# Lectures on Supersymmetry


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**Abstract**

These notes are based on a number of Graduate Lectures I gave over the last six years. The aim of these lectures is to provide a concise introduction to supersymmetry including some classic material such as the Coleman-Mandula theorem, supersymmetry algebra and non-renormalisation theorems. Then, specializing to $\mathcal{N}=1$ and $\mathcal{N}=2$ Yang-Mills theory, I will present simple derivations of recent non-perturbative results in supersymmetric field theory. In particular, low energy effective actions, electric-magnetic duality and superconformal Ward identities are explained.

# 1 Introduction

A supersymmetric Lagrangian is invariant under a transformation which relates bosonic degrees of freedom, $\varphi$ say to Fermionic degrees of freedom e.g.

$$\delta\varphi = \epsilon^\alpha \psi_\alpha .$$

For consistency $\epsilon$ must have mass dimension $-\frac{1}{2}$ and must be Grassmann-valued, i.e. anti-commuting. The corresponding transformation of a fermion would then have to be of the form

$$\delta\psi_\alpha = (\epsilon \partial\!\!\!/ \varphi)_\alpha + \quad \text{possibly other fields.}$$

Imposing invariance under such a transformation would then put conditions on the most general form of the Lagrangian. In particular one would find that the number of bosonic- and fermionic degrees of freedom would have to be the same and that Bosons and Fermions would have to have the same masses.

What is the physical motivation for imposing such a symmetry? One motivation for demanding supersymmetry is to improve the ultraviolet behaviour of the theory. For example if we consider the vacuum energy for free fields of mass $m_j$ and spin $j$

$$\begin{aligned} E_j &\propto (-1)^{2j}(2j+1)\int \mathrm{d}^3k \sqrt{k^2+m_j^2} \\ &= (-1)^{2j}(2j+1)\int \mathrm{d}^3k\ |k|\Big(1+\frac{1}{2}\frac{m_j^2}{k^2}+O(k^{-4})\Big), \end{aligned} \tag{1}$$

we find that a necessary condition for the absence of power-like divergences is that both, integer and half-integer spin fields are present and that these fields have the same masses.

Supersymmetry may also be required to solve the *hierarchy problem*, which expresses our ignorance in explaining the value of the Higgs mass. Indeed, the Higgs mass is not protected by any symmetry. Therefore if $m_H$ is different from zero its natural value would be of the order of the Plank scale. In a supersymmetric version of the Standard Model however, the Higgs field is accompanied by its fermionic super partner which could acquire a mass through anomalous chiral symmetry breaking with condensate $\langle\bar{\psi}\psi\rangle \simeq \Lambda_W^3 \simeq m_H^3$ where the last identity follows from the fact that the Higgs field and its fermionic partner $\psi$ are in the same supersymmetry multiplet. This

Figure 1: n-particle scattering

explains the continued interest in supersymmetry despite of the fact that no experimental evidence in favour of supersymmetry has been found so far.

Another motivation is of conceptual nature. It originates in the attempt to unify in a non-trivial way internal symmetries with the Poincaré group. In the early 60's much effort was put into finding such a unified symmetry group. This came to an end after a series of no-go results culminating in the Coleman and Mandula no-go theorem [1] which excludes this possibility, at least within the context of Lie-groups.

More recently supersymmetry has become increasingly important as a tool to analyse strongly coupled non-abelian Yang-Mills (YM) theories and string theories. Supersymmetrising ordinary YM-theories has lead to a number of new results on the strong-coupling, and therefore non-perturbative sector of these theories. In particular it lead to the discovery of explicit realisations of electromagnetic duality and generalisations of it to string theory. This electric-magnetic equivalence in turn provides evidence for the equivalence of a number of a priory different field theories. To what extend these properties rely on supersymmetry is still unclear.

The first part of these lectures, sections 2-5, reviews some textbook material, in particular, Coleman-Mandula theorem, supersymmetry algebra, representation theory, superspace, action formulas and non-renormalisation theorems. The second part, sections 5-7, contains more recent results. In particular, we derive the superconformal Ward-identities for $\mathcal{N}=1$ and $\mathcal{N}=2$ Yang-Mills theory and obtain the quantum corrected central charge of the $\mathcal{N}=2$ SUSY-algebra. Low energy effective actions for supersymmetric $QCD$ and $\mathcal{N}=2$ Yang-Mills theory are then derived by integrating the superconformal anomaly.

# 2 Coleman-Mandula Theorem

The Coleman-Mandula theorem is concerned with the unification of symmetries in interacting quantum field theory. Concretely it applies to the internal symmetries of the $S$-matrix. Let us first recall some elements of *scattering theory:*

The Hilbert space of scattering theory, $\mathcal{H}$ is the infinite direct sum of $n$-particle sub-spaces

$$\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \cdots,$$

where $\mathcal{H}^{(n)}$ is the (symmetrised) subspace of the tensor product of one-particle Hilbert spaces. The $S$-matrix is then a unitary operator on $\mathcal{H}$ describing all possible scattering processes in a given theory. It is usually written as

$$S = 1 - i(2\pi)^4 \delta^4(p_\mu - p'_\mu)T,$$

where the delta function ensures the conservation of energy and momentum during the scattering process. A unitary operator $U$ on $\mathcal{H}$ is a symmetry transformation of the $S$-matrix if

- 1) It maps one-particle states into one-particle states;
- 2) $U$ acts on many particle states by the tensor product representation of one-particle states;
- 3) $U$ commutes with $S$.

*Internal symmetries* of the $S$-matrix are symmetries which do not act on space-time coordinates, $x$ ( e.g. $U(1) \times SU(2) \times SU(3)$). However, if we want to include gravity into the scattering process, then the symmetry group has to be extended to include the Poincaré group $\mathcal{P}$, which acts on the vierbein $e^a_\mu$. The question we will need to answer is then whether the Poincaré group can be combined in a non-trivial way with other internal symmetries of the $S$-matrix.

In what follows we restrict ourselves to theories for which all scattering states are in *positive mass* representations of the Poincaré group $P$. We shall further assume that for any finite mass $M$ there are only a finite number of particle types with mass smaller than $M$ (*particle finiteness assumption*)

*Theorem:*

Let a Lie group $G$ be a symmetry group of the $S$-matrix which contains the Poincaré group $P$ and which connects a finite number of particles in a supermultiplet. Assume furthermore that

- 4) Elastic scattering amplitudes are analytic in the centre of mass energy $s = (p+q)^2$ and the momentum transfer $t = (p - p')^2$ (see fig. 2).
- 5) $T|p, q> \neq 0$ for almost all $s$

*Then,* $G$ is (locally) isomorphic to a direct product of an internal symmetry group and the Poincaré group $G$.

*Proof* (sketch):

Figure 2: Scattering of two particles with momentum $p$ and $q$ into two particles with momentum $p'$ and $q'$.

Let $\mathcal{D}$ be the subset of one-particle states whose momentum space wave-functions are test functions. For two two-particle states $\phi_1 \otimes \phi_2$ and $\psi_1 \otimes \psi_2$ in $\mathcal{D} \otimes \mathcal{D}$ the variation of their scalar product under infinitesimal $G$-transformations is then given by

$$(\psi_1, A\phi_1) + (\psi_2, A\phi_2)$$

where $A$ is an infinitesimal generator of $G$. The $G$-invariance of the $S$-matrix is then equivalent to

$$(S\ \psi_1 \otimes \psi_2, A\ S\ \phi_1 \otimes \phi_2) = (\psi_1 \otimes \psi_2, A\ \phi_1 \otimes \phi_2) \tag{2}$$

For a given test function $f$ with support in a region $R$ not containing the origin (of momentum space) we then consider the distribution

$$f \cdot A \equiv \int d^4a\ U^\dagger(1,a) A\ U(1,a) \tilde{f}(a),$$

where $\tilde{f}(a)$ is the Fourier transform of $f(p)$. Since

$$U(1,a)|p\rangle = e^{-ip\cdot a}|p\rangle$$

$f \cdot A$ has matrix elements

$$f \cdot A(p', p) = f(p - p')A(p', p).$$

By the finite particle assumption, if $R$ is sufficiently small not containing the origin, there exist regions $U_i$ in a given mass hyperboloid, such that for $p \in U_i$ and $k$ in $R$, the sum $p+k$ is not in any mass hyperboloid (see fig. 3) and hence this state is annihilated by $f \cdot A$. We now choose $p$ to be in the complement of these regions and $q, p', q' \in \cup_i U_i$ such that

$$p + q = p' + q'.$$

Figure 3: A sketch of a two-dimensional mass hyperboloid with four scattering states satisfying $p+q=p'+q'$. If $p$ is outside the region $U$ then this state is annihilated by $f \cdot A$.

The centre of mass energy and invariant momentum transfer are defined as usual by

$$s=(p+q)^2 \quad \text{and} \quad t=(p-p')^2$$

and are chosen to be below the threshold for pair production. By (2) the $S$-matrix element must be zero for such states. Using the analyticity in $s$ and $t$, the $S$ matrix is then zero in the whole region of analyticity. Repeating the argument for multi particle scattering one obtains the result:
*Lemma* 1 *:* The support of $A(p,p')$ is restricted to $p{=}p'$.
*Corollary:* $A$ cannot connect states on different mass hyperboloids (O'Raifeartaigh's theorem).

We make the technical assumption that $A$ is a matrix valued distribution. Then, it follows that $A(p)$ is a polynomial in the differential operator on the mass hyperboloid,

$$\nabla_\mu = \frac{\partial}{\partial p_\mu} - \frac{p_\mu p_\nu}{m^2}\frac{\partial}{\partial p_\nu}.$$

In other words

$$A \;=\; \sum_{n=0}^{N} A^{(n)}(p)_{\mu_1,\cdots,\mu_n}\frac{\partial}{\partial p_{\mu_1}}\cdots\frac{\partial}{\partial p_{\mu_n}}, \tag{3}$$

with $[A, p^\mu p_\mu]=0$, acting on any state on $\mathcal{D}$. To complete the argument we need another result which we state without proof.
*Lemma* 2*:*
Let $\mathcal{B}$ be the subset of $G$-transformations which commute with space-time translations then for $B(p)\in\mathcal{B}$

$$B(p)=a_\mu p^\mu + b,$$

where $a_\mu$ is a constant four vector (i.e. no internal indices) and $b$ is a constant Hermitian matrix which does not involve spin indices. Note hat lemma 2 assumes a Lie-multiplication [ , ] for generators of $\mathcal{B}$. In particular the result does not hold for anti-commuting generators. Now, taking the $N$-fold commutator of $A$ with $p_\mu$:

$$[p_{\mu_1}, [p_{\mu_2} \cdots, A] \cdots] = A^{(N)}_{\mu_1 \cdots \mu_N}(p)$$

we obtain an object which is in $\mathcal{B}$, hence by lemma 2,

$$A^{(N)}_{\mu_1 \cdots \mu_N}(p) = a_{\lambda \mu_1 \cdots \mu_N} p^\lambda + b_{\mu_1 \cdots \mu_N}.$$

However, (3) and the symmetry properties of $N$-fold commutators with $p^\mu$ imply

$$b_{\mu_1 \cdots \mu_N} = 0,$$

unless $N = 0$. Similarly

$$a_{\lambda \mu_1 \cdots \mu_N} = 0,$$

unless $N = 0$ or $N = 1$. For $N = 0$ we then showed that $A$ is the sum of a translation and an internal symmetry transformation and for $N = 1$ the antisymmetry of $a_{\lambda\mu}$ implies that $A$ is just an infinitesimal Lorentz transformation. This then completes the proof of the theorem.

# 3 Supersymmetry

A way out of this dilemma was discovered by Golfand and Likhtman [2] in 1971. They avoided the conclusions of Coleman and Mandula by introducing a graded Lie-algebra, that is, to the standard generators of a Lie-algebra, satisfying commutation relations they added (fermionic) generators $Q^{\mathbf{i}}_\alpha$ with the corresponding anti-commutation relations[1]. We have

$$[A, B\} \equiv AB - (-1)^{ab} BA,$$

[1]Without restricting the generality we can assume the $Q^{\mathbf{i}}_\alpha$ to be Majorana fermions

where $a, b$ is the number of fermionic generators in $A$ and $B$ respectively. There is a corresponding generalised Jacobi identity (exercise)

$$[A,[B,C\}\} + (-)^{a(b+c)}[B,[C,A\}\} + (-)^{c(a+b)}[C,[A,B\}\} = 0.$$

## 3.1 SUSY-algebra

Let us now determine the structure constants of the general super algebra. Because of $[odd, even\} = odd$ we must have

$$[Q^{\mathbf{i}}_\alpha, B_a] = -(h_a)_{\alpha\mathbf{j}}^{\beta\mathbf{i}} Q^{\mathbf{j}}_\beta.$$

The $(Q, B_a, B_b)$-Jacobi identity then requires that the matrices $h_a$ represent the bosonic symmetry algebra i.e.

$$[h_a, h_b] = if_{ab}{}^c h_c.$$

This implies in particular the commutation relations

$$\left[Q^{\mathbf{i}}_\alpha, P_\mu\right] = 0, \quad \text{and} \quad \left[Q^{\mathbf{i}}_\alpha, J_{\mu\nu}\right] = (b_{\mu\nu})_\alpha{}^\beta Q_\beta{}^{\mathbf{i}}, \tag{4}$$

with

$$[b_{\mu\nu}, b_{\rho\lambda}] \;=\; \eta_{\mu\lambda}b_{\nu\rho} + \eta_{\nu\rho}b_{\mu\lambda} - \eta_{\mu\rho}b_{\nu\lambda} - \eta_{\nu\lambda}b_{\mu\lambda}, \tag{5}$$

showing that $b_{\mu\nu}$ forms a representation of the of the Lorentz transformations. We shall assume that $Q^{\mathbf{i}}_\alpha$ be in the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation of the Lorentz group and hence $b_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$. Similarly for the generators of the internal symmetry group $T^a$

$$[Q_\alpha{}^{\mathbf{i}}, T^a] = (l^a)^{\mathbf{i}}{}_{\mathbf{j}} Q^{\mathbf{j}}{}_\alpha + i(t^a)^{\mathbf{i}}{}_{\mathbf{j}}(\gamma_5)_\alpha{}^\beta Q^{\mathbf{j}}{}_\beta,$$

where $l^a + it^a\gamma_5$ represents the the Lie algebra of the internal symmetry group. Here we have used that $\delta_\alpha{}^\beta$ and $\gamma_5$ are the only (pseudo-) scalar invariant tensors.

