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# Solitons and soliton–antisoliton pairs of a Goldstone model in $3 + 1$ dimensions

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## Abstract

We study finite energy topologically stable static solutions to a global symmetry breaking model in  $3 + 1$  dimensions described by an isovector scalar field. The basic features of two different types of configurations are studied, corresponding to axially symmetric multisolitons with topological charge  $n$ , and unstable soliton–antisoliton pairs with zero topological charge.

## 1 Introduction

The familiar solitons in  $3 + 1$  spacetime dimensions are the monopoles [1] of the Yang–Mills–Higgs (YMH) model and the Skyrmions [2] of the  $O(4)$  nonlinear sigma model. The first of these [1] is a solution of a gauged scalar (Higgs) field model, while the second [2] is not related to a gauge field. But this is not a very strict distinction, since it is also possible to find solitons of the  $SO(3)$  gauged  $O(4)$  nonlinear sigma model [3]. These are all models that pertain to physically rather different contexts but are nonetheless closely related insofar as they are all classical field theories whose static energies are bounded from below by topological charges. On this rather technical level therefore, one can ask whether there might be an ungauged Higgs analogue of the the (ungauged) Skyrme soliton? We refer to such a symmetry–breaking field theory, as a Goldstone model in  $3 + 1$  dimensions.

It is known on the other hand that such models do exist [4] not just in  $3 + 1$  but in all dimensions. These are the gauge decoupled versions of the  $SO(D)$  gauged Higgs models [5] in  $D + 1$  spacetime dimensions. That such models should support solitons follows from the simple fact that certain truncated versions of these have such solutions in closed form [6]. The solitons of these models have the salient feature that the asymptotic behaviour of their solitons feature the same properties as gauged Higgs models, and hence afford a simple background for the study of Dirac equations [7] in all dimensions.

While the existence of solitons to generic such Goldstone models were known for sometime, a concrete and detailed construction of these has not been presented in the literature to date. This is what we propose to do in this paper, and since  $3 + 1$  is physically the most relevant dimension, we have chosen this for our example. The solitons we construct are in a sense alternatives to the usual Skyrmions, though we have not pushed this analogy here. Our strategy here is instead to expose the generic properties of such solitons. To this end, we construct the topological charge-1 spherically symmetric soliton, the axially symmetric winding number  $n$  multisolitons (MS) and examine the possibility of the existence of bound states. We also construct an (axially

symmetric) topological charge-0 soliton-antisoliton (SAS) pair, to highlight the analogy of the model studied with the usual YMH model.

In Section 2 we define the flat space energy density functional of the model and the topological charge density presenting its lower bound. In Section 3 we present the charge-1 solitons, the charge- $n$  MS, and the charge-0 SAS pairs, in successive Subsections respectively, and in Section 4 we summarise our results.

## 2 The model and the topological charge

The symmetry breaking model in 3 spatial dimensions, to which we refer as a Goldstone model, is described by a scalar isovector field  $\phi^a$ ,  $a = 1, 2, 3$ . There is such a hierarchy of models [4] that arise from the gauge decoupled limit of the three dimensional  $SO(3)$  gauged Higgs model descended from the  $p$ -th member of the Yang-Mills (YM) hierarchy on  $\mathbb{R}_3 \times S^{4p-3}$ . Here we have chosen the simplest of these, namely that descended from the 2-nd member of the YM hierarchy. Using the notation

$$\phi_i^a = \partial_i \phi^a \quad , \quad \phi_{ij}^{ab} = \partial_{[i} \phi^a \partial_{j]} \phi^b \quad , \quad \phi_{ijk}^{abc} = \partial_{[i} \phi^a \partial_j \phi^b \partial_k] \phi^c \quad ,$$

with the brackets  $[ij\dots]$  implying total antisymmetrisation, the static energy density functional is

$$\mathcal{E}_{(p=2)} = \lambda_0 (\eta^2 - |\phi^a|^2)^4 + \lambda_1 (\eta^2 - |\phi^b|^2)^2 |\phi_i^a|^2 + \lambda_2 |\phi_{ij}^{ab}|^2 \quad , \quad (1)$$

which implies a total mass  $M = 1/(4\pi) \int \mathcal{E} dV$ .

