + \lambda \Sigma_{\Lambda \not\ni i} B(i, \Lambda)\Sigma_{j \in \Lambda}\, \delta_j f + \varepsilon \quad ,\end{aligned} \tag{4.44}$$

whence follows (4.41). Next define the generator on X, G^Λ, by

$$G^\Lambda = \Sigma_{\Lambda' \subset \Lambda}\, G_{\Lambda'} \quad , \tag{4.45}$$

whose comparison matrix is denoted B^Λ. From 1^o we conclude that $B - \alpha$ is a bounded dissipative operator on ℓ^1 and that $B^\Lambda - \alpha - 2K$ is such for each Λ. For $\lambda > 0$ such that

$$\lambda(\alpha + 2K) < 1 \quad , \tag{4.46}$$

$(1 - \lambda B)^{-1}$ and $(1 - \lambda B^\Lambda)^{-1}$ exist as bounded linear operators on ℓ^1, and we have for $v \in \ell^1$

$$\lim_{\Lambda \to S}\, (1 - \lambda B^\Lambda)^{-1}\, v = (1 - \lambda B)^{-1}\, v. \tag{4.47}$$

Henceforth we assume $\lambda > 0$ and (4.46). By power series expansion one can show that all the matrix elements of $(1 - \lambda B)^{-1}$ and $(1 - \lambda B^{\Lambda})^{-1}$ are positive.

For $g \in F_0^*(X)$, $\|\delta g\|_1 < \infty$, set

$$f^{\Lambda} = (1 - \lambda G^{\Lambda})^{-1} g, \tag{4.48}$$

which is well defined since G^{Λ} acts as a bounded linear dissipative operator on $F(X)$. The result for G^{Λ} corresponding to (4.41) is valid, and it is not necessary to assume that $\|\delta f^{\Lambda}\|_1 < \infty$ as only finite sums are involved. Then

$$(1 - \lambda B^{\Lambda})\delta f^{\Lambda} \leqslant \delta g. \tag{4.49}$$

This implies that $\|\delta f^{\Lambda}\|_1 < \infty$, and from the positivity of the elements of $(1 - \lambda B^{\Lambda})^{-1}$,

$$\delta f^{\Lambda} \leqslant (1 - \lambda B^{\Lambda})^{-1} \delta g. \tag{4.50}$$

It follows from (4.47) and (4.39) that

$$\lim_{\Lambda \to S} \| (G - G^{\Lambda}) f^{\Lambda} \| = 0. \tag{4.51}$$

From this we deduce

$$\lim_{\Lambda \to S} (1 - \lambda G) f^{\Lambda} = g. \tag{4.52}$$

The dissipativity of $(1 - \lambda G)$ implies that $\{f^{\Lambda}\}$ is a Cauchy net in $F_0^*(X)$ so there is an $f \in F_0^*(X)$ such that

$$\lim_{\Lambda \to S} \| f - f^{\Lambda} \| = 0. \tag{4.53}$$

It is not difficult to verify that

$$\delta f \leqslant (1 - \lambda B)^{-1} \delta g, \tag{4.54}$$

$$(1 - \lambda G) f = g. \tag{4.55}$$

We have shown that the dissipative operator G has in its domain the set of $g \in F_0^*(X)$ such that $\|\delta g\|_1 < \infty$, and that the image of $(1 - \lambda G)$ also contains this set. That the action of G on functions for which $\|\delta g\|_1 < \infty$ can be obtained from the closure of the action of G on $F_0^*(X)$, can be proved by using the approximations of the next section. The existence and uniqueness of T_t then follow from the Hille-Yosida Theorem 2.7. From (4.53), (4.55) and the Trotter-Kato Theorem 2.8 we have (4.36). Let $f \in F_0^*(X)$, $\|\delta f\|_1 < \infty$. $T_t f$ can be evaluated by

$$T_t f = \lim_{n \to \infty} (1 - tG/n)^{-n} f . \tag{4.56}$$

From (4.54) we deduce

$$\delta T_t f \leqslant \exp(tB)\, \delta f. \tag{4.57}$$

Now we assume condition 3^0 of the Theorem. For $f \in F_0^*(X)$, $\|\delta f\|_1 < \infty$ and $0 < t < s$,

$$\begin{aligned} \|T_t f - T_s f\| &= \|\textstyle\int_t^s G(T_r f)\, dr\,\| \\ &\leqslant K\, \|\delta f\|_1 \textstyle\int_t^s e^{\alpha r}\, dr = K \|\delta f\|_1 (e^{\alpha s} - e^{\alpha t})/\alpha\ . \end{aligned} \tag{4.58}$$

Thus $\{T_t f\}$ satisfies a Cauchy condition, so there is a limit function $f_\infty \in F_0^*(X)$ satisfying (4.37). From (4.57) we see that $f_\infty \in F_\infty(X)$. The limit (4.38) follows by approximating g by functions for which (4.37) is applicable.

In the event of G being local, $\exp tG^\Lambda$ maps $F_0(X)$ into itself for each Λ, so T_t does as well. For $g \in F_0(X)$ we have $g_\infty \in F_\infty(X) \cap F_0(X)$, so g_∞ is constant. Similar considerations apply in the case that G is continuous.

The original result of this kind is due to Dobrushin (1971). The above treatment is based on Sullivan (1974), using ideas from Lanford(1971) and Liggett (1972). Most of these results can be generalized by making comparisons on ℓ^1_β rather than ℓ^1 (see Sullivan (1974)); for simplicity we have omitted this generalization in the present work.

5. APPROXIMATE EVOLUTIONS.

From the previous section we know that a generator $G = \Sigma G_\Lambda$ on X satisfying certain conditions gives rise to a semigroup T_t on $F_o^*(X)$. We now show that, if G is local, then T_t on $F_o(X)$ is the evolution of a Markov kernel $P(t,x,\cdot)$. In general, one has a parametrized family of Markov kernels corresponding to the evolution T_t of a generator G on X.

5.1 LEMMA. *Let* $f \in F_o^*(X)$. *Then for* $x \in X$

$$\lim_{\Lambda \to S} f({}_y x) = f^x(y) \tag{5.1}$$

exists in the $\|\cdot\|$ *norm and* $f^x(y) \in F_o(X)$.

Proof. For $\varepsilon > 0$ there is a Λ and an $h \in F^*(X|\Lambda)$ such that

$$\| f - h \| < \varepsilon. \tag{5.2}$$

If $z = x$ except on a finite set

$$| f({}_y x) - f({}_y z) | < 2\varepsilon . \tag{5.3}$$

The functions of y, $\{f({}_y x) : \Lambda\}$ form a Cauchy net in $F_o(X)$ so we have (5.1). The modification by y is on Λ.

5.2 LEMMA. *Let* $f \in X$, $\| \delta f \|_1 < \infty$. *Then* $f \in F_o^*(X)$.

Proof. We have used this result implicitly in the proof of Theorem 4.8. An argument similar to the one above shows that the limit (5.1) exists when $\| \delta f \|_1 < \infty$. In this notation, select a fixed $z \in X$ and consider $f^x({}_x z) \in F^*(X|\Lambda')$ with the modification on Λ'. We then have

$$\lim_{\Lambda' \to S} \| f^x({}_x z) - f(x) \| = 0. \tag{5.4}$$

5.3 LEMMA. *Let* $G = \Sigma G_\Lambda$ *be a generator on* X *such that for each fixed* Λ

$$\Sigma_{i \in S} \; \delta_i \, G_\Lambda < \infty . \tag{5.5}$$

Then for each $x \in X$ *there is a local generator* $G^x = \Sigma G^x_\Lambda$ *on* X *with*

$$G^x_\Lambda(y,\cdot) = \lim_{\Lambda' \to S} G_\Lambda({}_y x, \cdot\ ), \tag{5.6}$$

where the modification of x *is on* Λ', *and the limit exists in the* $||\cdot||$ *norm.*

The proof is as in Lemmas 5.1 and 5.2.

5.4 LEMMA. *Let* G *and* H *be generators on* X *which satisfy the hypothesis of Theorem 4.3 with the same* K. *Let the associated semigroups be* T_t *and* U_t *respectively, and let* C *denote the matrix of the Theorem corresponding to* G. *Then for* $f \in F^*_o(X)$ *we have*

$$|| T_t f - U_t f|| \leq \Sigma_{i,j \in S} \; h_i \, D_{ij} \; \delta_j f, \tag{5.7}$$

with

$$D = \int_o^t \exp s\, C\, ds, \tag{5.8}$$

and

$$h_j = \Sigma_{\Lambda \ni j} || G_\Lambda - H_\Lambda|| \leq 2K. \tag{5.9}$$

Proof. For $f \in F(X)$, $|| \delta f||_1 < \infty$

$$\begin{aligned} T_t f - U_t f &= \int_o^t \frac{d}{dt} (U_{t-s} T_s) f \, ds \\ &= \int_o^t U_{t-s}(G - H) T_s f. \end{aligned} \tag{5.10}$$

From (4.16), (5.10) and the result corresponding to (4.39) with G - H, one obtains (5.7). For the general $f \in F^*_o(X)$ we may approximate f by functions of the type used in Lemma 5.2.

5.5 THEOREM. *Let* G *satisfy the hypothesis of Theorem 4.3. Then for fixed* Λ" *and fixed* $t_o > 0$

$$\lim_{\Lambda \to S} \; \sup_{0 \leq t \leq t_o} \; \sup_{\substack{f \in F^*(X|\Lambda") \\ ||f|| \leq 1}} || T_t f - \exp(t \Sigma_{\Lambda' \subset \Lambda} G_{\Lambda'}) f|| = 0, \tag{5.11}$$

and there is a probability kernel parametrized by t, P(t,x,·), *such that for* $f \in F^*_o(X)$

$$T_t f(x) = \int P(t,x,dy) f^x(y), \tag{5.12}$$

where f^x *is given by* (5.1). *If* G *is local, then* P(t,x,•) *is a Markov kernel.*

Proof. The estimate (5.11) follows from (5.7) by setting $H = \Sigma_{\Lambda' \subset \Lambda} G_{\Lambda'}$. Note that each D_{ij} is an increasing function of t. For the set E whose indicator function $I_E \in F(X|\Lambda")$ define

$$P(t,x,E) = (T_t\, I_E)(x). \tag{5.13}$$

P(t,x,•) is a positive, finitely additive, set function with P(t,x,X) = 1. By the Daniell Extension Theorem (see Loomis (1953)), to show that P(t,x,•) is a measure on sets whose indicator functions are in F(X|Λ"), it is sufficient to show that whenever $f_n \downarrow 0$ in F(X|Λ"), then $(T_t f_n) \downarrow 0$. This follows for $\exp(tH) f_n$, as only bounded measures are involved. Then (5.11) implies that $(T_t f_n) \downarrow 0$. Since X is a Polish space, the Kolmogorov Extension Theorem yields a unique probability measure on X which agrees with the above definition on sets whose indicator functions are in $F_f(X)$. If x and y differ at most on a finite set, then $f^x(y) = f(y)$ and

$$\exp\,(tH^x)\; f^x(y) = \exp\,(tH)\; f(y). \tag{5.14}$$

This and (5.11) give (5.12). In the case in which G is local, from the action of T_t on $F_o(X)$ we get that P(t,x,•) is a Markov kernel.

5.6 COROLLARY. *Assume* G *satisfies the hypothesis of Theorem 4.8 with* $\alpha < 0$. *For* $f \in F_o^*(X)$ *write the limit function* f_∞ *as* $T_\infty f$. *Then* (5.11) *holds with* $t_o = \infty$, *and for each* $x \in X$ *there is a probability measure* $P(\infty,x,\cdot)$ *such that*

$$T_\infty f\,(x) = \int P(\infty,x,dy) f^x(y). \tag{5.15}$$

Henceforth, whenever G is local and satisfies the hypothesis of Theorem 4.3, we shall consider that the associated time development T_t, $t \geq 0$, is given by (1.3), where P is the Markov kernel of Theorem 5.5. We can then use the adjoint action T_t' given by (1.4) on measures.

Theorem 5.5 can be interpreted as follows. If G is local and satisfies the hypothesis of Theorem 4.3, then the evolution T_t of G on $F_o(X)$ is the same as that of a uniquely defined Markov kernel P(t,x,•). When G is not local, for each $x \in X$ there is a Markov kernel - the one corresponding to the local generator G^x - so that the evolution corresponding to T_t is given by (5.12).

Let $\Gamma \subset S$. Suppose that the generator $G = \Sigma G_\Lambda$ on X is such that when $\Lambda \subset \Gamma$, $\delta_j G = 0$ unless $j \in \Gamma$, and that when $\Lambda \subset \Gamma^c$, $\delta_j G = 0$ unless $j \in \Gamma^c$, and also that $G_\Lambda = 0$ if both $\Lambda \cap \Gamma$ and $\Lambda \cap \Gamma^c$ are nonempty. Then if $f \in F_0^*(X|\Gamma)$ and $g \in F_0^*(X|\Gamma^c)$, from (5.11) we deduce that $T_t(fg) = (T_t f)(T_t g)$. This property holds in an approximate sense for a wide class of generators.

We use $|\Lambda|$ to denote the number of elements in Λ.