Let us now consider the $odd - odd$ commutator $\{Q^{\mathbf{i}}_\alpha, Q^{\mathbf{j}}_\alpha\}$. The result must be composed of even generators and must be symmetric under simultaneous exchange of $\mathbf{i} \leftrightarrow \mathbf{j}$ and $\alpha \leftrightarrow \beta$. The most general possibility, compatible with the Coleman-Mandula theorem is then

$$\begin{aligned}\{Q^{\mathbf{i}}_\alpha, Q^{\mathbf{j}}_\beta\} &= r\delta^{\mathbf{ij}}(\gamma^\mu C)_{\alpha\beta}P_\mu + s\delta^{\mathbf{ij}}(\gamma^{\mu\nu}C)_{\alpha\beta}J_{\mu\nu} \\ &\quad + C_{\alpha\beta}U^{\mathbf{ij}} + (\gamma_5 C)_{\alpha\beta}V^{\mathbf{ij}}.\end{aligned} \tag{6}$$

The $(Q,Q,Q)$ Jacobi-identity together with $(Q,Q,B)$ Jacobi-identity then imply that $U$ and $V$ commute with everything and are therefore *central charges*. The latter identity also implies that $s=0$. The remaining undetermined constant $r$ can be absorbed in a redefinition of $P_\mu$. Here we take $r=-2$.

We end this section by noting that the above supersymmetry algebra is the most general superalgebra consistent with an interacting quantum field theory [3]. Note also that in the presence of fermionic charges the first part of the proof of the no-go theorem (O'Raifeartaigh's theorem) goes still through. In particular, supersymmetry transformations cannot connect different mass hyperboloids.

## 3.2 Irreducible Representations

Before constructing explicit irreducible representations of the algebra explained in the last section we recall some representation independent consequences of the algebra.
*i)* The Hamiltonian of a supersymmetric theory is positive definite.
*Proof:*
From the Majorana property $\bar{Q}^{\mathbf{i}} = Q^{T\mathbf{i}}C$ we have with (6,61)

$$\{Q^{\mathbf{i}}_\alpha, \bar{Q}^{\mathbf{j}}_\beta\} = 2\delta^{ij}\not{P}_{\alpha\beta} - \delta_{\alpha\beta}U^{ij} - (\gamma_5)_{\alpha\beta}V^{ij}.$$

Multiplying with $\gamma_0$ and taking the trace we then obtain

$$8NP_0 = Q^{\mathbf{i}}_\alpha(Q^{\mathbf{i}}_\alpha)^\dagger + h.c. \geq 0.$$

*ii)* The number of fermionic and bosonic states with non-zero energy in a supermultiplet is the same.
*Proof:*

Define the fermion number operator $(-1)^{N_F}$. Then, using $(-1)^{N_F}Q^{\mathbf{i}}_\alpha = -Q^{\mathbf{i}}_\alpha(-1)^{N_F}$,

$$0 = \mathrm{tr}\left[(-1)^{N_F}\{Q^{\mathbf{i}}_\alpha, \bar{Q}^{\mathbf{i}}_\beta\}\right] = 2\not{P}_{\alpha\beta}\mathrm{tr}\left[(-1)^{N_F}\right],$$

where the trace is over a single supermultiplet. For $P_0 \neq 0$ this leads to the result. For $P_0 = 0$ the result does not hold. We then have

$$\mathrm{tr}\left[(-1)^{N_F}\right] = \#\text{zero energy bosons} - \#\text{zero energy fermions}$$

This is the *Witten index.* We note in passing that if this index is non-vanishing spontaneous SUSY breaking is excluded since we have from *i)* that $Q|0\rangle \neq 0$ implies $P_0 > 0$, i.e. the zero energy modes are lifted. But if their number is not equal then *ii)* is violated.

The irreducible representations of the super-Poincaré group can be found using the Wigner method for constructing irreducible representations of the Poincaré group. This method consists of finding a representation of the little group leaving a given four-momentum invariant and boosting it up to a representation of the full Poincaré group. In this way one obtains irreducible representations labelled the values of the Casimir operators $P_\mu P^\mu = -m^2$ and $W_\mu W^\mu = m^2 s(s+1)$, where $W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}J^{\nu\rho}P^\lambda$ is the Pauli-Lubanski vector. The two Casimir operators of the super Poincaré group are $p^2$ and $\hat{W}^2$, with $\hat{W}_\mu$ a linear combination of $W_\mu$ and $\bar{Q}\gamma_\mu\gamma_5 Q$.

### 3.2.1 Massless Supermultiplet

We choose $p_\mu = (E, 0, 0, E)$. The little group (in the Lorentz group) of that vector is generated by

$$J_1 = J_{10} + J_{13}, \quad J_2 = J_{20} + J_{23} \quad \text{and} \quad J = J_{12}.$$

They satisfy the commutation relations of $E_2$ i.e.

$$[J_1, J] = -iJ_2, \qquad [J_2, J] = iJ_1, \qquad [J_1, J_2] = 0.$$

As in any finite dimensional unitary representation, $J_1$ and $J_2$ are trivially represented. The remaining operator $J = J_{12}$ has the helicity-eigenstates

$$J|\lambda\rangle = \lambda|\lambda\rangle,$$

with $\lambda \in \frac{1}{2}\mathbf{Z}$.

On the other hand for vanishing central charge the *odd-odd* commutator (6) simplifies to

$$\{Q^{\mathbf{i}}_{\alpha}, \bar{Q}^{\mathbf{i}}_{\beta}\} = 2E\delta^{\mathbf{ij}}(\gamma^0 + \gamma^3)_{\alpha\beta}. \tag{7}$$

Making use of the Majorana condition one then easily shows that (7) contains only one independent non-vanishing commutator i.e.

$$\{Q^{\mathbf{i}}_2, Q^{\dagger\mathbf{j}}_2\} = 4E\delta^{\mathbf{i}}_{\mathbf{j}}.$$

But from (4) and (5)

$$\left[J, Q^{\dagger\mathbf{i}}_2\right] = -\frac{1}{2}Q^{\dagger\mathbf{i}}_2 \quad \text{and} \quad \left[J, Q^{\mathbf{i}}_2\right] = \frac{1}{2}Q^{\mathbf{i}}_2,$$

and therefore $Q^{\mathbf{i}}_2$ and $Q^{\dagger\mathbf{i}}_2$ act as creation and annihilation operators on the Hilbert space spanned by the eigenvectors of $J$. Positivity then requires that that the remaining supercharges are trivially realised on the physical states (eg. [4]). Starting from the highest weight vector $|\Omega\rangle$ defined by $Q^{\mathbf{i}}_2|\Omega\rangle = 0$ we then obtain the full Hilbert space upon acting with $Q^{\dagger\mathbf{i}}_2$, $\mathbf{i} = 1, \cdots, \mathcal{N}$. i.e.

$$|\Omega\rangle, \quad Q^{\dagger\mathbf{i}}_2|\Omega\rangle, \quad \cdots\cdots, \quad Q^{\dagger\mathbf{i}_1}_2 \cdots Q^{\dagger\mathbf{i}_{\mathcal{N}}}_2|\Omega\rangle$$

with helicities

$$\lambda, \quad \lambda - \frac{1}{2}, \quad \cdots\cdots, \quad \lambda - \frac{\mathcal{N}}{2}$$

and multiplicities

$$1, \quad \binom{\mathcal{N}}{1}, \quad \cdots\cdots, \quad \binom{\mathcal{N}}{\mathcal{N}}.$$

The total number of states is then $\sum\limits_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}}$. Because of the $CPT$-theorem, a physical massless state contains always the two helicities $\lambda$ and $-\lambda$ and therefore a single supermultiplet can contain all massless states for $\lambda = \frac{\mathcal{N}}{4}$. Otherwise the states must be doubled starting with a second vacuum $\lambda' = \frac{\mathcal{N}}{2} - \lambda$.

Examples:

$\mathcal{N}=1$ :

1. chiral multiplet: The chiral multiplet consists of two irreducible submultiplets which are related by the $CPT$ symmetry.

$$\Phi = (|\frac{1}{2}\rangle, |0\rangle)\,, \qquad \bar{\Phi} = (|0\rangle, |-\frac{1}{2}\rangle)$$

   Field theory realisation:

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix}\,, \qquad \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2P^\mu(\sigma_\mu)_{\alpha\dot{\beta}}\ .$$

   Start with $\varphi \equiv |0\rangle$, then

$$\begin{aligned} Q_\alpha\varphi = &\ \chi_\alpha \quad , \quad \bar{Q}_{\dot{\alpha}}\varphi &= 0 \quad \text{chiral !} \\ Q_\beta\chi_\alpha = &\ \tfrac{1}{2}\epsilon_{\beta\alpha}F, \bar{Q}_{\dot{\alpha}}\chi_\alpha &= -2i\partial\!\!\!/_{\dot{\alpha}\alpha}\varphi \\ Q_\alpha F = &\ \ 0 \quad , \quad \bar{Q}_{\dot{\alpha}}F &= 4i\partial\!\!\!/_{\dot{\alpha}\beta}\chi^\beta \end{aligned}$$

   Note that the field $F$ is an auxiliary field to match the number of degrees of freedom off-shell. It has no dynamical degrees of freedom.

   In order to complete the field theory representation of the $\mathcal{N}=1$ chiral multiplet we need to construct an invariant action for this multiplet. This leads to the *Wess-Zumino Model*

$$S = \int d^4x \left\{\partial_\mu\bar{\varphi}\partial^\mu\varphi - \frac{i}{4}\bar{\chi}^{\dot{\alpha}}\bar{\partial\!\!\!/}_{\dot{\alpha}}{}^{\beta}\chi_\beta - \frac{i}{4}\left(\partial\!\!\!/_\beta{}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}\right)\chi_\beta + \frac{1}{16}|F|^2\right\} \qquad (8)$$

2. vector multiplet: We denote the two irreducible multiplets by $W_\alpha$ and $W_{\dot{\alpha}}$.

$$W_\alpha = (|1\rangle, |\frac{1}{2}\rangle), \qquad W_{\dot{\alpha}} = (|-\frac{1}{2}\rangle, |-1\rangle)$$

Field theory realisation:

$$
\begin{array}{rcll}
Q_\alpha \lambda_\beta = & F_{\beta\alpha} - \frac{1}{2}\epsilon_{\alpha\beta} D, & \bar{Q}_{\dot{\alpha}} \lambda_\alpha & = 0 \\
Q_\beta A_\mu = & (\sigma_\mu)_\beta{}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}, & \bar{Q}_{\dot{\beta}} A_\mu & = (\bar{\sigma}_\mu)_{\dot{\beta}}{}^{\gamma} \lambda_\gamma \\
Q_\alpha D = & -2i(\partial\!\!\!/)_\alpha{}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}, & \bar{Q}_{\dot{\alpha}} D & = 2i(\bar{\partial\!\!\!/})_{\dot{\alpha}}{}^{\alpha} \lambda_\alpha
\end{array}
$$

where $F_{\beta\alpha} = F_{\mu\nu}(\sigma^{\mu\nu})_{\beta\alpha}$.

We will postpone the construction of an invariant action for the vector multiplet to the next section and continue instead with representations of the $\mathcal{N}=2$ SUSY algebra.

$\underline{\mathcal{N} = 2:}$

The internal symmetry of the $\mathcal{N}=2$ algebra is given by $U(2) = U(1)_I \times SU(2)_I$. This is an example of the so-called $R$-symmetry in supersymmetric theories.

1. Yang-Mills multiplet: The irreducible $\mathcal{N}=2$ multiplet with helicity $\leq 1$ contains already all states required by $CPT$ invariance. Thus

$$
\mathcal{A} = \Big(|1\rangle, |\frac{1}{2}\rangle + |\frac{1}{2}\rangle, |0\rangle + |0\rangle, |-\frac{1}{2}\rangle + |-\frac{1}{2}\rangle, |-1\rangle\Big)
$$

$$
\begin{array}{ccc}
 & A_\mu & \\
\psi^1_\alpha & & \psi^2_\alpha \\
 & \varphi, \bar{\varphi} &
\end{array}
$$

We note in passing that this multiplet can be realised as a combination of $\mathcal{N} = 1$ chiral- and vector multiplets. The invariant action for this multiplet is given by

$$
\begin{aligned}
S \;=\; & -\frac{1}{g^2}\mathrm{Tr}\int \Big\{\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2}\varphi^\dagger \Delta \varphi \\
& + \frac{i}{2}\bar{\psi}^{\mathbf{i}} \bar{\sigma}^\mu D_\mu \psi_{\mathbf{i}} + \frac{i}{2}\psi^{\mathbf{i}} \sigma^\mu D_\mu \bar{\psi}_{\mathbf{i}} \\
& - \frac{i}{2}\varphi^\dagger \{\psi^{\mathbf{i}}, \psi_{\mathbf{i}}\} + \frac{i}{2}\varphi\{\bar{\psi}^{\mathbf{i}}, \bar{\psi}_{\mathbf{i}}\} - \frac{1}{4}[\varphi^\dagger, \varphi]^2\Big\}, \\
& + \frac{i\theta}{8\pi^2}\mathrm{Tr}\int \frac{1}{4} F_{\mu\nu}\tilde{F}^{\mu\nu}
\end{aligned}
\tag{9}
$$

Twisting:

When working in Euclidean space, $E_4$ it is sometimes convenient to identify the various $SU(2)$ symmetries of $\mathcal{N}=2$ YM-theory in a non-standard way. Indeed the 'Lorentz' group in $E_4$ is $L = SO(4) \simeq SU(2)_L \times SU(2)_R$. Combining this with the internal $R$-symmetry the total symmetry group contains

$$H = SU(2)_L \times SU(2)_R \times SU(2)_I$$

with the various fields transforming as follows

| | $SU(2)_L$ | $SU(2)_R$ | $SU(2)_I$ |
|---|---|---|---|
| $A_\mu$ | 1/2 | 1/2 | 0 |
| $\psi^{\mathbf{i}}{}_\alpha$ | 1/2 | 0 | 1/2 |
| $\bar{\psi}_{\dot{\alpha}\mathbf{i}}$ | 0 | 1/2 | 1/2 |
| $\varphi, \bar{\varphi}$ | 0 | 0 | 0 |

The standard embedding of the Lorentz group is, of course,

$$L = SU(2)_L \times SU(2)_R$$

Alternatively we can embed the Lorentz and $R$-symmetry as

$$\begin{aligned} L' &= SU(2)_L \times SU(2)'_R \\ SU(2)'_R &= \mathrm{diag}\left(SU(2)_R \times SU(2)_I\right) \end{aligned}$$

Of course, $L'$ and $L$ describe the same $\mathcal{N}=2$ Yang-Mills theory in $E_4$, but the transformation properties are now different:

$$\left(\psi^{\mathbf{i}}{}_\alpha, \bar{\psi}_{\dot{\alpha}\mathbf{i}}\right) \to \begin{cases} \psi & \text{scalar} \\ \psi_\mu & \text{vector} \\ \psi_{\mu\nu} & \text{antisymm. tensor} \end{cases}$$

This alternative embedding is called the *twisted* $\mathcal{N}=2$ Yang-Mills multiplet. While the two embeddings are equivalent in flat space they lead to different couplings to gravity.

