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Point interactions in one dimension and holonomic quantum fields

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Abstract

We introduce and study a family of quantum fields, associated to δ-interactions in one dimension. These fields are analogous to holonomic quantum fields of M. Sato, T. Miwa and M. Jimbo. Corresponding field operators belong to an infinite-dimensional representation of the group $SL(2,\mathbb{R})$ in the Fock space of ordinary harmonic oscillator. We compute form factors of such fields and their correlation functions, which are related to the determinants of Schroedinger operators with a finite number of point interactions. It is also shown that these determinants coincide with tau functions, obtained through the trivialization of the det*-bundle over a Grassmannian associated to a family of Schroedinger operators.

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1 Introduction

The study of holonomic quantum fields [10] has led to important advances in both integrable quantum field theory and analytic theory of linear differential equations. In physical language, these fields represent a particular case of the canonical Bogolyubov transformations (see, for instance, [2]), which makes possible the exact computation of their form factors and correlation functions.

The main physical examples of holonomic quantum fields are given by the exponential fields of the sine-Gordon theory at the free-fermion point (SG_{ff}) [3, 4, 11] and order/disorder variables of the two-dimensional Ising model [5, 6]. Both models possess an underlying free-fermion structure. However, their observables are nonlinear in terms of free fields (they are represented by ordered exponentials of fermion bilinears) and have nontrivial braiding relations with them. This nonlinearity leads to interesting correlation functions.

Correlators of holonomic quantum fields are usually called tau functions. In the special case of the SG_{ff}-theory they have the meaning of determinants of Dirac operators with branching points on the Euclidean plane. An attempt to give a geometric definition of such tau functions was made in [9]. It was based on the approach, developed in the article [8], where the tau function of the Schlesinger system was related to the determinant of a singular Cauchy-Riemann operator. The idea of [9] was to consider a family $\mathcal{A}$ of Dirac operators, parametrized by the coordinates of branchpoints, and to associate to each of these operators a subspace of boundary values of local solutions of the Dirac equation. These subspaces are then embedded into an infinite-dimensional grassmannian. One can construct *à la* Segal-Wilson [12] the $\det^*$-bundle over this grassmannian and its canonical section σ. Next, using the Green function of the Dirac operator, one may endow the $\det^*$-bundle with another, trivializing section. This latter allows to identify σ with a (tau) function on $\mathcal{A}$.

The main goal of the present paper is to explain the concept of holonomic quantum fields and the above definition of the tau function with a simple example. It appears that these two notions naturally emerge in the calculation of the resolvent of the Schroedinger operator with δ-interactions in one dimension. Though such operators and their resolvents have already been extensively discussed in physical and mathematical literature (see, for example, the monograph [1] and references therein), holonomic quantum fields allow to examine this fairly classical subject from a new point of view.

Moreover, we believe that the ideology developed in this paper applies as well to certain unsolved quantum mechanical problems, the most appealing one being the computation of the fermionic vacuum quantum numbers, induced by a finite number of magnetic vortices in $2+1$ dimensions. In practice, the latter problem reduces to the calculation of the resolvent of the Dirac hamiltonian with point sources. At present, the answer is known only in the case of a single vortex [13]. It seems, however, that the multivortex resolvent can be obtained from the correlation functions and form factors of certain Bogolyubov transformations, generalizing exponential fields of the SG_{ff}-theory.

This paper is organized as follows. After introducing basic notations and terminology in Section 2, we turn in the next section to the calculation of the resolvent of the Schroedinger operator with δ-interactions. It is expressed (by the formula (3.3)) through the ratio of correlation functions of certain local fields in the 1D quantum field theory of free massive real bosons. These correlation functions are computed in the lagrangian approach by an auxiliary integration method. They comprise the fields of two types: the free ones and interacting fields, associated to delta-sources. These latter represent the simplest prototypes of holonomic quantum fields, since the operators, corresponding to them in the hamiltonian picture, realize certain canonical Bogolyubov transformations of the Heisenberg algebra. In Section 4, we compute form factors and correlation functions of the fields, which correspond to more general Bogolyubov transformations and belong to an infinite-dimensional representation of the group $SL(2,\mathbb{R})$ in the Fock space of harmonic oscillator. Section 5 is devoted to the definition and calculation of the tau function. It is obtained through the trivialization of the $\det^*$-bundle over a finite-dimensional grassmannian of boundary conditions, associated to a family of Schroedinger operators with point interactions. We conclude with a brief discussion of possible generalizations and open questions.

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2 Schroedinger operator without point interactions

Let us first consider Schroedinger operator without point interactions,

$$L = -\frac{d^2}{dx^2},$$

acting on functions from the Sobolev space $H^2(\mathbb{R})$ as the second derivative. Its resolvent $(L-E)^{-1}$ is given by the integral operator with the kernel

$$G_E(x,y) = \frac{e^{-m|x-y|}}{2m}, \tag{2.1}$$

which can be easily evaluated by Fourier transformation. Here we have introduced the notation $E = -m^2$ and assumed that $\mathrm{Re}\, m > 0$.

Note that for real $E < 0$ the resolvent kernel $G_E(x,y)$ coincides by construction with the two-point correlation function in the relativistic euclidean quantum field theory of free massive real bosons in one dimension. In particular, if we define the action

$$S_0[\varphi] = \frac{1}{2}\int\limits_{-\infty}^{\infty} dx\, \varphi(x)\left(-\frac{d^2}{dx^2}+m^2\right)\varphi(x), \tag{2.2}$$

then $G_E(x,y)$ can be formally written through the ratio of two functional integrals:

$$G_E(x,y) = \langle\varphi(x)\varphi(y)\rangle = \frac{\int \mathcal{D}\varphi\; \varphi(x)\varphi(y)\; e^{-S_0[\varphi]}}{\int \mathcal{D}\varphi\; e^{-S_0[\varphi]}}. \tag{2.3}$$

The theory, described by the action (2.2), admits two natural interpretations:

- If we interpret our single dimension as space, then the action (2.2) coincides with the energy functional of the infinite string in a parabolic well. The integral in the denominator of (2.3) represents string partition function, and two-point correlator $\langle\varphi(x)\varphi(y)\rangle$ is the thermodynamic average of the product of transverse coordinates of two different points of the string.

- On the other hand, the action (2.2) describes the dynamics of harmonic oscillator in imaginary time. In this setting, correlation function $\langle\varphi(x)\varphi(y)\rangle$ may be interpreted as the vacuum expectation value of the ordered product of coordinate operators at two different times.

