Sgríbhinní Institiúid Árd-Léighinn
Bhaile Átha Cliath. Sraith A Uimh. 3

Communications of the Dublin Institute for
Advanced Studies. Series A No. 3

DEVELOPMENTS IN QUANTUM ELECTRODYNAMICS

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DUBLIN

THE DUBLIN INSTITUTE FOR ADVANCED STUDIES

64-5 MERRION SQUARE

1946

Reprinted 1968

DEVELOPMENTS IN QUANTUM ELECTRODYNAMICS.

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§ 1. *Recapitulation.*

Quantum electrodynamics is based on analogy with classical electrodynamics. We start out with the equations of classical electrodynamics, consisting of Maxwell's equations for the electromagnetic field and Lorentz's equations for the motion of charged particles, the latter equations including terms to represent the effect of radiation damping. In order that we may have a precise mathematical theory, we picture the particles as point charges for the purpose of determining their action on the electromagnetic field. This classical theory is in exact agreement with the laws of conservation of energy and momentum and with the requirements of relativity, and provides a satisfactory foundation on which to build, although there are some difficulties in its application, which will be discussed at the end of § 3.

Before one can pass to the quantum theory one must put the classical equations into Hamiltonian form. The method of doing this has been described in my previous lectures[1] and the results only will be given here. There is one Hamiltonian for each particle, that for the i-th being

$$F_i \equiv -\frac{1}{2m_i}\left\{\left(\mathbf{p}_i - e_i\mathbf{A}(\mathbf{z}_i)\right)^2 - m_i^2\right\} = 0 \qquad (1\cdot1)$$

where $\mathbf{z}_i$ and $\mathbf{p}_i$ are 4-vectors which give the position in space-time and the momentum-energy of the i-th particle, in terms of units which make the velocity of light unity, and e_i and m_i are the charge and rest-mass of this particle. These Hamiltonians determine the dependence of any dynamical variable ξ on the proper time s_i of the i-th particle by means of the equation

$$\frac{d\xi}{ds_i} = [\xi, F_i], \qquad (1\cdot2)$$

[1] Communications of the Dublin Institute for Advanced Studies, Series A, No. 1. This will be referred to later by I. A mistake in this paper may be pointed out here, namely, the μ's introduced on page 7 should be proved to be equal and cannot be arranged by definitions to be equal. The equation

$$\langle b \mid (\xi \mid a)\rangle = (\langle b \mid \xi) \mid a\rangle$$

gives

$$\mu_1 b_1 a_0 + \mu_2 b_2 a_1 + \mu_3 b_3 a_2 + \ldots = \mu_0 b_1 a_0 + \mu_1 b_2 a_1 + \mu_2 b_3 a_2 + \ldots$$

This must hold for all values of the b's and a's, and hence the μ's must be equal. We can now redefine the a's and b's to make the μ's unity.

where the condition that $F_i = 0$ must not be used before evaluating the Poisson Bracket. We assume the P.B.'s of the particle variables $z_{\mu i}$, $p_{\mu i}$ with one another to have their usual values, the P.B. of a particle variable and a field variable $A_\mu(x)$ to be zero, and the P.B. of two field variables $A_\mu(\mathbf{x})$, $A_\nu(\mathbf{x}')$, where $\mathbf{x}$ and $\mathbf{x}'$ are any two points in space-time, to be

$$[A_\mu(\mathbf{x}),\ A_\nu(\mathbf{x}')] = \tfrac{1}{2} g_{\mu\nu} \{ \Delta(\mathbf{x} - \mathbf{x}' + \lambda) + \Delta(\mathbf{x} - \mathbf{x}' - \lambda) \}. \quad (1{\cdot}3)$$

The λ here is a small time-like 4-vector, which is ultimately to be made to tend to zero. The equations

$$\Box A = 0 \quad (1{\cdot}4)$$

$$\frac{\partial A_\mu(x)}{\partial x_\mu} = -\tfrac{1}{2} \Sigma_i e_i \{ \Delta(\mathbf{x} - \mathbf{x}' + \lambda) + \Delta(\mathbf{x} - \mathbf{x}' - \lambda) \} \quad (1{\cdot}5)$$

are also needed, and as they are not deducible from the equations of motion (1·2), although they are consistent with these equations of motion, they must be assumed as supplementary conditions. The equation (1·5), like the equation $F_i = 0$, must not be used before evaluating a P.B.

The foregoing scheme of Hamiltonians, P.B.'s and supplementary conditions reproduces the Maxwell Lorentz equations in the limit $\lambda \to 0$ when the times of all the particles z_{0i} are put equal, and it is of a suitable form from which one can make a passage to the quantum theory According to the usual rules for this passage, one must make all the dynamical variables into linear operators satisfying commutation relations corresponding to their P.B. relations in the classical theory, and must use the Hamiltonians (1·1) to provide wave equations

$$\{ (\mathbf{p}_i - e_i \mathbf{A}(\mathbf{z}_i))^2 - m_i^2 \} | \rangle = 0, \quad (1{\cdot}6)$$

$| \rangle$ denoting a vector, called a ket-vector, in a many-dimensional space, the coordinates of which vector form the Schrödinger wave function. Also the equation (1·5) must be used to provide supplementary conditions on the wave function or on $| \rangle$, namely

$$\left\{ \frac{\partial A_\mu(\mathbf{x})}{\partial x_\mu} + \tfrac{1}{2} \Sigma_i e_i \{ \Delta(\mathbf{x} - \mathbf{z}_i + \lambda) + \Delta(\mathbf{x} - \mathbf{z}_i - \lambda) \} \right\} | \rangle = 0. \quad (1{\cdot}7)$$

The equation (1·4), however, is to be retained unchanged in the quantum theory. The difference in the treatments of (1·4) and (1·5) is due to their different status, (1·4) being an equation such that one can take the P.B. of both sides of it with any dynamical variable and get a correct result, but not so (1·5).

The wave equations (1·6) and supplementary conditions (1·7) are easily seen to be self-consistent, provided one restricts oneself to consider

only positions for the particles z_i which are all outside each other's light-cones, or rather which satisfy the slightly more stringent conditions

$$(\mathbf{z}_i - \mathbf{z}_j \pm \lambda)^2 < 0 \qquad i \neq j. \tag{1·8}$$

To prove the self-consistency, one has only to verify that the linear operators operating on $|\ \rangle$ in the various equations (1·6) and (1·7) obtained by putting different values for i in (1·6) and different values for $\mathbf{x}$ in (1·7) all commute with each other, provided the condition (1·8) holds. The condition (1·8) does not restrict the range of applicability of the theory, since in practice one is interested in all the times z_{0i} being equal and then the condition (1·8) merely prevents two particles being closer together than a distance of order λ, and since λ is to be considered as excessively small and is ultimately made to tend to zero, this does not matter.

The wave equations (1·6) refer to particles without spin, for which one must use the Klein-Gordon interpretation of the wave functions, involving sometimes negative probabilities for the particles to be in certain positions. One can modify the theory to make it refer to particles of spin $\frac{1}{2}\hbar$ and at the same time eliminate these particular negative probabilities by replacing the wave equations (1·6) by

$$\{p_{0i} - e_i A_0(\mathbf{z}_i) + \Sigma_r\, a_{ri}\,(p_{ri} - e_i A_r(\mathbf{z}_i)) + a_{mi}\, m_i\}\ |\ \rangle = 0 \tag{1·9}$$

the variables a_{ri}, a_{mi} being spin variables of the usual kind belonging to the i-th particle. This replacement does not affect the self-consistency of the equations. The present lectures will be concerned mainly with the treatment of the field variables and with general properties of the theory, for which it does not matter whether one uses the equations (1·6) or (1·9).

Equation (1·4) shows that the field can be resolved into waves all moving with the velocity of light. The usual Maxwell field cannot be so resolved when there are charges present, but the field occurring in the Hamiltonian formulation of electrodynamics is defined somewhat differently from the usual Maxwell field. It is called the Wentzel field to distinguish it, and its properties were dealt with in I.

The Wentzel field may be split up into two parts, a part consisting of longitudinal waves and a part consisting of transverse waves. By making a certain mathematical transformation one can eliminate the variables describing the longitudinal waves from all the equations, in either classical or quantum theory, as was shown in I. Some new terms appear in the Hamiltonians to replace the longitudinal waves, and these terms just give rise to the Coulomb forces between the particles, while the supplementary conditions (1·5) or (1·7) are automatically satisfied and drop out. The new formulation of the theory so obtained is much simpler than the previous one from the practical point of view, since it

contains only half as many field variables. But the new formulation is not of relativistic form, since the splitting of the field into longitudinal and transverse parts is not Lorentz invariant, and so it may be unsuitable for certain theoretical investigations.

The development of the theory so far has been fairly straightforward and has led to a scheme of equations for quantum electrodynamics in which one can have confidence. It remains to solve the equations and to get a physical interpretation of the solution, and here the difficulties arise. The obvious way of solving the equations is to use a perturbation method, after eliminating the longitudinal waves, treating the interaction between the particles and the transverse waves as small and working in a representation referring to the numbers of photons in the various Fourier components of the field, but this way leads to divergent integrals. A deeper analysis of the problem is therefore necessary.

§ 2. *The Method of Redundant Variables.*

A powerful way of handling the wave equations is provided by the method of redundant variables. A form of the method was given in I and was there connected with a special assumption for the physical interpretation of the wave functions. The method will be here formulated in a more general way, in which it can be applied independently of any special assumptions for physical interpretation.

We express the field $A_\mu(\mathbf{x})$ as the sum of two fields

$$A_\mu(\mathbf{x}) = M_\mu(\mathbf{x}) + N_\mu(\mathbf{x} + \lambda), \tag{2·1}$$

where the fields $\mathbf{M}$ and $\mathbf{N}$ are both resolvable into waves travelling with the velocity of light, i.e.

$$\Box \mathbf{M} = 0, \qquad \Box \mathbf{N} = 0, \tag{2·2}$$

and satisfy the commutation relations

$$\left.\begin{aligned} [M_\mu(\mathbf{x}),\ M_\nu(\mathbf{x}')] = 0, \qquad [N_\mu(\mathbf{x}),\ N_\nu(\mathbf{x}')] = 0, \\ [M_\mu(\mathbf{x}),\ N_\nu(\mathbf{x}')] = \tfrac{1}{2} g_{\mu\nu} \Delta(\mathbf{x} - \mathbf{x}'). \end{aligned}\right\} \tag{2·3}$$

Those commutation relations make

$$\begin{aligned} [M_\mu(\mathbf{x}) + N_\mu(\mathbf{x} + \lambda),\ M_\nu(\mathbf{x}') + N_\nu(\mathbf{x}' + \lambda)] = \\ = \tfrac{1}{2} g_{\mu\nu} \{\Delta(\mathbf{x} - \mathbf{x}' + \lambda) + \Delta(\mathbf{x} - \mathbf{x}' - \lambda)\}. \end{aligned} \tag{2·4}$$

It is legitimate to express $A_\mu(\mathbf{x})$ in the form (2·1), since this leads, from (2·4), to the correct commutation relations (1·3) for $A_\mu(\mathbf{x})$, and also (1·4) follows from (2·2). The field variables $M_\mu(\mathbf{x})$, $N_\nu(\mathbf{x}')$ may be assumed to commute with all the particle variables $z_{\mu i}$, $p_{\mu i}$.