2. Massive Hypermultiplet All fields of a super multiplet will have the same mass $M$. We choose the rest frame $p_\mu = (M, 0, 0, 0)$. Using again the Majorana property we have in the absence of central charges

$$\{Q^{\mathbf{i}}_\alpha, Q^{\mathbf{j}\dagger}_\beta\} = 2M\delta^{\mathbf{ij}}\delta_{\alpha\beta}$$

that is

$$\{Q_1^{\mathbf{i}}, Q_1^{\mathbf{j}\dagger}\} = \{Q_2^{\mathbf{i}}, Q_2^{\mathbf{j}\dagger}\} = 2M\delta^{\mathbf{ij}},$$

with all other commutators vanishing. One then proceeds as in the massless case with the difference however that the number of generators is twice that of the massless representations. Correspondingly the dimension of a supersymmetry multiplet is now $2^{2\mathcal{N}}$. Note that in the presence of central charges there is a possibility of constructing massive representations that are in 'small' i.e. $2^{\mathcal{N}}$-dimensional representations of the SUSY algebra.

*Exercise:* What is the condition between mass and central charge for a massive representation of a $\mathcal{N}=2$ SUSY algebra to be $2^{\mathcal{N}}$, rather than $2^{2\mathcal{N}}$-dimensional.
*Solution:* Making use of the Majorana property and the $\gamma_5$-invariance (6) can be written as

$$\{Q_1^{\mathbf{i}}, Q_1^{\mathbf{j}\dagger}\} = \{Q_2^{\mathbf{i}}, Q_2^{\mathbf{j}\dagger}\} = 2M\delta^{\mathbf{ij}} + \epsilon^{\mathbf{ij}}\gamma^0_{\alpha\beta}Z, \tag{10}$$

where $Z$ is a real number. The second term on the right of (10) has eigenvalues $\pm Z$. In the eigenbasis the it is therefore proportional to $2M \pm Z$. In particular the combinations $Q_1^1 \pm Q_2^{2\dagger}$ diagonalise the r.h.s. For $Z = \pm 2M$ there is then only one linearly independent, non-vanishing commutator, reducing the representation to a $2^{\mathcal{N}}$-dimensional one. The little group of $p_\mu = (M, 0, 0, 0)$ is generated by $J_{12}, J_{23}, J_{13}$, i.e. the $SO(3) \sim SU(2)$ algebra. The states are labelled by the spin which is an $SU(2)$-Casimir. The combinations $Q_1^1 \pm Q_2^{2\dagger}$ and $Q_1^{1\dagger} \pm Q_2^2$ form a doublet with spin $\frac{1}{2}$. This follows from (6) and its hermitian conjugate. For $\mathcal{N} > 2$, $Z^{\mathbf{ij}}$ is a totally antisymmetric $\mathcal{N}\times\mathcal{N}$ matrix. For $\mathcal{N}$ even there exists a unitary transformation $U$ that brings $Z$ into the form $\epsilon^{\mathbf{ij}}\otimes \mathrm{diag}(\lambda_1, \cdots, \lambda_{\frac{\mathcal{N}}{2}})$. Multiplets which form a $2^{\mathcal{N}}$-dimensional representation are called *short multiplets*. They are related to $BPS$-configurations in field theory and play an important role for dualities in field theory and string theory as we will explain in the sequel.

# 4 Superspace

In order formulate manifestly SUSY-invariant field theories we first generalise ordinary Minkowski space to super-Minkowski space (or *super space*) $\mathbb{R}^{4|2\mathcal{N}}$. It can be realised as the coset $SP/L$ of the super-Poincaré group and the Lorentz group. Indeed, any element of the $\mathcal{N}=1$ superpoincaré group can be written in the form[2]

$$g = \exp\{a^\mu P_\mu + \bar{\epsilon}_{\dot{\alpha}} Q^{\dot{\alpha}} + \epsilon^\alpha Q_\alpha\} \exp\{\frac{1}{2}\omega^{\mu\nu} J_{\mu\nu}\},$$

where we have changed to Weyl notation ($Q_\beta = (Q_\alpha, \bar{Q}^{\dot{\alpha}});\ \alpha, \dot{\alpha} = 1, 2$) for the supercharges. The coset space is then parametrised by coordinates $(x^\mu, \theta^\alpha, \theta_{\dot{\alpha}}) \equiv z^A$ corresponding to the group element

$$\exp\{x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\}.$$

Superspace is therefore the tensor product of Minkowski space with the Grassmann algebra of anticommuting variables

$$\Lambda = \oplus_{l=0}^{n} (\wedge V)^l$$

where $V$ is a vector space over $\mathbf{C}$. The left action by a group element $g_0$ is equivalent to the coordinate transformation

$$\begin{aligned}
x^\mu &\rightarrow x^\mu + a^\mu + i\epsilon^\alpha (\sigma^\mu)_\alpha{}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} - i\theta^\alpha (\sigma^\mu)_\alpha{}^{\dot{\alpha}} \bar{\epsilon}_{\dot{\alpha}} + \omega^{\mu\nu} x_\nu \\
\theta^\alpha &\rightarrow \theta^\alpha + \epsilon^\alpha + \frac{1}{4} (\sigma^{\mu\nu})^\alpha_\beta \theta^\beta \omega_{\mu\nu} \\
\bar{\theta}_{\dot{\alpha}} &\rightarrow \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}} - \frac{1}{4} (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\theta}_{\dot{\beta}} \omega_{\mu\nu}\ .
\end{aligned}$$

*Exercise:* How does $z^{\mathcal{A}}$ transform under the product of two left multiplications?

We define left- and right differentiation on the algebra by

$$\frac{\partial}{\partial \theta^\alpha} \theta^\beta = \delta_\alpha{}^\beta,$$

or, in two component notation[3]

$$\frac{\partial}{\partial \theta^\alpha} \theta^\beta = \delta_\alpha{}^\beta \quad \text{and} \quad \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}},$$

[2]For simplicity we restrict ourself here to $\mathcal{N}=1$ supersymmetry

[3]The lowering of the the internal index $j$ for the antichiral fermions is purely conventional. The dotted indices indicate that $\bar{Q}_{\mathbf{j}}$ transforms according to the $(0, \frac{1}{2})$-representation

with a sign change for left differentiation. Similarly we can define an integration on the algebra as the functional

$$\int d\theta\ f(\theta) = \frac{\partial}{\partial\theta} f(\theta)|_{\theta=0}.$$

In analogy with usual $x$-space integrals we may define $\delta$-functions by

$$\int d\theta\delta(\theta-\theta')f(\theta) = f(\theta'),$$

and correspondingly for more that one Grassmann variable. For example, the delta function for $\mathcal{N}=1$-superspace, $\delta^4(x-x')\delta^4(\theta-\theta')$ can be represented by

$$\frac{1}{4}\delta^4(x-x')(\theta-\theta')^2(\bar{\theta}-\bar{\theta}')^2.$$

From the above it follows in particular that

$$\int d^8z\ f(z) = \int d^4x \mathrm{d}^2\theta d^2\bar{\theta} f(x,\theta,\bar{\theta}),$$

is an invariant integration over the coset space $SP/L$. Functional differentiation in superspace is then simply

$$\frac{\delta f(x',\theta',\bar{\theta}')}{\delta f(x,\theta,\bar{\theta})} = \delta^4(x-x')\delta^4(\theta-\theta').$$

## 4.1 Superfields

We now consider functions (superfields) on the superspace introduced above. A scalar superfield is defined by its transformation property:

$$\phi'(z') = \phi(z), \tag{11}$$

or, under an infinitesimal transformation

$$\delta\phi(z) = \phi(z+\delta z) - \phi(z) = \zeta^A X_A \phi(z), \tag{12}$$

where

$$\begin{aligned}
X_\mu &= \partial_\mu \\
X_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu)_\alpha{}^{\dot\alpha}\bar\theta_{\dot\alpha}\partial_\mu \\
X_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i(\sigma^\mu)^{\dot\alpha}{}_\alpha\theta^\alpha\partial_\mu \\
X_{\mu\nu} &= -(x_\mu\partial_\nu - x_\nu\partial_\mu) - \frac{1}{2}\theta^\beta(\sigma^{\mu\nu})_\beta{}^\alpha\frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\bar\theta_{\dot\beta}(\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}},
\end{aligned} \tag{13}$$

are the Killing vector fields of flat superspace and $\zeta^A = (a^\mu, \epsilon^\alpha, \bar\epsilon^{\dot\alpha}, \omega^{\mu\nu})$ are the infinitesimal parameters of the transformation.

A general superfield will be of the form $\phi^{\mathbf{i}}_A$ where $i$ denotes the collective index for the internal indices and $A$ for the space-time indices. The natural generalisation of (11) is then

$$\phi'^i_A(z^g) = [D(g)]^{iB}_{Aj}\ \phi^{\mathbf{j}}_B(z).$$

Next we introduce the covariant derivatives, that is derivatives that (anti-) commute with supersymmetry transformations. It is easy to see (exercise) that these are obtained from the first three Killing vectors in (13) by simply changing the sign of the second term, i.e.

$$\begin{aligned}
D_\mu &= \partial_\mu \\
D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \\
D_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu
\end{aligned} \tag{14}$$

Note that the connection appearing in the covariant derivatives (14) is not torsion free. Indeed computing the (anti-) commutators of the different covariant derivatives in (14) we obtain

$$[D_A, D_B\} = T_{AB}{}^C D_C,$$

where

$$T_{\alpha\dot{\alpha}}{}^{\mu} = -2i(\sigma^{\mu})_{\alpha\dot{\alpha}},$$

and all other elements of the torsion vanishing.

## 4.2 Superfield Actions

We are now ready to construct supersymmetric field theories choosing different superfield functionals that is, superspace integrals of suitable combinations of superfields. The simplest example, is

$$\int d^4x d^2\theta d^2\bar{\theta}\ \phi(x, \theta, \bar{\theta}). \tag{15}$$

However if we want irreducible representations of the supersymmetry algebra, (15) is not a valid action. Indeed, the constraint

$$D_{\dot{\alpha}}\phi = 0 \tag{16}$$

is supersymmetry invariant and hence a general $\phi$ is not an irreducible multiplet. We can remedy this by considering only chiral scalar fields $\phi$ defined by (16). We shall see that chiral super fields form an irreducible representation of the $\mathcal{N}=1$ SUSY-algebra. Substitution of a chiral super field into (15) however leads to a zero result. The next simplest possibility is then

$$\int d^4x d^2\theta d^2\bar{\theta}\ \bar{\phi}(x, \theta, \bar{\theta})\phi(x, \theta, \bar{\theta}), \tag{17}$$

which defines a valid supersymmetric field theory. To see what it corresponds to in terms of bosons and fermions we need to disentangle the component field content of the chiral superfield $\phi$. Due to the constraint (16) the only independent degrees of freedom in $\phi$ are

$$\begin{aligned} \varphi &\equiv \phi|_{\theta=0}, \\ \chi_\alpha &\equiv D_\alpha\phi|_{\theta=0}, \\ F &\equiv D^2\phi|_{\theta=0}, \end{aligned} \tag{18}$$

where $D^2 \equiv D^\alpha D_\alpha$. Higher derivatives of the superfield vanish identically. Assuming that the fields fall off fast enough at infinity we then use $d\theta^\alpha = D_\alpha$ and hence (17) becomes up to a global normalisation

$$\begin{aligned}\frac{1}{16}\int d^4x\ \bar{D}^2 D^2 \bar{\phi}\phi|_{\theta,\bar{\theta}=0} &= \frac{1}{16}\int d^4x\ \bar{D}^2(\bar{\phi}D^2\phi)|_{\theta,\bar{\theta}=0} \\ &= \int d^4x \left\{\partial_\mu\bar{\varphi}\partial^\mu\varphi - \frac{i}{2}\bar{\chi}^{\dot{\alpha}}\bar{\partial\!\!\!/}_{\dot{\alpha}}{}^{\beta}\chi_\beta + \frac{1}{16}|F|^2\right\}\end{aligned}$$

where we have used $\{D_\alpha, D_{\dot{\alpha}}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu$ and $(\sigma^\mu)^{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)_{\dot{\alpha}\alpha} = -\eta^\mu_\nu$. The component field action is therefore just the Wess-Zumino model (8) describing a free Weyl fermion and a complex scalar field. Note again the presence of the auxiliary field $F$. It has two purposes: i) to match the counting between fermionic and bosonic degrees of freedom of shell and ii) to linearise the supersymmetry transformations. We can read off the supersymmetry transformations of the component fields repeating (18) after making an infinitesimal super translation (12). Alternatively, using that the covariant derivatives differ from the Killing fields only by a term that vanishes when $\theta^\alpha = \bar{\theta}_{\dot{\alpha}} = 0$ we have

$$\begin{aligned}\delta\varphi = (\epsilon^\alpha D_\alpha + \bar{\epsilon}^{\dot{\alpha}}D_{\dot{\alpha}})\phi|_{\theta,\bar{\theta}=0} &= \epsilon^\alpha\chi_\alpha \\ \delta\chi_\alpha = (\epsilon^\alpha D_\alpha + \bar{\epsilon}^{\dot{\alpha}}D_{\dot{\alpha}})D_\alpha\phi|_{\theta,\bar{\theta}=0} &= \frac{1}{2}\epsilon^\alpha F - 2i(\partial\!\!\!/)_{\dot{\beta}\alpha}\bar{\epsilon}^{\dot{\beta}}\varphi \\ \delta F = -\frac{1}{2}(\epsilon^\alpha D_\alpha + \bar{\epsilon}^{\dot{\alpha}}D_{\dot{\alpha}})D^2\phi|_{\theta,\bar{\theta}=0} &= 4i(\partial\!\!\!/)_{\dot{\beta}\alpha}\bar{\epsilon}^{\dot{\beta}}\chi^\alpha\end{aligned} \tag{19}$$

which agrees with our previous result. Let us now consider a Yang-Mills multiplet. As the lowest component is a spinor we expect the corresponding superfield to be a spinorial one. Indeed the chiral superfield $W_\alpha$ with $D_{\dot{\alpha}}W_\alpha = 0$ has the components

$$\lambda_\alpha, F_{\alpha\beta} = F_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta} \quad \text{and} \quad D$$

respectively. In order to identify $F_{\mu\nu}$ with the field strength we impose the Bianchi constraint $\partial_{[\mu}F_{\nu\lambda]} = 0$ which, in superspace becomes (exercise)

$$D^\alpha W_\alpha = D_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}.$$

This constraint can be solved introducing a real superfield $V$ as

$$W_\alpha = \bar{D}^2(e^{-\frac{g}{2}V}D_\alpha e^{\frac{g}{2}V}). \tag{20}$$

Under gauge transformations the *prepotential* $V$ transforms like

$$e^V \to e^{i\Lambda^\dagger} e^V e^{-i\Lambda}, \tag{21}$$

where $\Lambda$ is a chiral super field[4]. The Yang-Mills action is simply

$$\frac{1}{4}\mathrm{Tr}\int d^4x\, d^2\theta\ (W^\alpha W_\alpha) + c.c..$$

Adding the chiral multiplet to this action we then obtain the superfield action for $N = 1$ SUSY $QCD$

$$\begin{aligned} S_{QCD} &= \frac{1}{16}\int d^8z\ \left(\phi^\dagger e^{gV}\phi - \tilde{\phi}^\dagger e^{-gV}\tilde{\phi}\right) \\ &\quad -\frac{1}{4}\int d^6z\ \mathcal{W}(\phi,\tilde{\phi}) + \text{c.c.} \\ &\quad +\frac{1}{4}\mathrm{Tr}\int d^6z\ W^\alpha W_\alpha + \text{c.c.} \end{aligned}$$

where $\mathrm{d}^8z$ is a short hand notation for $\mathrm{d}^4x\mathrm{d}^2\theta d^2\bar{\theta}$. Thich describes the strong interactions of gluons, gluinos, quarks, and squarks. Here $\tilde{\phi}$ transforms in the anti-fundamental representation of the gauge group so that

$$\Phi = \begin{pmatrix} \phi \\ \tilde{\phi} \end{pmatrix}$$

contains a Dirac spinor. The superpotential $\mathcal{W} = \frac{1}{2}m\phi\tilde{\phi} + \cdots$ contains the mass term for the quarks and squarks.