All the dimensionless constants  $\lambda_0$ ,  $\lambda_1$  and  $\lambda_2$  must be positive if the topological lower bound to be introduced below is to be valid. Moreover, any of these constants can also vanish, provided that the absence of the corresponding term in (2) does not violate the Derrick scaling requirement. That soliton solutions to this model exist is obvious since for particular choices of these dimensionless constants, explicit solution [6] are known. Pushing our freedom of choosing the numerical values of  $\lambda_0$ ,  $\lambda_1$  and  $\lambda_2$  further, we can add any other positive definite term to (1) multiplying a new dimensionless coefficient, as long as the scaling properties remain satisfied. In 3 spatial dimensions, there is one such possible kinetic term which for which a canonical momentum field exists, and that is the *sextic* term<sup>1</sup>. Thus the most general model we can consider is the following extension of (1)

$$\mathcal{E} = \lambda V(\eta, |\phi^a|) + \tau \left[ (\eta^2 - |\phi^b|^2)^2 |\phi_i^a|^2 + \frac{1}{4} |\phi_{ij}^{ab}|^2 \right] + \frac{\kappa^4}{36} |\phi_{ijk}^{abc}|^2 \quad , \quad (2)$$

$\tau$  and  $\lambda$  being dimensionless constants,  $\kappa$  is with dimension of length, and  $V(\eta, |\phi^a|)$  a generic symmetry breaking potential.

It is perhaps in order to point out that (2) is an *ad hoc* model, rather than a dimensionally descended model like (1). Indeed, a *sextic* term does appear in the next one to (1)

$$\mathcal{E}_{(p=3)} = \lambda_0 (\eta^2 - |\phi^a|^2)^6 + \lambda_1 (\eta^2 - |\phi^b|^2)^4 |\phi_i^a|^2 + \lambda_2 (\eta^2 - |\phi^b|^2)^2 |\phi_{ij}^{ab}|^2 + \lambda_3 |\phi_{ijk}^{abc}|^2 \quad , \quad (3)$$

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<sup>1</sup>The corresponding *sextic* term in the Skyrme model was considered in [8] in some detail.

descended from  $p = 3$  YM, with which we are not concerned. We will restrict our attention to (2), and mainly to the particular case where  $\lambda = \kappa = 0$ , which like the Skyrme model captures the main qualitative features of the soliton.

The model (2) has certain remarkable similarities to the Skyrme [2] model, and other properties that differ fundamentally. On the similarity side, there is the obvious shared feature of the scaling of the distinct terms in both three dimensional models. Also, for the important special case with  $\lambda = \kappa = 0$ , the Bogomol'nyi equations are overdetermined like in the Skyrme model and there exist no solutions saturating the Bogomol'nyi bound.

On the contrasting side, the order parameter field  $\phi^a$  here is a relic of a Higgs field and has the same dimensions ( $L^{-1}$ ) as a connection, and the finite energy conditions require the symmetry breaking boundary condition

$$\lim_{r \rightarrow \infty} |\phi^a| = \eta. \quad (4)$$

For the (more standard) case of multisolitons centred at the origin, the boundary condition there is

$$\lim_{r \rightarrow 0} |\phi^a| = 0. \quad (5)$$

For the unit charge spherically symmetric soliton, when the system is described by a single function  $h(r)$ , the conditions (5) and (4) (see (10) below) result in the *monopole* like asymptotics of our solitons, which are qualitatively different from the *instanton* like asymptotics of the Skyrmons <sup>2</sup>.

It is straightforward to show that (2) is bounded from below by the density

$$\begin{aligned} \varrho &= \frac{1}{4\pi} \varepsilon_{ijk} \varepsilon^{abc} (\eta^2 - |\phi^d|^2) \phi_i^a \phi_j^b \phi_k^c \\ &= \frac{1}{4\pi} \varepsilon_{ijk} \varepsilon^{abc} \partial_i \left[ \left( \eta^2 - \frac{3}{5} |\phi^d|^2 \right) \phi^a \phi_j^b \phi_k^c \right] \end{aligned} \quad (6)$$

whose volume integral is the topological charge, which is just the *winding number*.

Models like (1), (2) and (3) support *global monopoles* in the sense that their topological charges are the winding numbers of the scalar (Higgs) field on the 2-sphere at infinity. These solitons however have finite energy, unlike the usual *global monopoles*. This is due to two reasons. Firstly we have included the *quartic* kinetic term to satisfy the required scaling, and secondly, we have employed a non standard *quadratic* kinetic term which decays fast enough to satisfy finite energy requirements. These two features are guaranteed by the fact that these models are dimensionally descended from higher dimensional Yang–Mills models, the latter

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<sup>2</sup>The values of the scalar Higgs function of a unit charge *monopole* on the boundaries  $[r = 0, r = \infty]$  are  $[h(0) = 0, h(\infty) = 1]$ . The values of the scalar function  $w(r)$  of the spherically symmetric Yang–Mills (YM) *instanton* on the boundaries  $[r = 0, r = \infty]$  are  $[w(0) = \pm 1, w(\infty) = \pm 1]$ . The one dimensional reduced action density of the scale invariant YM system in  $4p$  dimensions is proportional to the corresponding  $O(2p + 1)$  sigma model reduced action in  $2p$  dimensions, described by the function  $f(r) \equiv \arccos w(r)$ . Hence the values of the scalar function  $\cos f(r)$  of the unit charge sigma model soliton on the boundaries  $[r = 0, r = \infty]$  are  $[\cos f(0) = \pm 1, \cos f(\infty) = \pm 1]$ , like an *instanton*.

being endowed with the corresponding topological properties, as explained in [4] and references therein.

