Features of gravity-Yang-Mills hierarchies in $d-$dimensions

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Abstract

Higher dimensional, direct analogues of the usual $d = 4$ Einstein–Yang-Mills (EYM) systems are studied. These consist of the gravitational and Yang-Mills hierarchies in $d = 4p$ dimensional spacetimes, both consisting of $2p$-form curvature terms only. Regular and black hole solutions are constructed in $2p + 2 \leq d \leq 4p$, in which dimensions the total mass-energy is finite, generalising the familiar Bartnik-McKinnon solutions in EYM theory for $p = 1$. In $d = 4p$, this similarity is complete. In the special case of $d = 2p + 1$, just beyond the finite energy range of d, exact solutions in closed form are found. Finally, $d = 2p + 1$ purely gravitational systems, whose solutions generalise the static $d = 3$ BTZ solutions, are discussed.

1 Introduction

Regular and black hole solutions of gravitating gauge field systems have been studied for a long time since the pioneering work of Bartnik and McKinnon (BK) [1], where regular solutions of the Yang–Mills (YM) and Einstein–Hilbert (EH) systems were presented for $d = 4$ spacetime dimensions. Other models of gravitating non-Abelian gauge fields usually share a number of common features with the BK particles, this example becoming canonical. After that discovery, there has been a great deal of numerical and analytical work on various aspects of Einstein-Yang-Mills (EYM) theory and a variety of self-gravitating structures with non-Abelian fields have been found (for a review see [2]). These include hairy black holes solutions, which led to certain revisions of some of the basic concepts of black hole physics based on the uniqueness and no-hair theorems.

In the last years, gravitating solutions with nonabelian fields enjoyed renewed interest in the AdS/CFT correspondence, and potentially in the study of Dp-branes of superstring theory. It is therefore pertinent to extend the study of such solutions to higher dimensions other than that of $d = 4$ spacetime.

Recently regular solutions were found in $d = 6, 7, 8$ in [3]; both regular and black hole solutions in five dimensions were presented in [4], the $d = 6, 7, 8$ black hole solutions being discussed in [5]. These were the classical solutions to systems consisting of higher order terms in both the YM and the gravitational curvatures, which do in fact appear in the low energy effective action of string theory. Such terms employed in [3, 4] were those constructed from the totally antisymmetrised $2p$-forms in both the YM and the Riemann curvature 2-forms, namely the YM and the gravitational hierarchies labelled by integers p. The $p = 1$ members in each case are the usual YM and the EH systems respectively, while the $p = 2$ gravitational member is the familiar Gauss-Bonnet term.

These asymptotically flat configurations differ in many respects from the $d = 4$ BK solutions. In particular, all $d > 4$ solutions have only one node in the gauge potential. Also, the black holes exist only up to a maximal value of the event horizon radius.

The salient property of these solutions [3, 4] is that they exist only for a limited range of the coupling parameter α^2 (which gives the strength of the gravitational interaction), and exhibit critical behaviours in α^2. This is not surprising since such composite models necessarily feature more than one dimensional constant, analogous to gravitating monopoles [6]. (In the latter case dimensionful Higgs vacuum expectation value (VEV) plays this role.)

More recently, a complete analysis of these critical behaviours in α^2 in the higher dimensional gravitating YM systems was presented in [8]. The methods used in [8] were those of fixed point analysis developed previously in [7] for proving the existence of the BK solutions [1] to the usual EYM system in $d = 4$. Concerning the fairly complicated landscape of critical points here [8], we restrict our comments only to pointing out that in addition to the two types of critical points occurring in gravitating monopoles [6], namely the ones associated with the *end point* and the *Reissner-Nordstrøm* types, there is also a *conical* fixed point, so described since at that scaling point an angular deficit appears in the gravitational metric function. We refer the reader to [8] for a full account of these critical behaviours.