Next we consider an $\mathcal{N}=2$ multiplet. As explained in section 3, a $\mathcal{N}=2$ supermultiplet containing the gauge field contains furthermore two Weyl fermions and a complex scalar. In terms of the $\mathcal{N}=1$-superfields, $\phi$ introduced earlier the $\mathcal{N}=2$-superfield $\mathcal{A}$ can be written as

$$\mathcal{A} = \phi + \theta^{\in\alpha}W_\alpha + \theta^{\in\alpha}\theta^{\in}_\alpha D^2\phi,$$

or equivalently

$$\phi = \mathcal{A}|_{\theta^\in=0}, \quad W_\alpha = D^2_\alpha\mathcal{A}|_{\theta^\in=0}.$$

[4]The chirality of $\Lambda$ is required in order to preserve the chirality of $\phi$.

In order to satisfy the chirality and Bianchi condition for the $\mathcal{N}=1$ superfields $\mathcal{A}$ must satisfy the $\mathcal{N}=2$-constraints

$$D^{\mathbf{i}}_{\dot\alpha}\mathcal{A}=0,\quad i=1,2\quad\text{and}\quad D^{\mathbf{ij}}\mathcal{A}=\bar D^{\mathbf{ij}}\bar{\mathcal{A}},$$

where $D^{\mathbf{ij}}=D^{\mathbf{i}\alpha}D^{\mathbf{j}}_{\alpha}$. Unlike the $\mathcal{N}=1$ constraints, a closed solution for the $\mathcal{N}=2$-constraints is available only for abelian fields. The action of the $\mathcal{N}=2$-YM-theory then is given by

$$\begin{aligned}
S &= \frac{1}{8\pi}\mathrm{ImTr}\int d^4xd^2\theta_1 d^2\theta_2\ \tau\mathcal{A}^2 \qquad (22)\\
&= \frac{1}{16\pi}\mathrm{ImTr}\int d^4x\int d^2\theta\ \tau W^\alpha W_\alpha+\frac{\mathrm{Im}(\tau)}{16\pi}\mathrm{Tr}\int d^4x\int d^2\theta\ d^2\bar\theta\ \phi^\dagger\phi\\
&= -\frac{1}{g^2}\mathrm{Tr}\int\Big\{\frac14(F_{\mu\nu})^2-\frac12\varphi^\dagger\Delta\varphi+\frac{i}{2}\bar\psi^{\mathbf{i}}\bar\sigma^\mu D_\mu\psi_{\mathbf{i}}+\frac{i}{2}\psi^{\mathbf{i}}\sigma^\mu D_\mu\bar\psi_{\mathbf{i}}\\
&\quad -\frac{i}{2}\varphi^\dagger\{\psi^{\mathbf{i}},\psi_{\mathbf{i}}\}+\frac{i}{2}\varphi\{\bar\psi^{\mathbf{i}},\bar\psi_{\mathbf{i}}\}-\frac14[\varphi^\dagger,\varphi]^2-X^2\Big\},\\
&\quad +\frac{i\theta}{8\pi^2}\mathrm{Tr}\int\frac14 F_{\mu\nu}\tilde F^{\mu\nu}
\end{aligned}$$

where $\tau=\frac{i4\pi}{g^2}+\frac{\theta}{2\pi}$ and $X$ is an auxiliary field. We thus reproduce (9). The prepotential $V$ has been absorbed in the definition of the chiral super field $\phi$ which is therefore *covariantly chiral* i.e. $[\nabla_{\dot\alpha},\phi]=0$ where $\nabla_{\dot\alpha}=e^{-\frac{g}{2}V}D_\alpha e^{\frac{g}{2}V}$. Note that the chiral fields are in the adjoint representation. An important consequence of this fact is the presence of flat directions in the potential for the scalar component $\varphi$ of the $\mathcal{N}=2$-multiplet, $V(\varphi)=\frac{1}{4g^2}\mathrm{Tr}[\varphi,\varphi^\dagger]^2$, where $g$ is the coupling constant. The potential vanishes for constant $\varphi$ taking its value in the Cartan subalgebra of the gauge group. For $\varphi\neq 0$, the Higgs mechanism breaks the gauge symmetry spontaneously down to $U(1)^l$, where $l$ is the rank of the Cartan subalgebra. In particular for gauge group to be $SU(2)$, the classical theory is then parametrised by two real parameters, $g^2$ and $|a|$, were $a=\mathrm{Tr}(\langle\varphi\rangle\sigma_3)$ is the expectation value of the scalar component of the $\mathcal{N}=2$ superfield[5]. The mass of the charged fields (with respect to the remaining $U(1)$) is proportional to $a$. In addition, the equations of motion have solitonic solutions with magnetic- or dyonic charge. The simplest of them is the so-called t'Hooft-Polyakov monopole. It is obtained starting with a radially symmetric ansatz:

$$\varphi_a=\frac{x^a}{gr^2}H(agr)\qquad A^0_a=0\qquad A^i_a=-\epsilon_{aij}\frac{x^j}{gr^2}\left[1-K(agr)\right] \qquad (23)$$

[5]The phase of $a$ is irrelevant because of the $R$-symmetry $\mathcal{A}\to e^{i\alpha}\mathcal{A}$

Insertion of (23) in the equations of motions yields the system of coupled differential equations

$$\begin{aligned} r^2\frac{d^2K}{dr^2} &= KH^2 + K(K^2-1) \\ r^2\frac{d^2H}{dr^2} &= 2K^2H. \end{aligned} \tag{24}$$

The boundary conditions are determined by imposing finite energy and $\varphi^2 \to a^2$ for $r \to \infty$. The solutions of (24) are given by

$$K(\xi) = \frac{\xi}{\sinh \xi} \qquad\qquad H(\xi) = \frac{\xi}{\tanh \xi} - 1 \tag{25}$$

The mass of these states can be obtained by computing the their energy. Substituting (23,25) into the energy functional yields

$$E = \int d^3x \partial^i \left[B^i_a \varphi_a\right] \equiv a g_m$$

where $\mathbf{B^a}$ is the non-abelian magnetic field and $g_m = 4\pi/g$ is the magnetic charge of the monopole solution (25). Corrigan et al. [5] have shown that the most general solution of the previous equations is of the form

$$A^\mu_a = \frac{1}{a^2 e}\epsilon_{abc}\varphi_b\partial^\mu\varphi_c + \frac{1}{a}\varphi_a B^\mu$$

where $B^\mu$ is arbitrary. The corresponding non-abelian magnetic field

$$B^i_a = -\frac{1}{2}\epsilon_{ijk}F_{a\ jk} = -\frac{1}{2a^4 e}\varphi_a\epsilon_{ijk}\epsilon^{bcd}\varphi^b\partial^j\varphi^c\partial^k\varphi^d$$

is aligned with the Higgs field. The magnetic charge is therefore

$$g_m = \frac{4\pi}{g}\epsilon_{abc}\int \mathrm{d}\varphi^a \wedge \mathrm{d}\varphi^b \wedge \mathrm{d}\varphi^c. \tag{26}$$

It is clear from (26) that the magnetic charge is topological since it counts the times that the two-sphere, defined by $\varphi^2 = a^2$ is covered when the two-sphere at infinity in space is covered once. To summarise we get

$$M = ag \quad \text{and} \quad M = a g_m,$$

for the masses of the electrically and magnetically charged states respectively. This structure generalises to states with both electric and magnetic charge (dyons) [6]. In the quantum theory the monopoles and dyons correspond to a new particles of the spectrum which are extended objects (with size $\sim 1/a$) located in the region where the energy density is appreciably different from zero. The quantum Hamiltonian is normally taken to be the Laplace operator on the moduli space of the classical solutions [7].

According our discussion of the massless and massive representations of the supersymmetry algebra in section 3, assuming continuity in $a$, the central extension for these states must be equal to their mass. To see that this is indeed the case we determine the central extension in these charge sectors. It is convenient to combine the two Majorana supercharges $Q_1$ and $Q_2$ into a Dirac supercharge $Q = Q_1 - iQ_2$. For the classical action (22) $Q$ is then found to be

$$Q = \frac{1}{\sqrt{2}g^2}\mathrm{Tr}\int \mathrm{d}^3x\ \mathcal{S}\psi,$$

where $\mathcal{S} = X + iY$ and

$$\begin{aligned} X &= -i\gamma\cdot(g^2\mathbf{\Pi} + i\gamma_5\mathbf{B}) - i\gamma^0\gamma_5[A,B], \\ Y &= g^2(\pi_A - i\gamma_5\pi_B) + \gamma\cdot\mathbf{D}(A + i\gamma_5 B)\gamma^0. \end{aligned}$$

Here $A$ and $B$ are the real- and imaginary part of the scalar field $\varphi$ respectively and $\psi$ is a Dirac Fermion. The central charges $U$ and $V$ in turn are related to the Dirac super charges by

$$U = \frac{1}{4i}\mathrm{Tr}\{Q,\bar{Q}\} \quad \text{and} \quad V = \frac{1}{4i}\mathrm{Tr}(\{Q,\bar{Q}\}\gamma_5),$$

respectively. Evaluating the Poison brackets and taking the trace one one finds

$$U + V = 2\mathrm{Tr}\int \mathrm{d}^3x\ [\mathbf{\Pi}\cdot\mathbf{D}\varphi + 2[A,B]\pi_\varphi^\dagger + i\{\psi,\pi_\psi\}], \tag{27}$$

where $\pi_\varphi^\dagger = \frac{1}{2}(\pi_A + i\pi_B)$. Now, using the Bianchi identity $\mathbf{D}\cdot\mathbf{B} = 0$ and Gauss's law

$$\mathbf{D}\cdot\mathbf{\Pi} = i[\varphi,\pi_\varphi] + i[\varphi^\dagger,\pi_\varphi^\dagger] + i\{\psi,\pi_\psi\}$$

the centre (27) can be written as a pure boundary term

$$U + V = 2\mathrm{Tr} \int \mathrm{d}\Omega \, \varphi \left( \mathbf{\Pi} + \frac{\tau}{4\pi} \mathbf{B} \right) . \tag{28}$$

Using the asymptotic behaviour of the fields appearing in (28) the centre becomes

$$U + V = 2a(n_e + \frac{i}{g^2} n_m),$$

where $n_e$ and $n_m$ are the electric and magnetic charges respectively. This last formula shows that the classical spectrum of $\mathcal{N}=2$ YM-theory indeed coincides with the BPS-states.

# 5 Quantisation

In this section we consider the quantisation of supersymmetric Yang-Mills theory. The quantum theory is encoded in the set of all Green functions that is the Schwinger functional $\mathcal{W}[J]$, or, equivalently in the 1PI effective action $\Gamma[\varphi, W_\alpha]$, which is the Legendre transform of $\mathcal{W}$. Evaluated at zero momentum the effective action reduces to the effective potential. It characterises the ground state of the quantum theory and, in particular determines the value of the different order parameters[6]. What distinguishes supersymmetric theories is that the effective superpotential is an integral over only half of superspace In other words it is a holomorphic function (or more precisely, a section) of the superfield. This strongly restricts the general form of the super potential and often allows to determine it exactly.

## 5.1 Background Field Effective Action

The background field effective action is best adapted to supersymmetric theories. In contrast to the standard definition via Legendre transform it is (formally) defined by

$$e^{\frac{i}{\hbar}\Gamma[\phi]} \equiv \int [D\Phi] e^{\frac{i}{\hbar} S[\Phi+\phi] - \frac{i}{\hbar} \int d^4x \, \frac{\delta \Gamma}{\delta \phi} \Phi + S_{gf}[\Phi,\phi]}, \tag{29}$$

where $\phi$ is a general background field and $\Phi$ is the fluctuation field and is integrated out. In a loop-wise ($\hbar$) expansion the second term in the exponent of (29) is is of one order lower in $\hbar$ and therefore the definition makes sense. The last term in (29) is

[6]Because a superfield also contains derivatives of the component fields, the super potential sometimes contains derivative terms as well.

the gauge-fixing term which is necessary when $\phi$ is a gauge field. It can be shown on general grounds [8] that the two definitions agree provided the Schwinger functional is evaluated in a particular gauge.

In a supersymmetric theory the functional integral which corresponds to a sum over all possible configurations includes, in particular, also all configurations which are related to a given configuration by supersymmetry transformations. On the other hand, because the action is invariant under such transformations the integration over supersymmetry orbit is equivalent to multiplication with the super-volume

$$V_s = \int \mathrm{d}^4 a d^2 \epsilon d^2 \bar{\epsilon} = 0$$

unless the background field absorbs the $\epsilon$-integration. Here, $a$ and $\epsilon, \bar{\epsilon}$ are the translation and supersymmetry transformations respectively. This qualitative argument suggests the following picture

$$\begin{array}{rrcl} \text{Partition function:} & Z & = & e^{\frac{i}{\hbar}\Gamma[0]} = \sum\limits_{\{c\,:\,Qc=\bar{Q}c=0\}} e^{\frac{i}{\hbar}S[c]} \\ \text{Super potential:} & e^{\frac{i}{\hbar}\Gamma[\phi]_\theta} & = & \sum\limits_{\{c\,:\,\bar{Q}c=0\}} e^{\frac{i}{\hbar}S[c+\phi]} \\ \text{D-term:} & e^{\frac{i}{\hbar}\Gamma[\phi]_{\theta,\bar{\theta}}} & = & \sum\limits_{\{c\}} e^{\frac{i}{\hbar}S[c+\phi]}, \end{array} \tag{30}$$

where $\{c\}$ denotes the set of configurations corresponding to the functional integral. This simple picture is only approximately correct. This is because the relation between functional integrals and the above sum over configurations is formal. In particular, regularisation and gauge fixing lead to additional complications. The purpose of the following discussion is to make these properties more precise.