While we are exclusively concerned with the classical properties of the model (2) here, it is nevertheless interesting to comment on its possible quantum aspects. It is believed that the Skyrme model is a reliable approximate theory of the Nucleons [9] and our model shares many similarities with the latter. The main difference between the models is that while the Skyrme field is a constrained field and hence the procedure of quantisation must take account of the constraint, the order parameter field in our model is unconstrained. On the other hand, the *quadratic* kinetic term in (2) is quite unconventional, featuring the field dependent factor  $(\eta^2 - |\phi^b|^2)^2$ . The effect of the latter is to prevent the definition of a propagator, this difficulty of quantisation replacing the *constraint* problem in the Skyrme model. But it is also well known that the most efficient practical method of quantising Skyrme theory is that of *collective coordinate* quantisation, employed in [9]. This method applies is equally well to the system (2), subject to making the essential change of field coordinates

$$\phi^a \rightarrow \Phi = \phi^a \tau_a \quad ,$$

$\tau_a$  being the three Pauli matrices.

### 3 The solitons

Our aim in this Section is to demonstrate the close similarities of the solitonic solutions in this model, with the various monopole (and dipole) solutions of the YMH model. The Section is divided into three Subsections. The first involves the charge-1 soliton analogous to the 't Hooft–Polyakov monopole [1]. The second is concerned with axially symmetric solutions satisfying standard boundary conditions, with arbitrary winding number  $n$ , namely the multisolitons (MS) of this model. A question of interest raised in this Subsection is that of the mutual attraction or repulsion two 1-solitons. In the last Subsection we impose those boundary conditions on the axially symmetric fields, which result in zero charge (unstable) solutions representing soliton–antisoliton (SAS) pairs situated on the symmetry axis. These boundary conditions are those employed in [10, 11] for the corresponding  $SO(3)$  YMH model.

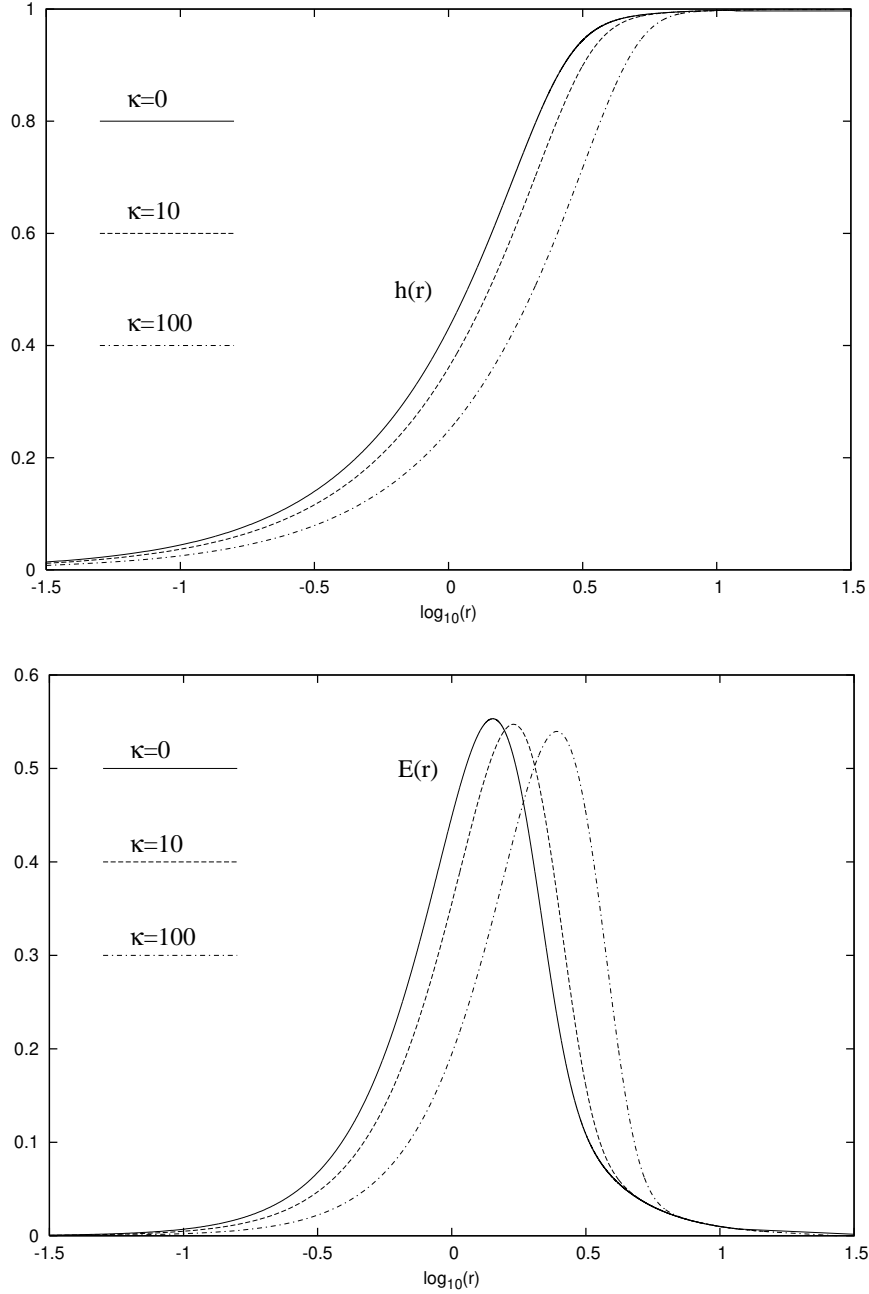
#### 3.1 Charge-1 soliton: spherically symmetric