5.7 THEOREM. *Let the generator* $G = \Sigma G_\Lambda$ *on* X *satisfy the hypothesis of Theorem 4.3 and also satisfy for each* $j \in S$

$$\Sigma_{\Lambda \ni j} |\Lambda| \; \| G_\Lambda \| \leqslant K, \tag{5.16}$$

$$\Sigma_\Lambda \; |\Lambda| \; \delta_j G_\Lambda \leqslant K. \tag{5.17}$$

Then for each $f \in F_0^*(X)$ *and each* $t_0 \geq 0$

$$\lim_{\Lambda \to S} \sup_{\substack{g \in F_0^*(X|\Lambda^c) \\ \|g\| \leqslant 1}} \sup_{0 \leqslant t \leqslant t_0} \| T_t(fg) - (T_t f)(T_t g) \| = 0 . \tag{5.18}$$

Before proving the above we give some corollaries. A probability measure μ on X is said to be *mixing*, if for each Borel set E and each $\varepsilon > 0$ there is a finite set $\Gamma \subset S$ such that

$$| \mu(E \cap F) - \mu(E)\, \mu(F) | < \varepsilon \tag{5.19}$$

for each set F whose indication function is in $F(X|\Gamma^c)$.

5.8 COROLLARY. *If* μ *is a mixing probability measure and* G *is a local generator satisfying the hypothesis of Theorem 5.7, then* $T_t' \mu$ *is mixing for each real* $t > 0$.

One frequently considers systems for which $S = Z^d$, the points with integer coordinates in d-dimensional Euclidean space, and the generator G is translation invariant. In this case it is natural to consider probability measures which are *ergodic* in the sense of the Birkhoff Theorem (see Pitt(1942)).

5.9 COROLLARY. *Let* $S = Z^d$ *and let the* G *of Theorem 5.7 be local and translation invariant. If* μ *is a translation invariant and Birkhoff ergodic probability measure on* X, *then so is* $T_t' \mu$ *for each real* $t > 0$.

Proof of Theorem 5.7. Select a fixed element $x^* \in X$. For finite $\Gamma \subset S$ define the generator H on X as follows:

$$H_\Lambda(x,\cdot) = \begin{cases} G_\Lambda^x(y,\cdot) \text{ if } \Lambda \subset \Gamma \text{ with } y = x \text{ on } \Gamma,\ y = x^* \text{ on } \Gamma^c \\ G_\Lambda(z,\cdot) \text{ if } \Lambda \subset \Gamma^c \text{ with } z = x \text{ on } \Gamma^c,\ z = x^* \text{ on } \Gamma \\ 0 \text{ otherwise.} \end{cases} \tag{5.20}$$

It follows that H satisfies the hypothesis of Theorem 4.3. Let U_t be the associated semigroup. For $f \in F^*(X|\Gamma)$, $g \in F_0^*(X|\Gamma^c)$, $U_t(fg) = (U_t f)(U_t g)$. We note that T_t and U_t satisfy (5.7), and we can estimate the $\{h_j\}$ of (5.9) as follows:

$$j \in \Gamma:\quad h_j \leqslant \sum_{\substack{\Lambda \ni j \\ \Lambda \cap \Gamma^c \neq \phi}} \|G_\Lambda\| + \sum_{k \in \Gamma^c} \sum_{\Lambda \ni j} \delta_k G_\Lambda. \tag{5.21}$$

$$j \in \Gamma^c:\quad h_j \leqslant \sum_{\substack{\Lambda \ni j \\ \Lambda \cap \Gamma \neq \phi}} \|G_\Lambda\| + \sum_{k \in \Gamma} \sum_{\Lambda \ni j} \delta_k G_\Lambda . \tag{5.22}$$

$$\sum_{j \in \Gamma^c} h_j \leqslant \sum_{\Lambda \cap \Gamma \neq \phi} |\Lambda|\ \|G_\Lambda\| + \sum_{k \in \Gamma} \sum_\Lambda |\Lambda|\ \delta_k G_\Lambda . \tag{5.23}$$

The matrix D of (5.8) satisfies, for each $k \in S$,

$$\sum_{j \in S} D_{jk} \leqslant K^*,\ \sum_{j \in S} D_{kj} \leqslant K^* = \int_0^t e^{sK} ds. \tag{5.24}$$

We proceed to prove (5.18). It is sufficient to do so when $f \in F^*(X|\Lambda_1)$. Given $\varepsilon > 0$ and Λ_1, by (5.24) we can find $\Lambda_2 \supset \Lambda_1$, so that

$$\sum_{j \in \Lambda_2^c} \sum_{k \in \Lambda_1} D_{jk} < \varepsilon. \tag{5.25}$$

From (5.21), (4.13) and (4.14) we can select $\Lambda_3 = \Gamma \supset \Lambda_2$ so that

$$\Sigma_{j \in \Lambda_2} h_j < \varepsilon. \qquad (5.26)$$

From (5.23), (5.16) and (5.17) we can select $\Lambda_4 \supset \Lambda_3$ so that

$$\Sigma_{j \in \Lambda_4^c} h_j < \varepsilon. \qquad (5.27)$$

Finally, from (5.24) we can find $\Lambda_5 \supset \Lambda_4$ so that

$$\Sigma_{j \in \Lambda_4} \Sigma_{k \in \Lambda_5^c} D_{jk} < \varepsilon. \qquad (5.28)$$

From these estimates we have, for $f \in F^*(X|\Lambda_1)$, $g \in F_o^*(X|\Lambda_5^c)$,

$$|| T_t(fg) - U_t(fg)|| \leqslant 2 \varepsilon || f|| \; || g|| (2K^* + 4K), \qquad (5.29)$$

$$|| (T_t f)(T_t g) - U_t(fg)|| \leqslant 2 \varepsilon || f|| \; || g|| (2K^* + 4K). \qquad (5.30)$$

Since each D_{ij} is an increasing function limit (5.18) follows.

The results of this section are based on Sullivan (1975b).

6. GENERATORS FROM CONDITIONAL PROBABILITIES AND POTENTIALS.

There are many examples of interacting Markov processes which fall within and somewhat beyond the framework of the two preceding sections. The reader should consult in particular the works of Spitzer, Liggett, Holley and Harris. In this section we give certain generators obtained from conditional probabilities and from potentials.

6.1 DEFINITION. Let μ be a probability measure on X and let $\Gamma \subset S$. A Γ-*conditional probability distribution of* μ is a function $\mu_\Gamma(E|x)$ defined for each Borel subset E of W^Γ and each $x \in X$, such that for fixed E, $\mu_\Gamma(E|x) \in F(X|\Gamma^c)$, for fixed x, $\mu_\Gamma(\cdot|x)$ is a positive measure on W^Γ, and

$$\int \left(\int f({}_y x) \quad \mu_\Gamma(dy|x) \right) \mu(dx) = \int f d\mu \tag{6.1}$$

for each $f \in F(X)$.

Since we have assumed X to be a Polish space, conditional probability distributions always exist (see Doob (1953)). Condition (6.1) does not determine the conditional probability distribution uniquely, but only within μ-a.e. equivalence. In the results below we must select a certain representative of μ_Γ so that the conditions of the theorems of the previous sections can be satisfied. When Γ is a single point set, $\Gamma = \{j\}$, we use the simplified notation μ_j, G_j.

6.2 DEFINITION. Let μ be a probability measure on X. A *type-I generator* of μ is a generator $G = \Sigma G_\Lambda$ on X with $G_\Lambda = 0$ unless $\Lambda = \{j\}$ for some $j \in S$, and

$$G_j(x,\cdot) = \mu_j(\cdot|x). \tag{6.2}$$

With a type-I generator we associate the matrix $M = (M_{ij})$, with $M_{ii} = 0$ and

$$M_{ij} = \frac{1}{2} \delta_i \mu_j = \frac{1}{2} \delta_i G_j, \; i \neq j . \tag{6.3}$$

The ambiguity in the conditional probability distribution results in a corresponding ambiguity for the type-I generator. However, in many cases of interest there is a natural choice for μ_j in (6.2).

From Definition 4.7 we find that the comparison matrix B of the type-I generator G is B = M - 1, with 1 denoting the identity matrix. From Theorem 4.8 we have the following result.

6.3 THEOREM. *Let* μ *be a probability measure on* X *possessing a type-I generator* $G = \Sigma G_j$, *which is local and such that the* M *of* (6.3) *satisfies for each* $j \in S$

$$\Sigma_{i \in S} M_{ij} \leqslant \eta , \tag{6.4}$$

where η *is a fixed real number satisfying* $0 \leqslant \eta < 1$. *Then the closure* $\overline{G}$ *is the infinitesimal generator of a strongly continuous, positive, linear semigroup of contractions* T_t, $t \geqslant 0$, *on* $F_o(X)$. *For each* $f \in F_o(X)$

$$\lim_{t \to \infty} \| T_t f - \int f \, d\mu \| = 0 , \tag{6.5}$$

and, if $\| \delta f \|_1 < \infty$,

$$\| T_t f - \int f \, d\mu \| \leqslant e^{t(\eta - 1)} \| \delta f \|_1 . \tag{6.6}$$

It is interesting to see what can happen if the assumption of locality does not obtain. Let ρ be a probability measure on W, and use the same symbol ρ to denote the probability measure on X which is the product of ρ on each factor space W. Let σ be similarly defined with $\sigma \neq \rho$. Consider $\mu = \frac{1}{2}(\rho + \sigma)$. There exists a Borel set E, whose indicator function $I_E \in F_\infty(X)$, and $\rho(E) = 1$, $\sigma(E) = 0$. Use the particular μ_j

$$\mu_j(\cdot | x) = \begin{cases} \rho(\cdot) & \text{if } x \in E, \\ \sigma(\cdot) & \text{if } x \in E^c . \end{cases} \tag{6.7}$$

This gives a type-I generator of μ for which M is the zero matrix. $T_t f$ converges exponentially when $\| \delta f \|_1 < \infty$, but (6.5) is not satisfied.

6.4 THEOREM. *Let* μ *be a probability measure on* X *with a type-I generator* $G = \Sigma G_j$ *which satisfies the hypothesis of Theorem 6.3. Let* $\{\alpha_\Lambda\}$ *be a family* *of positive numbers parametrized by the nonempty finite subsets of* S *such that*

$$1 \leq \Sigma_{\Lambda \ni j} \alpha_\Lambda \leq K \tag{6.8}$$

for each $j \in S$ *with* K *a fixed real number. Then the closure of the generator* $H = \Sigma H_\Lambda$ *on* X *, with*

$$H_\Lambda(x,dy) = \alpha_\Lambda \, \mu_\Lambda \, (dy|x), \tag{6.9}$$

(The particular choice of μ_Λ *is given in the proof.) is the infinitesimal generator of a strongly continuous, positive, linear, semigroup of contractions* U_t, $t \geq 0$, *on* $F_o(X)$. *For each* $f \in F_o(X)$

$$\lim_{t \to \infty} \| U_t f - \int f d\mu \| = 0, \tag{6.10}$$

and, if $\| \delta f \|_1 < \infty$,

$$\| U_t f - \int f d\mu \| \leq e^{t(\eta - 1)} \| \delta f \|_1 . \tag{6.11}$$

Proof. We use G as in Theorem 6.3. Write $G^\Lambda = \Sigma_{j \in \Lambda} G_j$. Define $\mu_\Lambda(\cdot|x)$ in terms of

$$\int \mu_\Lambda \, (dy|x) \; f(_y x) = \lim_{t \to \infty} \exp(tG^\Lambda) \; f(x) \tag{6.12}$$

for each $f \in F_o(X)$. Using (6.1), Theorem 5.5 and Corollary 5.6, it is not difficult to show that the right hand side of (6.12) defines a Λ-conditional probability distribution of μ which is a local function of x. The proof now proceeds like that of Theorem 4.8, with slightly different estimates on H_Λ. The terms for $i \in \Lambda$ are straightforward. If we can show for $i \in \Lambda^c$, $f \in F_o(X)$,

$$| [\Delta_i^c , H_\Lambda] \, f(x) | \leq \alpha_\Lambda \, \Sigma_{j \in \Lambda} \, C_{ij} \, \delta_j \, f, \tag{6.13}$$

where (C_{ij}) is a matrix with positive elements, and

$$\Sigma_{i \in \Lambda^c} \, C_{ij} \leq \eta \tag{6.14}$$

for each $j \in \Lambda$, then the proof goes as before.

Consider the matrix M partitioned

$$M = \begin{pmatrix} N & P \\ Q & R \end{pmatrix} , \tag{6.15}$$

so that N - 1 is the comparison matrix of G^{Λ} . By the method of Lemma 5.4, we obtain for $f \in F_o(X)$,

$$\left|\int f({}_y x)(\mu_{\Lambda}(dy|x) - \mu_{\Lambda}(dy|{}_c x))\right| \leqslant \tfrac{1}{2}\Sigma_{j,k \in \Lambda} \delta_i \mu_j (1-N)^{-1}_{jk} \delta_k f, \tag{6.16}$$

where the c modification of x is at the i-position. The factor ½ can be inserted because $\mu_j(W|x) = \mu_j(W|{}_c x)$. Then (6.16) implies (6.13) with the matrix C equal to $Q(1 - N)^{-1}$. Let(1*, 1') be the partition of the identically 1 row vector corresponding to (6.15). Condition (6.4) can be written, in part,

$$1^* N + 1' Q \leqslant \eta 1^* . \tag{6.17}$$

From this we deduce

$$1' Q(1 - N)^{-1} \leqslant \eta 1^* , \tag{6.18}$$

which yields (6.14). The "comparison matrix" B of H obtained by the above method satisfies, for each $j \in S$,

$$B_{jj} = - \Sigma_{\Lambda \ni j} \alpha_{\Lambda} , \tag{6.19}$$

$$\Sigma_{i \neq j} B_{ij} \leqslant -\eta B_{jj}, \quad \Sigma_i B_{ij} \leqslant \eta - 1. \tag{6.20}$$

The rest follows as in the proof of Theorem 4.8.