### Super Feynman rules

The derivation of the superfield Feynman rules proceeds in close analogy to the non-supersymmetric case by expanding the Schwinger functional about the free field action. That is, for a general action in $\mathcal{N}=1$ superspace

$$S = \int d^4x d^2\theta d^2\bar{\theta}\; \mathcal{L}(V, \phi, \bar{\phi}) + \int d^4x d^2\theta\; V(\phi) + h.c.,$$

we define the Schwinger functional $\mathbf{W}[J,j]$ by

$$
\begin{aligned}
e^{\frac{i}{\hbar}W[J,j]} &\equiv \int [DVD\phi D\bar{\phi}]\, e^{\frac{i}{\hbar}S[V,\phi,\bar{\phi}]}\, e^{\frac{i}{\hbar}\int d^4x\, d^2\theta\, d^2\bar{\theta} VJ + \frac{i}{\hbar}\int d^4x\, d^2\theta \phi j + h.c} \\
&= \exp\left[ iS_{int}\left(\frac{1}{i}\frac{\delta}{\delta j}, \frac{1}{i}\frac{\delta}{\delta J}\right)\right] e^{\frac{i}{\hbar}\mathbf{W}_0[J,j]} \qquad (31)
\end{aligned}
$$

where $\mathbf{W}_0[J,j]$ is the corresponding Schwinger functional for the free theory. It is important to note that functional integrals are defined only for <u>unconstrained</u> fields[7]. Functional differentiation of both expressions in (31) with respect to the currents then yields the Feynman rules in the usual way:

1. Propagators

$$
\begin{aligned}
\langle \phi(1)\bar{\phi}(2)\rangle &= \frac{-i\hbar\delta^4(\theta_1-\theta_2)}{p^2+m^2} \\
\langle \phi(1)\phi(2)\rangle &= \frac{i\hbar m D^2\delta^4(\theta_1-\theta_2)}{4p^2(p^2+m^2)} \\
\langle V(1)V(2)\rangle &= \frac{i\hbar(\Pi_{\frac{1}{2}}+\alpha\Pi_0)\delta^4(\theta_1-\theta_2)}{p^2}
\end{aligned}
$$

   where $\Pi_{\frac{1}{2}} = \frac{D^\beta\bar{D}^2D_\beta}{8p^2}$, $\Pi_0 = -\frac{\bar{D}^2D^2+D^2\bar{D}^2}{16p^2}$ are projection operators [4] and $\alpha$ is the gauge fixing parameter.

2. Vertices such as $ig\bar{\phi}V^{ext}\phi, \quad i\lambda\phi^k \;\cdots$ are represented by

$$
ig\int d^2\theta d^2\bar{\theta} V^{ext} D^2\bar{D}^2, \quad i\lambda\int d^2\theta d^2\bar{\theta}(\bar{D}^2)^{k-1}, \cdots
$$

3. Loop integrals are given by $\int \frac{d^4k}{(2\pi)^4}$

4. There is an overall factor $(2\pi)^4\delta^4\left(\sum_{p_{ext}} p_{ext}\right)$

After these preparations we are now ready to formulate the non-renormalisation theorems in the next paragraph.

[7]In the case of the chiral field $\phi$ both, the constrained and the unconstrained integral lead to the same result. This is not so for the YM-multiplet.

## 5.2 Non-Renormalisation Theorems

In this subsection we discuss two different but related approaches to the non-renormalisation properties. The first is based on simple properties of the super Feynman rules [4]. The second uses the holomorphicity [10, 11] and the global symmetries of the classical action.

### 5.2.1 Feynman Graphs

Let us begin by formulating a theorem based on Feynman diagrams:

*Any counter term in the effective action must be a full superspace integral of local functionals of the fields corresponding to the external lines.*

The proof of this theorem follows form the structure of the various propagators given in the last paragraph and making use of the properties of the $D_\alpha{}'s$ [4].

If we restrict ourself to graphs with no external lines, then these functionals depend only on the background fields. This then implies that counter terms, holomorphic in the background fields cannot arise in perturbation theory. What about SUSY $QED$? First note that the above result does not apply at one loop in super YM-theories. This is due to the gauge fixing where ghost terms must be included. In a supersymmetric theory however the ghosts themselves have a gauge-invariance, and in this way one obtains ghosts for ghosts. This can only be avoided by introducing prepotential for the background fields which invalidates the above results. However, provided there are no massless charged fields in the theory, one can show [9, 13] that the situation is unchanged beyond the the one-loop corrections. Let me illustrate the above remarks for the two-point function on $\mathcal{N}=1$ YM-theory. Using the Feynman rules listed above one finds that the vacuum polarisation contributes to the effective action a term (see fig. 4)

$$\begin{aligned}
&\frac{g^2}{2}\int\frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}d^2\theta_1 d^2\bar\theta_1 d^2\theta_2 d^2\bar\theta_2 V(-p,\theta_1)\frac{\bar D_1^2 D_1^2}{16}\frac{\delta_{12}}{(p^2+k^2)}\frac{D_1^2\bar D_1^2}{16}\frac{\delta_{12}}{k^2}V(p,\theta_2)\\
&\propto\left(\frac{1}{\epsilon}\int\frac{d^4p}{(2\pi)^4}d^2\theta d^2\bar\theta V(-p,\theta)D^\alpha\bar D^2 D_\alpha V(p,\theta)+O(\epsilon)\right),
\end{aligned}\qquad(32)$$

where $\delta_{12}=\delta^4(\theta_1-\theta_2)$ and we have used dimensional regularisation to regulate the

Figure 4: Feynman diagram for vacuum polarisation

divergent momentum integral. Now, using (20) the last integral can be written as

$$\int \frac{d^4p}{(2\pi)^4} d^2\theta \, W^\alpha(-p,\theta) W_\alpha(p,\theta),$$

and therefore this is indeed a counter term for the super potential leading to the 1-loop running of the gauge coupling. Similarly, contributions of the form

$$\int \frac{d^4p}{(2\pi)^4} d^2\theta d^2\bar{\theta} \, W^\alpha(-p,\theta) \frac{D^\beta D_\beta}{p^2} W_\alpha(p,\theta)$$

arise at higher orders. They contribute to the effective action unless an infrared cut-off $\mu$ is introduced in the momentum integral. This leads to the so-called Wilsonian effective action [9]. The problem of construction of such a cut-off without destroying gauge-invariance is to date unsolved.

### 5.2.2 Holomorphicity [10, 11]

The starting point is the assumption that superpotential is a holomorphic section of the superfield (and the coupling constants), excluding essential singularities. If the classical theory has furthermore continuous global symmetries and if the fate of these upon quantisation is under control, one can make use of these symmetries to derive *selection rules*. Let us illustrate this approach with a few examples.

1.) *Wess-Zumino Model:*
The Wess-Zumino model is defined by the classical super-field Lagrangian

$$S = \frac{1}{16} \int d^4x \mathrm{d}^2\theta d^2\bar{\theta} \; \bar{\phi}\phi - \frac{1}{4} \int d^4x \mathrm{d}^2\theta \; \left( m\phi^2 + g\phi^3 \right).$$

This Lagrangian is invariant under $U(1) \times U(1)_R$-transformations, provided the 'fields' $\phi, m, g$ are given the charges

$$\begin{array}{rclclcl} U(1) & : & Q_\phi = 1 & , & Q_m = -2 & , & Q_g = -3 \\ U(1)_R & : & Q_\phi = -\frac{2}{3} & , & Q_m = -\frac{2}{3} & , & Q_g = 0. \end{array}$$

Provided there are no holomorphic anomalies the effective super potential must then be of the form

$$\Gamma[\phi]_\theta = \int d^4x \mathrm{d}^2\theta \; m\phi^2 \; f\left(\frac{g\phi}{m}\right),$$

where $f(z)$ is some holomorphic section. For small $g$ perturbation theory is valid and therefore $\Gamma[\phi]$ has can be expanded as

$$\Gamma[\phi] = \sum_{n=0}^{\infty} c_n \frac{g^n \phi^{n+2}}{m^{n-1}}, \tag{33}$$

corresponding to a tree-diagram with $n+2$ external legs, $n$-vertices and $n-1$ propagators. For $n > 1$ such a diagram would however not be 1-particle irreducible and does therefore not contribute to the effective action. Loop corrections are not compatible with the structure of (33) and must therefore be absent. We therefore conclude that in the Wess-Zumino model the classical potential is not modified by quantum corrections.

2.) *SUSY QED*:
Using (21) we find that the minimal gauge invariant combination of chiral fields and YM-fields is given by

$$S = \frac{1}{16\pi}\mathrm{Im}\int \; d^4x \; d^2\theta \tau W^\alpha W_\alpha + \frac{1}{16}\int \; d^4x \; d^2\theta \; d^2\bar{\theta}\left(\phi^\dagger e^V \phi + \tilde{\phi}^\dagger e^{-V}\tilde{\phi}\right),$$

where $\tau = \frac{i4\pi}{g^2} + \frac{\theta}{2\pi}$ and we have used that

$$\mathrm{Im}\int \; d^4x \; d^2\theta \; W^\alpha W_\alpha = 4\int \; d^4x \; F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

The matter fields $\phi$ and $\tilde{\phi}$ have electric charge 1 and $-1$ respectively. The effective coupling $\tau_{eff}$ can only depend $\phi$ and $\tilde{\phi}$. The classical theory is invariant under $U(1)_R$-transformations. The $R$-charges of $W_\alpha$, $\phi$ and $\tilde{\phi}$ are 1, 2 and 2 respectively. Due to the axial anomaly the continuous $U(1)_R$ is broken down to the discrete subgroup $\mathbf{Z}_2$. In order to reproduce this anomaly $\tau_{eff}$ must be a linear combination of $\log(\phi)$ $\log(\tilde{\phi})$ and $\log(\phi + \tilde{\phi})$. Higher log's are incompatible with the Adler-Bardeen theorem and therefore absent. This in turn implies that the $\beta$-function for the gauge coupling gets no contribution above 1-loop. On the other hand an explicit computation shows that the $\beta$-function gets contributions at all orders. This discrepancy is due to the presence of massless charged fields which typically lead to holomorphic anomalies [9]. This can be cured by introducing an explicit infrared cut-off. The corresponding effective action is the *Wilsonian effective action* and is in general different from the 1PI-effective action. This shows the limitations of the holomorphic approach.

3.) *SUSY QCD*:
Let us now consider a supersymmetric version of $QCD$ with chiral matter in the adjoint representation

$$\begin{aligned} S &= S_{YM} + \frac{1}{16}\mathrm{tr}\int d^8z\, \left(\bar{\phi}e^{gV}\phi\right) \\ &\quad -\frac{1}{4}\int d^6z\, \mathcal{W}(\phi) + \text{c.c.} \end{aligned} \tag{34}$$

with a general superpotential:

$$\mathcal{W}(\phi) = \sum_{k=0}^{n} \frac{g_k}{k+1}\mathrm{tr}\,\phi^{k+1}$$

If the non-abelian gauge symmetry is unbroken by the superpotential we expect that the non-abelian theory is confining and that the low energy degrees of freedom are described in terms of gauge singlets. Such a singlet is the *glueball superfield* $S = \frac{1}{32\pi^2}\mathrm{tr}\,(W_\alpha W^\alpha)$. In what follows we will try to constrain the general form of the effective potential for the glueball field, $\mathcal{W}(S)$ upon integrating out the massive chiral multiplet $\phi$. Following [11] we will again use a combination of dimensional analysis and invariance properties under $U(1)\times U(1)_R$. The dimensions $\Delta$ and charges $Q_\phi$, $Q_R$ of the different multiplets are given by

| | $\Delta$ | $Q_\phi$ | $Q_R$ |
|---|---|---|---|
| $\phi$ | $1$ | $1$ | $-2/3$ |
| $W_\alpha$ | $3/2$ | $0$ | $-1$ |
| $g_l$ | $2-l$ | $-(l+1)$ | $\frac{2}{3}(l-2)$ |

Assuming that these symmetries are conserved we then conclude that $\mathcal{W}(g_k, S)$ depends only on the combination $g_k/g_1^{(k+1)/2}$. Thus,

$$\sum_k (k+1)g_k\frac{\partial}{\partial g_k}\mathcal{W} = 0 \tag{35}$$

Figure 5: A planar diagram with vertices of degree 3. The double line notation indicates that the fields are in the adjoint representation with each line representing one index of $\phi_{ij}$

Using dimensional analysis we then find

$$\mathcal{W} = W_\alpha^2 f\left(\frac{g_k \mathcal{W}_\alpha^{k-1}}{g_1^{(k+1)/2}}\right) ,$$

so that ($k \geq 1$)

$$\left(\sum_k (2-k) g_k \frac{\partial}{\partial g_k} + \frac{3}{2} W_\alpha \frac{\partial}{\partial W_\alpha}\right) \mathcal{W} = 3\mathcal{W} \tag{36}$$

Finally, combining (35) and (36) we end up with

$$\left(\sum_k (1-k) g_k \frac{\partial}{\partial g_k} + W_\alpha \frac{\partial}{\partial W_\alpha}\right) \mathcal{W} = 2\mathcal{W} \tag{37}$$

To continue we count the number of loops $L$ of diagrams with vertices $i$ of degree $k_i + 1$ (see fig 5). For a planar diagram $P_L$ we have

$$L = 2 + \frac{1}{2} \sum_i (k_i - 1)$$

so that we can make the substitution

$$\sum_k (1-k) g_k \partial / \partial g_k \to (4 - 2L)$$

Combining this with (37) we then find that the power of $S$ coming from a planar diagram $P_L$ equals $L - 1$. Such terms arise by the same mechanism form as in (32) from covariant derivatives with a trace for each closed index loop in fig. 5.

For non-planar diagrams (e.g. fig. 6), on the other hand, we have

$$L = 2 - 2g + \frac{1}{2} \sum_i (k_i - 1) \tag{38}$$

which requires at least one double trace within a single index loop in fig. 6. Therefore non-planar diagrams cannot contributes to $\mathcal{W}(S)$. It turns out that this result is unchanged when gauge-loops are included. Furthermore this result holds as well for (spontaneously) broken gauge symmetry.

Figure 6: A non-planar genus $g = 1$ diagram

# 6 Currents in Supersymmetric Theories

In this section we first describe how the different Noether currents are assembled in super multiplets. We then discuss the different anomalies and the corresponding Ward-Identities for the effective action. This serves as a preparation for the next and final section where we will 'integrate' these Ward identities to derive low-energy effective actions.

## 6.1 Noether Currents

The conserved Noether currents for a Poincaré invariant theory are given by

$$T_{\mu\nu} \quad \text{and} \quad L_{\mu\nu\rho} = -x_\nu T_{\mu\rho} + x_\mu T_{\nu\rho}$$

respectively. Apart from general relativity all fundamental theories are in fact invariant under a bigger group of transformations at least if the bare masses are taken to be zero. This is the group of conformal transformations which contains in addition to the Poincaré transformations the dilatation $D$ and the special conformal transformations $K_\nu$. The corresponding conserved currents are

$$x^\nu T_{\mu\nu} \quad \text{and} \quad 2x_\nu x^\lambda T_{\lambda\mu} - x^2 T_{\mu\nu}.$$

These currents are conserved provided the stress tensor is traceless i.e. $T^\mu_\mu = 0$.