Subjecting (2) to spherical symmetry via the Ansatz

$$\phi^a = \eta h(r) \hat{x}^a \quad , \quad (7)$$

and taking into account the factor  $r^2$  in the volume element, the reduced one dimensional energy functional, after some rescalings, is

$$E = \lambda \eta^{-2} r^2 V(\eta, h) + \tau \eta^6 \left[ (1 - h^2)^2 (r^2 h'^2 + 2h^2) + \frac{h^2}{r^2} (2r^2 h'^2 + h^2) \right] + (\eta\kappa)^4 \frac{h^4}{r^2} h'^2. \quad (8)$$



**Figure 1.** The profiles of the function  $h(r)$  and the mass-energy  $E(r)$  of typical spherically symmetric solutions are shown for several values of  $\kappa$ .

Mostly, we will retain only the terms multiplying  $\tau$ . In that case the lower bound on the integral of (8) is

$$Q = \int \varrho d^3x = 6 \int_0^\infty \frac{d}{dr} \left( \frac{h^3}{3} - \frac{h^5}{5} \right) dr. \quad (9)$$

Substituting into the limits of the definite integral (9), the asymptotic values

$$\lim_{r \rightarrow 0} h(r) = 0 \quad , \quad \lim_{r \rightarrow \infty} h(r) = 1 \quad (10)$$

following from (5) and (4), one finds

$$Q = \frac{4}{5}$$

which was verified numerically. The solutions of the field equation can be constructed numerically. We follow the usual approach and, by using a standard ordinary differential equation solver, we evaluate the initial condition

$$h = br - \frac{b^3}{5(b^2 + 2)}r^3 + O(r^5)$$

at  $r = 10^{-6}$  for global tolerance  $10^{-14}$ , adjusting for fixed shooting parameter and integrating towards  $r \rightarrow \infty$ . The behaviour of finite energy solutions as  $r \rightarrow \infty$  is

$$h \sim 1 + ce^{-2r} - \frac{1}{4r^2} - \frac{15}{32r^4} + O(1/r^6),$$

where  $c$  is a free parameter. For all considered cases, solutions with the correct asymptotics occurs only when the first derivative of the scalar function  $h(r)$  evaluated at the origin,  $h'(0) = b$ , takes on a certain value. For example  $b \sim 0.443613$  for a model without a sextic term ( $\kappa = 0$ ), while  $b \sim 0.367479$  for  $\kappa = 10$  (the symmetry breaking potential is vanishing in both cases).

The profiles of typical solutions are presented in Figure 1 for several values of the parameter  $\kappa$  and no symmetry breaking potential. The energy functional, as given by (8) is also exhibited. No multinode radial solutions were found, although we have no analytical argument for their absence. However, our preliminary numerical results indicate that, similar to the case of monopoles and sphalerons, the gravitating Goldstone model also presents radial excitations with an arbitrary number of nodes of the function  $h(r)$ . A study of these solutions will be presented elsewhere.

### 3.2 Charge- $n$ multisoliton: axially symmetric MS

The axially symmetric Ansatz for the scalar field

$$\phi^a = (\phi^\alpha, \phi^3)$$

is

$$\phi^\alpha = \eta \varphi_1(\rho, z) n^\alpha \quad , \quad \phi^3 = \eta \varphi_2(\rho, z) \quad , \quad (11)$$

where  $\rho^2 = |x_\alpha|^2 = x_1^2 + x_2^2$ ,  $z = x_3$ , denoting  $x_i = (x_\alpha, x_3)$ , and  $n^\alpha$  is the unit vector

$$n^\alpha = (\cos n \phi, \sin n \phi) \quad (12)$$

with azimuthal winding  $n$ .