Next we consider certain generators on X which come from potentials. We assume that a probability measure σ is specified for the individual component phase space W. For spaces of the form W^{Λ} we denote by σ(dy) the corresponding product probability measure. For certain of the estimates below, the product nature of σ is not essential. However, in order for the relationship to conditional probabilities to be valid, it is necessary for σ to be a product measure. The relationship between potentials and conditional probabilities is discussed in Preston (1974b) and Sullivan (1973).

6.5 DEFINITION. A *potential* U is a family of functions $\{U_\Lambda(x)\}$ parametrized by the nonempty finite subsets of S, such that $U_\Lambda(x) \in F(X|\Lambda)$ for each Λ. The *energy* E^U of the potential U is the family of functions $\{E^U_\Lambda(x)\}$ given by

$$E^U_\Lambda(x) = \Sigma_{\Lambda' \cap \Lambda \neq \phi} U_{\Lambda'}(x) \qquad (6.21)$$

whenever the series converges.

Given the potential U, with a well-defined energy E^U, we obtain a generator term P_Λ as follows:

$$p_\Lambda(x) = \exp - E^U_\Lambda(x). \qquad (6.22)$$

$$P_\Lambda(x) = p_\Lambda(x)/\int p_\Lambda({}_y x)\ \sigma(dy). \qquad (6.23)$$

The generator term P_Λ is the parametrized probability measure $P_\Lambda({}_y x)\ \sigma(dy)$, with the modification of x on Λ.

6.6 LEMMA. *Let* μ *be a probability measure,* D *a bounded Borel function on* W^Λ . *Then*

$$\| \mu - e^{D(y)}\mu(dy)/ \int e^{D(\bar{y})}\ \mu(d\bar{y})\ \|_m \leq 2\|D\|\, e^{2\|D\|} . \qquad (6.24)$$

Proof. The left hand side of (6.24) equals

$$\int \left| 1 - e^{D(y)}/ \int e^{D(\bar{y})}\ \mu(d\bar{y}) \right| \mu(dy). \qquad (6.25)$$

By comparing power series one obtains the inequality

$$| 1 - e^s | \leq |s|\, e^{|s|}. \qquad (6.26)$$

Then (6.24) follows from (6.25) and (6.26), since

$$\int e^{D(y)}\ \mu(dy) = e^{D^*}, \text{ with } |D^*| \leq \|D\|. \qquad (6.27)$$

6.7 PROPOSITION. *Let* K *be a real number and* U *a potential which satisfies, for each* $j \in S$,

$$\Sigma_{\Lambda \ni j} |\Lambda| \; \|U_\Lambda\| \; \leqslant \; K. \tag{6.28}$$

Let $\{\alpha_\Lambda\}$ *be a family of positive numbers which satisfies for each* $j \in S$

$$\Sigma_{\Lambda \ni j} |\Lambda| \, \alpha_\Lambda \; \leqslant \; K. \tag{6.29}$$

Then the generator on X, $G = \Sigma\, G_\Lambda$, *with*

$$G_\Lambda(x, dy) \; = \; \alpha_\Lambda \, P_\Lambda({}_y x) \, \sigma(dy), \tag{6.30}$$

is local and satisfies the hypotheses of Theorems 4.3 and 5.7.

Proof. Let x and z differ only at $i \in \Lambda^c$. We have

$$|E^U_\Lambda({}_y x) - E^U_\Lambda({}_y z)| \; = \; |\Sigma_{\substack{\Lambda' \cap \Lambda \neq \phi \\ \Lambda' \ni i}} U_{\Lambda'}({}_y x) - U_{\Lambda'}({}_y z)| \leqslant 2K. \tag{6.31}$$

By Lemma 6.6 this gives

$$\delta_i P_\Lambda \; \leqslant \; 4e^{4K} \, \Sigma_{\substack{\Lambda' \cap \Lambda \neq \phi \\ \Lambda' \ni i}} \; \|U_{\Lambda'}\| \, . \tag{6.32}$$

Then

$$\begin{aligned} \Sigma_{i \in S} \, \delta_i P_\Lambda \; &\leqslant \; 4e^{4K} \, \Sigma_{\Lambda' \cap \Lambda \neq \phi} |\Lambda' \backslash \Lambda| \; \|U_{\Lambda'}\| \\ &\leqslant \; 4Ke^{4K} \, |\Lambda| \, . \end{aligned} \tag{6.33}$$

$$\Sigma_{i \in S} \, \Sigma_{\Lambda \ni j} \, \delta_i G_\Lambda \; \leqslant \; 4K^2 \, e^{4K} \, . \tag{6.34}$$

Next, for i fixed,

$$\Sigma_\Lambda \, \alpha_\Lambda |\Lambda| \Sigma_{\substack{\Lambda' \cap \Lambda \neq \phi \\ \Lambda' \ni i}} \|U_{\Lambda'}\| = \Sigma_{\Lambda' \ni i} \|U_{\Lambda'}\| \; \Sigma_{\Lambda \cap \Lambda' \neq \phi} \, \alpha_\Lambda |\Lambda| \leqslant K^2, \tag{6.35}$$

$$\Sigma_\Lambda \, |\Lambda| \, \delta_i G_\Lambda \leqslant \; 4K^2 \; e^{4K} \, . \tag{6.36}$$

It is straightforward to verify the remaining hypotheses of Theorems 4.3 and 5.7, and to check locality.

Next we consider a different type of generator obtained from the potential V. The generator term Q_Λ is of the form $Q_\Lambda(x)\ \sigma(dy)$, where

$$Q_\Lambda(x) = q_\Lambda(x)/\| q_\Lambda(x) \| , \tag{6.37}$$

$$q_\Lambda(x) = \exp - E_\Lambda^V(x). \tag{6.38}$$

By techniques like those employed above, we get the following:

6.8 PROPOSITION. *Let* K *be a real number, and* V *a potential which satisfies for each* $j \in S$

$$\Sigma_{\Lambda \ni j} |\Lambda|\ \ \| V_\Lambda \| \leq K. \tag{6.39}$$

Let $\{\alpha_\Lambda\}$ *be a family of positive numbers satisfying* (6.29). *Then the generator* $G = \Sigma\, G_\Lambda$ *with*

$$G_\Lambda(x, dy) = \alpha_\Lambda Q_\Lambda(x)\ \sigma(dy) \tag{6.40}$$

is local and satisfies the hypotheses of Theorems 4.3 and 5.7.

It is not difficult to verify

$$\delta_i(Q_\Lambda(x)\ P_\Lambda({}_y x)\ \sigma(dy)) \leq \delta_i Q_\Lambda + \delta_i P_\Lambda , \tag{6.41}$$

so we have the following:

6.9 PROPOSITION. *Let* K *be a real number and* U, V *two potentials which satisfy* (6.28) *and* (6.39). *Let* $\{\alpha_\Lambda\}$ *be a family of positive numbers satisfying* (6.29). *Then the generator* $H = \Sigma\, H_\Lambda$ *with*

$$H_\Lambda(x, dy) = \alpha_\Lambda\ Q_\Lambda(x)\ P_\Lambda({}_y x)\ \sigma(dy) \tag{6.42}$$

is local and satisfies the hypotheses of Theorems 4.3 and 5.7.

7. Reversibility and Relaxation.

The study of fluxuations from equilibrium forms an important class of time dependent problems in both theory and experiment. One considers a system which is perturbed, either externally or by internal variations, and observes the rate and manner of return to equilibrium. It is natural to prefer models of fluxuation from equilibrium which do not distinguish the direction of time. This property for Markov time development amounts to *reversibility*. Considerations of reversibility have long been employed by physicists under the title of the *Principle of Detailed Balance*.

Considerations of rates of *relaxation* (return to equilibrium) for random field models involve certain subtleties. Theorem 4.8 gives one relaxation rate based on the "norm" $\|\delta f\|_1$. This convergence is, in general, applicable only to functions f for which $\|\delta f\|_1$ is finite; furthermore, the result is strongly dependent on the specifics of the evolution. In this section we consider a different type of relaxation rate based on spectral properties of the generator.

We recall that the probability measure μ is said to be *reversible* for the Markov kernel P, if, for each t > 0, $P(t,x,dy)\mu(dx)$ is a symmetric measure in x, y. An equivalent definition is that the action T_t of P be a symmetric operator on $L^2(\mu)$ for each t > 0. Either of these two definitions can be applied to jump generators and the Markov kernels obtained from them. Conditions for reversibility of a generator G on X are more involved. Throughout this section we shall assume that generators are local and satisfy conditions of the form

$$\Sigma_{\Lambda \ni k} \; G_\Lambda < \infty \tag{7.1}$$

for each k ε S, so that the domain of G will contain $F_f(X)$. We emoloy the notation $(\cdot,\cdot)_\mu$ to denote

$$(f, g)_\mu = \int f \, g \, d\, \mu. \tag{7.2}$$

7.1 DEFINITION. Let $G = \Sigma G_\Lambda$ be a generator on X which is local and satisfies (7.1) and let μ be a probability measure on X. μ is called *reversible for* G if

$$(Gf,g)_\mu = (f,Gg)_\mu \tag{7.3}$$

for each f, g ε $F_f(X)$. μ is called *strictly reversible for* G if μ is reversible for each G_Λ.

7.2 THEOREM. Let the generator G on X be of the form $G = \Sigma G_k$, that is, $G_\Lambda = 0$ unless $\Lambda = \{k\}$, with k ε S. If the probability measure μ is reversible for G, then μ is strictly reversible for G.

Proof. We need to show that each G_k is symmetric in the $(\cdot,\cdot)_\mu$ inner product. It is sufficient to show that the measure on W x X given by $G_k({}_ax,db)d\mu({}_ax)$ is symmetric in a and b. Let f, g ε $F(X|\{k\})$, and let h be the indicator function of a set such that h ε $F(X|\Lambda)$, where k ε Λ^c. Then

$$\begin{aligned}(Gfh, gh)_\mu - (Ggh, fh)_\mu &= \\ (G_kf, gh)_\mu + (Gh, fgh)_\mu - (G_kg, fh)_\mu &- (Gh, f\,gh)_\mu .\end{aligned} \tag{7.4}$$

By hypothesis, the left hand side of (7.4) is zero. Thus $(G_kf,gh)_\mu = (G_kg,fh)_\mu$ when f, g and h are as above. This implies the required symmetry for $G_k({}_ax, db)\, d\mu({}_ax)$.

For the first result of the above kind, as well as for certain generalizations, see Logan (1974).

In Sections 4 and 5 it was convenient to consider the action of G on functions in $F_o^*(X)$ which are not in $F_o(X)$. Here this generalization proves inconvenient. Sufficient assumptions will be made so that the generators below correspond uniquely to Markov kernels. The semigroups below are those which correspond to the action of the Markov kernels.

7.3 THEOREM. *Let the generator* $G = \Sigma G_\Lambda$ *on* X *be local and satisfy the hypothesis of Theorem 4.3. Let* μ *be a probability measure which is reversible for* G. *Let* $\bar{G}$ *denote the* $L^2(\mu)$ *closure of* G *as defined on* $F_f(X)$. *Then* G *is* *self-adjoint, and its spectrum consists of negative real numbers. Let* $-\lambda$ *be the supremum of the strictly negative spectrum of* G *and let* $\mathcal{P}_0$ *denote the projection on the zero eigenspace of* G. *Then for each* $f \in L^2(\mu)$, $t \geq 0$,

$$\lim_{t \to \infty} \| T_t f - \mathcal{P}_0 f \|_2 = 0, \tag{7.5}$$

$$e^{\lambda t} \| T_t f - \mathcal{P}_0 f \|_2 \leq \| f - \mathcal{P}_0 f \|_2 \quad , \tag{7.6}$$

with $\| \cdot \|_2$ *denoting the norm in* $L^2(\mu)$. *Further,* μ *is reversible for the Markov kernel associated with* G.

Proof. For G as defined on $F_f(X)$ in $L^2(\mu)$, the $L^2(\mu)$ closure $\bar{G}$ is a densely defined, closed, symmetric, dissipative, linear operator. The $L^2(\mu)$ closure contains the $\| \cdot \|$ closure, so each $f \in F_0(X)$ with $\| \delta f \|_1 < \infty$ is in the domain of $\bar{G}$. From the proof of Theorem 4.8, we find that $(1 - \lambda\bar{G})^{-1}$ is a symmetric bounded linear operator on $L^2(\mu)$, so $(1 - \lambda\bar{G})^{-1}$ is selfadjoint. Then $\bar{G}$ is selfadjoint. We deduce (7.5) and (7.6) from spectral representation of the dissipative, selfadjoint operator $\bar{G}$. The reversibility of μ for the Markov kernel of G follows from the symmetry of T_t with respect to $(\cdot,\cdot)_\mu$.