In order to introduce several important notations, let us briefly recall the hamiltonian approach to oscillator dynamics. Fields $\varphi(x)$ and $\pi(x) = \frac{1}{i}\frac{d\varphi(x)}{dx}$ here become operators, obeying the commutation relation $[\hat{\varphi},\hat{\pi}] = i$. The hamiltonian, being expressed in terms of $\hat{\varphi}$ and $\hat{\pi}$, has the form

$$\hat{H} = \frac{1}{2}\left(\hat{\pi}^2 + m^2\hat{\varphi}^2\right) - \frac{m}{2},$$

where the constant term is subtracted for future convenience. Imaginary time evolution of an arbitrary operator $\hat{\mathcal{O}}$ is given by the equation

$$\hat{\mathcal{O}}(x) = e^{-\hat{H}x}\hat{\mathcal{O}}(0)\, e^{\hat{H}x}. \tag{2.4}$$

It is customary to define the creation-annihilation operators

$$a = \sqrt{\frac{m}{2}}\left(\hat{\varphi}(0) + \frac{i\,\hat{\pi}(0)}{m}\right), \qquad a^\dagger = \sqrt{\frac{m}{2}}\left(\hat{\varphi}(0) - \frac{i\,\hat{\pi}(0)}{m}\right),$$

satisfying canonical commutation relation $[a, a^\dagger] = 1$. The hamiltonian can then be rewritten in terms of these operators as $\hat{H} = m\, a^\dagger a$. Vacuum vector $|0\rangle$ is fixed by conditions $a|0\rangle = 0$, $\langle 0|0\rangle = 1$. The operators that we wish to consider act in the Fock space $\mathcal{F}$, spanned by the orthonormal vectors

$$|k\rangle = \frac{\left(a^\dagger\right)^k}{\sqrt{k!}}\,|0\rangle, \qquad k = 0, 1, 2\ldots, \tag{2.5}$$

constituting the set of hamiltonian eigenstates: $\hat{H}|k\rangle = km\,|k\rangle$.

The computation of correlation functions of local fields in the hamiltonian approach is equivalent to the calculation of form factors, i. e. matrix elements of the corresponding field operators in the orthonormal basis of eigenstates of $\hat{H}$. For instance, form factor expansion of the two-point correlator $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle$ is written as

$$\begin{aligned}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle &= \langle 0|\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)|0\rangle = \langle 0|\hat{\mathcal{O}}_1(0)\, e^{-\hat{H}(x_2-x_1)}\hat{\mathcal{O}}_2(0)|0\rangle = \\ &= \sum_{k=0}^{\infty} \langle 0|\,\hat{\mathcal{O}}_1(0)|k\rangle\,\langle k|\hat{\mathcal{O}}_2(0)|0\rangle\, e^{-km(x_2-x_1)},\end{aligned} \tag{2.6}$$

where we have assumed that $x_2 \geq x_1$. One can obtain analogous expressions for the multipoint correlation functions, simply using the formula (2.4), and inserting the appropriate number of times the resolution of the identity operator, $\mathbf{1} = \sum_{k=0}^{\infty} |k\rangle\,\langle k|$, into the correlators.

Example. It is easiest to see how this scheme works on the example of the two-point correlation function $\langle \varphi(x)\varphi(y)\rangle$. Note that since the operators $\hat{\varphi}$ and $\hat{\pi}$ are given by

$$\hat{\varphi}(0) = \frac{1}{\sqrt{2m}}\left(a^\dagger + a\right), \qquad \hat{\pi}(0) = i\sqrt{\frac{m}{2}}\left(a^\dagger - a\right),$$

the only non-zero form factors are

$$\begin{aligned}\langle k+1|\,\hat{\varphi}(0)|k\rangle &= \langle k|\,\hat{\varphi}(0)|k+1\rangle = \sqrt{\frac{k+1}{2m}}\,, \\ \langle k+1|\,\hat{\pi}(0)|k\rangle &= -\langle k|\,\hat{\pi}(0)|k+1\rangle = i\sqrt{\frac{k+1}{2m}}\,,\end{aligned}$$

that is, the numbers of *in*- and *out*-particles should differ by 1. Therefore, in the expansion over intermediate states (2.6) for $\langle \varphi(x)\varphi(y)\rangle$ only the terms with $k = 1$ will remain, reproducing thus the formula (2.1).

3 Introducing δ-interactions

Let us now see what happens if instead of L we consider Schroedinger operator with a finite number of δ-interactions,

$$L_{a,V} = -\frac{d^2}{dx^2} + V(x), \qquad V(x) = \sum_{i=1}^{N} V_i\,\delta(x - a_i)\,.$$

The most known way of calculating the resolvent $(L_{a,V} - E)^{-1}$ is to expand it formally in a series in powers of V. Summing up this series, one obtains a compact expression for the resolvent kernel $G_{E,V}(x, y)$:

$$G_{E,V}(x, y) = G_E(x, y) - \sum_{i,j=1}^{N} G_E(x, a_i)\, U_{ij}^{-1}\, G_E(a_j, x), \tag{3.1}$$

Here $G_E(x, y)$ denotes the unperturbed resolvent (2.1) and the matrix U is defined as

$$U_{ij} = \frac{1}{V_i}\,\delta_{ij} + G_E(a_i, a_j), \qquad i, j = 1, \ldots, N.$$

An alternative simple proof of this result follows from field-theoretic considerations. Again, for real negative $E = -m^2$ the resolvent $G_{E,V}(x, y)$ coincides with the two-point correlator in the one-dimensional quantum field theory, described by the action

$$S_V[\varphi] = \frac{1}{2}\int_{-\infty}^{\infty} dx\; \varphi(x) \left(-\frac{d^2}{dx^2} + m^2 + V(x) \right) \varphi(x) = S_0[\varphi] + S_d[\varphi], \tag{3.2}$$

where

$$S_d[\varphi] = \frac{1}{2}\sum_{i=1}^{N} V_i\, \varphi^2(a_i).$$

Thus we have

$$G_{E,V}(x, y) = \langle \varphi(x)\varphi(y) \rangle_V = \frac{\int \mathcal{D}\varphi\; \varphi(x)\varphi(y)\; e^{-S_V[\varphi]}}{\int \mathcal{D}\varphi\; e^{-S_V[\varphi]}}.$$

The action (3.2) reproduces the energy of a one-dimensional string with N masses attached to the points $a_1, \ldots, a_N$. We will assume for definiteness that these masses are all positive.

The idea is to represent the factor $e^{-S_d[\varphi]}$, appearing in the functional integrals, as a gaussian integral over N auxiliary variables $\mu_1, \ldots, \mu_N$:

$$e^{-S_d[\varphi]} = \frac{1}{(2\pi)^{N/2} \prod_{j=1}^{N} \sqrt{V_j}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\mu_1 \ldots d\mu_N\; \exp\left\{ -\sum_{j=1}^{N} \mu_j^2 / V_j + \sum_{j=1}^{N} i\mu_j\, \varphi(a_j) \right\}.$$

After interchanging the order of integration over φ's and μ's, the corresponding integrals can be readily evaluated, since the quadratic form in φ is now given by the unperturbed action $S_0[\varphi]$. At the end of this calculation one finds the the formula (3.1).