We denote the two functions  $(\varphi_1, \varphi_2) \equiv \varphi_A$  by labeling  $\varphi^A$  with  $A = 1, 2$ . Subjecting (2) to (11), the  $\lambda = 0$  system reduces to

$$E = 2\pi \rho \left\{ \tau_1^2 \eta^6 (1 - |\varphi_B|^2)^2 \left[ (|\partial_\rho \varphi_A|^2 + |\partial_z \varphi_A|^2) + \left( \frac{n\varphi_1}{\rho} \right)^2 \right] \right\}$$

$$\begin{aligned}
& +\tau_2^2\eta^4 \left[ (\varepsilon_{AB} \partial_\rho\varphi_A \partial_z\varphi_B)^2 + \left( \frac{n\varphi_1}{\rho} \right)^2 (|\partial_\rho\varphi_A|^2 + |\partial_z\varphi_A|^2) \right] \\
& +\kappa^4\eta^6 \left( \frac{n\varphi_1}{\rho} \right)^2 (\varepsilon_{AB} \partial_\rho\varphi_A \partial_z\varphi_B)^2 \Big\}, \tag{13}
\end{aligned}$$

which in terms of the more useful variables  $(r, \theta)$  is

$$\begin{aligned}
E & = 4\pi\eta^6\tau_1^2 \sin\theta \left\{ (1 - |\varphi_B|^2)^2 \left[ (r^2|\partial_r\varphi_A|^2 + |\partial_\theta\varphi_A|^2) + \frac{n^2\varphi_1^2}{\sin^2\theta} \right] \right. \\
& + \left( \frac{\tau_2}{\tau_1\eta} \right)^2 \left[ (\varepsilon_{AB} \partial_r\varphi_A \partial_\theta\varphi_B)^2 + \left( \frac{n\varphi_1}{r\sin\theta} \right)^2 (r^2|\partial_r\varphi_A|^2 + |\partial_\theta\varphi_A|^2) \right] \\
& \left. + \left( \frac{\kappa^2}{\tau_1} \right)^2 \left( \frac{n\varphi_1}{r\sin\theta} \right)^2 (\varepsilon_{AB} \partial_r\varphi_A \partial_\theta\varphi_B)^2 \right\}, \tag{14}
\end{aligned}$$

and rescaling  $r$  as

$$\tau r \equiv \left( \frac{\tau_2}{\tau_1\eta} \right) r \rightarrow r$$

removes the coupling constant in front of the *quartic* term.

For the multisoliton (MS) solutions at hand, the boundary values of the functions  $\varphi_A$  in the  $r \gg 1$  region are

$$\lim_{r \rightarrow \infty} \varphi_1(r, \theta) = \sin\theta \quad , \quad \lim_{r \rightarrow \infty} \varphi_2(r, \theta) = \cos\theta, \tag{15}$$

while at the origin we find

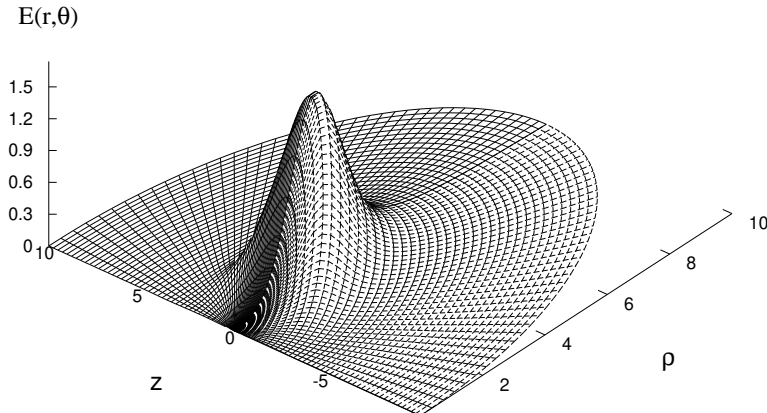
$$\varphi_1|_{r=0} = \varphi_2|_{r=0} = 0. \tag{16}$$

Since our imposition of axial symmetry requires also  $z \rightarrow -z$  reflection symmetry, the actual (numerical) integration need be performed only over the range  $0 \leq \theta \leq \frac{\pi}{2}$ . The field equations have been solved by imposing the following boundary conditions along the axes

$$\varphi_1|_{\theta=0} = \partial_\theta\varphi_2|_{\theta=0} = 0, \quad \partial_\theta\varphi_1|_{\theta=\pi/2} = \varphi_2|_{\theta=\pi} = 0. \tag{17}$$

We solve numerically the set of two coupled non-linear elliptic partial differential equations arising from the variation of the functional (14), subject to the above boundary conditions, employing a compactified radial coordinate  $x = r/(1+r)$ . To obtain axially symmetric solutions, we start with the  $n = 1$  solution discussed above as initial guess (corresponding to  $\varphi_1 = h(r) \sin\theta$ ,  $\varphi_2 = h(r) \cos\theta$ ) and increase the value of  $n$  slowly. The iterations converge, and repeating the procedure one obtains in this way solutions for arbitrary  $n$ . The physical values of  $n$  are integers. The typical numerical error for the functions is estimated to be lower than  $10^{-3}$ . The numerical calculations for  $n > 1$  were performed with the software package CADSOL/FIDISOL, based on the Newton-Raphson method [12]. In Figure 2 we show the local mass-energy as given by (14) of the  $\kappa = 0$   $n = 2$  MS solution as function of the coordinates  $z = r \cos\theta$  and  $\rho = r \sin\theta$ . In Figure 3 the profiles scalar functions  $\varphi_1$  and  $\varphi_2$  of the same solution are shown for several angles as a function of the radial coordinate  $r$ .