Since the variables $M_\mu(\mathbf{x})$ for different values of μ and $\mathbf{x}$ all commute with one another, we can set up a representation in which they are all

diagonal. The $M_\mu(\mathbf{x})$ now play the rôle of field coordinates and the $N_\mu(\mathbf{x})$, or rather certain linear combinations of the $N_\mu(\mathbf{x})$, play the rôle of field momenta. To complete the determination of the representation we may take the particle coordinates $z_{\mu i}$ to be also diagonal, so that a wave function appears as $\langle z_{\mu i}, M_\mu(\mathbf{x})|\rangle$, a function of the variables $z_{\mu i}$ for all μ and i and the variables $M_\mu(\mathbf{x})$ for all μ and $\mathbf{x}$. These wave functions are not the ones needed for physical interpretation, since in practice one is interested in the probability of various numbers of photons being in various states and not in the probability of the field variables $M_\mu(\mathbf{x})$ having certain values, but the representation is a good one for examining the wave equations mathematically, on account of its simplicity and the directness with which it can be introduced.

When we express the $A_\mu(\mathbf{x})$ in the form (2·1) we double the number of field variables occurring in the theory, since we have eight functions of position $M_\mu(\mathbf{x})$, $N_\mu(\mathbf{x})$ ($\mu = 0, 1, 2, 3$) instead of the four composing $A_\mu(\mathbf{x})$. This doubling means that we have introduced some unnecessary variables that have no physical meaning. Put

$$B_\mu(\mathbf{x}) = M_\mu(\mathbf{x}) - N_\mu(\mathbf{x} - \lambda). \tag{2·5}$$

Then

$$[A_\mu(\mathbf{x}),\ B_\nu(\mathbf{x}')] = [N_\mu(\mathbf{x} + \lambda),\ M_\nu(\mathbf{x}')] - [M_\mu(\mathbf{x}),\ N_\nu(\mathbf{x}' - \lambda)] = 0$$

from (2·3). The $B_\nu(\mathbf{x}')$ thus commute with the $A_\mu(\mathbf{x})$ and therefore, since they also commute with the particle variables, *they commute with all the variables occurring in the wave equations and supplementary conditions.* These are then the unnecessary variables without physical meaning. We call them redundant variables. Their commutation relations amongst themselves are

$$[B_\mu(\mathbf{x}),\ B_\nu(\mathbf{x}')] = -\tfrac{1}{2} g_{\mu\nu}\{\Delta(\mathbf{x} - \mathbf{x}' + \lambda) + \Delta(\mathbf{x} - \mathbf{x}' - \lambda)\}, \tag{2·6}$$

the same form as (1·3) except for a difference in sign.

Let us examine in a general way the effect of redundant variables in quantum mechanics. Consider a general dynamical system treated with redundant variables. Set up a representation which separates the redundant and the physical variables, so that some of the variables occurring in the wave function are physical variables and the others are redundant variables. Calling the physical variables in the wave function q and the redundant variables r, the wave function will be written $\langle q r|\rangle$, and will satisfy a wave equation in which the operators operate only on the q variables. The r variables now appear as parameters inserted arbitrarily into the solution of the wave equation, each set of numerical values that one assigns to the r's giving one solution of the wave equation. To get a physical interpretation for the wave function, we must integrate over all values for the variables r. Thus we can form

$\int |\langle q\,r\,|\,\rangle|^2 dr$ and consider it as giving the probability distribution for the physical variables q, and we can also form $\int \langle q\,r\,|\,\rangle\, dr\, \langle\,|\,q'\,r\,\rangle$ and consider it as a matrix in q and q' giving the density of a Gibbs ensemble.

When we do not have redundant variables in the treatment, to get a general Gibbs ensemble we need several wave functions. However, with the redundant variables, one wave function is sufficient, since the one wave function $\langle q\,r\,|\,\rangle$ contains several solutions of the wave equation without redundant variables, one arising with each set of numerical values for the r's.

We can get a more general interpretation for the wave function $\langle q\,r\,|\,\rangle$ by introducing a weight function $\rho(r)$ in the redundant variables. Thus we can consider $\int |\langle q\,r\,|\,\rangle|^2\,\rho(r)\,dr$ as giving the probability distribution for the q's, and $\int \langle q\,r\,|\,\rangle\,\rho(r)\,dr\,\langle\,|\,q'r\rangle$ as giving the density of the Gibbs ensemble. The weight function $\rho(r)$ is completely arbitrary, except for the condition that it must be aways positive or zero if we want to restrict our Gibbs density to be positive definite and to give only positive probabilities. We thus see that, *in a treatment with redundant variables, the physical interpretation of the wave functions is not determined until we choose a weight function in the redundant variables.* This general conclusion must still hold if we are working in a representation which mixes up the redundant and physical variables, as must the previous conclusion that a single wave function can give a general Gibbs ensemble, since a mere change of representation cannot affect such questions.

With a weight function $\rho(r)$, the general rule for physical interpretation is that the average value of an observable ξ for the Gibbs ensemble corresponding to the ket vector $|\,A\,\rangle$ is

$$\langle A\,|\,\xi\rho\,|\,A\,\rangle / \langle A\,|\,\rho\,|\,A\,\rangle \qquad (2{\cdot}7)$$

Let us return to quantum electrodynamics. Instead of introducing the redundant variables by means of equations (2·1), (2·2), (2·3), we might first eliminate the longitudinal waves and then introduce the redundant variables. This will lead to fewer redundant variables being introduced into the theory. The elimination of the longitudinal waves leaves wave equations involving only a transverse field, $A^t{}_\mu(\mathbf{x})$ say, satisfying

$$A^t{}_0 = 0\,, \quad \Box A^t{}_r = 0\,, \quad \frac{\partial A^t{}_r(\mathbf{x})}{\partial x_r} = 0 \quad \cdot\ (r = 1, 2, 3)\,. \qquad (2{\cdot}8)$$

The commutation relations for this transverse field are, corresponding to (1·3),

$$[A^t{}_r(\mathbf{x})\,,\ A^t{}_s(\mathbf{x}')] = \tfrac{1}{2}\{g_{rs}\Delta(\mathbf{x} - \mathbf{x}' + \lambda) + g_{rs}\Delta(\mathbf{x} - \mathbf{x}' - \lambda)\}^t\,, \qquad (2{\cdot}9)$$

where $\{g_{rs}\Delta(\mathbf{x})\}^t$ means the transverse part of $g_{rs}\Delta(\mathbf{x})$, obtained by resolving $\Delta(\mathbf{x})$ into its Fourier components and multiplying each

component moving in the direction k_μ (with $k_\mu k^\mu = 0$) by $g_{rs} + k_r k_s / k_0^2$. A straightforward calculation gives

$$\{g_{rs}\,\Delta(\mathbf{x})\}^t = g_{rs}\,\Delta(\mathbf{x}) + \frac{\partial^2}{\partial x_r\,\partial x_s}\begin{pmatrix} 1 \\ -x_0/|x| \\ -1 \end{pmatrix} \tag{2·10}$$

$$= \left(\frac{x_r\,x_s}{|x|^2} - \delta_{rs}\right)\Delta(\mathbf{x}) + \left(\frac{x_0\,\delta_{rs}}{|x|^3} - \frac{3\,x_0\,x_r\,x_s}{|x|^5}\right)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2·11}$$

where $|x| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$ and for the triplets such as $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ the first member is to be taken when $\mathbf{x}$ is inside the past light-cone, the second when $\mathbf{x}$ is outside the light-cone, and the third when it is inside the future light-cone. Note that

$$\frac{\partial}{\partial x_r}\left\{g_{rs}\,\Delta(\mathbf{x})\right\}^t = \frac{\partial}{\partial x_s}\left\{g_{rs}\,\Delta(\mathbf{x})\right\}^t = 0$$

We may now express $A^t{}_r(\mathbf{x})$ as

$$A_r{}^t(\mathbf{x}) = M_r{}^t(\mathbf{x}) + N_r{}^t(\mathbf{x} + \boldsymbol{\lambda}) \,, \tag{2·12}$$

where $M_r{}^t$ and $N_r{}^t$ are transverse fields resolvable into waves travelling with the velocity of light, i.e.

$$\left.\begin{aligned} \Box\, M_r{}^t &= 0\,, & \Box\, N_r^t &= 0 \\ \frac{\partial\, M_r{}^t(\mathbf{x})}{\partial\, x_r} &= 0\,, & \frac{\partial\, N_r{}^t(\mathbf{x})}{\partial\, x_r} &= 0\,, \end{aligned}\right\} \tag{2·13}$$

and satisfy the commutation relations

$$\left.\begin{aligned} [M_r{}^t(\mathbf{x}),\ M_s{}^t(\mathbf{x}')] &= 0\,, & [N_r{}^t(\mathbf{x}),\ N_s{}^t(\mathbf{x}')] &= 0, \\ [M_r{}^t(\mathbf{x}),\ N_s{}^t(\mathbf{x}')] &= \tfrac{1}{2}\,\{g_{rs}\,\Delta(\mathbf{x} - \mathbf{x}')\}^t. \end{aligned}\right\} \tag{2·14}$$

This is legitimate, since equations (2·12), (2·13), (2·14) make $A_r{}^t(\mathbf{x})$ satisfy all the conditions it has to according to (2·8) and (2·9). Putting

$$B_r{}^t(\mathbf{x}) = M_r{}^t(\mathbf{x}) - N_r{}^t(\mathbf{x} - \boldsymbol{\lambda}), \tag{2·15}$$

we find that the $B_r{}^t(\mathbf{x})$ commute with $A_s{}^t(\mathbf{x}')$ and thus with all the variables occurring in the Hamiltonians, so they are the redundant variables. They satisfy the commutation relations

$$[B_r{}^t(\mathbf{x}),\ B_s{}^t(\mathbf{x}')] = -\tfrac{1}{2}\{g_{rs}\,\Delta(\mathbf{x} - \mathbf{x}' + \boldsymbol{\lambda}) + g_{rs}\,\Delta(\mathbf{x} - \mathbf{x}' - \boldsymbol{\lambda})\}^t, \tag{2·16}$$

the same form as (2·9) except for a difference in sign.