Let us consider yet another approach to the calculation of the resolvent $G_{E,V}(x,y)$. On the one hand, it is equal to pair correlation function in the theory with the action (3.2). However, it may also be expressed through the ratio of certain multipoint correlators in the theory, described by the unperturbed action (2.2). Namely, one can write

$$G_{E,V}(x,y) = \langle \varphi(x)\varphi(y)\rangle_V = \frac{\langle \mathcal{O}_{V_1}(a_1)\ldots\mathcal{O}_{V_N}(a_N)\varphi(x)\varphi(y)\rangle}{\langle \mathcal{O}_{V_1}(a_1)\ldots\mathcal{O}_{V_N}(a_N)\rangle}\,, \tag{3.3}$$

where the local fields $\mathcal{O}_V(a)$ are defined as

$$\mathcal{O}_V(a) = \exp\left\{-\frac{1}{2}\,V\varphi^2(a)\right\}. \tag{3.4}$$

It should be pointed out that the correlation function $\langle \mathcal{O}_{V_1}(a_1)\ldots\mathcal{O}_{V_N}(a_N)\rangle$, standing in the denominator of (3.3), is equal to the ratio of partition functions of the string with and without attached point masses. It can be formally expressed through the determinants of Schroedinger operators:

$$\langle \mathcal{O}_{V_1}(a_1)\ldots\mathcal{O}_{V_N}(a_N)\rangle = \sqrt{\frac{\det(L-E)}{\det(L_{a,V}-E)}}\,. \tag{3.5}$$

Repeating the above trick with auxiliary fields, one may also compute this correlator explicitly:

$$\langle \mathcal{O}_{V_1}(a_1)\ldots\mathcal{O}_{V_N}(a_N)\rangle = \left(\det\left\|\,\delta_{ij}+\sqrt{V_iV_j}\,G_E(a_i,a_j)\,\right\|\right)^{-1/2}, \qquad i,j=1,\ldots,N. \tag{3.6}$$

Consider now the fields $\mathcal{O}_V$ in the hamiltonian picture. The operator $\hat{\mathcal{O}}_V = \exp\left\{-\frac{1}{2}\,V\hat{\varphi}^{\,2}\right\}$ has remarkable equal-time commutation relations with the operators $\hat{\varphi}$ and $\hat{\pi}$:

$$\begin{aligned}\hat{\mathcal{O}}_V(0)\hat{\varphi}(0)-\hat{\varphi}(0)\hat{\mathcal{O}}_V(0) &= 0, && (3.7)\\ \hat{\mathcal{O}}_V(0)\hat{\pi}(0)-\hat{\pi}(0)\hat{\mathcal{O}}_V(0) &= -iV\hat{\varphi}(0)\hat{\mathcal{O}}_V(0). && (3.8)\end{aligned}$$

This is a manifestation of the fact that any correlator $f(x)=\langle\ldots\mathcal{O}_V(a)\varphi(x)\ldots\rangle$, being considered as a function of x, is a local solution of the Schroedinger equation $(L-E)f=0$, satisfying at the point a the boundary condition

$$f(a+0)-f(a-0)=0, \qquad f'(a+0)-f'(a-0)=Vf(a).$$

One can also rewrite the commutation relations (3.7)–(3.8) in terms of the creation-annihilation operators:

$$\hat{\mathcal{O}}_V(0)\begin{pmatrix} a \\ a^\dagger\end{pmatrix}\hat{\mathcal{O}}_V^{-1}(0) = \begin{pmatrix} 1+\frac{V}{2m} & \frac{V}{2m} \\ -\frac{V}{2m} & 1-\frac{V}{2m}\end{pmatrix}\begin{pmatrix} a \\ a^\dagger\end{pmatrix}. \tag{3.9}$$

Therefore, $\hat{\mathcal{O}}_V$ realizes a Bogolyubov transformation, i. e. linear transformation of the Heisenberg algebra, preserving canonical commutation relation $[a,a^\dagger]=1$. The formula (3.9) determines the operator $\hat{\mathcal{O}}_V$ almost completely. More precisely, $\hat{\mathcal{O}}_V$ is fixed by (3.9) up to a constant numerical factor.

In the next section, we will compute form factors and correlation functions of the fields, corresponding to more general Bogolyubov transformations. It should be emphasized that we will use only the relations of type (3.9) and no reference will be made to the explicit formula (3.4). The reason for doing so is that it is not always clear, which operator should be associated to a given field and vice versa. This is the case, for instance, in the analysis of δ'-interactions. Even more severe difficulties arise in some two-dimensional problems: the fields, realizing relevant Bogolyubov transformations, being themselves local, are not mutually local with the free fields. The most known example of this kind is given by the exponential fields in the SG_{ff}-theory.

4 Form factors and correlation functions of Bogolyubov fields

Consider a linear transformation of the creation-annihilation operators

$$\Lambda : \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \mapsto \begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{4.1}$$

It preserves canonical commutation relation iff the real parameters α, β, γ, δ satisfy the condition $\alpha\delta - \beta\gamma = 1$. In the following, this condition is assumed to hold. We want to represent Λ as a similarity transformation. Namely, we are looking for the invertible operator $\hat{\mathcal{O}}_\Lambda$, such that

$$\hat{\mathcal{O}}_\Lambda \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \hat{\mathcal{O}}_\Lambda^{-1} = \Lambda \begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{4.2}$$

These operators are called Bogolyubov transformations, and the corresponding local fields will be called Bogolyubov fields in the rest of this paper[1]. The operators $\hat{\mathcal{O}}_\Lambda$ are determined by (4.2) up to a constant, realizing thus an infinite-dimensional projective representation of the group $SL(2,\mathbb{R})$ in the Fock space of our harmonic oscillator.

In order to construct $\hat{\mathcal{O}}_\Lambda$ in terms of a and $a^\dagger$, one should examine the properties of basic Bogolyubov transformations:

$$\hat{P}_\lambda = e^{\frac{1}{2}\lambda (a^\dagger)^2}, \qquad \hat{R}_\nu = e^{\frac{1}{2}\nu a^2},$$

$$\hat{Q}_\mu = :e^{\mu a^\dagger a}: \stackrel{def}{=} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \left(a^\dagger\right)^n a^n.$$

It is easy to check that induced linear transformations have the form

$$\hat{P}_\lambda \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \hat{P}_\lambda^{-1} = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{4.3}$$

$$\hat{Q}_\mu \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \hat{Q}_\mu^{-1} = \begin{pmatrix} \frac{1}{1+\mu} & 0 \\ 0 & 1+\mu \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{4.4}$$

$$\hat{R}_\nu \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \hat{R}_\nu^{-1} = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{4.5}$$

[1] Actually, there already exist several names for some special fields of this type, to mention only 'holonomic quantum fields', 'exponential fields' and 'monodromy fields'. All these names, however, do not reflect common structure of such fields, and are related to the particularities of different two-dimensional problems. That is why we believe that the name 'Bogolyubov fields' would be more appropriate.