7.4 THEOREM. *Let* G *and* H *be generators on* X *which are local and satisfy the hypothesis of Theorem 4.3. Let the corresponding evolutions be denoted* T_t , U_t. *Let* μ *be a probability measure on* X , *which is reversible and ergodic for the Markov kernel associated with* G *by Theorem 5.5. Assume*

$$(H f, f)_\mu \leq (G f, f)_\mu \tag{7.7}$$

for each $f \in F_f(X)$. *Then* μ *is an ergodic invariant probability measure for the Markov kernel associated with* H. *If* $\lambda \geq 0$ *is such that*

$$e^{\lambda t} \| T_t f - \int f d\mu \|_2 \tag{7.8}$$

is bounded in $t \geq 0$ *for each* $f \in L^2(\mu)$, *then*

$$e^{\lambda t} \| U_t f - \int f d\mu \|_2 \leq \| f - \int f d\mu \|_2 \ . \tag{7.9}$$

Proof. From (7.7) we deduce that H is dissipative in $L^2(\mu)$. It follows easily that μ is invariant under the Markov kernel corresponding to U_t. Let I_E be the indicator function of a set which is invariant under U_t. Then $\bar{H}\, I_E = 0$. From (7.7) and reversibility we conclude that I_E is in the domain of $(-\bar{G})^{\frac{1}{2}}$, and $(-\bar{G})^{\frac{1}{2}} I_E = \bar{G}\, I_E = 0$. Hence I_E is a fixed point of T_t in $L^2(\mu)$. This implies that E is μ-almost invariant under T_t. The ergodicity assumption on T_t implies that $\mu(E)$ is zero or one. Hence μ is also ergodic for U_t. Let $L_o^2(\mu)$ denote the subspace of $L^2(\mu)$ consisting of those functions whose μ-integral vanishes. From the boundedness of (7.8), using the spectral representation, we find that the restriction of $G + \lambda$ to $F_f(X) \cap L_o^2(\mu)$ is dissipative. From (7.7), we have that the restriction of $H + \lambda$ to $F_f(X) \cap L_o^2(\mu)$ is dissipative. Then $e^{\lambda t}\, U_t$ is a contraction semigroup on $L_o^2(\mu)$, which yields (7.9).

The inequality (7.7) is satisfied in the important case in which H can be written $H = G + G'$, where all three are generators on X, satisfy the hypothesis of Theorem 4.3, and possess μ as an invariant probability measure.

8. PROJECTIONS.

It is frequently useful to compare time evolutions of different systems. In this section we give two methods of relating interacting Markov processes on different spaces. The first, due to Dobrushin (1971), approximates evolutions on $X = W^S$ by evolutions on W^Λ.

8.1 DEFINITION. Let μ be a probability measure on X with a specified family $\{\mu_\Gamma(\cdot|x)\}$ of Γ-conditional probability distributions of μ, where Γ runs through the cofinite subsets of S. Let $G = \Sigma\, G_\Lambda$ be a local generator on X such that, for each $j \in S$,

$$\Sigma_{\Lambda \ni j} \|G_\Lambda\| < \infty \tag{8.1}$$

For each Λ, the $\mu\Lambda$-*projection* of G, $G^{\mu\Lambda}$, is the generator on W^Λ with terms $G^{\mu\Lambda}_{\Lambda'}$, for $\Lambda' \cap \Lambda \neq \phi$, so that for $z \in W^\Lambda$

$$G^{\mu\Lambda}_{\Lambda'}(z,\cdot) = \int G_{\Lambda'}({}_x z,\cdot)\ \mu_{\Lambda^c}(dx|z), \tag{8.2}$$

with the modification on Λ^c.

In working with the $\mu\Lambda$-projection of G, we employ implicitly certain natural mappings between various functions and measure spaces. Also, the definition depends on the particular choice of conditional probability distributions.

From assumption (8.1) we can regard $G^{\mu\Lambda}$ as acting as a jump generator on W^Λ. We deduce easily the following:

8.2 LEMMA. *In the notation of Definition 8.1 and of (7.2), for each* $f, g \in F(X|\Lambda)$

$$(G f, g)_\mu = (G^{\mu\Lambda} f, g)_\mu , \tag{8.3}$$

with the natural identification of $F(X|\Lambda)$ *and* $F(W^\Lambda)$.

8.3 PROPOSITION. *Let* G *be local and satisfy the hypothesis of Theorem 4.3. Let the probability measure* μ *be invariant under the Markov kernel of* G .

Then the restriction of μ *to* W^{Λ} *is invariant under the Markov kernel of* $G^{\mu\Lambda}$. *If* μ *is reversible for* G, *then its restriction is reversible for* $G^{\mu\Lambda}$.

Proof. By differentiation of $\int T_t f \, d\mu$, it is not difficult to show that μ is invariant under the Markov kernel of G if and only if

$$\int G f \, d\mu = 0 \tag{8.4}$$

for all $f \in F_f(X)$. By (8.3) this carries over to $G^{\mu\Lambda}$. Similar considerations apply for reversibility.

The second comparison technique we discuss is based on embedding two Markov processes in a third Markov process. For historical details of this method see Harris (1974).

8.4 PROPOSITION. *Let* $P(t,x,\cdot)$ *be a Markov kernel on the Polish space* X, *and let* $\pi : X \to X^*$ *and* $\tau : X^* \to X$ *be Borel mappings such that* $\pi(\tau(z)) = z$ *for each* z *in the Polish space* X^*. *Assume that for each Borel* $E \subset X^*$ *and each* $t > 0$,

$$P(t, x, \pi^{-1}(E)) = P(t, y, \pi^{-1}(E)), \tag{8.5}$$

whenever $\pi(x) = \pi(y)$, $x,y \in X$. *Then* $P^*(t,z,E)$, *defined by*

$$P^*(t,z,E) = P(t, \tau(z), \pi^{-1}(E)), \tag{8.6}$$

is a Markov kernel on X^*.

Proof. We must show that

$$\int P^*(t,z,dw) \, P^*(s,w,E) = P^*(t+s, z,E). \tag{8.7}$$

The left hand side of (8.7) is

$$\int P(t, \tau(z), \pi^{-1}(dw)) \, P(s, \tau(w), \pi^{-1}(E)). \tag{8.8}$$

By the usual change of variable formula, (8.8) equals

$$\int P(t, \tau(z), dx) \, P(s, \tau[\pi(x)], \pi^{-1}(E)). \tag{8.9}$$

From assumption (8.5) we can then replace $\tau(\pi(x))$ by x in (8.9), which gives the desired conclusion.

Now we specialize to random field generators. We consider that we are given Borel mappings

$$\pi : W \to W^*, \ \tau : W^* \to W; \tag{8.10}$$

$$\pi(\tau(b)) = b \quad \text{all } b \in W^*. \tag{8.11}$$

We take the natural extensions

$$\pi : X = W^S \to X^* = W^{*S}, \ \tau : X^* \to X, \tag{8.12}$$

using the same symbols for the extended mappings, which are also Borel.

8.5 PROPOSITION. *Let* $G = \Sigma\, G_\Lambda$ *be a local generator on* X *which satisfies the hypothesis of Theorem 4.3. Assume, for each* Λ *and each Borel subset* E *of* $W^{*\Lambda}$*, that*

$$G_\Lambda(x, \pi^{-1}(E)) = G_\Lambda(y, \pi^{-1}(E)) \tag{8.13}$$

whenever $\pi(x) = \pi(y)$*, where* π *and* τ *are the natural extensions of* (8.10) *to the appropriate spaces. Then* $G^* = \Sigma\ G_\Lambda^*$*, defined by*

$$G_\Lambda^*(z, dw) = G_\Lambda(\tau(z), \pi^{-1}(dw)), \tag{8.14}$$

is local and satisfies the hypothesis of Theorem 4.3. Let T_t, T_t^* *and* P, P^* *be the associated evolutions and Markov kernels. For* $f \in F_o(X^*)$ *we have*

$$(T_t^* f) \circ \pi = T_t(f \circ \pi), \tag{8.15}$$

while P *and* P^* *are related by* (8.6).

Proof. It is straightforward to verify that G^* is local and satisfies the hypothesis of Theorem 4.3. It is sufficient to prove (8.15) for $f \in F_f(X)$, and this also implies that P and P^* are related by (8.6). Further, we may assume that $G = \Sigma\ G_\Lambda$ has only a finite number of terms, since the general result follows from finite approximations.

We consider (8.13) as a condition on the positive measure kernel $G_\Lambda(x,\cdot)$. It follows that the jump generator corresponding to the action of G_Λ satisfies the condition corresponding to (8.13). So we consider the bounded measure kernel $H(x, dy)$ which represents a finite sum of jump generators corresponding to G_Λ's. We assume H satisfies

$$H(x, \pi^{-1}(E)) = H(y, \pi^{-1}(E)) \tag{8.16}$$

whenever $\pi(x) = \pi(y)$ and E is a Borel subset of X^*. Define $H^*(z, dw)$ by

$$H^*(z, dw) = H(\tau(z), \pi^{-1}(dw)). \tag{8.17}$$

Let H^n represent the n-fold composition of H with itself. We show that

$$H^{*n}(z, E) = H^n(\tau(z), \pi^{-1}(E)) . \tag{8.18}$$

Assume (8.18) holds for $n - 1$. Then

$$\begin{aligned} H^n(\tau(z), \pi^{-1}(E)) &= \int H^{n-1}(\tau(z), dy)\, H(\tau \circ \pi(y), \pi^{-1}(E)) \\ &= \int H^{n-1}(\tau(z), \pi^{-1}(dw))\, H(\tau(w), \pi^{-1}(E)) \qquad (8.19) \\ &= \int H^{*n-1}(z, dw)\, H^*(w, E). \end{aligned}$$

Then we obtain, for $f \in F(X^*)$,

$$\{\exp(t H^*) f\} \circ \pi = \exp(t H)(f \circ \pi). \tag{8.20}$$

This completes the proof.

Condition (8.13) requires, in effect, that, whenever E is in π^{-1} of the Borel field of X^*, $G_\Lambda(x, E)$ must be measurable with respect to π^{-1} of this field. In this formulation, the proofs of Propositions 8.4 and 8.5 are simply diagram chasing.

9. EVOLUTIONS ON ORDERED SPACES.

The spaces of certain interacting random processes are intrinsically equipped with an underlying order structure, *e.g.* the spaces associated with birth-death processes and ferromagnetic, dynamic Ising models. In this section we consider certain properties of evolutions related to partial orderings.

We assume that the individual component phase space W is equipped with a relation $\leqslant$ which satisfies

$$a \leqslant b,\ b \leqslant c \Rightarrow a \leqslant c, \tag{9.1}$$

$$a \leqslant b,\ b \leqslant a \Leftrightarrow a = b. \tag{9.2}$$

Further, we assume that the graph of $\leqslant$,

$$\{(a, b) \in W \times W : a \leqslant b\}, \tag{9.3}$$

is a Borel set.

For certain of the results below, we assume that W has a maximum element w and a minimum element 0, such that for each $a \in W$

$$0 \leqslant a, \quad a \leqslant w. \tag{9.4}$$

On the product space $X = W^S$ we use the product ordering

$$x \leqslant y \quad \Leftrightarrow \quad x_i \leqslant y_i \text{ , all } i \in S. \tag{9.5}$$

When W has extremal elements, w, 0, then spaces of the form W^Γ, $\Gamma \subset S$, also have extremal elements. We shall use the same symbols w, 0 for extremal elements in different spaces, the specific meaning coming from context.

9.1 DEFINITION. The symbol i subscripted to a function space, *e.g.* $F_{oi}(X)$, denotes those functions in the space which are increasing. Let μ, ν be measures on W^Λ. We say that ν is *inferior* to μ, written

$$\mu \geqslant^i \nu , \tag{9.6}$$

if for each $f \in F_i(X|\Lambda)$,

$$\int f\, d\mu \geqslant \int f\, d\nu. \tag{9.7}$$

When μ and ν are measures on X, the requirement is that (9.7) hold for each $f \in F_{oi}(X)$.

9.2 DEFINITION. Let $G = \Sigma\, G_\Lambda$ be a local generator on X. G is called *attractive*, if for each G_Λ there is a $\lambda > 0$ such that

$$(1 - \lambda\, G_\Lambda(x, W^\Lambda))\varepsilon_x^\Lambda + \lambda\, G_\Lambda(x,\cdot) \geqslant^i \tag{9.8}$$
$$(1 - \lambda\, G_\Lambda(y, W^\Lambda))\varepsilon_y^\Lambda + \lambda\, G_\Lambda(y,\cdot)$$

whenever $x \geqslant y$, where ε_x^Λ and ε_y^Λ denote unit measures concentrated at the points of W^Λ which have the same coordinates as x and y respectively.

9.3 PROPOSITION. *Let* $G = \Sigma\, G_\Lambda$ *be a local generator on* X *which is attractive and satisfies the hypothesis of Theorem 4.3. Then* T_t *maps* $F_{oi}(X)$ *into itself.*

Proof. Let the measure kernel M(x, dz) denote the left hand side of (9.8). For $f \in F_{oi}(X)$,

$$\int M(x,dz) f({}_z x) \geqslant \int M(y,dz)\, f({}_z x) \geqslant \int M(y,dz)\, f({}_z y), \tag{9.9}$$

so $1 + \lambda G$ maps $F_{oi}(X)$ into itself. Next, define the operator $N(\lambda, \Lambda)$ by

$$N(\lambda, \Lambda) = 1 + \lambda\, \Sigma_{\Lambda' \subset \Lambda}\, G_{\Lambda'} . \tag{9.10}$$

It follows that, for sufficiently small $\lambda > 0$, $N(\lambda, \Lambda)$ maps $F_{oi}(X)$ into itself. Then

$$\exp(t\, \Sigma_{\Lambda' \subset \Lambda}\, G_{\Lambda'}) = \lim_{n \to \infty} N(t/n, \Lambda)^n \tag{9.11}$$

maps $F_{oi}(X)$ into itself. From (4.15) we deduce that T_t maps $F_{oi}(X)$ into itself.