We assume the field variables $M_r{}^t(\mathbf{x})$, $N_s{}^t(\mathbf{x}')$ commute with all the particle variables $z_{\mu i}$, $p_{\mu i}$. We can now set up a representation in which the field variables $M_r{}^t(\mathbf{x})$ and the particle coordinates $z_{\mu i}$ are diagonal, so that a wave function appears as $\langle z_{\mu i},\ M_r{}^t(\mathbf{x}) \mid \rangle$. This wave function has a point in its domain corresponding to each set of values for the variables $z_{\mu i}$ satisfying (1·8) together with a set of functions $M_r{}^t(\mathbf{x})$ satisfying the conditions given for $M_r{}^t$ in (2·13), so that the wave function is a *functional* of the functions $M_r{}^t(\mathbf{x})$. The operator $p_{\mu i}$ applied to $\mid \rangle$ results in the operator $i\hbar \dfrac{\partial}{\partial z_i{}^\mu}$ being applied to the wave function, as is usual in quantum mechanics, while the operator $N_s{}^t(\mathbf{x}')$ applied to $\mid \rangle$ results in a certain operator of functional differentiation being applied to the wave function, defined by

$$\langle \mathbf{z}_i,\ M_r{}^t(\mathbf{x}) \mid N_s{}^t(\mathbf{x}') \mid \rangle = \lim_{\epsilon \to 0} \epsilon^{-1} \left[\langle \mathbf{z}_i,\ M_r{}^t(\mathbf{x}) - \tfrac{1}{2} i \hbar \epsilon \{ g_{rs} \Delta(\mathbf{x} - \mathbf{x}') \}^t \mid \rangle - \langle \mathbf{z}_i,\ M_r{}^t(\mathbf{x}) \mid \rangle \right]. \tag{2·17}$$

The presence of redundant variables in the theory results in a great deal of arbitrariness in the solutions of the wave equations, since we can operate on any solution with any function of the redundant variables and get another solution. There is a greater generality in the solutions of the wave equations than is needed for the treatment of physical problems. It becomes desirable to restrict the solutions of the wave equations as far as possible without spoiling their adequacy for the treatment of all physical problems. The following restrictions will be found suitable, namely, we consider only those solutions $\mid \rangle$ of the wave equations that satisfy

$$N_s{}^t(\mathbf{x}') \mid \rangle = 0 \quad \text{for } \mathbf{x}' \text{ inside future light-cone of all } \mathbf{z}_i. \tag{2·18}$$

Let us examine the consistency of the restrictions (2·18). There is one condition for each point $\mathbf{x}'$ that lies inside the future light-cone of all the particle points $\mathbf{z}_i$, and these conditions are all consistent with one another since the operators $N_s{}^t(\mathbf{x}')$ all commute, from (2·14). Again, $N_s{}^t(\mathbf{x}')$ commutes with all the variables occurring in the Hamiltonians except $A_r{}^t(\mathbf{z}_i)$ and its P.B. with this is, according to (2·12) and (2·14),

$$\tfrac{1}{2} \{ g_{rs} \Delta(\mathbf{z}_i - \mathbf{x}') \}^t .$$

From (2·11) this vanishes for $\mathbf{x}'$ inside the future light-cone of $\mathbf{z}_i$, and hence the conditions (2·18) are consistent with the wave equations.

Let us now see what effect the restrictions (2·18) will have on the wave function $\langle \mathbf{z}_i,\ M_r{}^t(\mathbf{x}) \mid \rangle$. The restictions will read

$$\langle \mathbf{z}_i,\ M_r{}^t(\mathbf{x}) \mid N_s{}^t(\mathbf{x}') \mid \rangle = 0$$

for $\mathbf{x}'$ inside the future light-cone of all $\mathbf{z}_i$. From (2·17) we see that the wave function must remain unaltered if we substitute

$$M_r^t(\mathbf{x}) - \tfrac{1}{2} i\hbar\varepsilon \{g_{rs}\Delta(\mathbf{x}-\mathbf{x}')\}^t$$

for $M_r^t(\mathbf{x})$, where $\mathbf{x}'$ is any point inside the future light-cone of all $\mathbf{z}_i$. More generally, the wave function must remain unaltered if we change $M_r^t(\mathbf{x})$ by any field $K_r^t(\mathbf{x})$ expressible linearly in terms of the various quantities $\{g_{rs}\Delta(\mathbf{x}-\mathbf{x}')\}^t$ obtained by giving different values to s and taking $\mathbf{x}'$ to be different points inside the future light-cone of all $\mathbf{z}_i$. Now if $K_r^t(\mathbf{x})$ is any field of transverse waves which vanishes at every point inside the past light-cone of a $\mathbf{z}_i$, it is expressible linearly in this form, as is shown by the following argument.

It is sufficient to consider the case when $K_r^t(\mathbf{x})$ at any time lies entirely in a finite volume, since if the required result can be proved for this case, it can be proved also for the case when $K_r^t(\mathbf{x})$ extends to infinity, by the application of a limiting process. With $K_r^t(\mathbf{x})$ lying entirely in a finite volume at a certain time, we can choose a time x'_0 sufficiently far in the future so that K_r^t then consists only of outward going waves. These outward going waves must lie entirely within the future light-cone of every $\mathbf{z}_i$, if $K_r^t(\mathbf{x})$ vanishes at every point inside the past light-cone of a $\mathbf{z}_i$. We can express $K_r^t(\mathbf{x})$ in the form

$$K_r^t(\mathbf{x}) = -\frac{1}{4\pi}\iiint\Big[K_r^t(\mathbf{x}')\frac{\partial}{\partial x'_0}\Delta(\mathbf{x}-\mathbf{x}') + \frac{\partial K_r^t(\mathbf{x}')}{\partial x'_0}\Delta(\mathbf{x}-\mathbf{x}')\Big]dx'_1\,dx'_2\,dx'_3, \qquad (2\cdot19)$$

as may be verified by observing that the right-hand side of (2·19) gives, with the help of well-known properties of the Δ function (see equation (4·7) of I), the correct value for $K_r^t(\mathbf{x})$ and $\partial K_r^t(\mathbf{x})/\partial x_0$ at the time $x_0 = x'_0$, which is sufficient to ensure that it gives the correct value of $K_r^t(\mathbf{x})$ throughout space-time. (2·19) leads to

$$K_r^t(\mathbf{x}) = -\frac{1}{4\pi}\iiint\Big[K_s^t(\mathbf{x}')\frac{\partial}{\partial x'_0}\{g_r{}^s\Delta(\mathbf{x}-\mathbf{x}')\}^t + \frac{\partial K_s^t(\mathbf{x}')}{\partial x'_0}\{g_r{}^s\Delta(\mathbf{x}-\mathbf{x}')\}^t\Big]dx'_1\,dx'_2\,dx'_3 \qquad (2\cdot20)$$

with the help of (2·10) and an integration by parts, and of the transverse wave conditions

$$\frac{\partial}{\partial x'_s}K_s^t(\mathbf{x}') = 0, \qquad \frac{\partial}{\partial x'_s}\frac{\partial K_s^t(\mathbf{x}')}{\partial x'_0} = 0.$$

Since $K_s^t(\mathbf{x}')$ and $\partial K_s^t(\mathbf{x}')/\partial x'_0$ vanish except when $\mathbf{x}'$ lies inside the future light-cone of every $\mathbf{z}_i$, equation (2·20) gives $K_r^t(\mathbf{x})$ expressed linearly in terms of $\{g_{rs}\Delta(\mathbf{x}-\mathbf{x}')\}^t$ with $\mathbf{x}'$ inside the future light-cone of all $\mathbf{z}_i$.

We can now conclude that the wave function $\langle \mathbf{z}_i, M_r^t(\mathbf{x}) \mid \rangle$ remains unaltered if we change $M_r^t(\mathbf{x})$ by any field of tranverse waves which vanishes at every point inside the past light-cone of some $\mathbf{z}_i$. This means that *the wave function depends only on* $\mathbf{z}_i$ *and on the field variables* $M_r^t(\mathbf{x})$ *at points* $\mathbf{x}$ *that lie inside the past light-cone of some* $\mathbf{z}_i$. The field M_r^t may thus be considered as a physical field which affects the state of the particles only when it reaches them. This consequence of the restrictions (2·18) is the chief reason why these particular restrictions are interesting to incorporate into the theory, rather than any others which are mathematically possible.

The conditions (2·18) give, with the help of (2·12) and (2·15),

$$\{A_s^t(\mathbf{x}') - B_s^t(\mathbf{x}')\} \mid \rangle = 0 \quad \text{for } \mathbf{x}' \pm \lambda \text{ inside future light-cone of all } \mathbf{z}_i, \tag{2·21}$$

so that

$$\langle \mathbf{z}_i, M_r^t(\mathbf{x}) \mid A_s^t(x') - B_s^t(\mathbf{x}') \mid \rangle = 0 \qquad \text{for} \quad z_{0i} = -\infty \tag{2·22}$$

with any $\mathbf{x}'$. The conditions $\mathbf{z}_{0i} = -\infty$ mean taking the initial wave function. Thus equation (2·22) shows that the $\mathbf{A}^t$ field and the $\mathbf{B}^t$ field are equal when applied to the initial wave function. This is sufficient to fix the form of the initial wave function completely, so far as concerns the field variables, as is shown by the following argument.

Let a be one of the Fourier amplitudes of the $\mathbf{A}$ field, i.e. the (vector) component in a transverse direction of the coefficient of one of the terms $e^{i(\mathbf{kx})}$ in the Fourier expansion of $\mathbf{A}^t(\mathbf{x})$. We may suppose the Fourier components to be discrete, by enclosing the field in a finite box or some similar device. Then a satisfies with its conjugate $\bar{a}$ a commutation relation of the form

$$\bar{a}a - a\bar{a} = c, \tag{2·23}$$

where c is a real number, depending on how close together the Fourier components are and also on λ. We arrange to have c positive, by interchanging a and $\bar{a}$ if it is not. Let $\bar{\beta}$ be the Fourier amplitude of the $\mathbf{B}^t$ field corresponding to a, so that β is the amplitude corresponding to $\bar{a}$. Then, since the commutation relations for the $\mathbf{A}^t$ and $\mathbf{B}^t$ fields are the same except for a difference in sign, as is shown by equations (2·9) and (2·16),

$$\bar{\beta}\beta - \beta\bar{\beta} = c. \tag{2·24}$$

Equation (2.22) gives

$$\left.\begin{aligned} \langle \mathbf{z}_i, M_r^t(\mathbf{x}) \mid a - \bar{\beta} \mid \rangle = 0 \\ \langle \mathbf{z}_i, M_r^t(\mathbf{x}) \mid \bar{a} - \beta \mid \rangle = 0 \end{aligned}\right\} \quad \text{for} \quad z_{oi} = -\infty. \tag{2·25}$$

Let us pass to a representation in which the variables a and β for the various Fourier components are diagonal, instead of $M_r^t(\mathbf{x})$. This will be a representation of the Fock type (see § 2 of I) for all the harmonic

oscillators that make up the $\mathbf{A}^t$ and $\mathbf{B}^t$ fields, and the new wave function will be an ascending power series in the variables α, β. Equations (2·25) transform into

$$\left.\begin{aligned}\langle \mathbf{z}_i,\ \alpha,\ \beta \mid \alpha - \bar{\beta} \mid \rangle &= 0\\ \langle \mathbf{z}_i,\ \alpha,\ \beta \mid \bar{\alpha} - \beta \mid \rangle &= 0\end{aligned}\right\} \quad \text{for} \quad z_{oi} = -\infty. \qquad (2\cdot26)$$

Equation (2·23) shows that $\bar{\alpha}$ applied to a $\mid \rangle$ is equivalent to $c\,\partial/\partial\alpha$ applied to the representative of the $\mid \rangle$ and equation (2.24) shows similarly that $\bar{\beta}$ applied to a $\mid \rangle$ is equivalent to $c\,\partial/\partial\beta$ applied to its representative. Thus equations (2·26) give

$$\left.\begin{aligned}(\alpha - c\,\partial/\partial\beta)\,\langle \mathbf{z}_i,\ \alpha,\ \beta \mid \rangle &= 0\\ (\beta - c\,\partial/\partial\alpha)\,\langle \mathbf{z}_i,\ \alpha,\ \beta \mid \rangle &= 0\end{aligned}\right\} \quad \text{for} \quad z_{oi} = -\infty.$$

Hence

$$\langle \mathbf{z}_i,\ \alpha,\ \beta \mid \rangle = \kappa\, e^{\alpha\beta/c} \quad \text{for} \quad z_{oi} = -\infty. \qquad (2\cdot27)$$

where κ is independent of α and β. This shows that the form of the initial wave function is fixed, so far as concerns the variables α, β. Since this applies to all the Fourier components, the form of the initial wave function in the original representation with $\mathbf{z}_i$, $M_r^t(\mathbf{x})$ diagonal must also be fixed so far as concerns the field variables $M_r^t(\mathbf{x})$.