General Bogolyubov transformation (4.2) is then given by

$$\hat{\mathcal{O}}_{\lambda,\mu,\nu} = \hat{P}_\lambda \hat{Q}_\mu \hat{R}_\nu = :\exp\left\{\frac{1}{2}\,\lambda\left(a^\dagger\right)^2 + \mu\, a^\dagger a + \frac{1}{2}\,\nu\, a^2\right\}: \tag{4.6}$$

Parameters α, β, γ, δ of the corresponding linear map $\Lambda = \Lambda_{R_\nu}\Lambda_{Q_\mu}\Lambda_{P_\lambda}$ are related to λ, μ and ν by the following formulas:

$$\alpha = \frac{1}{1+\mu}, \qquad \beta = -\frac{\lambda}{1+\mu}, \qquad \gamma = \frac{\nu}{1+\mu}, \qquad \delta = 1 + \mu - \frac{\lambda\,\nu}{1+\mu}\,. \tag{4.7}$$

Note that writing $\hat{\mathcal{O}}_\Lambda$ in the form (4.6), we adopt the convention $\langle \mathcal{O}_\Lambda \rangle \overset{def}{=} \langle 0|\hat{\mathcal{O}}_\Lambda|0\rangle = 1$. This normalization will be used hereafter. Together with the relation (4.2), it completely fixes the operator $\hat{\mathcal{O}}_\Lambda$.

Remark. One-point function of the Bogolyubov field $\mathcal{O}_V$, associated to a δ-interaction, is *not* equal to 1. It may be determined from the formula (3.6) by setting $N = 1$:

$$\langle \mathcal{O}_V \rangle = \left(1 + \frac{V}{2m}\right)^{-1/2}. \tag{4.8}$$

It can also be computed in the hamiltonian approach, being rewritten as the vacuum expectation value

$$\langle \mathcal{O}_V \rangle = \langle 0|\hat{\mathcal{O}}_V|0\rangle = \langle 0|e^{-\frac{1}{2}V\hat{\varphi}^2}|0\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{V}{4m}\right)^n \langle 0|\left(a + a^\dagger\right)^{2n}|0\rangle. \tag{4.9}$$

Using Wick's theorem, one obtains $\langle 0|\left(a+a^\dagger\right)^{2n}|0\rangle = (2n-1)!!$. Now remark that the Taylor expansion of the function

$$(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!}\left(-\frac{x}{2}\right)^n \tag{4.10}$$

coincides with (4.9), if we set $x = \frac{V}{2m}$. Thus we recover the formula (4.8).

Parameters λ, μ and ν, which correspond to the field $\mathcal{O}_V$, are determined from the comparison of (3.9) and (4.7). The result is

$$\lambda = \mu = \nu = -\frac{V/2m}{1 + V/2m}. \tag{4.11}$$

This leads to the following representation of the operator $\hat{\mathcal{O}}_V$:

$$\hat{\mathcal{O}}_V = \left(1 + \frac{V}{2m}\right)^{-1/2} :\,\exp\left\{-\frac{1}{2}\,\frac{V/2m}{1+V/2m}\,\left(a + a^\dagger\right)^2\right\}: \tag{4.12}$$

We now turn to the calculation of form factors $\langle k|\hat{\mathcal{O}}_{\lambda,\mu,\nu}|l\rangle$ of the operator (4.6). It is clear that such form factors will be non-zero only if the numbers of particles in *in*- and *out*-state are simultaneously even or odd. It is convenient to use instead of $\langle k|\hat{\mathcal{O}}_{\lambda,\mu,\nu}|l\rangle$ the auxiliary variables

$$F_{k,l} = \sqrt{k!}\,\sqrt{l!}\,\langle k|\hat{\mathcal{O}}_{\lambda,\mu,\nu}|l\rangle = \langle 0|a^k\,\hat{P}_\lambda\hat{Q}_\mu\hat{R}_\nu\left(a^\dagger\right)^l|0\rangle.$$

Using the relations (4.3)–(4.5) in the last expression, one can pull the operator $\hat{P}_\lambda$ through a^k, and the operators $\hat{Q}_\mu$, $\hat{R}_\nu$ through $\left(a^\dagger\right)^l$:

$$F_{k,l} = \langle 0|\hat{P}_\lambda \left(a + \lambda\, a^\dagger\right)^k \hat{Q}_\mu \hat{R}_\nu \left(a^\dagger\right)^l |0\rangle = \langle 0|\hat{P}_\lambda \left(a + \lambda\, a^\dagger\right)^k \hat{Q}_\mu \left(\nu\, a + a^\dagger\right)^l \hat{R}_\nu |0\rangle =$$
$$= \langle 0|\hat{P}_\lambda \left(a + \lambda\, a^\dagger\right)^k \Big(\nu(1+\mu)^{-1} a + (1+\mu)\, a^\dagger\Big)^l \hat{Q}_\mu \hat{R}_\nu |0\rangle.$$

Next, since we have

$$\langle 0|\hat{P}_\lambda = \langle 0|\hat{Q}_\mu = \langle 0|, \qquad \hat{Q}_\mu|0\rangle = \hat{R}_\nu|0\rangle = |0\rangle,$$

the variable $F_{k,l}$ may be rewritten as the vacuum expectation value of the product of certain linear combinations of the creation-annihilation operators:

$$F_{k,l} = \langle 0|\left(a + \lambda\, a^\dagger\right)^k \Big(\nu(1+\mu)^{-1} a + (1+\mu)\, a^\dagger\Big)^l |0\rangle.$$

Wick's theorem allows to express this vacuum expectation value through the sum over all possible pairings between the linear combinations. There will be only three types of such pairings:

$$\langle 0|\left(a + \lambda\, a^\dagger\right)^2 |0\rangle = \lambda, \tag{4.13}$$
$$\langle 0|\Big(\nu(1+\mu)^{-1} a + (1+\mu)\, a^\dagger\Big)^2 |0\rangle = \nu, \tag{4.14}$$
$$\langle 0|\left(a + \lambda\, a^\dagger\right)\Big(\nu(1+\mu)^{-1} a + (1+\mu)\, a^\dagger\Big)|0\rangle = 1+\mu. \tag{4.15}$$

Consider, for instance, the sum corresponding to $F_{2k,2l}$. If some term of this sum contains $2j$ pairings of type (4.15) (it is easy to understand that this number should be even and satisfy $0 \le j \le \min\{k,l\}$), then it also contains $k-j$ pairings of type (4.13) and $l-j$ pairings of type (4.14). The total number of such terms is equal to

$$C_{2j}^{2k} \times C_{2j}^{2l} \times (2j)! \times \frac{(2k-2j)!}{2^{k-j}(k-j)!} \times \frac{(2l-2j)!}{2^{l-j}(l-j)!}\,.$$

Simplifying this combinatorial factor and taking into account the above remarks, we obtain a general formula for the even-even form factor:

$$F_{2k,2l} = \sum_{j=0}^{\min\{k,l\}} \frac{(2k)!\,(2l)!}{(2j)!\,(k-j)!\,(l-j)!}\,(\lambda/2)^{k-j}\,(1+\mu)^{2j}\,(\nu/2)^{l-j}\,. \tag{4.16}$$

Analogously, the odd-odd form factor is given by

$$F_{2k+1,2l+1} = \sum_{j=0}^{\min\{k,l\}} \frac{(2k+1)!\,(2l+1)!}{(2j+1)!\,(k-j)!\,(l-j)!}\,(\lambda/2)^{k-j}\,(1+\mu)^{2j+1}\,(\nu/2)^{l-j}\,. \tag{4.17}$$