The analysis in [8] was carried out for models whose gravitational part consisted only of the $p = 1$ member of the gravitational hierarchy, namely the Einstein–Hilbert (EH) term. The physical reason for this was that the $p \geq 2$ Gauss-Bonnet like terms play only a quantitative role, confirmed numerically in [3], and have no effect on the existence of finite mass solutions. By contrast in the gauge field sector, various combinations of YM terms with $p \geq 1$ essential for the existence of such solutions were employed, as required by scaling arguments. Hence higher $p \geq 1$ YM terms were employed there, once d was greater than 4. Thus the analysis in [8] probed the effect of the higher $p \geq 2$ YM terms.

One of the two main aims of the present work is to probe the effect of higher order $p \geq 2$ gravitational terms in EYM models. But we know from the numerical results of [3] that once the $p = 1$ (EH) term is present, the effects of all higher $p \geq 2$ terms become masked. It is therefore the case that if one wishes to study the effects of the p-th gravitational term, all other gravitational terms with $p_i < p$ must be excluded. In practical terms, this means that we will restrict to models featuring only the p-th gravitational term.

The other one of our two main aims is to exhibit certain qualitative similarities of the solutions supported by a family of EYM models, the first member of which consists of the $p = 1$ gravitational (EH) and the $p = 1$ YM terms, supporting the BK solutions. It follows naturally that the YM term we must choose interacting with the (unique) p-th gravitational term, is the (unique) p-th YM term, in spacetime dimensions $d = 4p$. We expect to exhibit a recurring 'symmetry' in the properties of the solutions of this family of models *modulo* $4p$ dimensions. In the models studied here the complicated critical features of the gravitating monopole [6] and of the higher dimensional EYM solutions studied in [3, 4, 8] will be absent, since these are due to the presence of at least one additional (to the gravitational and YM couplings) dimensionful constant, e.g. the Higgs VEV, or, the higher curvature YM coupling constants.

In Section 2 we introduce the relevant gravitating YM models, and subject them to spherical symmetry. These models will be characterised by two equal integers $p_1 = p_2 = p$ specifying the model and the gauge group, and the dimension of the spacetime d. In Section 3 we specialise the dimension d to the values $d = 4p$, $d = 4p - 1$, ..., $d = 2p + 2$, which are the only dimensions in which asymptotically flat finite energy solutions exist. These solutions can be constructed only numerically so the results presented in Section 3 are mainly numerical. In Section 4 we specialise to spacetime dimensions $d = 2p + 1$ in which no asymptotically flat, finite energy solutions exist for $p_1 = p_2 = p$. The interest in the latter, inspite of their pathological properties, is that they can be given in closed form, which is a novel feature in gravitating gauge field theory. In addition to these we have supplied two Appendices, A and B, devoted to the extension of some the models studied in Sections 3 and 4, to feature a non-vanishing cosmological constant Λ. Like the solutions presented in Section 3, those in Appendices A and B are expressed in closed form. In Appendix A we study the geometric properties of the gauge decoupled limit models in $d = 2p + 1$ appearing in Section 3, supplemented with a cosmological term, which are the hierarchy of gravity solutions pertaining to the static BTZ solution [9], this last being the $p = 1$ member. The Appendix B presents a generalisation of the exact gravity-YM $p_1 = p_2 = p$ solution in $d = 2p + 1$ dimensions for a nonzero Λ. In Section 5, we summarise our results.

2 The model and imposition of spherical symmetry

The precise form of the gravitational and non-Abelian matter content of string effective actions beyond leading order is still an evolving research subject [10]. Our position in studying higher dimensional gravitating YM solutions, in [3, 4, 8], has been to choose what we have referred to as the superposition of the p-th members of the gravitational and YM hierarchies, consisting of all possible higher order curvature forms allowed in any given dimension.