9.4 THEOREM. *Let* W *have extremal elements* w, 0 *satisfying* (9.4). *Let the Markov kernel* P *have the semigroup* T_t *which maps* $F_{oi}(X)$ *into itself. Assume* *there are given a family* $\mathcal{F}$ *of functions in* $F_{oi}(X)$, *and a strictly positive, increasing function* $r(t)$, *such that for each* $f \in \mathcal{F}$

$$\lim_{t \to \infty} r(t)\,(T_t f(w) - T_t f(0)) = 0. \tag{9.12}$$

Then, for each function h *in the algebra generated by* $\mathcal{F}$,

$$\lim_{t \to \infty} \sup_{x,y \in X} r(t)\left|T_t h(x) - T_t h(y)\right| = 0, \tag{9.13}$$

and for each invariant probability measure μ *of* P,

$$\lim_{t \to \infty} r(t)\| T_t h - \int h\, d\mu\| = 0. \tag{9.14}$$

If $\mathcal{F}$ *generates the* σ*-field of Borel sets, then* P *has at most one invariant probability measure.*

Proof. Let f, g $\in F_{oi}(X)$ both satisfy relations of the form (9.12) and take values in the interval (0, 1). Then f g and f + g - fg are in $F_{oi}(X)$, so

$$0 \leqslant T_t(fg)(w) - T_t(fg)(0) \leqslant T_t(f+g)(w) - T_t(f+g)(0). \tag{9.15}$$

Hence the product function fg satisfies (9.12). Since each function h in the algebra generated by $\mathcal{F}$ can be written as a finite linear combination of such products, we deduce (9.13). Limit (9.14) follows easily from (9.13). Let E be a real open interval. For any f $\in \mathcal{F}$ and any invariant probability measure μ of P we can find $\mu(f^{-1}(E))$ as follows. Select a sequence of polynomials of f which converges monotonically to the indicator function of $f^{-1}(E)$. Limit (9.14) then gives μ of this set. In a similar manner each invariant probability measure has its values defined on the σ-field generated by $\mathcal{F}$. When $\mathcal{F}$ generates the Borel field, there can be at most one invariant probability measure.

In application, $\mathcal{F}$ is often chosen to be $C_{fi}(X)$. When W is compact and the graph of $\leqslant$ is closed, the algebra generated by $C_{fi}(X)$ is uniformly dense in C(X), so that $C_{fi}(X)$ generates the σ-field of Borel sets (see Nachbin (1965)).

The ordering $\geqslant^i$ has been most effectively employed in the context of random fields by Preston (1974b). For random field evolutions, another ordering of measures has been employed, particularly by Holley and Harris.

9.5 DEFINITION. Let μ and ν be positive measures on X. Then μ is said to *surpass* ν, written

$$\mu \geqslant^s \nu , \tag{9.16}$$

if there is a probability kernel N(x,dy) with

$$N(x, \{z \in X : z \leqslant x\}) = 1 \tag{9.17}$$

for each $x \in X$, and such that

$$\nu(E) = \int \mu(dx)\, N(x, E) \tag{9.18}$$

for each Borel subset E of X.

Alternatively, one may view the relation $\mu \geqslant^s \nu$ as meaning that there exists a positive measure ρ on $X \times X$, concentrated on $\{(x,y) : x \geqslant y\}$, whose restriction to the first factor is μ and to the second factor is ν.

9.6 LEMMA. *Let* $\mu \geqslant^s \nu$. *Let* $f, g \in F(X)$ *be such that* $f(x) \geqslant g(y)$ *whenever* $x \geqslant y$. *Then*

$$\int f\, d\mu \geqslant \int g\, d\nu. \tag{9.19}$$

Proof.

$$\begin{aligned} \int f\, d\mu &= \int_{x \geqslant y} f(x)\mu(dx)N(x,dy) \\ &\geqslant \int_{x \geqslant y} g(y)\mu(dx)N(x,dy) = \int g\, d\nu. \end{aligned} \tag{9.20}$$

9.7 THEOREM. *Let* W *be a finite partially ordered set. Let* μ *and* ν *be positive measures on* W. *Then* $\mu \geqslant^s \nu$ *if and only if* $\mu \geqslant^i \nu$.

Proof. Half the theorem follows from Lemma 9.6. To complete the proof, we assume $\mu \geqslant^i \nu$ and show $\mu \geqslant^s \nu$. From $\mu \geqslant^i \nu$ we deduce that $\mu(W) = \nu(W)$, by considering the integrals of positive and negative constant functions. We remark that a necessary and sufficient condition for $\mu \geqslant^i \nu$, when $\mu(W) = \nu(W)$, is that

$$\mu(E) \geqslant \nu(E) \tag{9.21}$$

for each increasing subset E of W. Define

$$\mathcal{E} = \{E \subset W : E \text{ is increasing and } \mu(E) = \nu(E)\}. \tag{9.22}$$

If $\mathcal{E}$ contains every increasing subset of W, then $\mu = \nu$, and we can let the probability kernel N, which in this case is represented by a stochastic matrix, be the identity matrix. Otherwise we construct a sequence $\mu_1, \mu_2, \ldots$, such that $\mu \geqslant^s \mu_1$, $\mu_1 \geqslant^i \nu$; $\mu_1 \geqslant^s \mu_2$, $\mu_2 \geqslant^i \nu$; and so on in such a way that $\mathcal{E}_i$, defined as in (9.22) using μ_i, is strictly increasing in i. Then for some finite n, $\mathcal{E}_n$ contains all increasing subsets of W, so $\mu_n = \nu$. Since the relation $\geqslant^s$ is transitive, $\mu \geqslant^s \nu$.

Let A, B $\in \mathcal{E}$. Then

$$\mu(A \cup B) \geqslant \nu(A \cup B),\ \mu(A \cap B) \geqslant \nu(A \cap B);$$

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B) = \nu(A) + \nu(B); \tag{9.23}$$

$$\mu(A \cup B) = \nu(A \cup B),\ \mu(A \cap B) = \nu(A \cap B).$$

Thus $\mathcal{E}$ is closed under union and intersection. $\mathcal{E}$ defines an equivalence relation on W as follows. The class $\tilde{x}$ of $x \in W$ consists of those points of W which are not separated from x by $\mathcal{E}$, *i.e.* $y \in \tilde{x}$ means that $\{x,y\} \cap E$ is either empty or $\{x,y\}$ for each $E \in \mathcal{E}$. Define

$$x\!\uparrow = \cap\{E \in \mathcal{E} : E \ni x\},\ (x\!\downarrow)^c = \cup\{E \in \mathcal{E} : E \not\ni x\}. \tag{9.24}$$

It follows that $\tilde{x} = x\!\uparrow \cap\, x\!\downarrow$. If $\tilde{x} = \{x\}$ for each $x \in W$, then $\mu = \nu$. Otherwise, select x such that $\tilde{x}$ has more than one element. Let E be a nonempty proper subset of $\tilde{x}$ such that $\tilde{x} \setminus E$ is increasing in $\tilde{x}$, *i.e.* when $z \in \tilde{x}$ and $z \leqslant y$ for for some $y \in E$, then $z \in E$. We have that $x\!\uparrow \setminus E$ is increasing: when $z \in x\!\uparrow$ and $z \leqslant y$ for some $y \in E$, then $z \in x\!\downarrow$; so $z \in \tilde{x}$ and hence $z \in E$. Since $x\!\uparrow \setminus E$

is an increasing subset of W which is not in $\mathcal{E}$, $\mu(x\uparrow \setminus E) > \nu(x\uparrow \setminus E)$, so $\mu(E) < \nu(E)$. This means that the restrictions μ^r, ν^r of μ, ν to $\tilde{X}$ satisfy $\mu^r \geq^i \nu^r$ in the ordering on $\tilde{X}$. Further, for each nonempty, increasing, proper subset E of $\tilde{X}$, $\mu(E) > \nu(E)$. Let y be a maximal element of $\tilde{X}$. Then $\mu(\{y\}) > \nu(\{y\})$. If y were also a minimal element of $\tilde{X}$, we would have $\mu(\{y\}) < \nu(\{y\})$. Thus there is some $z \in \tilde{X}$ with $z < y$. We consider the stochastic matrix (N_{xy}) with $N_{tt} = 1$ for $t \neq y$, $N_{yy} = 1 - \alpha$, $N_{yz} = \alpha$, where $0 < \alpha < 1$. Select the least α in this range for which there is an increasing set $E \notin \mathcal{E}$ such that $\mu_1(E) = \nu(E)$, where $\mu_1 = \mu N$. The proof then follows the argument given above.

9.8 Corollary. *Let W be a countable partially ordered set. Let μ and ν be (finite) positive measures on $X = W^S$. Then $\mu \geq^s \nu$ if and only if $\mu \geq^i \nu$.*

Proof. By Lemma 9.6 we need only show that $\mu \geq^i \nu \Rightarrow \mu \geq^s \nu$. There is no loss in generality in assuming that W contains extremal elements w, 0 satisfying (9.4), for these may be added to W if necessary. There exists a sequence $\{X_n : n = 1, 2, \ldots\}$ of subsets of X, such that each X_n is the product of finite subsets of W,

$$\{0,w\} \subset X_n \subset X_{n+1} \quad \text{all } n, \tag{9.25}$$

and also

$$\lim_{n \to \infty} \mu(X \setminus X_n) + \nu(X \setminus X_n) = 0. \tag{9.26}$$

Let X_n^Λ denote the image of X_n under the natural projection of X to W^Λ. Define the measures μ_n^Λ and ν_n^Λ on X_n^Λ by

$$\mu_n^\Lambda(\{x\}) = \mu(\{\omega = x \text{ on } \Lambda\}),\ x \in X_n^\Lambda \setminus \{w\}, \tag{9.27}$$

$$\nu_n^\Lambda(\{x\}) = \nu(\{\omega = x \text{ on } \Lambda\}),\ x \in X_n^\Lambda \setminus \{0\}, \tag{9.28}$$

while

$$\mu_n^\Lambda(\{w\}) = \mu(\{\omega = w \text{ on } \Lambda\} \cup (X \setminus X_n)), \tag{9.29}$$

$$\nu_n^\Lambda(\{0\}) = \nu(\{\omega = 0 \text{ on } \Lambda\} \cup (X \setminus X_n)). \tag{9.30}$$

From the assumption $\mu \geq^i \nu$, it follows that $\mu_n^\Lambda \geq^i \nu_n^\Lambda$. By theorem 9.7 we have

$\mu_n^\Lambda \geq^s \nu_n^\Lambda$, so there is a positive measure ρ_n^Λ concentrated on $\{(x,y) \in X_n^\Lambda \times X_n^\Lambda : x \geq y\}$ which restricts appropriately to μ_n^Λ and ν_n^Λ. We consider the measures $\{\rho_n^\Lambda\}$ to be defined on $\{(x,y) \in X_n \times X_n : x \geq y\}$ by taking products with the unit measures concentrated on the w-elements of $X_n^{\Lambda^c} \times X_n^{\Lambda^c}$. As $\Lambda \to S$, by compactness $\{\rho_n^\Lambda\}$ has a weak limit point ρ_n. The sequence $\{\rho_n\}$ is tight, so it has as a weak limit point the measure ρ on $\{(x,y) \in X \times X : x \geq y\}$. It is straightforward to check that ρ restricts appropriately to μ and ν.

9.9 THEOREM. *Let* $G = \Sigma G_\Lambda$ *and* $H = \Sigma H_\Lambda$ *be local generators on* X *which satisfy the hypothesis of Theorem 4.3, with corresponding semigroups* T_t *and* U_t. *Assume, for each* Λ, *there is a* $\lambda > 0$ *such that*

$$(1 - \lambda G_\Lambda(x, W^\Lambda))\, \epsilon_x^\Lambda + \lambda G_\Lambda(x,\cdot) \geq^s$$
$$(1 - \lambda H_\Lambda(z, W^\Lambda))\, \epsilon_z^\Lambda + \lambda H_\Lambda(z,\cdot), \tag{9.31}$$

whenever $x \geq z$. *Then for each* $f, g \in F_o(X)$ *such that* $f(x) \geq g(z)$ *whenever* $x \geq z$, *we have* $T_t f(x) \geq U_t g(z)$ *for each* $t > 0$.

Proof. Let $M(x,\cdot)$ denote the measure kernel of the left hand side of (9.31); $N(z,\cdot)$, the measure kernel of the right hand side. Then, for f, g as stated,

$$M^n f(x) \geq N^n g(z) \tag{9.32}$$

for each positive integer n, by Lemma 9.6. The proof is completed along the lines of the proof of Proposition 9.3.

Variants of the above result can be found in the works of Holley and Harris. They use a different method of proof, which goes as follows. Since $M(x,\cdot) \geq^s N(x,\cdot)$, there is a positive measure ρ on $\{(a,b) \in W \times W : a \geq b\}^\Lambda$, whose restriction to the first variable is $M(x,\cdot)$, and to the second variable is $N(z,\cdot)$. Define the generator $J = \Sigma J_\Lambda$ on $\{(x,z) \in X \times X : x \geq z\}$ with $J_\Lambda(x,z;\cdot) = \rho/\lambda$. When we project J_Λ to the first variable, we get a generator term on X whose operator is the same as that of G_Λ; the projection of J_Λ to the second variable yields a generator term whose operator coincides with that of H_Λ. Though we do not know whether J satisfies Theorem 4.3, finite approximations are sufficient to provide the desired result. One must, however, show that $J_\Lambda(x,z;\cdot)$ is a Borel function of (x, z).