One may also prove these results by induction, taking as its first step the obvious formulas

$$F_{2k,0} = \langle 0|a^{2k}\hat{P}_\lambda|0\rangle = \frac{(2k)!}{k!}\left(\frac{\lambda}{2}\right)^k, \qquad F_{0,2l} = \langle 0|\hat{R}_\nu \left(a^\dagger\right)^{2l}|0\rangle = \frac{(2l)!}{l!}\left(\frac{\nu}{2}\right)^l, \tag{4.18}$$

and applying at the next steps the recursion relations

$$\begin{aligned} F_{k+1,l} &= \frac{\lambda}{1+\mu}\, F_{k,l+1} + l\left(1+\mu-\frac{\lambda\nu}{1+\mu}\right) F_{k,l-1}\,,\\ F_{k,l+1} &= \frac{\nu}{1+\mu}\, F_{k+1,l} + k\left(1+\mu-\frac{\lambda\nu}{1+\mu}\right) F_{k-1,l}\,. \end{aligned}$$

Another problem to be considered in this section is the calculation of correlation functions of Bogolyubov fields in the hamiltonian approach. In other words, we want to compute the vacuum expectation values of the ordered products of time-dependent operators

$$\hat{\mathcal{O}}_{\lambda,\mu,\nu}(x) = e^{-\hat{H}x}\hat{\mathcal{O}}_{\lambda,\mu,\nu}\, e^{\hat{H}x}, \tag{4.19}$$

where the operator $\hat{\mathcal{O}}_{\lambda,\mu,\nu}$ is defined by (4.6). Let us start with the two-point correlation function $\langle \mathcal{O}_{\lambda_1,\mu_1,\nu_1}(a_1)\mathcal{O}_{\lambda_2,\mu_2,\nu_2}(a_2)\rangle$. Assuming that $a_2 \geq a_1$ and applying the formulas (2.6) and (4.18), one obtains

$$\langle \mathcal{O}_{\lambda_1,\mu_1,\nu_1}(a_1)\mathcal{O}_{\lambda_2,\mu_2,\nu_2}(a_2)\rangle = \sum_{k=0}^{\infty} \langle 0|\mathcal{O}_{\lambda_1,\mu_1,\nu_1}|2k\rangle\, \langle 2k|\hat{\mathcal{O}}_{\lambda_2,\mu_2,\nu_2}|0\rangle\, e^{-2km(a_2-a_1)} =$$

$$= \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2}\left(\frac{1}{4}\,\nu_1\lambda_2\, e^{-2m(a_2-a_1)}\right)^k = 1 + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{k!}\left(\frac{1}{2}\,\nu_1\lambda_2\, e^{-2m(a_2-a_1)}\right)^k.$$

The function characterized by such series had already appeared in the computation of the one-point function $\langle \mathcal{O}_V\rangle$. From the comparison of the last expression with the expansion (4.10) it follows that

$$\langle \mathcal{O}_{\lambda_1,\mu_1,\nu_1}(a_1)\mathcal{O}_{\lambda_2,\mu_2,\nu_2}(a_2)\rangle = \left[1-\nu_1\lambda_2\, e^{-2m(a_2-a_1)}\right]^{-1/2}. \tag{4.20}$$

Remark. In order to obtain two-point correlation function of fields, corresponding to δ-interactions, one should make in (4.20) the substitution

$$\nu_1 = -\frac{V_1/2m}{1+V_1/2m}, \qquad \lambda_2 = -\frac{V_2/2m}{1+V_2/2m},$$

and take into account the one-point functions (4.8). The result is then given by

$$\langle \mathcal{O}_{V_1}(a_1)\mathcal{O}_{V_2}(a_2)\rangle = \left[\left(1+\frac{V_1}{2m}\right)\left(1+\frac{V_2}{2m}\right) - \frac{V_1V_2}{4m^2}\, e^{-2m(a_2-a_1)}\right]^{-1/2}. \tag{4.21}$$

One may easily check that it coincides with the formula (3.6), specialized to the case $N=2$.

Though form factor series for the multipoint correlation functions of Bogolyubov fields have a more complicated structure, compact expressions for these correlators may also be found. Note first that the form factors of the operator $\hat{\mathcal{O}}_{\lambda,\mu,\nu}(x)$ are obtained from the form factors of $\hat{\mathcal{O}}_{\lambda,\mu,\nu}$ by the substitution

$$\lambda \to \lambda\, e^{-2mx}, \qquad \nu \to \nu\, e^{2mx}.$$

Next, the product of any two such operators is again an operator of the form (4.6), multiplied by a constant. More precisely, one has

$$\hat{\mathcal{O}}_{\lambda_1,\mu_1,\nu_1}\hat{\mathcal{O}}_{\lambda_2,\mu_2,\nu_2} = c_{12}\hat{\mathcal{O}}_{\lambda_3,\mu_3,\nu_3},$$

where the parameters λ_3, μ_3 and ν_3 are determined from the comparison of induced linear transformations,

$$\lambda_3 = \lambda_1 + \lambda_2\,\frac{(1+\mu_1)^2}{1-\nu_1\lambda_2}, \qquad \mu_3 = \frac{\mu_1+\mu_2+\mu_1\mu_2+\nu_1\lambda_2}{(1+\mu_1)(1+\mu_2)}, \qquad \nu_3 = \nu_2 + \nu_1\,\frac{(1+\mu_2)^2}{1-\nu_1\lambda_2},$$

and the coefficient c_{12} is given by

$$c_{12} = \langle 0|\hat{\mathcal{O}}_{\lambda_1,\mu_1,\nu_1}\hat{\mathcal{O}}_{\lambda_2,\mu_2,\nu_2}|0\rangle = [1-\nu_1\lambda_2]^{-1/2}.$$

Successively applying the above observations, one may reduce any product of time-dependent operators to a *single* operator of type (4.6), multiplied by a constant factor. This constant clearly gives the correlation function we are looking for.

5 Tau-function of the Schroedinger operator with point interactions

It was shown in Section 3 that the correlation function of certain Bogolyubov fields can be formally expressed by the formula (3.5) through the (Weinstein-Aronszajn) determinant of the Schroedinger operator with δ-interactions. Therefore, it is natural to assume that this correlator has a geometric interpretation. In this section, we construct a geometric invariant of the Schroedinger operator, simply related with the above correlation function.