The said gravitational system is the superposition of all possible (p,q)-Ricci scalars $R_{(p,q)}$

$$\mathcal{L}_{\rm grav}^{(P)} = \sum_{p=1}^{P} \frac{\kappa_p}{2p}\ e\ R_{(p,q)}\ , \tag{1}$$

where $R_{(p,q)}$ are constructed from the $2p$-form $R(2p) = R\wedge R\wedge ...\wedge R$ resulting from the totally antisymmetrised p-fold products of the Riemann curvature 2-forms R. We express $R_{(p,q)}$ in the notation of [11] as

$$e\,R_{(p,q)} = \varepsilon^{\mu_1\mu_2...\mu_{2p}\nu_1\nu_2...\nu_q} e_{\nu_1}^{n_1} e_{\nu_2}^{n_2} ... e_{\nu_q}^{n_q}\ \varepsilon_{m_1m_2...m_{2p}n_1n_2...n_q}\ R_{\mu_1\mu_2....\mu_{2p}}^{m_1m_2..m_{2p}}\ , \tag{2}$$

where e_ν^n are the *Vielbein* fields, $e = \det(e_\nu^n)$ in (1), and $R_{\mu_1\mu_2....\mu_{2p}}^{m_1m_2..m_{2p}} = R(2p)$ is the p-fold totally antisymmetrised product of the Riemann curvature, in component notation. This leads to the definition of the p-th Einstein tensor

$$G_{(p)}{}_\mu^a = R_{(p)}{}_\mu^a\ -\ \frac{1}{2p}\ e_\mu^a\ R_{(p)}\ . \tag{3}$$

One reads from (2) that

$$d = 2p + q\,.$$

Now the minimum nontrivial value of q is $q = 1$, since when $q = 0$ (2) is manifestly a total divergence, namely the Euler-Hirzebruch density. Thus the highest nontrivial value of P in the superposition (1) is

$$P_{\rm max} \le \frac{1}{2}(d-1).$$

The corresponding superposition of the members of the YM hierarchy is

$$\mathcal{L}_{\rm YM}^{(P)} = \sum_{p=1}^{P} \frac{\tau_p}{2(2p)!}\ e\ {\rm Tr}\,F(2p)^2\ , \tag{4}$$

in which the $2p$-form $F(2p)$ is the p-fold antisymmetrised product $F(2p) = F\wedge F\wedge...\wedge F$ of the YM curvature 2-form F. Here the maximum value of P in the superposition (4) is simply $P_{\rm max} \le \frac{1}{2}d$. We define the p-stress tensor pertaining to each term in (4) as

$$T_{\mu\nu}^{(p)} = {\rm Tr}\ F(2p)_{\mu\lambda_1\lambda_2...\lambda_{2p-1}} F(2p)_\nu{}^{\lambda_1\lambda_2...\lambda_{2p-1}} - \frac{1}{4p} g_{\mu\nu}\ {\rm Tr}\ F(2p)_{\lambda_1\lambda_2...\lambda_{2p}} F(2p)^{\lambda_1\lambda_2...\lambda_{2p}}\ . \tag{5}$$

The $P = 1$ systems (1) and (4) are the usual EH gravity and YM theories, respectively.

In [3, 4, 8], some convenient superpositions (1) and (4) were selected to be studied, taking into account the particular properties of the solutions that were being sought.

It is our aim here to truncate both (1) and (4) such that only one term is present in each. This is so that, analogously to the BK case, the solutions not be parameterized by one (e.g. α^2) or more parameters. The other criterion stated in Section 1 is that in the appropriate dimensions higher members of the gravitational hierarchy be employed. So far, subject to respecting the Derrick scaling requirements, one can choose any $P = p_1$ in (1) and any $P = p_2$ in (4). The final criterion is that of symmetry and analogy with the BK case in $d = 4$, namely that $p_1 = p_2 = p$ in $d = 4p$ dimensions [1].

[1] There is yet another apparently unrelated coincidence here. In Euclidean signature, the double-self-duality of the Riemann $2p$ form curvature leads to the vacuum p-Einstein equations being satisfied, and yields a self-dual $SO_\pm(4p)$ YM $2p$-form field strength.