**Figure 2.** A three-dimensional plot of the mass-energy  $E(r, \theta)$  of a  $n = 2$ ,  $\kappa = 0$  axially symmetric MS solution.

The analysis was carried out in the first place setting the constant  $\kappa = 0$  and for several values of  $n$ , which captures the main qualitative properties of the MS. The maximum of the mass-energy density (14) moves outwards with increasing  $n$ . However, for  $n > 3$ , the numerical errors start to increase, and for some  $n_{max}$  the numerical iterations fail to converge. The problem resides in the behaviour of the scalar function  $\varphi_2$ , which for large  $n$ , tends to develop a discontinuity for some value of the radial coordinate.

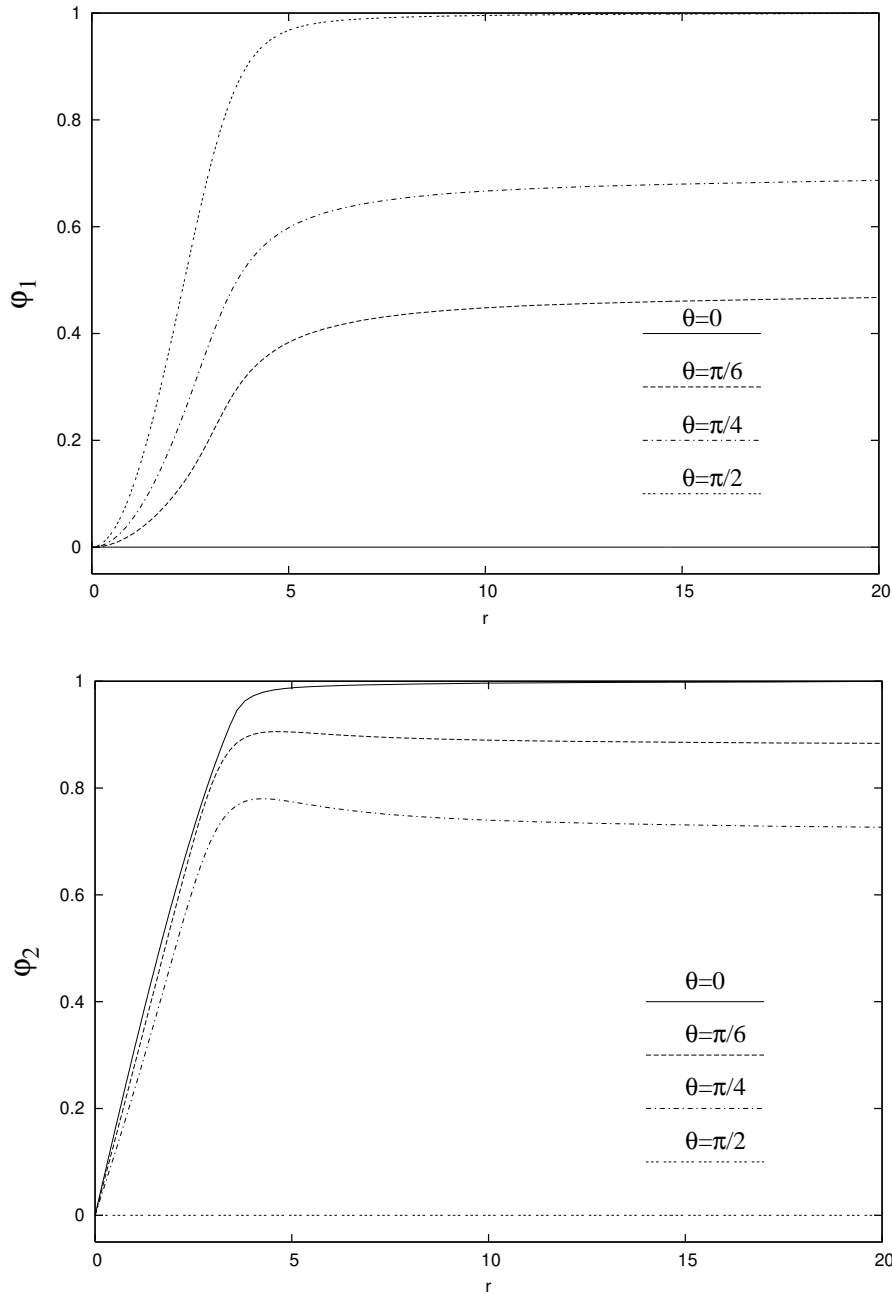
Because of the close analogy between our model and the Skyrme model, it is worthwhile checking one of the remarkable properties of axially symmetric multi-Skyrmions. The property in question is that up to vorticity ( $\equiv$  baryon number)  $n = 4$  the energy of the multi-Skyrmion is smaller than that of  $n$  infinitely separated 1-Skyrmions, i.e. that the MS can be regarded as a bound state [13].