Let us choose a collection $a = (a_1, \ldots, a_N)$ of N distinct points on the real line and suppose for definiteness that $a_1 < a_2 < \ldots < a_N$. Schroedinger operators with δ-interactions at the points $a_1, \ldots, a_N$ are rigorously defined as the elements of an N-parametric family of self-adjoint extensions of the operator $\dot{L}_a = -\frac{d^2}{dx^2}$ with the domain given by

$$\operatorname{dom}\dot{L}_a = \left\{ f \in H^2(\mathbb{R}) \,|\, f(a_j) = 0,\ j = 1, \ldots, N \right\}.$$

A general element of this family, $\dot{L}_{a,V}$, is written as

$$\dot{L}_{a,V} = -\frac{d^2}{dx^2}, \qquad \operatorname{dom}\dot{L}_{a,V} = \{ f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash a) \,|\, f'(a_j+0) - f'(a_j-0) = V_j f(a_j), \quad V_j \in \mathbb{R},\ j = 1, \ldots, N\}. \tag{5.1}$$

Self-adjointness of this operator implies the following important statement:

Proposition 5.1 *If $m^2 \in \mathbb{C}\backslash\mathbb{R}$, then the Schroedinger equation*

$$(\dot{L}_{a,V} + m^2)\psi = 0 \tag{5.2}$$

has no solutions in the domain of $\dot{L}_{a,V}$.

Let us now study the spaces of boundary values of certain *local* solutions of the equation (5.2). It will always be assumed that $m^2 \in \mathbb{C}\backslash\mathbb{R}$ and $\mathrm{Re}\, m > 0$. We isolate the points $a_1, \ldots, a_N$ in the union $S = \bigcup_{j=1}^N S_j$ of N disjoint open intervals $S_j = (x_j^L, x_j^R)$, chosen so that $a_j \in S_j$ for $j = 1, \ldots, N$. The set S is fixed once and for all, while the coordinates $a_1, \ldots, a_N$ are allowed to vary provided each a_j stays in S_j. The whole family of operators $\dot{L}_{a,V}$ that satisfy this condition will be denoted by $\mathcal{L}_S$. It depends on $2N$ parameters, including the positions of delta-interactions $\{a_j\}_{j=1,\ldots,N}$ and their strengths $\{V_j\}_{j=1,\ldots,N}$. Define an auxiliary map

$$\pi : \mathrm{dom}\, \dot{L}_{a,V} \to W = \underbrace{\mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2}_{2n \text{ times}},$$

$$\psi \mapsto \psi^{(1)} \oplus \ldots \oplus \psi^{(N)}, \qquad \psi^{(i)} = \begin{pmatrix} \psi_{R,+}^{(i)} \\ \psi_{L,-}^{(i)} \end{pmatrix} \oplus \begin{pmatrix} \psi_{R,-}^{(i)} \\ \psi_{L,+}^{(i)} \end{pmatrix}, \qquad i = 1, \ldots, N,$$

where

$$\psi_{L,\pm}^{(i)} = \psi(x_i^L) \pm m^{-1}\psi'(x_i^L), \qquad \psi_{R,\pm}^{(i)} = \psi(x_i^R) \pm m^{-1}\psi'(x_i^R).$$

Consider now the space of functions from the domain of $\dot{L}_{a,V} \in \mathcal{L}_S$, which solve the equation (5.2) in the exterior of S. We will denote by W^{ext} the image of this space in W under the map π. Note that the subspace $W^{ext} \subset W$ is actually independent of the choice of $\{a_j\}$ and $\{V_j\}$. Since for any $h \in W^{ext}$ there exists a function $\psi \in \mathrm{dom}\, \dot{L}_{a,V}$, such that $h = \pi(\psi)$ and $(\dot{L}_{a,V} + m^2)\psi = 0$ on $\mathbb{R}\backslash\overline{S}$, the coordinates of h should satisfy the relations

$$h_{L,-}^{(1)} = h_{R,+}^{(N)} = 0, \tag{5.3}$$

$$h_{L,-}^{(i+1)} = w_i\, h_{R,-}^{(i)}, \qquad h_{R,+}^{(i)} = w_i\, h_{L,+}^{(i+1)}, \qquad w_i = e^{-m(x_{i+1}^L - x_i^R)}, \qquad i = 1, \ldots, N-1. \tag{5.4}$$

One can define along the same lines the space $W_{a,V}^{int}$ of boundary values of functions from the domain of $\dot{L}_{a,V}$, satisfying the equation (5.2) in the interior of S. It is straightforward to check that the coordinates of any vector $g \in W_{a,V}^{int}$ verify

$$\begin{pmatrix} g_{R,-}^{(i)} \\ g_{L,+}^{(i)} \end{pmatrix} = N_i(V_i) \begin{pmatrix} g_{R,+}^{(i)} \\ g_{L,-}^{(i)} \end{pmatrix}, \qquad N_i(V_i) = \begin{pmatrix} \alpha_i(V_i) & \beta_i(V_i) \\ \gamma_i(V_i) & \delta_i(V_i) \end{pmatrix}, \qquad i = 1, \ldots, N. \tag{5.5}$$

where we have introduced the notation

$$\alpha_i(V_i) = -\frac{V_i/2m}{1 + V_i/2m}\, e^{-2m(x_i^R - a_i)}, \qquad \delta_i(V_i) = -\frac{V_i/2m}{1 + V_i/2m}\, e^{-2m(a_i - x_i^L)}, \tag{5.6}$$

$$\beta_i(V_i) = \gamma_i(V_i) = \frac{e^{-m(x_i^R - x_i^L)}}{1 + V_i/2m}. \tag{5.7}$$

Proposition 5.2 *The subspaces W^{ext} and $W_{a,V}^{int}$ are transverse in W.*

■ The subspaces W^{ext} and $W_{a,V}^{int}$ have zero intersection. In the opposite case one would be able to correspond to any nontrivial vector $w \in W^{ext} \cap W_{a,V}^{int}$ a global solution of the equation (5.2). However, the existence of such solutions is forbidden by the Proposition 5.1. Observing that $\dim W^{ext} = \dim W_{a,V}^{int} = 2N$, one may now conclude that $W^{ext} \cup W_{a,V}^{int} = W$. □

We can thus associate to any operator $\dot{L}_{a,V} \in \mathcal{L}_S$ a point $W^{int}_{a,V}$ in the grassmannian $Gr(2N, 4N)$ of $2N$-dimensional subspaces of $W \simeq \mathbb{C}^{4N}$. Already at the present stage one could define the tau function as an invariant of four points of this grassmannian. Such construction, which can be thought of as a generalized cross-ratio, has been proposed in [7]:

Definition 5.3 *Let W be a $2k$-dimensional complex vector space, and let $Gr(k, 2k)$ denote the grassmannian of its k-dimensional subspaces. Given four points $W_1, W_2, W_3, W_4 \in Gr(k, 2k)$ in general position, the tau function $\tau(W_1, W_2, W_3, W_4)$ is defined as*

$$\tau(W_1, W_2, W_3, W_4) = \frac{\det\left(W_1 \xrightarrow{W_3} W_2\right)}{\det\left(W_1 \xrightarrow{W_4} W_2\right)},$$

where $W_1 \xrightarrow{W_j} W_2$ denotes the projection of W_1 on W_2 along W_j ($j = 3, 4$).