For the following result we assume that W is a subset of a vector lattice. The expression $h \geqslant 0$ is meaningful within the vector lattice. The function f on W is called *convex* if

$$f(x + h) - f(x) \geqslant f(y + h) - f(y) \tag{9.33}$$

whenever x, y, x + h, y + h ε W and $x \geqslant y$, $h \geqslant 0$. When - f is convex, f is called *concave*. Clearly (9.33) is meaningful when f takes values in a partially ordered vector space.

9.10 THEOREM. *Let the generator* $G = \Sigma G_\Lambda$ *on X be local and satisfy the hypothesis of Theorem 4.3. Assume for each* Λ *there is a* $\lambda > 0$ *so that the measure kernel,*

$$M(x,\cdot) = (1 - \lambda G_\Lambda(x, W^\Lambda))\, \varepsilon_x^\Lambda + \lambda G_\Lambda(x,\cdot), \tag{9.34}$$

is an increasing concave (convex) function of x *in the ordering* $\geqslant^i$ *on measures on* W^Λ. *Then for each concave (convex) function* $f \;\varepsilon\; F_{oi}(X)$, $T_t f$ *is increasing and concave (convex) for each* $t > 0$.

Proof. By the argument of Proposition 9.3 it suffices to show that $M(x,\cdot)$ maps concave functions in $F_{oi}(X)$ to concave functions. We use the following notation:

$$(f_x)\,(y) = f({}_y x), \tag{9.35}$$

$$M_x\, f_z = \int M(x, dy)\, f({}_y z). \tag{9.36}$$

The modification is on Λ. For concave $f \varepsilon F_{oi}(X)$,

$$\begin{aligned} M_{x+h}\, f_{x+h} - M_x f_x &= M_{x+h}(f_{x+h} - f_x) + (M_{x+h} - M_x)\, f_x \\ &\leq M_{x+h}(f_{y+h} - f_y) + (M_{y+h} - M_y)\, f_x \\ &\leq M_{y+h}(f_{y+h} - f_y) + (M_{y+h} - M_y)\, f_y\,. \end{aligned} \tag{9.37}$$

In (9.37) we use the positivity of the measure M_{x+h} with the concavity of f, the concavity of M with the increasing property of f_x, and the increasing property of M with the decreasing property of $(f_{y+h} - f_y)$ and $(f_x - f_y)$. Inequality (9.37) implies the desired property for M(x, dy). For the convex case we have (9.37) with the inequalities reversed.

For further results of the type above, and some striking applications, the reader should consult Harris (1974).

10. THE GLAUBER MODEL WITHOUT TRANSLATION INVARIANCE.

In this section we consider a specific example which illustrates many previous results. The example is a generalization of the model proposed by Glauber (1963) for time development in the one dimensional Ising chain. The original Glauber model deals with translation invariant, nearest neighbour interactions in zero magnetic field; Felderhof (1970) gave an ingenious solution to the associated eigenvalue problem. We generalize the model by removing the requirement of translation invariance. This allows us to study in some detail dynamic Ising systems with distinct phases.

We shall be considering both equilibrium and time dependent models. An *equilibrium* state of the system is a probability measure on X corresponding to the potential by the usual rules of equilibrium random fields. An *invariant* state is a probability measure on X which is fixed under the given time development. For the generators we consider, each equilibrium state will be reversible and invariant. When the equilibrium state is unique, we shall show that it is the only invariant state. Whether, in general, there are invariant states which are not equilibrium states, is an open question.

We consider two cases: the first having $S = Z = \{0, \pm 1, \ldots\}$, and the second having $S = N = \{1, 2, \ldots\}$. The phase space of a single spin is $W = \{+1, -1\}$. The interaction is specified by a sequence of constants $\{J_k : k \in S\}$, with the interaction energy between sites k and k + 1 equal to $- J_k x_k x_{k+1}$ for the configuration $x \in X$.

Consider spins $x_1, x_2, \ldots, x_n$. In equilibrium calculations which do not involve spin $x_2, \ldots, x_{n-1}$, we can replace the constants $J_1, \ldots, J_{n-1}$ by a single constant $\tilde{J}$, such that

$$\tanh \tilde{J} = \Pi_{j=1}^{n-1} \tanh J_j , \qquad (10.1)$$

and regard x_1 as coupled directly to x_n by $\tilde{J}$. In this manner we can show that the equilibrium state is not unique when

$$\Pi_{j=1}^{\infty} \tanh J_j \neq 0. \qquad (10.2)$$

For $S = \mathbb{N}$ there are at most two extreme equilibrium states; for $S = \mathbb{Z}$ there are at most four. When $S = \mathbb{N}$, all equilibrium states are Markov chains; when $S = \mathbb{Z}$, this is *not* true in general. We omit the detailed, but elementary, calculations.

The time development is obtained from the generator $G = \Sigma\, G_k$, *i.e.* $G_\Lambda = 0$ unless $\Lambda = \{k\}$ for $k \in S'$, and

$$G_k(x, \{-x_k\}) = \tfrac{1}{2}(1 - \alpha_k\, x_{k-1}\, x_k - \beta_k\, x_k\, x_{k+1}),$$

$$\alpha_k = \tfrac{1}{2}(\tanh\,(J_{k-1} + J_k) + \tanh\,(J_{k-1} - J_k)), \qquad (10.3)$$

$$\beta_k = \tfrac{1}{2}(\tanh\,(J_{k-1} + J_k) - \tanh\,(J_{k-1} - J_k)).$$

It follows that G is local and satisfies Theorem 4.3.

If ν is any probability measure on X whose one point conditional probabilities satisfy

$$\nu_k(a|x) = \exp(J_{k-1}\, a\, x_{k-1} + J_k\, a\, x_{k+1})/Z_{kx}\ , \qquad (10.4)$$

for each $k \in S$, with Z_{kx} chosen so that $\nu_k(W|x) = 1$, then G is a type-I generator of ν. In particular, ν is strictly reversible for G. One such ν is the Markov chain μ satisfying

$$\mu(\{x_k = a\}) = \tfrac{1}{2},$$
$$\mu(x_{k+1}|x_k) = \tfrac{1}{2}(1 + x_k\, x_{k+1}\, \tanh J_k). \qquad (10.5)$$

Let s_k denote the element of $C(X)$ such that $s_k(x) = x_k$. From (10.3) we deduce

$$G\, s_k = -\, s_k + \alpha_k\, s_{k-1} + \beta_k\, s_{k+1}\ . \qquad (10.6)$$

Each element $v \in \ell^1(S)$ yields an element of $C(X)$ under the correspondence

$$v \to \Sigma_{k \in S}\ v_k\, s_k\ . \qquad (10.7)$$

This mapping is norm preserving, and the image of $\ell^1(S)$ in $C(X)$ is a closed linear subspace which we denote $\mathcal{F}$. G maps $\mathcal{F}$ into itself and gives the following differential equation for $v(t) \in \ell^1(S)$, via (10.7):

$$\frac{d}{dt} v(t) = B\, v(t), \tag{10.8}$$

$$(B\, v)_k = \beta_{k-1} v_{k-1} - v_k - \alpha_{k+1} v_{k+1}$$

One verifies that B is the comparison matrix of G, so Theorems 4.8 and 7.3 give the following:

10.1 THEOREM. *Let* $\{J_k : k \in S\}$ *satisfy*

$$|J_k| \leq J, \text{ all } k \in S . \tag{10.9}$$

Put $\lambda = 1 - \tanh 2J$. *Then the semigroup* T_t *of the* G *of* (10.3) *has a unique invariant probability measure* μ *which is the Markov chain of* (10.5). *For each* $g \in C_f(X)$,

$$e^{\lambda t} \| T_t g - \int g\, d\mu \| \text{ is bounded in } t \geq 0 , \tag{10.10}$$

and for each $f \in L^2(\mu)$,

$$\lim_{t \to \infty} e^{\lambda t} \| T_t f - \int f\, d\mu \|_2 = 0. \tag{10.11}$$

We go on to the case of unbounded J's. There is no loss of generality in assuming that $J_k \geq 0$ for each k, as the spins can be relabelled to achieve this. We shall assume, further, that

$$J_k > 0, \text{ all } k \in S, \tag{10.12}$$

leaving the case in which some J's are zero to the interested reader. We define $\{u_k : k \in S\}$ by

$$e^{-u_k} = \tanh J_k . \tag{10.13}$$

10.2 THEOREM. *Let* $\{J_k : k \in S\}$ *satisfy* (10.12) *and let* $\{u_k : k \in S\}$ *of* (10.13) *satisfy*

$$\lim_{k \to +\infty} u_k = 0, \tag{10.14}$$

$$\Sigma_{k > 0}\ u_k = +\infty, \tag{10.15}$$

$$\Sigma_{k < 0}\ u_k = +\infty. \tag{10.16}$$

Then the T_t *corresponding to* G *of* (10.3) *has a unique invariant probability measure, the* μ *of* (10.5). *The only* $\lambda \geqslant 0$ *such that*

$$\lim_{t \to \infty} e^{\lambda t} \| T_t f - \int f\, d\mu \|_2 = 0, \tag{10.17}$$

for all $f \in L^2(\mu)$, *is* $\lambda = 0$.

Proof. Let ν be an invariant probability measure of T_t. Define $w \in \ell^\infty(S)$ by

$$w_k = \int s_k\, d\nu. \tag{10.18}$$

The invariance of ν implies $\int G\, s_k\, d\nu = 0$, so

$$0 = -w_k + \alpha_k w_{k-1} + \beta_k w_{k+1}. \tag{10.19}$$

Any solution of the recurrence relation (10.19) is given by a linear combination of the two solutions w' and w" with components

$$w'_k = \begin{cases} 1/\ \Pi_{j=1}^{k-1} \tanh J_j, & k > 1, \\ 1, & k = 1, \\ \Pi_{j=k}^{0} \tanh J_j, & k < 1. \end{cases} \tag{10.20}$$

$$w''_k = 1/\ w'_k. \tag{10.21}$$

From (10.15) and (10.16) it follows that no nontrivial linear combination of w' and w" is bounded, so

$$\int s_k\, d\nu = w_k = 0, \quad \text{all } k \in S. \tag{10.22}$$

Then Theorem 9.4 implies $\mu = \nu$. We now show that 0 is a limit point of the spectrum of G in $L^2(\mu)$. Note that

$$(s_k, s_k)_\mu = 1, \quad (s_k, 1)_\mu = 0, \tag{10.23}$$

$$(G\, s_k, s_k) = -1 + \alpha_k (s_k, s_{k-1})_\mu + \beta_k (s_k, s_{k+1})_\mu . \tag{10.24}$$

Now as $k \to +\infty$, $(s_k, s_{k-1})_\mu \to 1$, $(s_k, s_{k+1})_\mu \to 1$, so $(G\, s_k, s_k)_\mu \to 0$. Since the constants are the only fixed points of T_t in $L^2(\mu)$, 0 is a limit point of the spectrum of G.

We now consider the situation in which there are distinct phases,

$$\Sigma_{k \in S} \; u_k < \infty . \tag{10.25}$$

We could carry out the analysis for S = Z, but we consider instead the slightly simpler case of S = N. To get G for this case, simply put $J_o = s_o = 0$ in (10.6). Each equilibrium state in this case is a Markov chain. There are exactly two extremal equilibrium states which will be denoted μ_+, μ_- (recall assumption (10.12)), with

$$\int s_j \, d\mu_+ = -\int s_j \, d\mu_- = \Pi_{i=j}^{\infty} \tanh J_i . \tag{10.26}$$

We specialize even further by considering the case in which $\{u_k\}$ is a geometric series, $u_k = Ar^{2k}$,

$$\tanh J_k = \exp - Ar^{2k}, \qquad A > 0, \quad 0 < r < 1. \tag{10.27}$$

The correlations

$$\int s_j \, s_k \, d\mu_+ - \int s_j \, d\mu_+ \int s_k \, d\mu_+ \tag{10.28}$$

decay as r^{2k} for $k \to \infty$ with j fixed.

10.3 Theorem. *Consider the time development* T_t *corresponding to* (10.3) *on* $X = \{+1, -1\}^N$, *with the* J*'s given by* (10.27). *Put* $\lambda = (1 - r)^2/(1 + r^2)$. *Then*

$$e^{\lambda t} \| T_t f - \int f \, d\mu_+ \|_2 \tag{10.29}$$

is bounded in $t > 0$ *for each* $f \in L^2(\mu_+)$.