In a scale-invariant, supersymmetric theory the set of conserved currents is supplemented by the supersymmetry current $j_{\mu\alpha}$ and the superconformal current $(\not{x} j_\mu)_\alpha$. The corresponding conservation equations are

$$\partial_\mu j^\mu_\alpha = \gamma_\mu j^\mu_\alpha = 0.$$

In addition, if the theory is $U(1)_R$-invariant then the $R$-current $j^{(5)}_\mu$ is also conserved. Note that only $T_{\mu\nu}, j_{\mu\alpha}$ and $j^{(5)}_\mu$ are fundamental objects the remaining currents are moments of the first set. The idea is then that in a supersymmetric theory the different currents form a super multiplet. Indeed the counting of degrees of freedom, taking into account the constraints we have

$$T_{\mu\nu} \quad : \quad 5, \qquad j_{\mu\alpha} \quad : \quad 8, \qquad j^{(5)}_{\mu} \quad : \quad 3.$$

In addition the dimensions are such that we can collect them into a super current $j_{\alpha\dot{\alpha}}$ which has $j^{(5)}_{\mu}$ as its lowest component.

## 6.2 Supercurrent

In a non-supersymmetric theory the conformal transformations are generated by

$$Q_h = \int d^3x J_0 \,,$$

where $J_\mu = h^\nu T_{\mu\nu}$ and $h^\nu(x)$ satisfies the Killing equation

$$h_{(\mu,\nu)} = \frac{d}{2}\eta_{\mu\nu}(h^\lambda{}_{,\lambda}).$$

In order to combine these transformations with supersymmetry and superconformal transformations we promote $h^\mu$ into a parameter superfield $h^{\alpha\dot{\alpha}}$. However, in order to get an irreducible multiplet of symmetry transformations the superfield $h^{\alpha\dot{\alpha}}$ must be be constraint. To see which constraints are the correct ones let us first consider a bigger group of transformations, that is the *superdiffeomorphisms*. For our purpose it will be sufficient to consider those superdiffeomorphisms which preserve chirality in superspace since this already includes the localised superpoincaré transformations necessary to obatin the Noether currents. Such chirality preserving superdiffeomorphisms are then parametrised by $h^{\alpha\dot{\alpha}}$ (and $\bar{h}^{\alpha\dot{\alpha}}$) subject to the constraints

$$\bar{D}^{(\dot{\beta}}h^{\alpha\dot{\alpha})} = 0 \;, \qquad\qquad D^{(\beta}\bar{h}^{\alpha)\dot{\alpha}} = 0 \;. \tag{39}$$

To keep our formulas simple we consider $\mathcal{N}=1$ superspace at present. We will give the necessary generalisations later in the next section[8]. The corresponding transformations of the chiral coordinates are given by

$$\begin{aligned} \delta x^\mu_+ &= h^\mu(z) + 2i\lambda(z_+)\sigma^\mu\bar{\theta} \quad , \quad \delta\theta^\alpha = \lambda^\alpha(z_+) \\ \delta x^\mu_- &= \bar{h}^\mu(z) - 2i\theta\sigma^\mu\bar{\lambda}(z_-) \quad , \quad \delta\bar{\theta}^{\dot{\alpha}} = \bar{\lambda}^{\dot{\alpha}}(z_-) \end{aligned}$$

[8]see [12] for more details.

where $x_\pm^{\alpha\dot\alpha} \equiv x^{\alpha\dot\alpha} \pm 2i\theta^\alpha\bar\theta^{\dot\alpha}$ and $\theta^\alpha$ respectively $\bar\theta^{\dot\alpha}$ form the coordinates $z_+$ and $z_-$ of chiral- and antichiral superspace respectively. Furthermore,

$$\lambda^\alpha(z_+) = -\frac{i}{8}\bar D_{\dot\alpha} h^{\alpha\dot\alpha} \ , \qquad \bar\lambda^{\dot\alpha}(z_-) = \frac{i}{8} D_\alpha \bar h^{\alpha\dot\alpha} \ .$$

We are now ready to impose the restriction to superconformal transformations. These are obtained by imposing

$$h^{\alpha\dot\alpha} = \bar h^{\alpha\dot\alpha} \tag{40}$$

To see that this is the correct constraint we can look at the components of the parameter superfield which are then just the usual superconformal transformations. Concretely we have

$$\begin{aligned} h^{\alpha\dot\alpha} = \ & a^{\alpha\dot\alpha} + 4i\varepsilon^\alpha\bar\theta^{\dot\alpha} + 4i\bar\varepsilon^{\dot\alpha}\theta^\alpha - \omega^\alpha{}_\beta x_-^{\beta\dot\alpha} + \bar\omega^{\dot\alpha}{}_{\dot\beta} x_+^{\alpha\dot\beta} - 4\eta\theta^\alpha\bar\theta^{\dot\alpha} - 6i\eta\theta^\alpha\bar\theta^{\dot\alpha} \\ & + \kappa x^{\alpha\dot\alpha} + x_-^{\beta\dot\alpha} b_{\beta\dot\beta} x_+^{\alpha\dot\beta} - x_-^{\beta\dot\alpha}\rho_\beta\theta^\alpha + \bar\theta^{\dot\alpha}\bar\rho_{\dot\beta} x_+^{\alpha\dot\beta} \ . \end{aligned}$$

The different parameters correspond to translations $a^{\alpha\dot\alpha}$, supersymmetry transformations $\varepsilon^\alpha$, Lorentz transformations $\omega^\alpha{}_\beta$ (with $\omega_{\alpha\beta} = \omega_{\beta\alpha}$ and $\omega^\alpha{}_\alpha = 0$), $U(1)_R$-transformations $\eta$, dilation $\kappa$, special conformal transformations $b_{\alpha\dot\alpha}$ and special superconformal transformations $\rho_\alpha$. Finally, super-Poincaré transformations are those for which the chiral superfield

$$\sigma = \frac{1}{6}\left(D^\alpha\lambda_\alpha - \frac{1}{2}\partial_{\alpha\dot\alpha} h^{\alpha\dot\alpha}\right)$$

vanishes.

In order to obtain the multiplet of conserved currents by the Noether procedure we also need the representations of these transformations on superfields. The representation on chiral scalar superfields, $\phi$ is given by

$$\mathcal{L}\phi = \left(h^\mu\partial_\mu + \lambda^\alpha D_\alpha^- 2q\sigma\right)\phi \tag{41}$$

where

$$\sigma = \frac{1}{6}\left(D^\alpha\lambda_\alpha - \frac{1}{2}\partial_{\alpha\dot\alpha} h^{\alpha\dot\alpha}\right) \tag{42}$$

The conformal dimension $\Delta$ and the $R$-weight $Q_R$ are the given the real- and imaginary parts of $q$ respectively, i.e.

$$\Delta = (q + \bar q) \ , \qquad Q_R = -\frac{2\mathcal{N}}{3}(q - \bar q) \ .$$

Note, that $[\bar{D}_{\dot{\alpha}}, \mathcal{L}] = 0$ as a consequence of (39).

For a supersymmetric theory with action $S[\phi]$, we then have $\delta S[\phi] = 0$ for any global super-Poincaré transformation. As in ordinary Noether procedure, we then consider the variation of the action under a local transformation, that is by letting the parameters have an arbitrary $x$-dependence. We will implement this by removing the reality condition (40) but still maintaining the chirality preserving constraint (39). Of course, there is always the ambiguity of adding terms to $\delta\phi$ which are proportional to derivatives of the parameters of super-Poincaré transformations. By construction, the terms in $\delta S[\phi]$ induced by them are of the form "derivatives of the parameters" times "equations of motions" and thus induce in the currents terms which vanish on-shell. We will make use of this freedom when dealing with constrained superfields. Using (41), the variation of the action can then be written as

$$\delta S[\phi] = \frac{i}{16} \int d^8 z \, \left( h^{\alpha\dot{\alpha}} - \bar{h}^{\alpha\dot{\alpha}} \right) T_{\alpha\dot{\alpha}} - \frac{1}{2} \int d^6 z_+ \, \sigma J - \frac{1}{2} \int d^6 z_- \, \bar{\sigma} \bar{J} \, , \tag{43}$$

with $T_{\alpha\dot{\alpha}}$ real and $J$ chiral. Note that if the theory has superconformal invariance then $\delta S$ vanishes off-shell for $h^{\alpha\dot{\alpha}} = \bar{h}^{\alpha\dot{\alpha}}$. Thus $J$ and $\bar{J}$ vanish identically for scale-invariant theories. Consequently the multiplet $T_{\alpha\dot{\alpha}}$ contains the complete set of Noether currents,

$$T_{\alpha\dot{\alpha}} \ni \{T_{\mu\nu}, j_{\mu\alpha}, j_\mu^{(5)}\} \; ,$$

where $T_{\mu\nu}$ and $j_{\mu\alpha}$ are traceless[9] . This *irreducible* multiplet of Noether currents is the *improved* multiplet.

In order to obtain the conservation equations we solve the constraint on $h^{\alpha\dot{\alpha}}$ in terms of an *unconstrained* superfield. This is achieved by

$$h^{\alpha\dot{\alpha}} = 2\bar{D}^{\dot{\alpha}} L^\alpha \; , \qquad\qquad \bar{h}^{\alpha\dot{\alpha}} = -2 D^\alpha \bar{L}^{\dot{\alpha}} \; .$$

where $L^\alpha$ is an unconstrained spinor superfield. Substitution into (43) and imposing the variation of the action to vanish, on shell, under an arbitrary local transformation parametrised by $L^\alpha$ then leads to the conservation equation

$$\bar{D}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} - \frac{1}{6} D_\alpha J = 0$$

where we have used (42). This equation contains at the same time the conservation equations of $T_{\mu\nu}$, $j_{\mu\alpha}$ and $j_\mu^{(5)}$, as well as the zero-trace condition of $T_{\mu\nu}$ and $j_{\mu\alpha}$. The superfield Noether procedure is thus an economical way to deal with the variety of

[9] By the trace of $j_{\mu\alpha}$ we mean $\gamma^\mu j_\mu$.

Noether currents. Note again that $J$ vanishes for a scale-invariant theory. Thus $J$ contains the trace of $T_{\mu\nu}$ , $j_{\mu\alpha}$ and the divergence of $j^{(5)}_\mu$. In particular, we have

$$\begin{aligned} T^\mu{}_\mu &= -2(D^2 J + \bar{D}^2\bar{J})|_{\theta=0} \\ j^{(5)\mu}{}_{,\mu} &= \frac{i}{12}(D^2 J - \bar{D}^2\bar{J})|_{\theta=0} \end{aligned}$$

The multiplet $J$ is thus called the *multiplet of anomalies*. It will play an important role in the next section.

As an illustration we give the supercurrents for two simple models. First we compute the supercurrent for the $\mathcal{N}=1$ sigma model defined in terms of arbitrary real Kähler potential $K(\phi,\bar{\phi})$ and superpotential $\mathcal{W}(\phi)$, where $\phi$ is a chiral scalar field. Such Lagrangians arise as the local part of quantum effective actions for supersymmetric field theories and string theory. The general action is given by

$$S = \frac{1}{16}\int d^8z\ K(\phi,\bar{\phi}) - \frac{1}{4}\int d^6z_+\ \mathcal{W}(\phi) - \frac{1}{4}\int d^6z_-\ \bar{\mathcal{W}}(\bar{\phi})\ ,$$

The variation of this action under an infinitesimal transformation (41) can then be written in the form (43) with

$$\begin{aligned} T_{\alpha\dot{\alpha}} &= \tfrac{1}{12}D_\alpha\phi\bar{D}_{\dot{\alpha}}\bar{\phi}K_{\phi\bar{\phi}} - \tfrac{i}{6}\partial_{\alpha\dot{\alpha}}\phi K_\phi + \tfrac{i}{6}\partial_{\alpha\dot{\alpha}}\bar{\phi}K_{\bar{\phi}}\ , \\ J &= -\tfrac{1}{4}\bar{D}^2(K - q\phi K_\phi) + 3\mathcal{W} - q\phi\mathcal{W}_\phi\ . \end{aligned} \tag{44}$$

The next example we consider is that of a single chiral multiplet $\Phi$ coupled to an abelian gauge multiplet $W_\alpha$. Applying the procedure outlined above we find (exercise)

$$\begin{aligned} T_{\alpha\dot{\alpha}} &= -\tfrac{1}{8}W_\alpha\bar{W}_{\dot{\alpha}} + \tfrac{1}{12}\nabla_\alpha\Phi\bar{\nabla}_{\dot{\alpha}}\bar{\Phi} - \tfrac{i}{6}\bar{\Phi}\overleftrightarrow{\nabla}_{\alpha\dot{\alpha}}\Phi\ , \\ J &= \tfrac{1}{4}(q-1)\bar{\nabla}^2\left(\bar{\Phi}\Phi\right)\ , \end{aligned}$$

where

$$\nabla_\alpha = D_\alpha - gD_\alpha V \quad ; \quad \nabla_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} - \frac{i}{2}g\bar{D}_{\dot{\alpha}}D_\alpha V$$

are gauge- and SUSY- covariant derivatives.

The alert reader will have noticed that in the discussion so far we have included the $U(1)_R$-symmetry but not the $U(1)$-symmetry introduced in the section on non-renormalisation theorems. The reason for this is that $U(1)$-transformation,

$$U_\phi(1) : \begin{pmatrix} \phi & \mapsto & e^{i\alpha}\phi \\ W_\alpha & \mapsto & W_\alpha \\ Q_\alpha & \mapsto & Q_\alpha \end{pmatrix}$$

commutes with all generators of the $\mathcal{N}=1$ supersymmetry algebra. The Nother currents associated to this symmetry therefore forms its own super multiplet which contains no other physical components. This multiplet is constructed in the same way as above by localising the $U(1)$-transformation as

$$\delta\phi = \sigma\phi, \qquad \bar{D}_{\dot{\alpha}}\sigma = 0$$

The variation of the action under the local transformation is then

$$\begin{aligned} \delta_\sigma S\left[\phi, W_\alpha\right] &= \bar{D}^2 J_\phi - \frac{1}{4}\phi\frac{\partial\mathcal{W}}{\partial\phi} \\ J_\phi &= \frac{1}{16}\bar{\phi}\, e^{gV}\phi \end{aligned}$$

In particular, the divergence of the $U_\phi(1)$-current resides in the superfield equation, $(D^2 - \bar{D}^2)J_\phi$, and is thus conserved if and only if $\phi\mathcal{W}'(\phi) = 0$, up to anomalies which we will discuss now.

## 6.3 Anomalies

In supersymmetric field theory quantum corrections are to a large extent due to anomalies. These, in turn, can often be computed exactly allowing to extract non-perturbative information about the quantum theory. In this paragraph we consider in particular

- scale anomaly: $T^\mu_{\ \mu} \neq 0$, $(\not{\partial} j_\mu)_\alpha \neq 0$
- $U_R(1)$-anomaly: $\partial^\mu j^{(5)}_\mu \neq 0$
- $U_\phi(1)$-anomaly: $\bar{D}^2 J_\phi \neq 0$

since they will play an important role for the calculation of low energy effective action in the next section. It is clear that the supermultiplet structure of Noether currents imposes constraints on anomalies. In particular, if the anomalies form a irreducible multiplet,it is enough to determine one anomaly explicitly. One can then deduce the remaining anomalies using the multiplet structure.