The form (2·27) shows, when one refers to the physical interpretation of wave functions in Fock's representation, that the number of photons in a Fourier component of the $\mathbf{B}^t$ field is equal to the number of photons initially in the corresponding Fourier component of the $\mathbf{A}^t$ field. This number is not restricted in any way, but can be any integer from zero to infinity. Thus the conditions (2·18) do not put any restriction on the number of incident photons in any Fourier component of the field, so they do not spoil the adequacy of the scheme of equations for the treatment of all physical problems.

One further question with regard to the conditions (2·18) should be looked into, namely, their Lorentz invariance. Since the relativistic form of our equations is destroyed when we eliminate the longitudinal waves, we cannot be sure whether any extra equations which are introduced subsequently are Lorentz invariant or not. In order to settle this question we must rearrange the work so that any extra equations needed are introduced before eliminating the longitudinal waves, and must then see that everything is all right. To do this we assume, instead of (2·18), the conditions

$$\left\{\frac{\partial N^\mu(\mathbf{x}')}{\partial x'_\nu} - \frac{\partial N^\nu(\mathbf{x}')}{\partial x'_\mu}\right\} \mid \rangle = 0 \quad \text{for } \mathbf{x}' \text{ inside future light-cone of all } \mathbf{z}_i \qquad (2\cdot28)$$

before eliminating the longitudinal waves. These conditions are obviously

Lorentz invariant. We have to verify that they are consistent, not only with one another and with the wave equations, but also with the supplementary conditions (1·7). This can be done by observing that the linear operators in the various equations all commute. For example, calling $R(\mathbf{x})$ the linear operator that occurs in (1.7),

$$[R(\mathbf{x}),\ N^{\nu}(\mathbf{x}')] = \left[\frac{\partial A_{\mu}(\mathbf{x})}{\partial x_{\mu}},\ N^{\nu}(\mathbf{x}')\right] = \tfrac{1}{2}\frac{\partial}{\partial x_{\nu}}\Delta(\mathbf{x}-\mathbf{x}'),$$

so that $$\left[R(\mathbf{x}),\ \frac{\partial N^{\mu}(\mathbf{x}')}{\partial x'_{\nu}} - \frac{\partial N^{\nu}(\mathbf{x}')}{\partial x'_{\mu}}\right] = 0.$$

We can now eliminate the longitudinal waves for both the **A** and **B** fields together. The result of this elimination is just our previous scheme with (2·18), which is thus obtained in a more roundabout but more logical way. The details of the elimination will not be given here, as they can best be expressed with the method of § 4. They will be given at the end of § 4.

§ 3. *Application to a Single Electron.*

For the further development of the theory it is convenient to restrict ourselves to the case of a single electron, as this removes some complications which are not really relevant to our main problem of radiation damping. The case of a single electron is sufficient for the discussion of all fundamental questions.

In this case there are no Coulomb forces arising from the elimination of the longitudinal waves, so the wave equation (1·9) reads, after elimination of the longitudinal waves,

$$[p_0 + \Sigma_r a_r \{p_r - e M_r^t(\mathbf{z}) - e N_r^t(\mathbf{z}+\lambda)\} + a_m m]\,|\,\rangle = 0. \qquad (3{\cdot}1)$$

for an electron with spin, with the help of (2·12). In our work up to the present λ has been a small 4-vector lying within the light-cone. We now impose on it the further condition that it shall lie within the future light-cone, i.e. $\lambda_0 > 0$. Then

$$N_r^t(\mathbf{z}+\lambda)\,|\,\rangle = 0$$

from (2·18), so that (3·1) reduces to

$$[p_0 + \Sigma_r a_r \{p_r - e M_r^t(\mathbf{z})\} + a_m m]\,|\,\rangle = 0. \qquad (3{\cdot}2)$$

The $\mathbf{N}^t$ field has completely disappeared from the equation. *We are left with a wave equation which is equivalent to the wave equation in the elementary theory of radiation in which the electromagnetic field is not quantized.* A similar simplification in the wave equation occurs for the Klein-Gordon electron without spin, though the reduction is not quite

so straightforward on account of the quadratic terms in the field quantities.

To get a solution of (3·2), we need solutions of the wave equation in the elementary theory with unquantized field—one solution for each field $M_r^t(\mathbf{x})$. The various solutions for different $M_r^t(\mathbf{x})$ must not be chosen arbitrarily, but must be made to satisfy the condition that the solutions for two fields $M_r^t(\mathbf{x})$ whose difference is the field $K_r^t(\mathbf{x})$ must be equal for the electron in that region of space-time which has not been reached by the field $K_r^t(\mathbf{x})$, i.e. for the electron at a point within whose past light-cone $K_r^t(\mathbf{x})$ vanishes. This will ensure that (2·18) is satisfied. The various solutions all put together to make a functional of $M_r^t(\mathbf{x})$ will then give a solution of (3·2) in the representation with $\mathbf{z}$ and $M_r^t(\mathbf{x})$ diagonal.

It remains to get a method of physical interpretation for this solution of (3·2). From the general discussion of redundant variables on page we see that we must choose a weight function ρ in the redundant variables and then each wave function will give us a Gibbs ensemble. ρ may be any function of the variables that describe the $\mathbf{B}^t$ field. In particular we may take ρ to be a function of the numbers of photons, n_B, say, in the various Fourier components of the $\mathbf{B}^t$ field. (The general ρ would be a function of the n_B's and of their conjugate phases.) If ρ is a function of the n_B's only, it will give Gibbs ensembles with a definite probability for any particular numbers of photons in the various Fourier components of the $\mathbf{B}^t$ field, and thus with a definite probability for any particular numbers of photons initially in the various Fourier components of the $\mathbf{A}^t$ field. The arbitrariness in the wave function cannot influence this probability, since, as we saw in connection with (2·27), the form of the wave function is initially fixed, so far as concerns the field variables. Thus with ρ a function of the n_B's only, we have a definite probability for any particular numbers of photons in the ingoing beams of radiation. For example, we may take ρ to be that function of the n_B's which is unity when every $n_B = 0$ and is zero otherwise, and it will give us Gibbs ensembles with no ingoing photons. (A Gibbs ensemble of this simple kind corresponds to a pure state, i.e. it could come from a single wave function without redundant variables.)

We can choose a solution of (3·2) to correspond to a given initial state for the electron. We can also choose ρ to correspond to given ingoing radiation. Thus we can get a Gibbs ensemble to correspond to any given initial physical conditions. We can then use the standard interpretation of quantum mechanics to calculate the probability for particular numbers of outgoing photons associated with a particular final state for the electron. This completes the general theory of the method of redundant variables.

The usual way of solving the wave equation of quantum electrodynamics without redundant variables leads to divergent integrals. The

question arises—What corresponds to these divergent integrals in the method of redundant variables? The divergent integrals occur when we try to get a solution of the wave equation without redundant variables as a power series in the electronic charge e. To see what corresponds to the divergent integrals in the method of redundant variables let us take, for definiteness, the case of no ingoing photons, and let us get the solution of equation (3·2) as a power series in e, say

$$| \rangle = | 0 \rangle + e | 1 \rangle + e^2 | 2 \rangle + e^3 | 3 \rangle + \ldots \qquad (3{\cdot}3)$$

Then the various terms on the right-hand side of (3·3) must correspond to the various terms in the solution of the wave equation without redundant variables for no ingoing photons, and the infinities that occur in the latter solution mean that the corresponding terms in (3·3) must be infinitely great ket-vectors, in fact each term in the right-hand side of (3·3) must usually be infinitely greater than the previous one. Thus the series (3·3) does not converge. It is now a relatively unimportant matter that, by using the representation with M^t (x) diagonal, we can represent the various terms in the right-hand side of (3·3) in a finite form, while, if we use the usual representation without redundant variables, the infinities show up directly. In any case we do not have a workable solution of the wave equation.

The squared length of a ket-vector $| n \rangle$ in our scheme with redundant variables is $\langle n | \rho | n \rangle$ and thus depends on ρ, so the question of the convergence of (3·3) depends on our choice of ρ. We considered above the case of ρ corresponding to no ingoing photons, but the argument would not be changed if we took ρ instead to correspond to certain definite ingoing photons, such as would occur in an actual physical problem. With ρ corresponding to any practical ingoing field, the expansion (3·3) always diverges.

This is how the fundamental difficulty of the divergent integrals appears in the redundant-variable theory. Two methods will now be proposed by which one might hope to get over it.

Method (i).—Instead of choosing ρ so as to correspond to the ingoing photons in the problem in which we are interested, we may try to choose it so as to make (3·3) converge, without in the first place paying any attention to its physical consequences. This leads to the theory given in I. The most suitable ρ for this method is

$$\rho = \Pi \{ (-)^{n_B} 2 \} \qquad (3{\cdot}4)$$

with one factor { } for each independent state of a photon of the $\mathbf{B}^t$ field. The alternating plus or minus sign in (3·4) introduces a new convergence factor into the expansion, but it leads to Gibbs ensembles with negative probabilities, which cannot apply to the actual world. One

can, however, use these Gibbs ensembles to calculate probability coefficients for transition processes, and since these probability coefficients turn out to be always positive, it is reasonable to assume that they are the same as those of the actual world.[2]

Method (ii).—We may keep ρ corresponding to the ingoing photons for the problem we are interested in and abandon the power series form for the solution of (3·2). The power series form must be a good one for weak fields $\mathbf{M}^t(\mathbf{x})$, but it is not good for strong fields, and as both weak and strong fields contribute to the importance of the various terms in the expansion (3·3), it might be that the lack of convergence of the expansion is due entirely to the inappropriateness of the power series form. To find out whether this is the case, we should see whether the accurate solution of (3·2) would give rise to any infinities when we proceed to its physical interpretation.

Suppose we have the accurate solution of (3·2) in the representation with $\mathbf{z}$ and $M_r^t(\mathbf{x})$ diagonal, i.e. we have the wave function $\langle \mathbf{z}, M_r^t(\mathbf{x}) | \rangle$. We may resolve $M_r^t(\mathbf{x})$ into its Fourier components and, calling a typical Fourier amplitude ξ and its conjugate complex $\bar{\xi}$, express the wave function in terms of the variables ξ, $\bar{\xi}$ instead of $M_r^t(\mathbf{x})$, so that it reads $\langle \mathbf{z}\, \xi\, \bar{\xi} | \rangle$. We can now pass to the representation in which $\mathbf{z}$ and the Fourier amplitudes α and β are diagonal, as we had in the previous section. In the limit $\lambda = 0$ the transformation reads

$$\langle \mathbf{z}\, \alpha\, \beta | \rangle = (\pi c)^{-1} \iint_{-\infty}^{\infty} e^{[-\alpha\beta + \sqrt{2}\alpha\bar{\xi} + \sqrt{2}\beta\xi - \xi\bar{\xi}]/c} \langle \mathbf{z}\, \xi\, \bar{\xi} | \rangle\, d\xi_R\, d\xi_I \qquad (3{\cdot}5)$$

where ξ_R and ξ_I are the real and pure imaginary parts of the complex variable ξ, as may be verified in the following way. The connection between the Fourier amplitudes α, β and the Fourier amplitudes ξ of the M^t field and η, say, of the $\mathbf{N}^t$ field is of the form

$$\left.\begin{aligned} \sqrt{2}\,\alpha &= \xi + \eta & \sqrt{2}\,\bar{\alpha} &= \bar{\xi} + \bar{\eta} \\ \sqrt{2}\,\bar{\beta} &= \xi - \eta & \sqrt{2}\,\beta &= \bar{\xi} - \bar{\eta} \end{aligned}\right\} \qquad (3{\cdot}6)$$

from (2·12) and (2·15) with $\lambda = 0$. The commutation relations (2·23), (2·24) show that $\bar{\alpha}$ and $\bar{\beta}$ are equivalent to the operators of differentiation $c\,\partial/\partial\alpha$ and $c\,\partial/\partial\beta$ in the α, β representation and the commutation relations connecting the $\mathbf{N}^t$ field with the $\mathbf{M}^t$ field show that η and $\bar{\eta}$ are equivalent to the operators of differentiation—$c\,\partial/\partial\bar{\xi}$ and $c\,\partial/\partial\xi$ in the ξ, $\bar{\xi}$ representation.