We have here checked that starting from  $n = 2$ , and up to  $n = 5$ , the energies of the  $n$ -MSs of our model are greater than that of  $n$  1-solitons. Moreover it turns out that this deficit of binding energy increases with increasing  $n$ , indicating that none of the MSs in this model can be regarded as bound states. For example we have found  $M(n = 2)/(2M(n = 1)) - 1 = 0.229$ ,  $M(n = 3)/(3M(n = 1)) - 1 = 0.381$  while  $M(n = 4)/(4M(n = 1)) - 1 = 0.492$  (where  $M(n = 1) = 1.188$ ).

It is for this reason that we have introduced the sextic <sup>3</sup> term in (14), to check whether its presence may reverse this trend and lead to MS bound states? However, we find that for  $0 \leq \kappa \leq 10$ , MSs with charges up to 3 the deficit of binding energy persists and increases with  $n$ , confirming that like charged solitons of this model are mutually repulsive in spite of the

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<sup>3</sup>The presence of higher order terms in the (covariant) derivatives of the Higgs field is known to result in the mutual attraction of like charged (monopoles) solitons [14, 15].

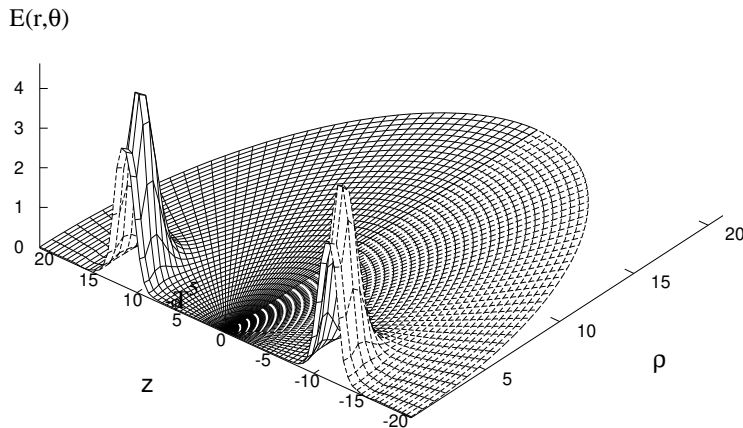


**Figure 3.** The profiles of the scalar functions  $\varphi_1$  and  $\varphi_2$  are shown for the  $n = 2$ ,  $\kappa = 0$  axially symmetric MS solution.

model having descended from the gauge decoupling of Higgs models [14, 15] supporting mutually attracting monopoles of like charges.

### 3.3 Charge-0 soliton–antisoliton: axially symmetric SAS

For simplicity we will restrict to winding number 1 SAS solutions, whence we set  $n = 1$  in the Ansatz (11).



**Figure 4.** A three-dimensional plot of the mass-energy  $E(r, \theta)$  of a  $n = 1$ ,  $\kappa = 0$  axially symmetric SAS solution.

The reduced two dimensional energy density functionals (13) and (14) are unchanged for the SAS solutions. Also the boundary conditions (17) arising from the requirements of analyticity on the  $z$ -axis and at the origin, remain unchanged. What are different between the MS and SAS solutions are the boundary conditions at  $r \rightarrow \infty$  and at  $r = 0$ .

For the SAS solutions in the region  $r \gg 1$ , instead of (15) we require

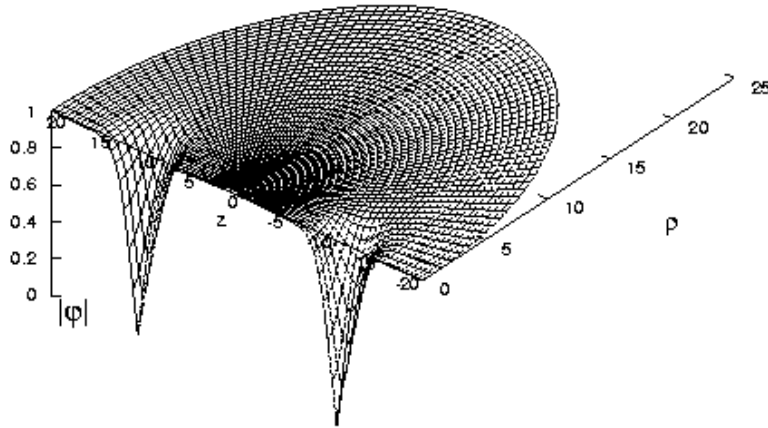
$$\lim_{r \rightarrow \infty} \varphi_1(r, \theta) = \sin m\theta \quad , \quad \lim_{r \rightarrow \infty} \varphi_2(r, \theta) = \cos m\theta \quad , \quad (18)$$

with  $m \geq 2$  for SAS chains analogous to the monopole antimonopole chains [16]. But we are here only interested in SAS pairs, therefore we restrict to  $m = 2$  in (18).