Proof. For J_k satisfying (10.27), one can verify by elementary calculus that α_k, β_k in (10.3) satisfy

$$0 < \alpha_k < r^2/(1 + r^2), \tag{10.30}$$

$$0 < \beta_k < 1/(1 + r^2). \tag{10.31}$$

To prove (10.29), it is sufficient to show that for each $f \in C_f(X)$ with $\int f \, d\mu_+ = 0$,

$$(Gf, f)_{\mu_+} \leq - \lambda(f, f)_{\mu_+} . \tag{10.32}$$

But this holds if, for each positive integer n, the $\mu_+\{1, \ldots , n\}$-projection of G given by (8.2) satisfies the result corresponding to (10.32). We use $G^{(n)}$ to denote this projection. For $k = 1, \ldots , n - 1$, $G^{(n)}s_k$ agrees with (10.6). For s_n we have

$$\begin{aligned} G^{(n)} s_n &= - s_n + \alpha_n s_{n-1} + \beta_n(\alpha^* s_n + \beta^*), \\ \alpha^* &= \tfrac{1}{2}(\tanh(J_n + J^*) + \tanh(J_n - J^*)), \\ \beta^* &= \tfrac{1}{2}(\tanh(J_n + J^*) - \tanh(J_n - J^*)), \\ \tanh J^* &= \Pi_{i=n+1}^{\infty} \tanh J_i . \end{aligned} \tag{10.33}$$

We consider $G^{(n)}$ on the space spanned by $\{\hat{s}_1, \ldots , \hat{s}_n\}$, where

$$\hat{s}_k = s_k - \int s_k \, d\mu_+ . \tag{10.34}$$

In this invariant subspace $G^{(n)}$ has a tridiagonal matrix $B^{(n)}$, with

$$B^{(n)}_{nn} = - 1 + \beta_n \alpha^* \leq - 1 + r^2/(1 + r^2); \tag{10.35}$$

$$B^{(n)}_{jj} = - 1, \qquad 1 \leq j < n; \tag{10.36}$$

$$0 < B^{(n)}_{j\,j+1} = \alpha_{j+1} < r^2/(1 + r^2); \tag{10.37}$$

$$0 < B^{(n)}_{j+1\ j} = \beta_j < 1/(1 + r^2); \qquad (10.38)$$

$$B^{(n)}_{ij} = 0 \quad \text{otherwise.} \qquad (10.39)$$

$B^{(n)}$ is similar to a symmetric tridiagonal matrix with the same diagonal elements as $B^{(n)}$ but with off-diagonal elements $(\beta_j\ \alpha_{j+1})^{\frac{1}{2}}$, which are less than $r/(1 + r^2)$. From diagonal dominance in this matrix, one knows that the eigenvalues of $B^{(n)}$ are all less than $-\lambda$. Because of the attractive property of $B^{(n)}$, by Theorem 9.4 this result on the subspace spanned by $\{\hat{s}_1, \ldots, \hat{s}_n\}$ extends to all functions in $C(X|\ \{1, \ldots, n\}\ )$ whose μ_+ integral vanishes. This implies (10.32).

Next we consider the case in which $u_k = A/k^s$,

$$\tanh J_k = \exp - A/k^s, \quad A > 0, \quad 0 < s < \infty. \qquad (10.40)$$

Here there is only one phase for $0 < s \leqslant 1$, but for $s > 1$, μ_+ and μ_- are distinct.

10.4 PROPOSITION. *Consider the time development* T_t *corresponding to* (10.3) *on* $X = \{+1, -1\}^N$, *with the* J's *given by* (10.40). *For each* $\lambda > 0$ *there is an* $f \in C_f(X)$ *such that* $\int f\, d\mu_+ = 0$ *and*

$$e^{\lambda t}\ \| T_t f \|_2 \to \infty \qquad (10.41)$$

as $t \to \infty$.

Proof. With s_k as defined in (10.34), we have

$$\lim_{k \to \infty} (G\ \hat{s}_k, \hat{s}_k)_{\mu_+} / (\hat{s}_k, \hat{s}_k)_{\mu_+} = 0. \qquad (10.42)$$

This implies that, for sufficiently large k, the spectral projection of G corresponding to the interval $(-\lambda, 0)$ has a nontrivial action on $\hat{s}_k$. This implies (10.41), with $f = \hat{s}_k$.

11. FREE ENERGY AND THE VARIATIONAL PRINCIPLE.

In many applications the components are labelled by the points of a geometrical lattice in Euclidean space, and the basic structures are translation invariant. In this context the concepts of specific entropy and specific free energy are important. In particular, for lattice gas models in statistical mechanics, the *variational principle* characterizes the translation invariant equilibrium states of the potential as exactly those translation invariant probability measures which minimize the specific free energy (see Preston (1974b)).

The basic results on specific free energy for Markovian time development in finite range interaction lattice gas systems are due to Holley (1971). In this section we generalize some of Holley's results. The line of proof is similar to Holley's, but we introduce new techniques for the fundamental estimates.

Throughout this section we assume that $S = Z^d$, the points with integer coordinates in d-dimensional Euclidean space. We assume that the single component phase space W is endowed with a base probability measure σ. We also use σ to denote the probability measure on W^Γ, $\Gamma \subset S$, which is the product measure, restricting to σ on each factor space. By a *cube* we mean the intersection of Z^d with a translate of a set of the form $[a,b]^d$ with a, b real. The limit

$$\lim{}^* \quad \Lambda \to S \tag{11.1}$$

is to be taken on the collection of cubes of even side length centred about the origin.

The function *log* is the natural logarithm defined for positive argument and taking values in the extended real numbers, with the convention $0 \log 0 = 0$. The function Ψ is defined for positive z,

$$\Psi(z) = z \log z - z + 1. \tag{11.2}$$

We note that Ψ is positive, convex and nonzero except for $\Psi(1) = 0$.

11.1 DEFINITION. Let ρ and μ be probability measures on X. The *free energy of* ρ *with respect to* μ, $\Phi(\rho|\mu)$, is defined as follows. If ρ is absolutely continuous with respect to μ, $\rho = f\,\mu$, where f is a positive Borel function, then

$$\Phi(\rho|\mu) = \int \Psi(f)\, d\mu; \tag{11.3}$$

otherwise, $\Phi(\rho|\mu) = +\infty$. The Λ-*free energy of* ρ *with respect to* μ, $\Phi^{\Lambda}(\rho|\mu)$, is defined to be $\Phi((\rho|\Lambda)|\mu)$, where $\rho|\Lambda$ denotes the *restriction* of ρ to W^{Λ}. The *specific free energy of* ρ *with respect to* μ is defined to be

$$\phi(\rho|\mu) = \lim_{\Lambda \to S}{}^{*} \; \Phi^{\Lambda}(\rho|\mu)/|\Lambda|. \tag{11.4}$$

whenever the limit exists. The *specific entropy of* ρ is defined to be $-\phi(\rho|\sigma)$.

When $\rho|\Lambda$ is absolutely continuous with respect to μ, we write

$$\rho|\Lambda = f^{\Lambda}\,\mu \tag{11.5}$$

where f^{Λ} is a positive Borel function on X which depends only on the coordinates on Λ. It is convenient to identify $\rho|\Lambda$ with the right hand side of (11.5), which we consider to be a measure on X rather than just W^{Λ}.

11.2 LEMMA. *The function* $\Phi^{\Lambda}(\rho|\mu)$ *is increasing in* Λ *and*

$$\lim_{\Lambda \to S} \Phi^{\Lambda}(\rho|\mu) = \Phi(\rho|\mu) \tag{11.6}$$

as a positive number or $+\infty$. *If both probability measures* ρ *and* μ *are translation invariant and* μ *is the product of its single component restrictions, then*

$$\lim_{\Lambda \to \dot{S}}{}^{*} \; \Phi^{\Lambda}(\rho|\mu)/|\Lambda| = \sup_{\text{cubic } \Lambda} \; \Phi^{\Lambda}(\rho|\mu)/|\Lambda|. \tag{11.7}$$

Proof. Assume $\Lambda' \supset \Lambda$ and $\Phi^{\Lambda'}(\rho|\mu) < \infty$. Then, in the notation (11.5), f^{Λ} is obtained from $f^{\Lambda'}$ by conditional expectation. Jensen's inequality then yields

$$\Phi^{\Lambda}(\rho|\mu) \leq \Phi^{\Lambda'}(\rho|\mu). \tag{11.8}$$

When $\{\Phi^{\Lambda}(\rho|\mu)\}$ is bounded, the collection $\{f^{\Lambda}\}$ forms a uniformly integrable Martingale with respect to μ (see Doob (1953)). Limit (11.6) follows easily. Next consider a cube Λ and a larger cube Λ^* of the form $\Lambda^* = \Lambda_1 \cup \ldots \cup \Lambda_n$, where the union is disjoint and each Λ_k is a translate of Λ. Assume $\Phi^{\Lambda^*}(\rho|\mu) < \infty$. Write $f_k(x_1 \ldots x_k)$ for the Radon-Nikodym derivative of $\rho|\Lambda_1 \cup \ldots \cup \Lambda_k$ with respect to μ, where $x_k \in W^{\Lambda_k}$. Then, with $f_0 \equiv 1$,

$$f_{k-1}(x_1 \ldots x_{k-1}) = \int f_k(x_1 \ldots x_k)\, \mu(d\, x_k | x_1 \ldots x_{k-1}); \tag{11.9}$$

$$\Phi^{\Lambda^*}(\rho|\mu) = \Sigma_{k=1}^{n} \int f_k \log (f_k/f_{k-1})\, d\mu . \tag{11.10}$$

Next, define

$$f_k^*(x_k) = \int f_k(x_1 \ldots x_k)\, \mu(dx_1 \ldots dx_{k-1} | x_k). \tag{11.11}$$

If μ is the product of its single component restrictions, from Jensen's inequality we deduce

$$\Phi^{\Lambda^*}(\rho|\mu) \geq \Sigma_{k=1}^{n} \int \Psi(f_k^*(x_k))\, \mu(dx_k) = \Sigma_{k=1}^{n} \Phi^{\Lambda_k}(\rho|\mu). \tag{11.12}$$

Then, when both μ and ρ are translation invariant,

$$\Phi^{\Lambda^*}(\rho|\mu)/|\Lambda^*| \geq \Phi^{\Lambda}(\rho|\mu)/|\Lambda|. \tag{11.13}$$

For cubes Λ^* which are not exact multiples of Λ, a slight modification of the above technique yields an inequality slightly weaker than (11.13) but sufficient to imply (11.7).

We shall be computing free energies with respect to a basic equilibrium distribution, which we denote by μ. Henceforth we shall assume that μ satisfies the following: We assume that μ is a translation invariant probability measure on X which has conditional probability distributions $\{\mu_\Lambda(dy|x)\}$ of the form

$$\mu_\Lambda(dy|x) = p_\Lambda({}_y x)\ \sigma(dy),$$
$$p_\Lambda,\ \log p_\Lambda \ \varepsilon\ F_o(X), \qquad (11.14)$$

with the modification of x on Λ. We assume that, when Λ and Λ' are translates, then p_Λ and $p_{\Lambda'}$ are related by translation. We also assume that for each $\varepsilon > 0$, there is an integer N, such that for any cube Λ of side > N and any $x,\ z\ \varepsilon\ X,\ y\ \varepsilon\ W^\Lambda$,

$$|\ \log p_\Lambda({}_y x) - \log p_\Lambda({}_y z)\ | < \varepsilon\ |\Lambda|. \qquad (11.15)$$

In order for μ to satisfy these conditions, it is sufficient for μ to be a Gibbs' state of a translation invariant, absolutely summable potential (see Preston (1974b)).

11.3 LEMMA. *We assume μ as above and continue the notation of the proof of Lemma 11.2, with Λ of side > N so that (11.15) holds. Then*

$$\int f_k \log(f_k/f_{k-1})d\mu \ \geqslant\ \Phi^{\Lambda_k}(\rho|\mu) - \varepsilon|\Lambda_k|. \qquad (11.16)$$

Proof. From (11.14) we deduce that $\mu|\Lambda = p^\Lambda\ \sigma$ with

$$p^\Lambda(y) = \int p_\Lambda({}_y x)\ \mu(dx). \qquad (11.17)$$

In analogy with f_k we write $\mu|\Lambda_I \cup \ldots \cup \Lambda_k = p_k\ \sigma$.

Since σ is a product probability measure,

$$\int f_k\, p_k \log(f_k\, p_k/\ f_{k-1}p_{k-1}) - f_k\, p_k \log(p_k/p_{k-1})\ d\sigma$$
$$\geqslant \Phi^{\Lambda_k}(\rho|\sigma) - \int f_k \log(p_k/p_{k-1})\ d\mu \qquad (11.18)$$
$$= \Phi^{\Lambda_k}(\rho|\mu) - \int f_k\ (\log(p_k/p_{k-1}) - \log p^{\Lambda_k})\ d\mu.$$

Now

$$p_k(x_1\ldots x_k)/p_{k-1}(x_1\ldots x_{k-1}) = \int p_{\Lambda_k}({}_{x_k} x_1\ldots x_{k-1}y)\mu(dy|x_1\ldots x_{k-1}), \qquad (11.19)$$

where the coordinates of a point of X are represented by $(x_1 \ldots ,x_k,y)$. The assumption (11.15) then implies (11.16).

11.4 Lemma. *Let* μ *be as above and let* ρ *be a translation invariant probability measure. Then* $\phi(\rho|\mu)$ *exists as a positive real number or* $+\infty$.

Proof. In the notation of Lemmas 11.2 and 11.3 we deduce from (11.16) that

$$\Phi^{\Lambda^*}(\rho|\mu)/|\Lambda^*| \geq \Phi^{\Lambda}(\rho|\mu)/|\Lambda| - \varepsilon, \qquad (11.20)$$

when the cube Λ^* is a multiple of the cube Λ, and where $\varepsilon \to 0$ as the side of Λ becomes infinite. A similar argument gives a slightly more complicated inequality if Λ^* is not an exact multiple of Λ. From this we deduce the existence of the limit (11.4).