We will give a more specialized definition of the tau function. First, remark that our grassmannian has a distinguished point $W^{int}_F \stackrel{def}{=} W^{int}_{a,0}$, corresponding to Friedrichs extension of the operator $\dot{L}_a$, i. e. to ordinary second derivative operator on $H^2(\mathbb{R})$. This point can be used to construct the $\det^*$-bundle over $Gr(2N, 4N)$. Its fiber at the point $U \in Gr(2N, 4N)$ is a line $\lambda^*(U) \otimes \lambda(W^{int}_F)$, where $\lambda(U)$ denotes maximal exterior power of the space U.

Another distinguished point of the grassmannian is given by the subspace W^{ext}. The splitting $W = W^{ext} \oplus W^{int}_F$ allows to introduce the projection $U \xrightarrow{W^{ext}} W^{int}_F$ for any $U \in Gr(2N, 4N)$. This projection induces a linear map from $\lambda(U)$ to $\lambda(W^{int}_F)$, and thus defines a canonical section σ of the $\det^*$-bundle:

$$\sigma : U \mapsto \det\left(U \xrightarrow{W^{ext}} W^{int}_F\right) \in \lambda^*(U) \otimes \lambda(W^{int}_F).$$

Given another map $F : U \to W^{int}_F$, one would be able to construct in analogous manner a trivializing section $\delta : U \mapsto \det F$, and to define the tau function

$$\tau(U) = \frac{\sigma(U)}{\delta(U)} = \frac{\det\left(U \xrightarrow{W^{ext}} W^{int}_F\right)}{\det F}. \tag{5.8}$$

It is easy to see that this definition is independent of the choice of bases of U and W^{int}_F. Therefore, it indeed gives a function on the grassmannian and, consequently, on the family $\mathcal{L}_S$ of Schroedinger operators.

In order to define the map F, let us introduce the notion of auxiliary projections P_{S_j} ($j = 1, \ldots, N$). Suppose there are no delta-interactions at all and consider a single open interval $S' \in \mathbb{R}$. The space $W(S') = \mathbb{C}^2 \oplus \mathbb{C}^2$ is decomposed as above into the direct sum of two-dimensional subspaces $W^{int}(S')$ and $W^{ext}(S')$, consisting of boundary values of H^2-solutions of Schroedinger equation without point interactions on S' and $\mathbb{R} \backslash \overline{S'}$, correspondingly. We will denote by $P_{S'}$ the projection of $W(S')$ on $W^{int}(S')$ along $W^{ext}(S')$. The map $F : U \to W^{int}_F$ is now defined as the restriction to U of a direct sum of such projections: $F = \left(P_{S_1} \oplus \ldots \oplus P_{S_N}\right)\Big|_U$. It is straightforward to check that for any vector $\psi \in U$ we have

$$(F\psi)^{(j)} = P_{S_j}\psi^{(j)} = \begin{pmatrix} \psi^{(j)}_{R,+} \\ \psi^{(j)}_{L,-} \end{pmatrix} \oplus N_j(0) \begin{pmatrix} \psi^{(j)}_{R,+} \\ \psi^{(j)}_{L,-} \end{pmatrix}, \qquad j = 1, \ldots, N. \tag{5.9}$$

Let us now obtain the explicit form of the map F and of the canonical projection $P : U \stackrel{W^{ext}}{\longrightarrow} W_F^{int}$, putting $U = W_{a,V}^{int}$. First one should make some choice of coordinate bases. Remark that abitrary vectors $f \in W_{a,V}^{int}$ and $g \in W_F^{int}$ can be written (as elements of W) in the following way:

$$f^{(j)} = \tilde{f}_j \oplus N_j(V_j)\tilde{f}_j \,, \qquad g^{(j)} = \tilde{g}_j \oplus N_j(0)\tilde{g}_j \,, \qquad j = 1, \ldots, N, \tag{5.10}$$

where

$$\tilde{f}_j = \begin{pmatrix} f_{R,+}^{(j)} \\ f_{L,-}^{(j)} \end{pmatrix}, \qquad \tilde{g}_j = \begin{pmatrix} g_{R,+}^{(j)} \\ g_{L,-}^{(j)} \end{pmatrix},$$

and the matrices $\{N_j(V_j)\}_{j=1,\ldots,N}$ are defined by the formulas (5.5)–(5.7). Hence we can represent f and g by the columns

$$f = \left(\tilde{f}_1^T \ldots \tilde{f}_N^T\right)^T, \qquad g = \left(\tilde{g}_1^T \ldots \tilde{g}_N^T\right)^T.$$

It follows from (5.9)–(5.10) that the map F is given in these coordinates by the identity matrix.

In order to find the representation of P in such coordinates, one should be able to decompose any vector $f \in W_{a,V}^{int}$ as $f = g + h$, with $g \in W_F^{int}$ and $h \in W^{ext}$. Let us obtain the relation between f and g. Since the coordinates of $h = f - g$ should satisfy the conditions (5.3)–(5.4), one has

$$g_{L,-}^{(1)} = f_{L,-}^{(1)}, \qquad g_{R,+}^{(N)} = f_{R,+}^{(N)},$$

$$g_{L,-}^{(j+1)} - w_j\, g_{R,-}^{(j)} = f_{L,-}^{(j+1)} - w_j\, f_{R,-}^{(j)}, \qquad g_{R,+}^{(j)} - w_j\, g_{L,+}^{(j+1)} = f_{R,+}^{(j)} - w_j\, f_{L,+}^{(j+1)}, \qquad j = 1, \ldots, N-1.$$

Using the representations (5.10), we may eliminate from these relations the extra variables $f_{R,-}^{(j)}$, $f_{L,+}^{(j)}$, $g_{R,-}^{(j)}$, and $g_{L,+}^{(j)}$ $(j = 1, \ldots, N)$. Obtained system of linear equations has the form

$$(\mathbf{1} + M_{a,V})\, f = (\mathbf{1} + M_0)\, g, \tag{5.11}$$

where $2N \times 2N$ matrices $M_{a,V}$ and M_0 are defined as

$$M_{a,V} = \begin{pmatrix} 0 & Q_1(V_2) & 0 & . & 0 \\ T_1(V_1) & 0 & Q_2(V_3) & . & 0 \\ 0 & T_2(V_2) & 0 & . & . \\ . & . & . & . & Q_{N-1}(V_N) \\ 0 & 0 & . & T_{N-1}(V_{N-1}) & 0 \end{pmatrix},$$

$$M_0 = \begin{pmatrix} 0 & Q_1(0) & 0 & . & 0 \\ T_1(0) & 0 & Q_2(0) & . & 0 \\ 0 & T_2(0) & 0 & . & . \\ . & . & . & . & Q_{N-1}(0) \\ 0 & 0 & . & T_{N-1}(0) & 0 \end{pmatrix}.$$

Auxiliary 2×2 matrices $Q_j(V_{j+1})$ and $T_j(V_j)$, entering these formulas, are given by

$$Q_j(V_{j+1}) = \begin{pmatrix} -w_j\, \gamma_{j+1}(V_{j+1}) & -w_j\, \delta_{j+1}(V_{j+1}) \\ 0 & 0 \end{pmatrix}, \quad T_j(V_j) = \begin{pmatrix} 0 & 0 \\ -w_j\, \alpha_j(V_j) & -w_j\, \beta_j(V_j) \end{pmatrix}.$$