We have already encountered the scale anomaly originating the introduction of a renormalisation scale in the one loop photon propagator in SUSY QED in the previous section. Similarly, at one loop the quadratic term in the effective action $\Gamma^{(1)}$ for the photon in SUSY-QED receives a contribution of the form

$$-\frac{1}{2}A(p)\int d^2\theta\; W^\alpha(-p,\theta)W_\alpha(p,\theta)$$

where $A(p)$ diverges logarithmically with the ultraviolate cut-off $\Lambda$. Minimal subtraction the introduces a scale dependence in the effective action of the form

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(1)} = -\frac{1}{2}T^{\alpha}{}_{\alpha} = -\frac{1}{16\pi}\int d^6z\, W^\alpha W_\alpha \,+\, \text{h.c.}$$

If we now recall the identity

$$-\frac{1}{2}T^{\alpha}{}_{\alpha} \;=\; D^2 J + \bar{D}^2\bar{J}$$

we find that the anomaly multiplet $J$ is given by

$$J = -\frac{1}{32\pi}\, W^\alpha W_\alpha$$

The generalisation of this result to supersymmetric QCD with $N_c$ colour, $N_f$ flavours in the fundamental representation and $N_c$ chiral multiplets in the adjoint representation of the gauge group is the easily found to be

$$J = (3N_c - N_f - N_a N_c)\, S \quad , \qquad S = \frac{1}{32\pi}\text{tr}(W^\alpha W_\alpha)\ .$$

Let us now turn to the $U_R(1)$-anomaly. For this we recall the identity

$$\bar{D}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} = \frac{1}{6} D_\alpha J$$

which implies in particular,

$$\begin{aligned}\partial^\mu j^{(5)}_\mu &= \frac{i}{12}\left(D^2 J - \bar{D}^2\bar{J}\right)|_{\theta=0}\\ &\propto \epsilon^{\mu\nu\lambda\delta}F_{\mu\nu}F_{\lambda\delta}\end{aligned}$$

Note that the Adler-Bardeen Theorem implies that the $U_R(1)$-anomaly is 1-loop exact. On the other hand we saw that the scale anomaly receives contribution at higher order as well. How is that compatible with the multiplet structure? The solution of this puzzle lies in the fact that at 1-loop

$$\begin{aligned} j^{(5)}_\mu &\in T_{\alpha\dot{\alpha}}\\ T_{\mu\nu} &\in T_{\alpha\dot{\alpha}} \cup J\end{aligned}$$

therefore $j^{(5)}_\mu$ and $T_{\mu\nu}$ do no longer form an irreducible multiplet at one loop. This is not so in $\mathcal{N}=2$ SUSY as we will see later.

Figure 7: A planar diagram with vertices of degree 3

Finally we consider the $U(1)$, or Konishi-anomaly. We start with the action for SUSY QCD with adjoint matter (34). $U(1)$-symmetry at the classical level then requires vanishing super potential, $\mathcal{W} \equiv 0$. Dimensional analysis then implies

$$\bar{D}^2 J_\phi = U$$

where $U$ is a chiral superfield of dimension 3. Thus, $U \propto \mathrm{tr} W^\alpha W_\alpha$. To fix the coefficient we use Pauli-Villars regularisation, that is we introduce a second massive multiplet with mass $m$. For the difference $(J_\phi - J_{\phi_m})$ UV-divergences are absent so that we can use the equations of motion leading to

$$\lim_{m\to\infty} \bar{D}^2 \left(J_\phi - J_{\phi_m}\right) = -\lim_{m\to\infty} \frac{m}{4} \mathrm{Tr}\langle\phi_m\phi_m\rangle$$

The one-loop contribution to the right hand side of this equation comes from the diagram in fig. 7. Concretely we have

$$\mathrm{Tr}\langle\phi_m\phi_m\rangle \;\; = \;\; -8imN_cN_a \int \frac{d^4p}{(2\pi)^4} \{ \frac{\mathrm{Tr} W^\alpha(q) W_\alpha(-q)}{[p^2+m^2][p^2+m^2][((p+q)^2+m^2]} \}$$

Taking the large mass limit, this expression simplifies and we end up with

$$\lim_{m\to\infty} \frac{m}{4} \mathrm{Tr}\langle\phi_m\phi_m\rangle \;\; = \;\; \frac{1}{16\pi^2} \mathrm{Tr} W^\alpha W_\alpha$$

Thus,

$$U \;\; = \;\; -2N_cN_a\, S$$

We note in passing that, unlike the $R$-anomaly, there are higher loop corrections to the Konishi anomaly but these corrections cannot be expressed in terms of chiral superfields.

# 7 Effective Actions

In this section we will combine the results established so far to obtain low energy effective actions for supersymmetric field theories. Essentially we do this by running the superfield Noether procedure backwards. That is, rather than obtaining the anomaly multiplet by varying the action we obtain an effective action by demanding that the anomalies computed in the microscopic theory are reproduced by varying the effective action. This procedure is called *integrating anomalies.*

## 7.1 $\mathcal{N}=1$ Yang-Mills theory

To begin with we recall that the anomaly multiplet for $\mathcal{N}=1$ Yang-Mills theory with gauge group $SU(N_c)$ is given by

$$J = 3N_c S$$

We now want to integrate this equation to obtain an effective potential for the glueball superfield $S$. For this we note that $S$ has $R$-charge $Q_R = -3$. Substitution into the formula (44) with $K \equiv 0$ then gives

$$J = 3\mathcal{W} - 3S\mathcal{W}_S$$

Combining these two results we get

$$\begin{aligned} \int d^6z\, \mathcal{W}(S) &= \int d^6z \left( \log(\frac{\Lambda^3}{S^{N_C}}) + \delta \right) \\ &\equiv \Gamma_{VY}[S] \end{aligned}$$

This is the *Veneziano-Yankielowicz effective potential.* Several comments are in order.

1. The generalisation of this result to include adjoint matter is simply $3N_C \to (3N_c - N_a N_c)$.

2. The integration constant $\delta$ is undetermined. It can always be set to 1 by a redefinition of the dynamical scale $\Lambda$.

3. $\mathcal{W}(S)$ is exact up to $R$-invariant terms, that is

   $$\mathcal{W}(S) = \mathcal{W}_{VY}(S) + \mathcal{W}_2(S, \{g_k\})$$

   where $\mathcal{W}_2(S\{g_k\})$ receives contributions only from planar graphs as a consequence of the non-renormalisation theorem proved in the last section.

4. $\mathcal{W}(S)$ contains no kinetic terms and therefore does not describe the dynamics of the glueball superfield. It solely determines the vacuum expectation values in terms of the dynamical scale $\Lambda$, i.e.

   $$\langle S \rangle|_{\theta=0} = \Lambda^{3/N_c} \quad ; \quad \delta = 1$$

5. The present analysis does not prove that $S$ is the relevant low energy field in SUSY Yang-Mills theory. It simply determines the effective potential for this degree of freedom.

### 7.2 $\mathcal{N}=2$ Yang-Mills Theory

In this section we compute the low energy effective action for $\mathcal{N}=2$ YM theory with gauge group $SU(2)$ by integrating the superconformal anomaly. To begin with let us recall some aspects of the classical theory. The classical action in $\mathcal{N}=2$ superspace is given by ($\tau=\frac{i4\pi}{g^2}+\frac{\theta}{2\pi}$)

$$S = \frac{1}{8\pi}\mathrm{ImTr}\int d^4x d^2\theta_1 d^2\theta_2\, \tau\mathcal{A}^2$$

In component fields this action includes a potential for the scalar field

$$V(\varphi,\bar{\varphi}) = \frac{1}{g^2}\mathrm{Tr}[\varphi,\varphi^\dagger]^2$$

which has flat directions for $\varphi \in \mathrm{Cartan}\,(su(N))$. For $\varphi \neq 0$ the $SU(N)$ gauge symmetry is spontaneously broken down to $U(1)^{N-1}$. This is the sector of the theory we will consider and which is called the *Coulomb branch.* In what follows we will assume that the gauge group is $SU(2)$. The classical moduli space of the theory is then parametrised by $\{g^2,|a|\}$, where

$$a = \mathrm{Tr}\,(\langle\varphi\rangle\sigma_3)$$

and is thus a subset of $\mathbb{R}^2$. In particular R-symmetry implies that the phase of $a$ is irrelevant. Charged fields are $\sigma^\pm$ valued with mass $M = |a|g$. Let us now consider the effective action. When expressed in terms of the massless $\mathcal{N}=2$-superfield[10] $\mathcal{A} = \varphi+\theta\psi+\cdots$, the most general local $N=2$ supersymmetric low energy superpotential must be of the form

$$\Gamma[\mathcal{A}] = \frac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta_1 d^2\theta_2 \mathcal{F}(\mathcal{A}), \tag{45}$$

where the prepotential $\mathcal{F}$, to be determined, is the result of integrating out the massive (i.e. charged, root-valued) fields. Using the formalism developed in section 4, it is straight forward to show that in $\mathcal{N}=1$ notation (45) becomes

[10] As we neglect derivative terms, it is not meaningful to retain the massive (root-valued fields) in the super potential.

$$\Gamma[A,W_\alpha]=\frac{1}{8\pi}\mathrm{Im\,Tr}\int \mathrm{d}^4x\Big\{\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}(\phi_D\bar{\phi}-\bar{\phi}_D\phi)+\int \mathrm{d}^2\theta\tau(a)\, W^\alpha W_\alpha\Big\}, \tag{46}$$

where $\phi$ and $W_\alpha$ are the (covariantly) chiral- and vector $\mathcal{N}=1$ superfields respectively. Furthermore

$$\phi_D=\mathcal{F}'(\phi) \quad \text{and} \quad \tau(a)=\mathcal{F}''(a)=a'_D(a). \tag{47}$$

The spectrum of the quantum theory is contained in the set of BPS-states of the supersymmetry algebra. The quantum corrections to the centre is determined by the Poisson brackets of the effective supercharges i.e.

$$\{Q^{\mathrm{eff}},\bar{Q}^{\mathrm{eff}}\}=i(2\not{P}^{\,\mathrm{eff}}-Z^{\mathrm{eff}}),$$

where $Q^{\mathrm{eff}}_\alpha$ are the supercharges obtained from the low energy effective Lagrangian $\mathcal{F}(\sqrt{\mathcal{A}^a\mathcal{A}^a})$ and $Z=U+V\gamma_5$. Apart from the fermionic contributions, the effective Lagrangian (46) has the same structure in the classical theory with

$$\begin{array}{ccc}\dfrac{4\pi}{g^2} & \longrightarrow & \mathcal{I}_{ab}\equiv \mathrm{Im}\dfrac{\partial^2\mathcal{F}}{\partial\mathcal{A}^a\partial\mathcal{A}^b}\\ \dfrac{\theta}{2\pi} & \longrightarrow & \mathcal{R}_{ab}\equiv \mathrm{Re}\dfrac{\partial^2\mathcal{F}}{\partial\mathcal{A}^a\partial\mathcal{A}^b}.\end{array}$$

Accordingly one has

$$Q=\frac{\sqrt{2}}{8\pi}\int \mathrm{d}^3x\; \mathcal{S}^a\mathcal{I}_{ab}\psi^b,$$

To continue we write $\mathcal{S}=X+iY$ with

$$\begin{aligned} X^a &= -i\gamma\cdot(4\pi\mathcal{I}^{ab}\tilde{\mathbf{\Pi}}_b+i\gamma_5\mathbf{B}^a)-i\gamma^0\gamma_5[A,B]^a+O(\psi^2),\\ Y^a &= 4\pi\mathcal{I}^{ab}(\tilde{\pi}_A-i\gamma_5\tilde{\pi}_B)_b+\gamma\cdot\mathbf{D}(A+i\gamma_5B)^a\gamma^0+O(\psi^2),\end{aligned}$$

with $\mathcal{I}^{ab}=\mathcal{I}^{-1}_{ab}$, $\tilde{\mathbf{\Pi}}^{\mathbf{a}}=\frac{\mathbf{1}}{\mathbf{4\pi}}\mathcal{I}_{\mathbf{ab}}\mathbf{E}^{\mathbf{b}}$ and $\tilde{\pi}_{Aa}=\frac{1}{4\pi}\mathcal{I}_{ab}(D_0\mathbf{A})^b$. Repeating the steps outlined for the classical case one ends up with the simple result

$$U+V=2\int \mathrm{d}\Omega\left(\varphi^a\mathbf{\Pi}_a+\frac{1}{4\pi}\varphi_{Da}\mathbf{B}^a\right)=2\left(an_e+a_Dn_m\right),$$

where $a_D=\mathcal{F}'(a)$. Hence, independent, of the actual form of the effective potential$\mathcal{F}$, the spectrum of the quantum theory is entirely determined by the quantities $a_D$ and $a$! The effective coupling $\tau = \frac{\mathrm{d}a_D}{\mathrm{d}a}$ is the coefficient of the kinetic term in (46) . Its imaginary part must be positive. On the other hand its real part plays the role of an effective $\theta$-angle: Re $\tau = \frac{\theta}{2\pi}$. Thus a shift $\tau \mapsto T(\tau) = \tau + 1$ is a symmetry transformation of the effective theory. The invariance of the chiral part in (46) together with (47) requires then that this transformation induces on $(a, a_D)$ a linear transformation in $U(1) \times SL(2, \mathbf{Z})$. In general the mass spectrum and hence the low energy effective theory may have other invariances which are not symmetries of the effective action. These play an important role for the solution of the model.

We now proceed to compute the effective potential $\mathcal{F}(\mathcal{A})$ for the massless abelian vector multiplet $\mathcal{A}$ by 'integrating' the $\mathcal{N}=2$ superconformal anomaly. For this we first need to construct the $\mathcal{N}=2$ supercurrent. The starting point is again the parameter superfield $h^{\alpha\dot{\alpha}}$ introduced in section 6. There is, however, an important difference with $\mathcal{N}=1$ supersymmetry. This is that in $\mathcal{N}=2$ superspace the constraints on $h^{\alpha\dot{\alpha}}$ and $\bar{h}^{\alpha\dot{\alpha}}$ can be solved as

$$h^{\alpha\dot{\alpha}} = \frac{1}{2}[D^{\alpha\mathbf{i}}, \bar{D}^{\dot{\alpha}}_{\mathbf{i}}]H\,, \quad \bar{h}^{\alpha\dot{\alpha}} = \frac{1}{2}[D^{\alpha\mathbf{i}}, \bar{D}^{\dot{\alpha}}_{\mathbf{i}}]\bar{H}$$

where

$$H \;=\; \bar{D}^{\mathbf{ij}}L_{\mathbf{ij}}, \quad \bar{H} = D^{\mathbf{ij}}\bar{L}_{\mathbf{ij}}$$

and $L_{\mathbf{ij}} = L_{\mathbf{ji}}$ is an unconstrained $\mathcal{N}=2$ superfield.