[2] The notation of I differs from the present notation and may be obtained from the present notation by writing $\langle X |$ for $\langle X | \rho$ and defining this $\langle X |$ as the conjugate imaginary of $| X \rangle$. Thus ρ does not appear explicitly in I, but gets absorbed in the connexion between a $|\ \rangle$ and its conjugate imaginary $\langle 1$.

One now finds that if one operates on the wave function $\langle z \xi \bar{\xi} \mid \rangle$ with the right-hand side of any of the equations (3·6) and transforms the result by (3·5), one gets the wave function $\langle z \alpha \beta \mid \rangle$ operated on by the left-hand side of the equation. These conditions ensure that the transformation equation (3·5) is correct except for the numerical coefficient on the right-hand side. This numerical coefficient was chosen so as to make the unit wave function transform into simply $e^{\alpha\beta}$.

The wave function $\langle z \alpha \beta \mid \rangle$ expressed in the form of an ascending power series in α and β is very suitable for physical interpretation. For example, if we are dealing with a problem with no ingoing photons, we must take ρ to vanish except when there are no photons in the $\mathbf{B}^t$ field, and then only the part of $\langle z \alpha \beta \mid \rangle$ which is independent of β will be effective. This part will be a power series in α alone and the various coefficients will correspond to various numbers of photons being present at any time in the physical problem.

The necessary and sufficient condition that there should be no infinities in the physical interpretation is that all the coefficients of $\langle z \alpha \beta \mid \rangle$ expressed as an ascending power series in α and β should be finite. It seems very likely that this is so, on account of the strong convergence produced by the factor $e^{-\xi\bar{\xi}/e}$ in the integrand on the right-hand side of (3·5). For example, the term independent of α and β is

$$(\pi e)^{-1} \iint_{-\infty}^{\infty} e^{-\xi\bar{\xi}/e} \langle z \xi \bar{\xi} \mid \rangle \, d\xi_R \, d\xi_I \qquad (3{\cdot}7)$$

Now the wave function $\langle z \xi \bar{\xi} \mid \rangle$ should be normalized for all values of ξ, $\bar{\xi}$, and hence the expression (3·7) can be looked upon as the weighted average of a set of normalized wave functions, with the weight $(\pi e)^{-1} e^{-\xi\bar{\xi}/e}$ which is always positive. Such a weighted average is presumably finite. The other coefficients are of the form (3·7) with powers of ξ and $\bar{\xi}$ inserted as factors in the integrand and such factors are unlikely to disturb the convergence of the integral.

The coefficients of $\langle z \alpha \beta \mid \rangle$ expressed as a power series in α and β must be the same as the coefficients in the solution of the ordinary wave equation without redundant variables for various values of the ingoing numbers of photons. The above argument thus makes it likely that the solution of the ordinary wave equation would be finite if obtained by a correct method and that the infinities occurring in the usual method are due entirely to the unsuitability of the power series in e for the form of the solution.

To continue with method (ii) it is necessary to find a better way of solving (3·2). A suggestion for this is as follows: Let us work in the representation with z and $M_r^t(\mathbf{x})$ diagonal and let us consider first the case when $M_r^t(\mathbf{x})$ consists entirely of waves moving in one particular

direction, say the direction of the x_3 axis, so that $M_3^t(\mathbf{x}) = 0$, and $M_1^t(\mathbf{x})$ and $M_2^t(\mathbf{x})$ are functions of the single variable $x_0 - x_3$. With this simplification, the wave equation can be solved accurately. The solution for the electron initially at rest is

$$\langle \mathbf{z}, M_r^t(\mathbf{x}) \mid \rangle = \left[1 + \frac{e}{2m}\left\{a_1 M_1^t(\mathbf{z}) + a_2 M_2^t(\mathbf{z})\right\}(1 + a_3)\right] v\, e^{-imz_0/\hbar} e^{-i\chi}, \tag{3·8}$$

where

$$\chi = \frac{e^2}{2m\hbar}\int_{-\infty}^{z_0}\left\{M_1^t(x_0 z_1 z_2 z_3)^2 + M_2^t(x_0 z_1 z_2 z_3)^2\right\} dx_0, \tag{3·9}$$

and v is a wave function in the spin variables satisfying

$$(1 + a_m)\, v = 0.$$

Note that χ is an integral along a world-line parallel to the time axis ending at the electron point $\mathbf{z}$. The solution (3·8) may be verified directly by substitution in the wave equation.

For strong fields $M_r^t(\mathbf{x})$ the factor $e^{-i\chi}$ in (3·8) is the most important factor in the wave function. The existence of this factor in (3·8) shows that for strong fields it would not be suitable to express the wave function as a power series in the electronic charge e, as the first few terms in the expansion would give a very poor approximation.

With $M_r^t(\mathbf{x})$ not restricted to consist entirely of waves moving in one direction, the form of the solution of the wave equation for strong fields is not known, but probably, for an electron initially at rest, it contains as its most important factor $e^{-i\chi_1}$, where

$$\chi_1 = \frac{e^2}{2m\hbar}\int_{-\infty}^{z_0} \Sigma_r M_r^t(x_0 z_1 z_2 z_3)^2\, dx_0, \tag{3·10}$$

as this factor goes over into the correct factor when $M_r^t(\mathbf{x})$ consists entirely of waves moving in one direction, no matter what that direction is. It therefore seems reasonable to try to get a solution of the wave equation in the form

$$\langle z, M_r^t(x) \mid \rangle = \psi\, e^{-i\chi_1}, \tag{3·11}$$

with ψ a power series in e. One may hope that the form (3·11) will not lead to infinities in the physical interpretaton, i.e. that when this form is substituted into (3·5) it will lead to a $\langle \mathbf{z}\, a\, \beta \mid \rangle$ expressible as a power series in a and β with finite coefficients, but I have not yet worked this out.

Two methods have been proposed for getting over the difficulty of the divergence of the expansion (3·3) and the question presents itself,

which of these methods is to be preferred. Each method has its advantages and its disadvantages. Method (i) has the advantage that the mathematics is simple (relatively to the other method) and it connects directly with the relativistic theory of second quantization to form an elegant general scheme. But it has the disadvantage that its wave functions apply only to a hypothetical world, so that its physical interpretation is indirect. Method (ii) has the advantage of a direct physical interpretation according to the standard rules of quantum mechanics, but it has the disadvantage that the mathematics is extremely complicated. Even the problem of a single electron initially at rest, in the absence of any static electromagnetic field or any ingoing photons, leads to a very complicated wave equation which has not yet been solved. The complication is to be attributed to the interaction of the electron with its own field. I do not know which method to prefer and think that both are worth further study. One would like to find a new form of the theory which combines the advantages of both methods, but this is hardly possible without some drastic alterations of the foundations.

There is one further fundamental problem in quantum electrodynamics. In the classical theory the equations of motion of a charged particle in a given ingoing field (Lorentz's equations with radiation damping) involve $\ddot{z}$, so the motion is not determined by the initial position and velocity of the particle. The equations allow the particle, in the absence of an ingoing field, to build up an acceleration and to radiate energy. This, of course, disagrees with experiment. One can get over the difficulty in the classical theory by assuming the extra condition that only those solutions of the equations of motion are allowed for which the final acceleration of the particle is zero.

In the quantum theory treated by method (ii) we again have the possibility of a single particle by itself emitting radiation in the absence of an ingoing field. Some extra condition is thus needed to correspond to the classical condition that the final acceleration is zero, but it is not at all clear what form this condition should take. The wave function of the quantum theory is connected with a Hamilton's principal function in the classical theory, referring to a family of solutions of the classical equations of motion, most of which solutions are not allowed by the final condition on the acceleration. So it looks as though we have to pick out that part of the final wave function which corresponds to no acceleration of the particle and discard the rest as unphysical. However, such a procedure would not fit in with the general principles of quantum mechanics as at present formulated.

With method (i) in the quantum theory, the analogy with the classical theory is not so close and there does not seem to be any need for an extra condition.

§ 4. *The Transformation Law of the Wave Functions.*

In theoretical physics we have to deal with quantities embedded in space-time having a number of components which transform linearly when we make a change of axes. The quantities first used were tensors, and at one time it was thought that all physical quantities would be expressible in terms of tensors. However, when the equations for the spinning electron were set up, it was found that the wave function was not expressible in terms of tensors. It provided a new quantity with components which transform linearly. These new quantities were called spinors and their study was called spinor analysis. They are rather more general than tensors in that there is an ambiguity of sign attached to their components.

With tensors and spinors together, people again thought they had the complete scheme in terms of which all physical quantities would be expressible. However, the wave functions of quantum electrodynamics (in a representation referring to the numbers of photons in the various Fourier components) are not expressible in terms of tensors and spinors, so a further extension of the scheme is needed. A new kind of quantity with components which transform linearly under Lorentz transformations must be introduced, and I call it an *expansor*. It is rather more general than a tensor or a spinor in that the number of its components is infinite, but enumerable.

Let us resolve the field $A_\mu(\mathbf{x})$ into its Fourier components and take the number of components to be discrete, thus

$$A_\mu(\mathbf{x}) = \Sigma_{\mathbf{k}} \{ a_{\mathbf{k}\mu}\, e^{i(\mathbf{k}\mathbf{x})} + \bar{a}_{\mathbf{k}\mu}\, e^{-i(\mathbf{k}\mathbf{x})} \}, \qquad (4{\cdot}1)$$

where k_μ is a four-vector lying along a null line with $k_0 > 0$. Let us confine our attention to the variables $a_{\mathbf{k}\mu}$, $\bar{a}_{\mathbf{k}\mu}$ for one particular value of $\mathbf{k}$ and drop the suffix $\mathbf{k}$. The variables a_μ, $\bar{a}_\mu$ satisfy the commutation relations

$$\left.\begin{aligned} a_\mu a_\nu - a_\nu a_\mu = 0\,, \quad \bar{a}_\mu \bar{a}_\nu - \bar{a}_\nu \bar{a}_\mu = 0\,, \\ \bar{a}_\mu a_\nu - a_\nu \bar{a}_\mu = -g_{\mu\nu} c\,, \end{aligned}\right\} \qquad (4{\cdot}2)$$

where c is some positive number. The a of equation (2·23) is just a transverse component of our present a_μ.

We may set up a representation of the Fock type with the four a's diagonal. From (4·2) $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, will be equivalent to the operators $c\,\partial/\partial a_1$, $c\,\partial/\partial a_2$, $c\,\partial/\partial a_3$ applied to the wave function $\langle a_\mu | \rangle$, and $\bar{a}_0$ to the operator $-c\,\partial/\partial a_0$.