The system (5.11) implies that the tau function (5.8), evaluated at the point $W_{a,V}^{int} \in Gr(2N, 4N)$, is equal to the ratio of determinants

$$\tau(W_{a,V}^{int}) = \frac{\det(\mathbf{1} + M_{a,V})}{\det(\mathbf{1} + M_0)}. \tag{5.12}$$

However, this expression may be simplified. Since $T_j(0)Q_j(0) = 0$, the matrix $\mathbf{1} + M_0$ can be represented as a product of a lower triangular and an upper triangular matrix with identities on their diagonals. Therefore, the determinant in the denominator of (5.12) is equal to 1. Moreover, one can show[2] that our tau-function does not depend on the choice of localization, i. e. on the coordinates $\left\{x_j^{L,R}\right\}_{j=1,\ldots,N}$. In particular, we may put $x_j^L = x_j^R = a_j$ for $j = 1, \ldots, N$ and obtain the following representation:

$$\tau(W_{a,V}^{int}) = \det \begin{pmatrix} \mathbf{1} & \tilde{Q}_1 & 0 & . & 0 \\ \tilde{T}_1 & \mathbf{1} & \tilde{Q}_2 & . & 0 \\ 0 & \tilde{T}_2 & \mathbf{1} & . & . \\ . & . & . & . & \tilde{Q}_{N-1} \\ 0 & 0 & . & \tilde{T}_{N-1} & \mathbf{1} \end{pmatrix}, \tag{5.13}$$

where

$$\tilde{Q}_j = \frac{e^{-m(a_{j+1}-a_j)}}{1 + \frac{V_{j+1}}{2m}} \begin{pmatrix} -1 & \frac{V_{j+1}}{2m} \\ 0 & 0 \end{pmatrix}, \qquad \tilde{T}_j = \frac{e^{-m(a_{j+1}-a_j)}}{1 + \frac{V_j}{2m}} \begin{pmatrix} 0 & 0 \\ \frac{V_j}{2m} & -1 \end{pmatrix}, \qquad j = 1, \ldots, N-1.$$

Finally, it is worth mentioning that the tau function $\tau(W_{a,V}^{int})$ and the correlation functions of Bogolyubov fields, associated to δ-interactions, are related by

$$\tau(W_{a,V}^{int}) = \left[\frac{\langle \mathcal{O}_{V_1}(a_1) \ldots \mathcal{O}_{V_N}(a_N)\rangle}{\langle \mathcal{O}_{V_1}\rangle \ldots \langle \mathcal{O}_{V_N}\rangle}\right]^{-2}. \tag{5.14}$$

This formula may be checked explicitly for small values of N by comparison of (5.13) and (3.6).

Example. For the two-point tau function one has

$$\tau(W_{a,V}^{int}) = \det \begin{pmatrix} \mathbf{1} & \tilde{Q}_1 \\ \tilde{T}_1 & \mathbf{1} \end{pmatrix} = \det\left(\mathbf{1} - \tilde{T}_1 \tilde{Q}_1\right) = 1 - \frac{V_1/2m}{1 + V_1/2m} \frac{V_2/2m}{1 + V_2/2m} e^{-2m(a_2-a_1)}.$$

Comparing this expression with the formulas (4.8) and (4.21), we may check the validity of (5.14) for $N = 2$.

6 Discussion

It is well-known that in order to describe delta-interactions in two and three dimensions, one needs to renormalize the strengths of point sources. This obstacle complicates the construction

[2]The proof is based on Krein's formula for the resolvents of self-adjoint extensions of $\dot{L}_a$ and lies beyond the scope of this paper.

of fields, associated to delta-interactions in higher dimensions. In particular, a naive generalization, $\hat{\mathcal{O}}_V(t,\vec{x}) = \exp\left\{-\frac{1}{2}V\hat{\varphi}^2(t,\vec{x})\right\}$, does not work, since even the vacuum expectation value of such an operator is infinite. It would be interesting to understand how the renormalization can be described (if it indeed can) in terms of Bogolyubov fields.

It would also be interesting to consider instead of the Schroedinger operator the hamiltonian of Dirac fermions in $2+1$ dimensions in the background of magnetic vortices. Its resolvent is the principal ingredient in the computation of induced fermionic vacuum quantum numbers. It also contains the information about the scattering of fermions, their bound states, etc. It seems that this resolvent may be expressed along the lines of the present paper (at least, for some values of self-adjoint extension parameters) through the ratio of certain correlation functions in the two-dimensional euclidean quantum field theory, described by the Dirac action with a parity-breaking term. Thus one needs to find form factors and correlation functions of the Bogolyubov fields, associated to magnetic vortices. The absence of parity-breaking term means that the resolvent is calculated at zero energy. Corresponding Bogolyubov fields reduce in this case to the exponential fields of the SG_{ff}-theory.

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden, *Solvable models in quantum mechanics*, Springer, (1988).

[2] F. A. Berezin, *Method of secondary quantization*, Nauka, Moscow, (1965).

[3] D. Bernard, A. Leclair, *Differential equations for sine-Gordon correlation functions at the free fermion point*, Nucl.Phys. **B426**, (1994), 534-558; Erratum-ibid. **B498**, (1997), 619-621.

[4] S. Coleman, *Quantum sine-Gordon equation as the massive Thirring model*, Phys. Rev. **D11**, (1975), 2088–2097.

[5] C. Itzykson, J. B. Zuber, *Quantum field theory and the two-dimensional Ising model*, Phys. Rev. **D15**, (1977), 2875–2884.

[6] L. P. Kadanoff, H. Ceva, *Determination of an operator algebra for the two-dimensional Ising model*, Phys. Rev. **B3**, (1971), 3918–3938.

[7] L. J. Mason, M. A. Singer, N. M. J. Woodhouse, *Tau functions and the twistor theory of integrable systems*, J. Geom. Phys. **32**, (2000), 397–430.

[8] J. Palmer, *Determinants of Cauchy-Riemann operators as τ-functions*, Acta Appl. Math. **18**, (1990), 199–223.

[9] J. Palmer, *Tau functions for the Dirac operator in the Euclidean plane*, Pacific J. Math. **160**, (1993), 259–342.

[10] M. Sato, T. Miwa, M. Jimbo, *Holonomic quantum fields I–V*, Publ. RIMS, Kyoto Univ. **14**, (1978), 223–267; **15**, (1979), 201–278; **15**, (1979), 577–629; **15**, (1979), 871–972; **16**, (1980), 531–584.

[11] B. Schroer, T. T. Truong, *The order/disorder quantum field operators associated with the two-dimensional Ising model in the continuum limit*, Nucl. Phys. **B144**, (1978), 80–122.

[12] G. Segal, G. Wilson, *Loop groups and equations of KdV type*, Publ. Math. I.H.E.S. **63**, (1985), 1–64.

[13] Yu. A. Sitenko, *The Aharonov-Bohm effect and inducing of vacuum charge by a singular magnetic string*, Nucl. Phys. **B372**, (1992), 622–634.