For the SAS in the region  $r \ll 1$ , instead of (16) we require

$$\varphi_1|_{r=0} = 0 \quad , \quad \partial_r \varphi_2|_{r=0} = 0. \quad (19)$$

The field equations have been solved by using the same methods employed in the MS case. However, the SAS solutions exhibit a very different picture. The energy density  $\epsilon = -T_t^t$  possesses maxima at  $z = \pm d/2$  and a saddle point at the origin, and presents the typical form exhibited in the literature on MA solutions [10]. The modulus of the scalar field  $|\varphi| = \sqrt{\varphi_1^2 + \varphi_2^2}$  possesses always two zeros at  $\pm d/2$  on the  $z$ -symmetry axis. In Figure 4 we plot the mass-energy (14) and the modulus of the scalar field of a typical  $m = 2$  solution as a function of the coordinates  $\rho, z$ , for  $\kappa = 0$  (*i.e.* no sextic term). This solution has a mass  $M = 2.588$  which is smaller than that of two ( $n = 1, m = 1$ ) 1-solitons ( $M=2.92$ ), similar to the sphaleron describing a monopole-antimonopole pair [10].



**Figure 5.** The modulus of the scalar field  $|\varphi| = \sqrt{\varphi_1^2 + \varphi_2^2}$  is shown for the  $n = 1$ ,  $\kappa = 0$  axially symmetric SAS solution.

## 4 Summary

We have studied the finite energy topologically stable static solutions to a (global) symmetry breaking model in  $3 + 1$  dimensions described by an isovector scalar field. Such models can be constructed in arbitrary  $D + 1$  dimensions since they are the gauge decoupled versions of Higgs models in all dimensions. We have chosen here  $D = 3$  examples, since this is the dimensionality of most physical interest, like the usual Skyrme model, but very different from the latter in many essential respects.

Two classes of solutions are studied: Axially symmetric multisolitons (MS) with topological charge  $n$ , and unstable soliton–antisoliton (SAS) pairs with zero topological charge, both with finite energies. There are two pertinent questions that arise here. In the case of the MS solutions, the question is whether solitons of like charge attract or repel, and it was found that they always repel, even when the model is augmented with a sextic kinetic term. In the case of the SAS pairs, the question is whether they can support a nonzero angular momentum<sup>4</sup>? This task is deferred to some future work, and presumably it will involve a stationary Q-ball like features.

As a scalar theory supporting soliton solutions in 3 space dimensions this model is like the Skyrme model. Unlike the latter however this is a symmetry breaking model, as result of which the boundary values of the field are akin to that of a monopole rather than that of an instanton as is the case for the Skyrme (nonlinear sigma) model. From the viewpoint of physical properties, there is no question that it can be regarded as an alternative for the

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<sup>4</sup>The corresponding rotating solutions in YMH model have been studied recently in [17] and [18].

Skyrme model which is known to give a good description of Nucleons [9] at low energies. This can be seen from two clear viewpoints: (*i*) the fact that Skyrmions are capable of forming bound states [13] describing exotic states, while the MSs of our model display the opposite property, and (*ii*) because the Skyrmion can be gauged with the (Maxwell)  $U(1)$  field [19, 20] enabling the description of the electromagnetic properties of the Nucleons, while the topological lower bound on the energy of our MS is invalidated when the scalar field is gauged with  $U(1)$ . This is because Higgs models, from the  $p = 2$  member of which [5] the present model is extracted, can be gauged only with  $SO(D)$ , or, be completely gauge-decoupled as the model considered here. While it is true that a  $O(D + 1)$  sigma model in  $D$  dimensions can be gauged with all  $SO(N)$  with  $N \leq D$  [19] with its energy bounded from below by a gauge invariant topological charge, gauging Higgs models [5] with  $SO(N)$  with  $1 < N < D$  causes the collapse of the topological lower bound on the energy.

Technically the properties of the model studied here are analogous to those of the usual  $SU(2)$  Higgs model with symmetry breaking potential supporting monopoles [1]. Like in that case, the Bogomol'nyi bound cannot be saturated, and, like-charged solitons repel in the sense that a(n axially symmetric) multisoliton of charge  $n$  has higher energy than  $n$  infinitely separated 1-solitons. (It is possible that solutions with less than axial symmetry may have lower mass than the axially symmetric ones studied here, so the possibility exists that such solutions may form bound states, however unlikely.) Another point in this analogy is that the sphaleron describing soliton-antisoliton pairs here is lighter than two 1-solitons, just as it is in the Higgs case [10].

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