In order that the operators $\bar{a}_1 a_1$, $\bar{a}_2 a_2$, $\bar{a}_3 a_3$, $\bar{a}_0 a_0$ may have positive eigenvalues, the wave function $\langle a_\mu | \rangle$, expressed as a power series in the a's must contain positive powers of a_1, a_2, a_3, and negative powers of

a_0. The wave function will therefore be an ascending power series in the variables a_1, a_2, a_3, beginning with the power zero, like expression (2·5) of I and will be a descending power series in the variable a_0, beginning with the power -1, like expression (2·6) of I. It will thus be of the form

$$\langle a_\mu \mid \rangle = \Sigma_0^\infty A_{nrst}\, a_0^{-n-1}\, a_1^r\, a_2^s\, a_3^t . \qquad (4{\cdot}3)$$

We can add on here arbitrary terms containing zero or positive powers of a_0 (with zero or positive powers of a_1, a_2, a_3), for the same reason as we found we could add on an arbitrary ascending power series to (2·6) of I. These arbitrary terms are of no physical significance and should be regarded as corresponding to the ket-vector zero.

The coefficients A_{nrst} in (4·3) are the components of a new kind of quantity that I call an expansor. By subjecting the four a's to a Lorentz transformation and expressing the wave function $\langle a_\mu \mid \rangle$ in terms of the new a's, we get a new set of coefficients A_{nrst} which are linear functions of the original ones. The connexion between the new A_{nrst} and the original ones gives the law of transformation of the components of an expansor. This transformation has been studied in a paper by the author.[3]

The coefficients A_{nrst} are not arbitrary but are restricted by the supplementary condition (1·7). Expressed in terms of Fourier components, this supplementary condition reads

$$\left.\begin{aligned} (k^\mu a_{\mathbf{k}\mu} - b_{\mathbf{k}}) \mid \rangle &= 0 \\ (k^\mu \bar{a}_{\mathbf{k}\mu} - \bar{b}_{\mathbf{k}}) \mid \rangle &= 0 \end{aligned}\right\} \qquad (4{\cdot}4)$$

where $b_{\mathbf{k}}$ is a number depending on the positions of the electrons. Thus, dropping the suffix $\mathbf{k}$ again, we get

$$(k^\mu a_\mu - b) \langle a_\mu \mid \rangle = 0 \qquad (4{\cdot}5)$$

$$\left(c k_\mu \frac{\partial}{\partial a_\mu} + \bar{b}\right) \langle a_\mu \mid \rangle = 0 \qquad (4{\cdot}6)$$

Equation (4·5) shows that $\langle a_\mu \mid \rangle$ is of the form

$$\langle a_\mu \mid \rangle = (k^\mu a_\mu - b)^{-1} G , \qquad (4{\cdot}7)$$

where G is an ascending power series in a_1, a_2, a_3, and the factor $(k^\mu a_\mu - b)^{-1}$ is to be considered as expanded in a descending power

[3] Dirac, Proc. Roy. Soc., A., 183, p. 284 (1944).

series in a_0 and an ascending power series in a_1, a_2, a_3, b. [If (4·7) is substituted in (4·5) it gives for the left-hand side of (4·5) a set of terms independent of a_0, which terms are equivalent to zero when they occur in a wave function in the a_μ representation.] (4·7) substituted into (4·6) now gives, when one observes that the operator in (4·6) commutes with $(k^\mu a_\mu - b)^{-1}$

$$c k_\mu \partial G / \partial a_\mu + \bar{b} G = 0 , \tag{4·8}$$

showing that G is of the form

$$G = e^{-\bar{b} \Sigma_r k_r a_r / k_0^2 c} G_1 \tag{4·9}$$

where G_1 is a function of a_1, a_2 and a_3 satisfying

$$k_\mu \partial G_1 / \partial a_\mu = 0 . \tag{4·10}$$

This equation shows that G_1 is a function of only the transverse components of (a_1, a_2, a_3).

Substituting (4·9) into (4·7) and applying the argument to all **k**-values, we get the final result that the wave function $\langle \mathbf{z}_i a_\mu \mid \rangle$ in the particle positions $\mathbf{z}_i$ and field variables a_μ is of the form

$$\langle \mathbf{z}_i a_\mu \mid \rangle = \Pi_k \{ (k^\mu a_{\mathbf{k}\mu} - b_{\mathbf{k}})^{-1} e^{-\bar{b}_{\mathbf{k}} \Sigma_r k_r a_{\mathbf{k}r} / k_0^2 c_{\mathbf{k}}} \} G_2 , \tag{4·11}$$

where the continued product contains one factor for each **k**-value (with $k_0 > 0$) and G_2 is a function of the z_i and of the transverse components of the $a_{\mathbf{k}r}$. The right-hand side of (4·11) should be considered as expanded in a descending power series in all the $a_{\mathbf{k}0}$ and an ascending power series in all the $a_{\mathbf{k}1}$, $a_{\mathbf{k}2}$, $a_{\mathbf{k}3}$, and the various coefficients in it will then correspond to various numbers of photons being in the different Fourier components of the field. The coefficients will transform under a Lorentz transformation like a product of a large number of expansors, one expansor for each **k**-value.

The physically important part of the wave function (4·11) is the factor G_2 involving the transverse field variables. The other factors merely show how the wave function depends on the longitudinal field variables. The elimination of the longitudinal waves means passing to wave equations for which G_2 by itself is the wave function.

Equation (4·11) gives the form of the wave function when there are no redundant variables in the theory. To do the corresponding work when redundant variables are introduced before eliminating the longitudinal waves, we must resolve the redundant field $B_\mu(\mathbf{x})$ into its Fourier components thus,

$$B_\mu(\mathbf{x}) = \Sigma_k \{ \bar{\beta}_{\mathbf{k}\mu} e^{i(\mathbf{k}\mathbf{x})} + \beta_{\mathbf{k}\mu} e^{-i(\mathbf{k}\mathbf{x})} \} \tag{4·12}$$

like (4·1), with β and $\bar{\beta}$ the opposite way round to what would be the analogue of (4·1). Confining our attention as before to one particular **k**-value and dropping the suffix **k**, we have the commutation relations

$$\left.\begin{aligned} \beta_\mu \beta_\nu - \beta_\nu \beta_\mu = 0, \quad \bar{\beta}_\mu \bar{\beta}_\nu - \bar{\beta}_\nu \bar{\beta}_\mu = 0, \\ \bar{\beta}_\mu \beta_\nu - \beta_\nu \bar{\beta}_\mu = -g_{\mu\nu} c, \end{aligned}\right\} \quad (4\cdot13)$$

and, of course, the β's commute with the a's. The introduction of β and $\bar{\beta}$ the opposite way round compensates for the commutation relations between the $B_\mu(\mathbf{x})$, equations (2·6) involving the opposite sign to those between the $A_\mu(\mathbf{x})$, equations (1·3).

We set up a representation of the Fock type in the four β's as well well as in the four a's, so that the wave function reads $\langle a_\mu, \beta_\mu | \rangle$. The variables $\bar{\beta}$ applied to this wave function will be equivalent to certain operators of differentiation, like the $\bar{a}$'s, and the wave function will be an ascending power series in $\beta_1, \beta_2, \beta_3$ as in a_1, a_2, a_3 and a descending power series in β_0 as in a_0. The supplementary conditions (4·4) will still hold, showing that the new wave function $\langle a_\mu, \beta_\mu | \rangle$ is still of the form of the right-hand side of (4·7) with G given by (4·9) and (4·10), but G_1 will now involve the variables β as well as the transverse components of a_r.

For the initial wave function the conditions (2·28) will hold for all $\mathbf{x}'$. Resolving these conditions into their Fourier components we get, for the particular **k**-value under consideration,

$$\left.\begin{aligned} \{k_\nu (a_\mu - \bar{\beta}_\mu) - k_\mu (a_\nu - \bar{\beta}_\nu)\} | \rangle = 0 \\ \{k_\nu (\bar{a}_\mu - \beta_\mu) - k_\mu (\bar{a}_\nu - \beta_\nu)\} | \rangle = 0. \end{aligned}\right\} \quad (4\cdot14)$$

Multiplying by k^μ, we get

$$k^\mu k_\nu (a_\mu - \bar{\beta}_\mu) | \rangle = 0, \quad k^\mu k_\nu (\bar{a}_\mu - \beta_\mu) | \rangle = 0,$$

showing that

$$k^\mu (a_\mu - \bar{\beta}_\mu) | \rangle = 0, \quad k^\mu (\bar{a}_\mu - \beta_\mu) | \rangle = 0. \quad (4\cdot15)$$

For the initial wave function equations (4·4) will hold with $b_{\mathbf{k}}$ put equal to zero, since electrons in the infinite past will not contribute to $b_{\mathbf{k}}$. Combining these equations with (4·15) we get

$$k^\mu \bar{\beta}_\mu | \rangle = 0, \quad k^\mu \beta_\mu | \rangle = 0. \quad (4\cdot16)$$

These results hold in the first place only for the initial wave function, but since they involve only redundant variables they must hold for all times and thus be generally true.

Equations (4·16) show that the wave function $\langle a_\mu, \beta_\mu | \rangle$ is of the form

$$\langle a_\mu, \beta_\mu | \rangle = (k^\mu \beta_\mu)^{-1} G_3$$

where G_3 is a function of the transverse components of β_r and of the four α's. Combining this with our previous knowledge of how the wave function involves the α's, we find

$$\langle \alpha_\mu, \beta_\mu \mid \rangle = (k^\mu \alpha_\mu - b)^{-1} (k^\nu \beta_\nu)^{-1} e^{-\bar{b}\Sigma_r k_r \alpha_r / k_0^2 c} G_4, \quad (4{\cdot}17)$$

where G_4 is a function of the transverse components of α_r and of the transverse components of β_r. We have not used all the information contained in equations (4·14). The remaining information leads to the result that initially G_4 involves the transverse components of α_r and of β_r in the form shown by (2·27).

We may apply the above argument for all **k**-values and get the result that the complete wave function $\langle \mathbf{z}_i, \alpha_{\mathbf{k}\mu}, \beta_{\mathbf{k}\mu} \mid \rangle$ is of the form of the product of a number of factors like the coefficient of G_4 in (4·17) and a factor, G_5 say, which is a function of the $\mathbf{z}_i$ and of the transverse components of the $\alpha_{\mathbf{k}\nu}$ and $\beta_{\mathbf{k}\nu}$. The elimination of the longitudinal waves in the theory with redundant variables consists in passing to wave equations for which G_5 is the wave function. The Hamiltonians in these wave equations must be the same as the Hamiltonians in the corresponding wave equations for G_2 in the theory without redundant variables, since the factors $(k^\nu \beta_{\mathbf{k}\nu})^{-1}$ which cause the difference between G_5 and G_2 commute with the Hamiltonians. Also the conditions (2·28) must lead to the conditions (2·18) for the wave function G_5, since the factors $(k^\nu \beta_{\mathbf{k}\nu})^{-1}$ commute with the operators $N_r(\mathbf{x}')$ occurring in (2·18). Thus we get the same scheme of equations as we would get if we introduced the redundant variables after elimination of the longitudinal waves, as was done in § 2 from equations (2·8) onwards.

§ 5. *The Position of a Photon.*

In most practical problems involving radiation the momentum of the photons is defined and not their position. There are a few problems, however, for which the position of a photon is important, e.g. if a photon is emitted from an atomic nucleus it starts in a state in which its position is defined fairly accurately. From the point of view of general theory, a photon is always emitted by the motion of a charged particle, and initially the photon is in a state for which its position is the same as the position of the particle. This fact must correspond to some property of the wave equation which it is interesting to investigate. We can get in this way some detailed information about the wave function, without meeting the difficulty of the infinite integrals.