11.5 Theorem. *Let* $G = \Sigma\, G_\Lambda$ *be local, translation invariant and satisfy the hypothesis of Theorem 4.3. Let the* μ *specified above be an invariant probability measure of the associated semigroup* T_t'. *Assume that* ν *is a translation invariant probability measure such that* $\nu|\Lambda$ *is absolutely continuous with respect to* μ *for each* Λ. *Then, for each* $t > 0$, $\phi(T_t'\,\nu)$ *exists as a positive number or* $+\infty$ *and*

$$\phi(T_t'\,\nu|\mu) \leq \phi(\nu|\mu). \qquad (11.21)$$

Proof. From the comment following Theorem 1.9 we have

$$\Phi(T_t'(\nu|\Lambda^*)|\mu) \leq \Phi^{\Lambda^*}(\nu|\mu). \qquad (11.22)$$

We use the notation of the proof of Lemma 11.2 with $\rho = T_t'(\nu|\Lambda^*)$. It follows that

$$\begin{aligned}\Phi(T_t'(\nu|\Lambda^*)|\mu) &\geq \Phi^{\Lambda^*}(T_t(\nu|\Lambda^*)|\mu)\\ &= \Sigma_{k=1}^{n} \int f_k \log(f_k/f_{k-1})\, d\mu.\end{aligned} \qquad (11.23)$$

Then there is at least one k with

$$\int f_k \log(f_k/f_{k-1})\, d\mu/|\Lambda| \leq \Phi^{\Lambda^*}(\nu|\mu)/|\Lambda^*|. \qquad (11.24)$$

We select a sequence of increasing cubes Λ^{*j}, and a corresponding $f_k^{(j)}$ such that the distance from the Λ corresponding to $f_k^{(j)}$ to $(\Lambda^{*j})^c$ becomes infinite as $j \to \infty$. For the function $f_k^{(j)}$ we claim an inequality of the form (11.24) with the right hand side augmented by a positive term $\delta(\Lambda^*)$, which approaches 0 as $\Lambda^* \to S$. We can translate Λ^{*j} so that $f_k^{(j)}$ corresponds to a fixed set, denoted Λ. We assume that Λ is sufficiently large for (11.15) to be satisfied. Then the modified (11.24) takes the form

$$\Phi^{\Lambda}(T_t'(\nu|\Lambda^{*j})|\mu)/|\Lambda| - \varepsilon \leqslant \Phi^{\Lambda^{*j}}(\nu|\mu)/|\Lambda^{*j}| + \delta(\Lambda^{*j}), \tag{11.25}$$

using (11.16). Next we note that

$$\lim_{j \to \infty} \| T_t'(\nu|\Lambda^{*j})|\Lambda - (T_t'\nu)|\Lambda \|_m = 0. \tag{11.26}$$

It follows that $(T_t'\nu)|\Lambda$ is absolutely continuous with respect to μ. From Fatou's Lemma we deduce

$$\lim_{j \to \infty} \Phi^{\Lambda}(T_t'(\nu|\Lambda^{*j})|\mu) \geqslant \Phi^{\Lambda}(T_t'\nu|\mu). \tag{11.27}$$

From Lemma 11.4 and inequalities (11.25), (11.27) we obtain

$$\Phi^{\Lambda}(T_t' \nu|\mu)/|\Lambda| \leqslant \phi(\nu|\mu) + \varepsilon . \tag{11.28}$$

A second application of Lemma 11.4, noting that ε in (11.28) goes to zero as $\Lambda \to S$, yields (11.21).

11.6 LEMMA. *Let the hypothesis of Theorem 11.5 be satisfied and write*

$$T_t' \nu|\Lambda = f^{\Lambda}(t) \mu. \tag{11.29}$$

Then $f^{\Lambda}(t)$ *is a continuously differentiable,* $L^1(W^{\Lambda},\mu)$*-valued function of* $t \geqslant 0$. *For* $g \in F(X|\Lambda)$,

$$\int (\frac{d}{dt} f^{\Lambda}(t)) g \, d\mu = \int (G g) \, d (T_t'\nu). \tag{11.30}$$

Proof. Let $g \in F(X|\Lambda)$, and define $G(\Lambda)$ by

$$G(\Lambda) = \Sigma_{\Lambda' \cap \Lambda \neq \phi} G_{\Lambda'}. \tag{11.31}$$

From Lemma 5.4 we deduce that there is a real constant C such that

$$\| T_t\, g - \exp(t\, G(\Lambda))\, g \| \leqslant Ct^2 \|g\| \tag{11.32}$$

for all $t \geqslant 0$. Then

$$\| (T_t\, g - g)/t - G\, g \| \to 0 \tag{11.33}$$

as $t \to 0$, uniformly in g with $\|g\| \leqslant 1$. By considering difference quotients for $\int f^\Lambda(t)\, g\, d\mu$, we deduce that $f^\Lambda(t)$ is differentiable in $L^1(W^\Lambda,\mu)$ and that (11.30) is satisfied. There exists a $\nu \in \ell^1(S)$ such that $\delta(G\, g) \leqslant \|g\|\, \nu$; the continuity of the derivative of $f^\Lambda(t)$ follows from considerations based on Lemma 5.4.

For positive integer k and positive r we define

$$\text{klog } r = \begin{cases} -k & \text{if } r < e^{-k} \\ k & \text{if } r > e^{k} \\ \log r & \text{otherwise,} \end{cases} \tag{11.34}$$

$$\Psi^k(r) = r \text{ klog } r + 1 - \exp(\text{klog } r). \tag{11.35}$$

Each Ψ^k is positive, convex and differentiable; the derivative is klog. The family $\{\Psi^k\}$ is increasing in k and for positive r

$$\lim_{k \to \infty} \Psi^k(r) = \Psi(r). \tag{11.36}$$

Since $\Psi^k(f^\Lambda(t))$ is a continuously differentiable, $L^1(W^\Lambda,\mu)$-valued function of $t \geqslant 0$, the Monotone Convergence Theorem gives the following:

11.7 LEMMA. *Under the hypothesis of Lemma 11.6 we have*

$$\Phi^\Lambda(T_t'\nu|\mu) - \Phi^\Lambda(\nu|\mu) = \lim_{k \to \infty} \int_0^t h_k^\Lambda(s)\, ds, \tag{11.37}$$

where

$$h_k^\Lambda(t) = \int (G \text{ klog } f^\Lambda(t))\, d(T_t'\, \nu). \tag{11.38}$$

11.8 PROPOSITION. *Let the hypothesis of Theorem 11.5 be satisfied with* G *of the form*

$$G = \Sigma\, G_\Lambda({}_y x,\ z)\ \sigma(dz), \tag{11.39}$$

where each G_Λ *is a positive function in* $F_o(X \times W^\Lambda)$ *and*

$$\Sigma_{\Lambda \ni j} \| G_\Lambda({}_y x,\ z) \| \leqslant K, \tag{11.40}$$

and the modification of x *by* y *is on* Λ *. Then, in the notation of Lemma 11.7, for each* Λ *the integral*

$$h^\Lambda(t) = \int (G \log f^\Lambda(t))\ \ d(T_t^{\prime}\nu) \tag{11.41}$$

exists as a real number or $-\infty$*, and*

$$\Phi^\Lambda(T_t^{\prime}\ \nu|\mu) - \Phi^\Lambda(\nu|\mu) \leqslant \int_o^t h^\Lambda(s)\ ds. \tag{11.42}$$

Proof. We show that the positive part of the integral in (11.41) is finite. From this we deduce that h^Λ is the pointwise limit of $\{h_k^\Lambda\}$. Fatou's Lemma then gives (11.42). For convenience, we take the K in (11.40) to be the same as in Theorem 4.3. Consider t fixed and write $\rho = T_t^{\prime}\ \nu$, $\rho|\Lambda = q^\Lambda\sigma$, $\mu|\Lambda = p^\Lambda\sigma$, so that $q^\Lambda/p^\Lambda = f^\Lambda = f^\Lambda(t)$. In considering the $G_{\Lambda'}$ term, we first examine

$$\begin{aligned} &\int \rho(dy\ dx)\ \sigma(dz)\ \log(f^\Lambda({}_y x)/\ f^\Lambda({}_z x)) \\ &= \int q^\Lambda({}_y x)\ \sigma(dy\ dx\ dz)\ \log\ (q^\Lambda({}_y x)/q^\Lambda({}_z x)) \\ &\quad - \int \rho(dy\ dx)\ \sigma(dz)\ \log\ (p^\Lambda({}_y x)/p^\Lambda({}_z x)), \end{aligned} \tag{11.43}$$

where y and z are Λ' variables. From the assumption (11.14) it can be shown that there is a constant C such that

$$\int \rho(dy\ dx)\ \sigma(dz) |\ \log\ (p^\Lambda({}_y x)/p^\Lambda({}_z x))| \leqslant C|\Lambda \cap \Lambda'|. \tag{11.44}$$

The q-integral term in (11.43) is bounded below by an integral of an expression of the form r log r, so the q-integral is $\geqslant -1/e$. We obtain the $G_{\Lambda'}$ term of (11.41) by multiplying the integrand of the left hand side of (11.43) by $-\ G_{\Lambda'}({}_y x,\ z)$. Using a slight generalization of (11.44), we deduce that the positive part of the integral in (11.41) is bounded above by

$$\Sigma_{\Lambda' \cap \Lambda \neq \phi} \| G_{\Lambda'} \| /e + C|\Lambda' \cap \Lambda| \; \| G_{\Lambda'} \sigma \|_m \leqslant K|\Lambda| \; (1/e + C). \tag{11.45}$$

11.9 PROPOSITION. *Let the hypothesis of Proposition 11.8 be satisfied. Define* $h(t)$ *by*

$$h(t) = \lim_{\Lambda \to S}{}^{*} \quad \sup \; h^{\Lambda}(t)/|\Lambda|. \tag{11.46}$$

Then

$$\phi(T_t' \nu|\mu) - \phi(\nu|\mu) \leqslant \int_0^t h(s)\, ds. \tag{11.47}$$

If, for some Λ', $G_{\Lambda'}$ *is bounded away from zero, and if* $h(t) > -\infty$, *then the probability measure* $\rho = T_t' \nu$ *has a conditional probability distribution of the form*

$$\rho_{\Lambda}(dy|x) = f_{\Lambda}({}_y x) \; \mu_{\Lambda}(dy|x) \tag{11.48}$$

for each Λ, *where* $f_{\Lambda}({}_y x)$ *is a positive Borel function, and*

$$h(t) \leqslant \Sigma_{\Lambda \ni j} \frac{1}{|\Lambda|} \int \rho(dy\, dx)\, G_{\Lambda}({}_y x, z) \sigma(dz) \; \log(f_{\Lambda}({}_z x)/ f_{\Lambda}({}_y x)). \tag{11.49}$$

Proof. From (11.45) we know that $h^{\Lambda}(t)/|\Lambda|$ is bounded above, so (11.47) follows from Fatou's Lemma. The q-integral term in (11.43) equals

$$\phi^{\Lambda^*}(\rho(dy\, dx)\sigma(dz) | \rho(dz\, dx)\, \sigma(dy)), \tag{11.50}$$

where the basic space is $X \times W^{\Lambda'}$ and Λ^* is Λ augmented by the Λ points of the new $W^{\Lambda'}$. Expression (11.50) is increasing in Λ. Let $G_{\Lambda'}$ be bounded away from zero. It is not difficult to show that, when (11.50) is unbounded, then $h(t) = -\infty$. Next we assume (11.50) to be uniformly bounded in Λ. As in the proof of Lemma 11.2, the family of functions $\{q^{\Lambda}({}_y x)/q^{\Lambda}({}_z x)\}$ forms a uniformly integrable Martingale, which converges a.e. to a function we denote $u_{\Lambda'}({}_y x, z)$. The function $u_{\Lambda'}$ satisfies

$$\rho(dy\, dx)\sigma(dz) = u_{\Lambda'}({}_y x, z)\rho(dz\, dx)\sigma(dy). \tag{11.51}$$

From $u_{\Lambda'}$ and its translates it is not difficult to construct the corresponding Radon-Nikodym derivative u_Λ for any Λ. We define

$$f_\Lambda({}_y x) = \left(\int u_\Lambda({}_y x,\ z)\ \rho(dz|x)\right)/p_\Lambda({}_y x). \tag{11.52}$$

The remainder of the proof is standard convergence and counting arguments, using the Martingale Convergence Theorem.

11.10 PROPOSITION. *We assume the hypothesis of Proposition 11.9 and the condition of strict reversibility,*

$$p_\Lambda({}_y x)\ G_\Lambda({}_y x,\ z) = p_\Lambda({}_z x)\ G_\Lambda({}_z x,\ y), \tag{11.53}$$

for each Λ, y, $z \in W^\Lambda$, $x \in X$. *Then, wherever* $\phi(T_t'\ \nu) > 0$, *we have* $h(t) < 0$.

Proof. By the variational principle, $\phi(\rho) > 0$ implies that the $\rho(dx)\mu_{\Lambda'}(dy|x)$ measure of the set for which $f_{\Lambda'}({}_y x) \neq 1$ is nonzero (see Preston (1974b)). The $G_{\Lambda'}$ integral in (11.49) can be written in the form

$$- \tfrac{1}{2} \int \rho(dx)\mu_{\Lambda'}(dy|x)\mu_{\Lambda'}(dz|x)\ G_{\Lambda'}({}_y x,\ z)/p_\Lambda({}_z x) \cdot$$

$$(f_{\Lambda'}({}_y x) - f_{\Lambda'}({}_z x))\ \log(f_{\Lambda'}({}_y x)/f_{\Lambda'}({}_z x)). \tag{11.54}$$

Expression (11.54) gives a strictly negative contribution to (11.49), the other terms being negative.

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