In order to find the representation of these transformations on $\mathcal{A}$ we first solve the constraint

$$D^{\mathbf{ij}}\mathcal{A} = \bar{D}^{\mathbf{ij}}\bar{\mathcal{A}}\ .$$

This constraint can be solved by $\mathcal{A} = \bar{D}^4 D^{\mathbf{ij}} V_{\mathbf{ij}}$ [4] where the prepotential $V_{\mathbf{ij}}$ is a real superfield. The superconformal transformations can be represented on $V_{\mathbf{ij}}$ by

$$\delta V_{\mathbf{ij}} \equiv -\frac{i}{48}(\mathcal{A}L_{\mathbf{ij}} - \bar{\mathcal{A}}\bar{L}_{\mathbf{ij}}).$$

Using the definition of $\mathcal{A}$ and the constraints it satisfies, we then compute the transformation of $\mathcal{A}$. It leads to

$$\delta\mathcal{A} = -\frac{i}{24}\bar{D}^4\left(D^{\alpha\mathbf{j}}L_{\mathbf{ij}}D^{\mathbf{i}}_{\alpha}\mathcal{A}\right) - \frac{i}{48}\bar{D}^4 D^{\mathbf{ij}}L_{\mathbf{ij}}\ \mathcal{A} - \frac{i}{48}\bar{D}^4\left[(H - \bar{H})\bar{\mathcal{A}}\right].$$

Upon substitution into the effective action (45) we then have

$$\delta S = i\int d^{12}z\ (H-\bar{H})T - 144i\int d^8 z_+\ \sigma\mathcal{J} + 144i\int d^8 z_-\ \bar{\sigma}\bar{\mathcal{J}} \tag{48}$$

with

$$T = -\frac{i}{384\pi}(\mathcal{A}\bar{\mathcal{A}}_D - \bar{\mathcal{A}}\mathcal{A}_D), \quad \mathcal{J} = \frac{1}{192\pi}(\mathcal{F} - \frac{1}{2}\mathcal{A}\mathcal{A}_D)\ .$$

This result deserves some comments:

*1)* The invariance of the action under super-Poincaré transformations is explicit as $H=\bar{H}$ and $\sigma=\bar{\sigma}=0$ for these transformations. It is then clear that the theory is superconformal invariant if and only if $\mathcal{J}=0$. Hence, in analogy with the $\mathcal{N}=1$ case, $\mathcal{J}$ is the superconformal anomaly and therefore our method provides a derivation of the anomalous superconformal 'Ward identity'.

*2)* As for $\mathcal{N}=1$, the conservation equations are obtained from (48) by expressing $H$, $\bar{H}$, $\sigma$ and $\bar{\sigma}$ in terms of the free parameters $L_{\mathbf{ij}}$ and $\bar{L}_{\mathbf{ij}}$. This leads to

$$D^{\mathbf{ij}}T = -i\bar{D}^{\mathbf{ij}}\bar{\mathcal{J}}. \tag{49}$$

The anomalous superconformal Ward identity in $\mathcal{N}=2$-superspace is now

$$\mathcal{F}(\mathcal{A}) - \frac{1}{2}\mathcal{F}'(\mathcal{A})\mathcal{A} = \frac{i}{\pi}\mathcal{J}, \tag{50}$$

where $\mathcal{J}$ is the $\mathcal{N}=2$ anomaly multiplet. In $\mathcal{N}=2$-Yang-Mills one has $\mathcal{J}=\frac{i}{2\pi}\mathcal{A}^2$ to all orders in perturbation theory as a consequence of the non-renormalisation theorems in the last section. Indeed, because there are no massless charged fields present in the Coulomb phase of this model, the only perturbative contribution to the holomorphic part of the effective action arises from the vacuum polarisation, where the effective field must be expressed in terms of the prepotential in order to avoid an infinite tower of ghosts. The higher loop contributions are absent because for non-zero $a$ the theory has a natural infrared cut-off. Integrating eqn (50) we obtain

$$\mathcal{F}(\mathcal{A}) = \frac{i}{2\pi}\mathcal{A}^2\log\left(\frac{\mathcal{A}^2}{\Lambda^2}\right).$$

If non-perturbative effects are included the relation between $\mathcal{J}$ and $\mathcal{A}$ becomes more complicated. Integrating the corresponding equation will occupy the rest of these section. In the low energy *effective* potential the phase of $a$ plays a role because the

$R$-symmetry is anomalous at the quantum level. The moduli space of inequivalent vacua is therefore 1-complex-dimensional. The equivalence is here with respect to the mass spectrum. On the other hand $a$ cannot be a global parameter on the moduli space since in that case the monodromy group of the couple $(a_D, a)$ would be abelian, contradicting the positivity of $\mathrm{Im}(\tau)$. We therefore need to choose some other (uniformizing) variable to parametrise the theory. The only other complex parameter available is the superconformal anomaly $u = \mathcal{J}|_{\theta=0}$. Let us assume that this is indeed a uniformizing variable[11] on the moduli space and think of the couple $(a, a_D)$ and $\tau$ as functions (sections) of $u \in \mathbf{C} - \{u_i\}$ where $\{u_i\}$ is the set of singular points of these functions. One such singular point is at infinity, where $\tau(u)$ diverges logarithmically and the functions $a_D$ and $\tau$ are defined only up to a discrete transformation

$$a_D \to a_D + 2na, \; n \in \mathbf{Z} \quad \text{and} \quad \tau \to \tau + 2n. \tag{51}$$

Note that the latter identification corresponds to a shift of the $\theta$ angle by $4\pi$ and is therefore a symmetry of the effective action. Given this form at infinity there must be at least two more singular points $u_i$ to avoid that the imaginary part of $\tau$ becomes negative. On the other hand, if $u \in \mathbf{C}$ indeed parametrises the space of inequivalent vacua, the mass spectrum must be single-valued everywhere and hence the ambiguity in $a_D$ and $a$ at these points must be by a $SL(2, \mathbf{Z}) \times U(1)$-transformation. From this we conclude in particular that the system $a_D, a$ is equivalent to the solutions of some second order differential equation (Riemann-Hilbert Problem). In fact we can say more. Differentiating the anomaly equation (50) with respect to $u$ we obtain

$$\mathcal{W}(a_D, a) = const.$$

where $\mathcal{W}(a_D, a)$ is the Wronskian of $a_D$ and $a$. From this we conclude that there is no first derivative term in the differential equation for $\mathbf{a} = (a_D, a)$ i.e.

$$a'' + Va = 0,$$

for some potential $V$. Differentiating this equation once more we get for $b = a'$

$$b'' - \frac{V'}{V} b' + Vb = 0.$$

[11]This assumption can in fact be derived from more fundamental properties of the theory [16]

On the other hand it is a well known fact in the theory of conformal mappings [17] that any $SL(2,\mathbf{Z})$-section can be written as

$$\begin{aligned} \tau(u) &= \frac{y_1}{y_2} \quad \text{with} \\ y'' + Qy &= 0 \end{aligned} \tag{52}$$

where $Q$ is of the form

$$Q = \frac{1}{2}\sum_i \frac{1}{2}\frac{1-\alpha_i^2}{(u-u_i)^2} + \frac{\beta_i}{u-u_i}, \tag{53}$$

with $0 \le \alpha_i < 1$. Compatibility of the two differential equations and $\tau = \frac{a'_D}{a'}$ then requires

$$\mathbf{b} = V^{\frac{1}{2}}\mathbf{y} \quad \text{and} \quad \frac{V''}{V} - \frac{3}{2}\left(\frac{V'}{V}\right)^2 + 2V = 2Q, \tag{54}$$

supplemented by the boundary conditions $Q \to 1/4u^2$ and $V \to 1/4u^2$ for $|u| \to \infty$ in order to recover the logarithm at infinity. Near a singular point $u_0$ of $V$ we have $V \simeq V_0(u-u_0)^\gamma$ with $V_0, \gamma \neq 0$. Eqn (54) then implies

$$-\frac{1}{2}\gamma(\gamma+2)\frac{1}{(u-u_0)^2} + 2V_0(u-u_0)^\gamma \simeq 2Q(u).$$

Consistency with (53) requires that $u_0$ coincides with one of the singularities $u_i$ of $Q$ and

$$\begin{aligned} V_0 &= \frac{1}{4}(1-\alpha_i^2) \quad \text{for} \quad \gamma = -2 \quad \text{and} \\ \gamma &= -1 \pm \alpha_i \quad \text{for} \quad \gamma > -2. \end{aligned} \tag{55}$$

Consequently $V$ must be of the form

$$V(u) = V\prod_r (u-u_r)^{-2}\prod_s (u-u_s)^{\pm\alpha_i - 1}, \tag{56}$$

where $u_{r,s}$ are the singular points of $Q(u)$. On the other hand, substitution of (56) into (54) implies that $V$ is rational function with no zeros. The boundary condition then implies that $V$ has only simple and double poles. But because it must have at least two poles it can have only simple poles and hence it has precisely two simple poles. More precisely

$$V(u) = \frac{1}{4(u-a)(u-b)}.$$

We are free to choose $a = -b = 1$. Furthermore $Q$ takes the form

$$Q = \frac{1}{2}\left(\frac{1}{(u-1)^2} + \frac{1}{(u+1)^2}\right) - \frac{1/2}{(u-1)(u+1)}. \tag{57}$$

The equation for $a_D(u)$ and $a(u)$ therefore becomes a standard hypergeometric differential equation with solutions

$$\begin{aligned} a_D(u) &= i\frac{u-1}{2}F\left(\frac{1}{2},\frac{1}{2},2;\frac{1-u}{2}\right), \\ a(u) &= \sqrt{2}\sqrt{u+1}F\left(-\frac{1}{2},\frac{1}{2},1;\frac{2}{u+1}\right). \end{aligned} \tag{58}$$

The prepotential can then, at least in principle be obtained by solving for $a_D(a)$. Instead of pursuing this route we note that the monodromy group associated with (57) is

$$\Gamma_2 = \{M \in SL(2,\mathbf{Z}) \mid M - \mathbf{1}_2 = 0 \bmod 2\}.$$

This group is generated by

$$T^{-2} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (T^2)^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The first generator corresponds to the the singularity at infinity whereas the second to $u = 1$. Hence we conclude that the involution

$$u \to 1 - \frac{4}{1-u},$$

is equivalent to exchanging $a_D$ and $a$ i.e.

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} -a \\ a_D \end{pmatrix},$$

or, equivalently

$$\tau \longrightarrow \frac{-1}{\tau} \equiv \tau_D. \tag{59}$$

But this is precisely the electric-magnetic duality transformation $g \longrightarrow \frac{4\pi}{g}$ and $n_e \leftrightarrow n_m$. This is the manifestation of electric-magnetic duality in this model. It tells us that for $u \simeq 1$ the system can equivalently be described by strongly coupled electrically charged fields or, by weakly coupled monopole fields. The only $\mathcal{N}=2$ supersymmetric theory with the same number of degrees of freedom and which is weakly coupled in the infrared $u \simeq 1$ is $\mathcal{N}=2$ *QED* where we replace the electric charges by magnetic charges. The action for this theory is given by the usual kinetic terms plus a potential (in $\mathcal{N}=1$ formulation)

$$\int d^4x \; d^2\theta \; \tilde{M}\phi_D M,$$

where $M$ and $\tilde{M}$ are the $n=1$ monopole fields in the fundamental representation. We can now break $\mathcal{N}=2$ SUSY to $\mathcal{N}=1$ by adding a mass term to the chiral multiplet in the classical action (22). That is

$$S \longrightarrow S + m^2 \int d^4x \; d^2\theta \; \mathrm{Tr}(\phi^2).$$

The dual effective potential then becomes

$$\mathrm{Re}\int d^4x \; d^2\theta \; \left(\tilde{M}\phi_D M + m\langle \mathrm{Tr}\phi^2\rangle\right) = \mathrm{Re}\int d^4x \; \left(\tilde{\mathcal{M}} a_D \mathcal{M} + m^2\langle \mathrm{Tr}\varphi^\dagger\varphi\rangle\right),$$

where $\mathcal{M}$ is the scalar component of the chiral monopole multiplet. Now, it can be shown [18] that $\langle \mathrm{Tr}\varphi^2\rangle = u$. Using this we then obtain upon minimising the dual effective potential

$$\begin{aligned} a_D &= 0 \quad \text{and} \\ \tilde{M} = M &= \left(-\frac{1}{2} m \frac{\bar{u}}{u} \frac{\mathrm{d}u}{\mathrm{d}a_D}|_{a_D=0}\right)^{\frac{1}{2}} \neq 0. \end{aligned} \tag{60}$$

Hence the monopoles condense! In other words the vacuum of softly broken $\mathcal{N}=2$

YM-theory is similar to that of a dual superconductor. In particular the electric flux lines are confined to a tube, which is precisely what is necessary to create a linear potential for the quarks (here gluinos). We therefore have proved that this theory confines! This is the main result of Seiberg and Witten's seminal paper [19] on $\mathcal{N}=2$ Yang-Mills theory.

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# 8 Appendix: Conventions

We choose the "relativist's" signature for the metric

$$\eta^{\mu\nu} \;=\; (-,+,+,+)$$

The conventions for the gamma matrices are

$$\begin{aligned}
\gamma^\mu &= \begin{pmatrix} 0 & -\sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \\
(\sigma^\mu)_{\alpha\dot\alpha} &= \left(-1\!\!1_2, \sigma^i\right)_{\alpha\dot\alpha} \\
(\bar\sigma^\mu)^{\dot\beta\beta} &= \left(-I_2, -\sigma^i\right)^{\dot\beta\beta} \\
&= \left[(i\sigma_2)\left(-1\!\!1_2, \sigma^i\right)(i\sigma^2)^T\right]^{\dot\beta\beta}
\end{aligned}$$

<u>Relativistic covariance:</u>

$$\begin{aligned}
S(\Lambda)\gamma^\mu S(\Lambda)^{-1} &= \gamma^\nu \Lambda_\nu{}^\mu \\
S(\Lambda) &= 1\!\!1_4 - \frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu} + O(\omega^2)
\end{aligned}$$

with

$$\sigma_{\mu\nu} \;=\; \frac{i}{2}[\gamma_\mu, \gamma_\nu] \ .$$

Weyl spinors:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \in (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0) \quad \text{of} \qquad SL(2, \mathbf{C})$$
$$(SU(2) \text{ in } E_4)$$

$\alpha, \dot{\alpha} = 1, 2.$

Charge conjugation: $\Psi^c = C\bar{\Psi}^T$

$$C = \begin{pmatrix} i\sigma^2\bar{\sigma}^0 & 0 \\ 0 & i\bar{\sigma}^2\sigma^0 \end{pmatrix} \tag{61}$$

Majorana: impose $\Psi^c = \Psi$, i.e.

$$\Psi = \begin{pmatrix} \psi_\alpha \\ (i\bar{\sigma}^2)^{\dot{\alpha}\beta}\psi^*_\beta \end{pmatrix}$$

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