Let us confine our attention to one of the four potenials A_μ, say a spatial one ($\mu = 1$, 2 or 3), and let us drop the suffix. We resolve this potential into its Fourier components.

$$A(\mathbf{x}) = \hbar^{\frac{1}{2}} (2\pi)^{-1} \iiint \left\{ a_{\mathbf{k}} e^{i(\mathbf{kx})} + \bar{a}_{\mathbf{k}} e^{-i(\mathbf{kx})} \right\} k_0^{-1} \, dk_1 \, dk_2 \, dk_3, \quad (5{\cdot}1)$$

with $k_0 = (k_1^2 + k_2^2 + k_3^2)^{\frac{1}{2}}$, in agreement with the notation of I. The $a_{\mathbf{k}}$'s satisfy the commutation relations

$$\bar{a}_{\mathbf{k}}\, a_{\mathbf{k}'} - a_{\mathbf{k}'}\, \bar{a}_{\mathbf{k}} = k_0\, \delta(k_1 - k_1')\, \delta(k_2 - k_2')\, \delta(k_3 - k_3') \qquad (5{\cdot}2)$$

in the limit $\lambda = 0$.

We introduce the variables l_1, l_2, l_3 to give the direction cosines of the vector k_1, k_2, k_3 in three dimensional space, so that

$$k_1 = k_0 l_1\,, \quad k_2 = k_0 l_2\,, \quad k_3 = k_0 l_3\,, \quad l_1^2 + l_2^2 + l_3^2 = 1\,.$$

The suffix $\mathbf{k}$ may now be replaced by the two suffices k_0 and l, with l standing for l_1, l_2, l_3, so that $a_{\mathbf{k}}$ may be written $a_{k_0 l}$. Thus equation (5·2) may be written

$$\bar{a}_{k_0 l}\, a_{k_0' l'} - a_{k_0' l'}\, \bar{a}_{k_0 l} = k_0^{-1}\, \delta(k_0 - k_0')\, \delta(l - l')\,, \qquad (5{\cdot}3)$$

where $\delta(l - l')$ is defined so that

$$\delta(k_0 - k_0')\, \delta(l - l') = k_0^2\, \delta(k_1 - k_1')\, \delta(k_2 - k_2')\, \delta(k_3 - k_3')\,.$$

Note that this definition of $\delta(l - l')$ makes

$$\iint f(k_0, l)\, \delta(k_0 - k_0')\, \delta(l - l')\, dk_0\, dl = f(k_0', l')\,, \qquad (5{\cdot}4)$$

where dl denotes an element of solid angle for the direction l_1, l_2, l_3.

Now put

$$a_l^{\tau} = \int_0^{\infty} a_{k_0 l}\, e^{i k_0 \tau}\, k_0\, dk_0\,, \qquad \bar{a}_{\tau l} = \int_0^{\infty} \bar{a}_{k_0 l}\, e^{-i k_0 \tau}\, k_0\, dk_0\,, \qquad (5{\cdot}5)$$

the integrations being taken for fixed l_1, l_2, l_3. The variable τ appearing here is of the nature of the position variable of a photon measured along its direction of motion. If we identify τ with

$$\tau = x_0 - l_1 x_1 - l_2 x_2 - l_3 x_3\,, \qquad (5{\cdot}6)$$

so that $e^{i k_0 \tau} = e^{i(\mathbf{k}\mathbf{x})}$, equations (5·1) and (5·5) show that

$$A(\mathbf{x}) = \hbar^{\frac{1}{2}} (2\pi)^{-1} \int (a_l^{\tau} + \bar{a}_{\tau l})\, dl \qquad (5{\cdot}7)$$

From the first of the equations (5·5), for any real number κ

$$\begin{aligned} \int_{-\infty}^{\infty} a_l^{\tau} e^{-i\kappa t}\, d\tau &= \int_0^{\infty} a_{k_0 l}\, k_0\, dk_0 \int_{-\infty}^{\infty} e^{i k_0 \tau}\, e^{-i\kappa\tau}\, d\tau \\ &= \int_0^{\infty} a_{k_0 l}\, k_0\, dk_0\, 2\pi\, \delta(k_0 - \kappa) \\ &= 2\pi\kappa\, a_{\kappa l} \text{ for } \kappa > 0 \text{ and } 0 \text{ for } \kappa < 0\,. \end{aligned} \qquad (5{\cdot}8)$$

Similarly $\int_{-\infty}^{\infty} \bar{a}_{\tau l}\, e^{i\kappa\tau}\, d\tau = 2\pi\kappa \bar{a}_{\kappa l}$ for $\kappa > 0$ and 0 for $\kappa < 0$. (5·9)

The a_l^{τ}, $\bar{a}_{\tau l}$ variables satisfy the commutation relations that the a_l^{τ}'s all commute with each other, the $\bar{a}_{\tau l}$'s all commute with each other, and from (5·3)

$$\begin{aligned}\bar{a}_{\tau l}\, a_{l'}^{\tau'} - a_{l'}^{\tau'}\, \bar{a}_{\tau l} &= \iint_0^{\infty} (\bar{a}_{k_0 l}\, a_{k_0' l'} - a_{k_0' l'}\, \bar{a}_{k_0 l})\, e^{-ik_0\tau}\, e^{ik_0'\tau'} k_0 k_0'\, dk_0\, dk_0' \\ &= \delta(l - l') \int_0^{\infty} e^{-ik_0(\tau - \tau')}\, k_0\, dk_0 \\ &= -\delta(l - l')(\tau - \tau')^{-2}. \end{aligned} \tag{5·10}$$

We may set up a Fock representation in the variables $a_{\mathbf{k}}$. The wave function is then an ascending power series in the variables $a_{\mathbf{k}}$, the terms of degree n corresponding to the presence of n photons in states with definite momentum values. With the help of equation (5·8) for $\kappa > 0$, we can transform the wave function to the variables a_l^{τ}. It is then a power series in the variables a_l^{τ}, the terms of degree n corresponding to the presence of n photons in states with definite directions of motion and with positions connected in some way with the parameter τ. The nature of this connexion is what we have to investigate.

Let us consider the terms of the first degree, corresponding to one photon. In the $a_{\mathbf{k}}$ representation they will be of the form

$$\iiint a_{\mathbf{k}}\, a_{\mathbf{k}}\, k_0^{-1}\, dk_1\, dk_2\, dk_3, \tag{5·11}$$

determined by a coefficient $a_{\mathbf{k}}$. Transformed to the a_l^{τ} representation, expression (5·11) becomes, from (5·8),

$$(2\pi)^{-1} \iint a_{k_0 l}\, dk_0\, dl \int_{-\infty}^{\infty} a_l^{\tau}\, e^{-ik_0\tau}\, d\tau = \iint a_{\tau l}\, a_l^{\tau}\, d\tau\, dl, \tag{5·12}$$

where

$$a_{\tau l} = (2\pi)^{-1} \int_0^{\infty} a_{k_0 l}\, e^{-ik_0\tau}\, dk_0. \tag{5·13}$$

Expression (5·12), which gives the part of the wave function corresponding to one photon in the new representation, is determined by a coefficient $a_{\tau l}$ involving the position parameter τ and the direction parameter l. Now $|a_{\mathbf{k}}|^2$ multiplied by a suitable coefficient gives the probability of the photon being in the momentum state described by $\mathbf{k}$, and one might expect $|a_{\tau l}|^2$ to give similarly the probability of the photon being in the state with direction l and position τ. This is not so, however, on account of the special features of the a_l^{τ} representation. The a_l^{τ} variables

are not even all independent, being connected by the linear equations (5·8) with $\kappa < 0$, so that the coefficients of a linear expression in the $a_l{}^\tau$. are not uniquely determined.

The equation (5·13) may be looked upon as defining $a_{\tau l}$ also for complex values of τ, so that $a_{\tau l}$ becomes a function of the complex variable τ. This function must be regular for τ in the lower half-plane [i.e. for $i\,(\tau - \bar{\tau}) > 0$], since the integral in (5·13) then converges more strongly than for real τ. Further, $a_l{}^\tau$ defined by (5·5) may be looked upon as a function of the complex variable τ regular in the upper half-plane. All the functions of τ occurring in the present theory may be looked upon as functions of a complex variable and are then regular either in the upper half-plane or in the lower half-plane. In order to distinguish easily the two kinds of functions we make the rule that, when τ occurs as a suffix in a function which is regular for τ in the upper half-plane, the suffix is put in the upper position, and when in a function which is regular for τ in the lower half-plane, it is put in the lower position. Thus, for example, τ must be in the lower position in $\bar{a}_{\tau l}$ The various functions of τ that we shall deal with may have singularities on the real axis, but are then to be looked upon as limits of functions with singularities just off the real axis, just above the real axis in the case of functions which are regular in the lower half-plane and just below the real axis in the case of functions which are regular in the upper half-plane. With these rules, we have the general results

$$\int_{-\infty}^{\infty} X_\tau\, Y_\tau\, d\tau = 0 \qquad \int_{-\infty}^{\infty} X^\tau\, Y^\tau\, d\tau = 0,$$

while $\int_{-\infty}^{\infty} X_\tau\, Y^\tau\, d\tau$ has a definite value usually not zero.

In a linear function of the variables $a_l{}^\tau$ such as the right-hand side of (5·12), we may add on to the coefficient of $a_l{}^\tau$ any function of τ which is regular in the upper half-plane without changing the value of the expression. The coefficient of $a_l{}^\tau$ is thus arbitrary to this extent. If, however, we impose on it the further condition that it shall be regular in the lower half-plane (and shall satisfy a suitable condition at infinity) it is uniquely determined. We take the coefficient restricted in this way to be the wave function for a single photon in the variables τ, l. Owing to its property of being regular in the lower half-plane instead of being an arbitrary function of τ, it cannot be interpreted in the usual way to give the probability of τ having any value.

It is possible to add an infinite series of $a_{\tau l}$'s, each regular in the lower half-plane, to get a sum which is not regular in the lower half-plane. Such a sum must be reckoned as non-convergent from the physical

point of view, even though there is a region of the τ-plane in which it is mathematically convergent. It would certainly not be convergent if transformed to the **k** variables.

The situation is similar with the terms of higher degree in the wave function, corresponding to more than one photon. For example the terms of the second degree, expressed in the a_l^τ variables, would read

$$\iiiint a_{\tau\tau' ll'}\, a_l^\tau\, a_{l'}^{\tau'}\, d\tau\, d\tau'\, dl\, dl'$$

where the coefficient $a_{\tau\tau' ll'}$ is arbitrary to the extent that we may add on to it any function of τ, τ', l, l' which is regular in the upper half-plane for τ and τ', but becomes uniquely determined if we impose on it the condition that it shall be regular for τ and τ' in the lower half-plane (and also the condition that it shall be symmetrical between τ, l, and τ' l').

The operator $\bar{a}_{\tau l}$ can be considered as an operator of differentiation when applied to the wave function in the a_l^τ variables in the following way. Since the left-hand side of (5·10) is regular for τ in the lower half-plane and τ' in the upper half-plane, the right-hand side must be looked upon as the limit of

$$-\delta(l - l')(\tau - \tau' - i\epsilon)^{-2} \tag{5·14}$$

with ϵ a small positive number tending to zero. The operator $\bar{a}_{\tau l}$ applied to the linear wave function $\iint a_{\tau' l'} a_{l'}^{\tau'}\, d\tau'\, dl'$ then gives the result

$$\iint a_{\tau' l'} (\bar{a}_{\tau l}\, a_{l'}^{\tau'} - a_{l'}^{\tau'}\, \bar{a}_{\tau l})\, d\tau'\, dl' = -\int_{-\infty}^{\infty} a_{\tau' l} (\tau - \tau' - i\epsilon)^{-2}\, d\tau'$$
$$= 2\pi i\, d a_{\tau l}/d\tau . \tag{5·15}$$

for τ in the lower half-plane. The result of the operator $\bar{a}_{\tau l}$ applied to terms of higher degree in the wave function may be worked out in the same way. In the notation of functional differentiation

$$\bar{a}_{\tau l} = -\int_{-\infty}^{\infty} (\tau - \tau' - i\epsilon)^{-2}\, \frac{\partial}{\partial a_l^{\tau'}}\, d\tau' = 2\pi i \frac{d}{d\tau} \frac{\partial}{\partial a_l^\tau} . \tag{5·16}$$

To make a deeper investigation of the position variable, we must study the transformation function connecting the dynamical variables at one time with their values at an earlier time. We must first see how the idea of such a transformation function is to be made precise in a many-time theory. We start with a wave function $\langle z_{\mu i},\ \phi \mid \rangle$ involving the coordinates of the electrons $z_{\mu i}$ and certain field variables ϕ. Let us

look upon the electron times z_{oj} as independent variables and the other variables z_{ri}, ϕ as dependent on them, so that they may be written $z_{ri}(z_{oj})$, $\phi(z_{oj})$. The wave function may then be considered as a function of only the dependent variables and written

$$\langle z_{ri}(z_{oj}),\ \phi(z_{oj}) \mid \rangle . \tag{5·17}$$

It involves fewer explicit variables than before, but since these variables are themselves subject to variation with the z_{oj}'s, it is actually just as complicated as before. The $z_{ri}(z_{oj})$ in (5·17) are subject to certain restrictions, corresponding to the points $z_{\mu i}$ being all outside each other's light-cones. When one works with the wave-function (5·11), all the dynamical variables become operators which depend on the times z_{oj}. They vary with the z_{oj} according to equations of motion which are the analogues of the classical equations of motion. There will now be a transformation function

$$\langle z_{ri}(z_{oj}),\ \phi(z_{oj}) \mid z_{ri}(z_{oj}^0),\ \phi(z_{oj}^0)\rangle$$

connecting the dynamical variables at the times z_{oj} with their values at some other times z_{oj}^0. We may write this for brevity

$$\langle z_{ri},\ \phi \mid z_{ri}^0\ \phi^0 \rangle ,$$

using the index ^{0} for variables at the times z_{oj}^0 and no index for variables at the times z_{oj}.

Let us take the variables ϕ on the left of the transformation function to be the Fock variables a_k, so that the transformation function is an ascending power series in the a_k. If we take the variables ϕ^0 on the right to be the corresponding Fock variables a_k^0 at the times z_{oj}^0, the transformation function would be a descending power series in the a_k^0. It is more convenient to take the ϕ^0 to be the $\bar{a}_k^0$ so that the transformation function is an ascending power series in the $\bar{a}_k^0$, as well as in the a_k. It then reads

$$\langle z_{ri},\ a_k \mid z_{ri}^0,\ \bar{a}_k^0 \rangle .$$

To evaluate this transformation function we should have to solve the wave equation and would get into the difficulties discussed in §§ 2 and 3. We can, however, get some information about the transformation function just from general arguments. Let us suppose first that no change takes place in the field between the times z_{oj}^0 and z_{oj}, so that

$$a_k = a_k^0, \qquad \bar{a}_k = \bar{a}_k^0 .$$

Then

$$\langle z_{ri}, a_k \mid a_k - a_k^0 \mid z_{ri}^0, \bar{a}_k^0 \rangle = 0, \langle z_{ri}, a_k \mid \bar{a}_k - \bar{a}_k^0 \mid z_{ri}^0, a_k^0 \rangle = 0. \tag{5·18}$$

Interpreting a_k^0 and $\bar{a}_k$ as operators of functional differentiation in conformity with the commutation relations (5·2), we get from (5·18)

$$(a_k - k_0 \partial / \partial \bar{a}_k^0) \langle z_{ri}, a_k \mid z_{ri}^0, \bar{a}_k^0 \rangle = 0$$

$$(k_0 \partial / \partial a_k - \bar{a}_k) \langle z_{ri}, a_k \mid z_{ri}^0, \bar{a}_k^0 \rangle = 0 .$$

Hence

$$\langle z_{ri}, a_k \mid z_{ri}^0, \bar{a}_k^0 \rangle = e^{\iiint a_k \bar{a}_k^0 k_0^{-1} dk_1 dk_2 dk_3} \psi , \qquad (5·19)$$

where ψ is a function of the electron variables only.

In general, of course, the field does change between the times z_{oj}^0 and z_{oj}. But it is useful to express the transformation function always in the form (5·19) and to allow ψ to involve the field variables a_k, $\bar{a}_k^0$. It will be an ascending power series in these variables, and the term of degree zero in it will correspond to no change taking place in the field, the terms of the first degree will correspond to one photon being emitted or absorbed (emitted for terms involving a_k and absorbed for terms involving $\bar{a}_k^0$), the terms of the second degree will correspond to two-photon processes, and so on. By means of (5·8) and (5·9), equation (5·19) can be expressed in terms of the a_l^τ, $\bar{a}^{\tau l}$ variables and then gives us the form of $\langle z_{ri}, a_l^\tau \mid z_{ri}^0, \bar{a}^{\tau l} \rangle$.

Let us now bring in the condition that an electron influences the field only at the place (in three dimensions) where it is situated. This means that the j-th electron, in moving from the point $z_{\mu j}^0$ to $z_{\mu j}$ (assuming $z_{\mu j}^0$ is the earlier) can influence the field only in that region of space-time which lies inside the future light-cone of $z_{\mu j}^0$ but not inside the future light-cone of $z_{\mu j}$ and in that region which lies inside the past light-cone of $z_{\mu j}$ but not inside the past light-cone of $z_{\mu j}^0$. Outside these regions we must have, in the quantum theory as well as in the classical theory with the Wentzel field,

$$A(\mathbf{x}) = A^0(\mathbf{x}) . \qquad (5·20)$$

Expressed in terms of the variables a_l^τ, $\bar{a}_{\tau l}$, (5·20) gives

$$a_l^\tau + \bar{a}_{\tau l} = a_l^{\tau 0} + \bar{a}_{\tau l}^0 \quad \text{for} \quad \tau < \tau_1 \quad \text{or} \quad \tau > \tau_2 , \qquad (5·21)$$

where τ_1 and τ_2 are two τ-values depending on l, τ_1 being the minimum τ-value obtained by substituting each $z_{\mu j}^0$ for x_μ in (5·6) and $\tau_2 (> \tau_1)$ being the maximum τ-value obtained by substituting each $z_{\mu j}$ for x_μ.

The effect of the conditions (5·21) on the transformation function

$\langle z_{ri},\ a_l^{\tau} \mid z_{ri}^{0},\ \bar{a}_{\tau l}^{0} \rangle$ may easily be worked out. We have, for $\tau < \tau_1$ or $\tau > \tau_2$,

$$\langle z_{ri},\ a_l^{\tau} \mid a_l^{\tau} + \bar{a}_{\tau l} - a_l^{\tau 0} - \bar{a}_{\tau l}^{0} \mid z_{ri}^{0},\ \bar{a}_{\tau l}^{0} \rangle = 0. \quad (5{\cdot}22)$$

This becomes, with the help of (5·16) and of a similar equation giving the result of the operator $a_l^{\tau 0}$ applied to a wave function $\langle \mid z_{ri}^{0},\ \bar{a}_{\tau l}^{0} \rangle$,

$$\left(a_l^{\tau} + 2\pi i \frac{d}{d\tau}\frac{\partial}{\partial a_l^{\tau}} - 2\pi i \frac{d}{d\tau}\frac{\partial}{\partial \bar{a}_{\tau l}^{0}} - \bar{a}_{\tau l}^{0}\right) \langle z_{ri},\ a_l^{\tau} \mid z_{ri}^{0},\ \bar{a}_{\tau l}^{0} \rangle = 0. \quad (5{\cdot}23)$$

The first factor on the right-hand side of (5·19), expressed in terms of the variables a_l^{τ}, $\bar{a}_{\tau l}^{0}$, must evidently satisfy these equations, and indeed for all values of τ. The general solution of (5·23) is thus of the form of the right-hand side of (5·19) expressed in terms of the variables $a_{l\tau}$, $\bar{a}_{\tau l}^{0}$, with ψ satisfying

$$\frac{d}{d\tau}\frac{\partial \psi}{\partial a_l^{\tau}} - \frac{d}{d\tau}\frac{\partial \psi}{\partial \bar{a}_{\tau l}^{0}} = 0 \quad \text{for} \quad \tau < \tau_1, \quad \text{or} \quad \tau > \tau_2. \quad (5{\cdot}24)$$

Let us take the linear terms in ψ and suppose they are

$$\psi_1 = \iint (a_{\tau l}\, a_l^{\tau} + b_l^{\tau}\, \bar{a}_{\tau l}^{0})\, d\tau\, dl.$$

Substituting ψ_1 into (5·24), we get

$$\frac{d}{d\tau}(a_{\tau l} - b_l^{\tau}) = 0 \quad \text{for} \quad \tau < \tau_1 \quad \text{or} \quad \tau > \tau_2,$$

or

$$a_{\tau l} - b_l^{\tau} = 0 \quad \text{for} \quad \tau < \tau_1 \quad \text{or} \quad \tau > \tau_2, \quad (5{\cdot}25)$$

if we require $a_{\tau l}$ and b_l^{τ} to vanish at $\tau = \infty$ and $\tau = -\infty$. Now $a_{\tau l}$ is regular for τ in the lower half-plane and b_l^{τ} is regular for τ in the upper half-plane and equation (5·25) expresses that $a_{\tau l} = b_l^{\tau}$ along that part of the real axis for which $\tau < \tau_1$, and that part for which $\tau > \tau_2$. It follows that *$a_{\tau l}$ and b_l^{τ} must together form a function of τ which is regular and singlevalued over the whole plane except for a slit along the real axis joining τ_1 and τ_2.*

Processes which involve the emission of just one photon and no absorption of photons while the electrons move from the times z_{oj}^{0} to z_{oj} will be connected with that part of the transformation function which contains $a_{\tau l}$ and so $a_{\tau l}$ may be looked upon as the wave function of the

emitted photon, expressed in terms of the direction variables l and the position variable τ. We know physically that the position of the emitted photon must be between τ_1 and τ_2, and we have worked out mathematically that the wave function $a_{\tau l}$ can be considered as a function of the complex variable τ and is then regular over the whole τ plane except for a slit along the real axis joining τ_1 and τ_2. This then is the condition that the wave function must satisfy when the position of the photon is restricted to be between τ_1 and τ_2. It replaces the condition that one has in non-relativistic quantum theory that the wave function must vanish for those real values of the position variable that lie outside the domain in which the particle is restricted to be. It shows that the existence of a photon in a given position is connected, not with the non-vanishing of the wave function in τ, but with the occurrence of singularities in this wave function considered as a function of a complex variable τ.