Thus we define the gravitating YM models in $2p+2 \leq d \leq 4p$ spacetime dimensions, whose static finite energy solutions will be constructed numerically in the next section, by the Lagrangians

$$\mathcal{L}_{(p,d)} = e \left(\frac{\kappa_p}{2p} \ R_{(p,q)} + \frac{\tau_p}{2(2p)!} \ \mathrm{Tr}\, F(2p)^2 \right) \quad , \quad 2 \leq q \leq 2p \, , \tag{6}$$

κ_p and τ_p being constants giving the strength of the gravitational and YM interactions, respectively. Note that the gravitational part in (6) is described by the (p,q)-Ricci scalars $R_{(p,q)}$ in which $q = d-2p$, as defined in (2). For this system, the variational equations for the YM and gravitational fields are

$$\tau_p D_\mu \left(e \ F^{\mu\nu} \right) = 0, \quad \kappa_p G_{(p)}{}_\mu^a \ = \ \frac{\tau_p}{2(2p)!} T^{(p)}{}_\mu^a. \tag{7}$$

The model (6) directly generalises the usual EYM model in $d=4$ spacetime for $d=4p$, the latter being the $p=1$ case. In that case the dimensions of the gravitational part of (6) are L^{-2p}, versus the dimensions of the YM part L^{-4p}. This choice has been made on grounds of symmetry rather than physics, since the leading terms in the effective action of string theory are the $p=1$ EH and YM terms, which are both excluded. Since the $p=1$ EYM system is best known for its BK solutions, the solutions to the hierarchy defined by (6) for $d=4p$ might be described as the BK hierarchy.

The definition of the model becomes complete on specifying the gauge groups and their representations. Adopting the criterion of employing chiral representations, both for *even* and for *odd* d in (6), it is convenient to choose the gauge group to be $SO(\bar{d})$. We shall therefore denote our representation matrices by $SO_\pm(\bar{d})$, where $\bar{d}=d$ and $\bar{d}=d-1$ for *even* and *odd* d respectively.

In this unified notation (for odd and even d), the spherically symmetric Ansatz for the $SO_\pm(\bar{d})$-valued gauge fields then reads [3, 4]

$$A_0 = 0 \ , \quad A_i = \left(\frac{1-w(r)}{r} \right) \Sigma_{ij}^{(\pm)} \hat{x}_j \ , \quad \Sigma_{ij}^{(\pm)} = -\frac{1}{4} \left(\frac{1 \pm \Gamma_{\bar{d}+1}}{2} \right) [\Gamma_i, \Gamma_j] \ . \tag{8}$$

The Γ's denote the $\bar{d}$-dimensional gamma matrices and $1, \ j = 1, 2, ..., d-1$ for both cases.

The spherically symmetric metric Ansatz we use is parameterized by two functions $N(r)$ and $\sigma(r)$

$$ds^2 = -N(r)\sigma^2(r)dt^2 \ + \ N(r)^{-1}dr^2 \ + \ r^2 d\Omega^2_{(d-2)} \ . \tag{9}$$

Inserting (8) in (6), the resulting reduced one dimensional Lagrangian is

$$\begin{aligned} L_{(p,d)} \ &= \ \frac{(d-2)!}{(d-2p-1)!} \sigma \left\{ \frac{\kappa_p}{2^{2p-1}} \frac{d}{dr} \left[r^{d-2p-1} (1-N)^p \right] \right. \\ &+ \ \left. r^{d-2} \frac{\tau_p}{2 \cdot (2p)!} W^{p-1} \left[(2p) N \left(\frac{1}{r} \frac{dw}{dr} \right)^2 + (d-2p-1) \, W \right] \right\}, \end{aligned} \tag{10}$$

where we note

$$W = \frac{(1-w^2)^2}{r^4}. \tag{11}$$

3 A Bartnik-McKinnon hierarchy

It follows from the Derrick-type scaling arguments that static finite energy solutions to the field equations of the model (6) exist only in spacetime dimensions

$$2p+2 \leq d \leq 4p \, , \tag{12}$$

and can only be constructed numerically, which we present here. The $d=4p$ family is referred to as the BK hierarchy, but we study all possible cases allowed by (12). Note that for $p=1$, $d=4$ is the only possibility allowed by (12).

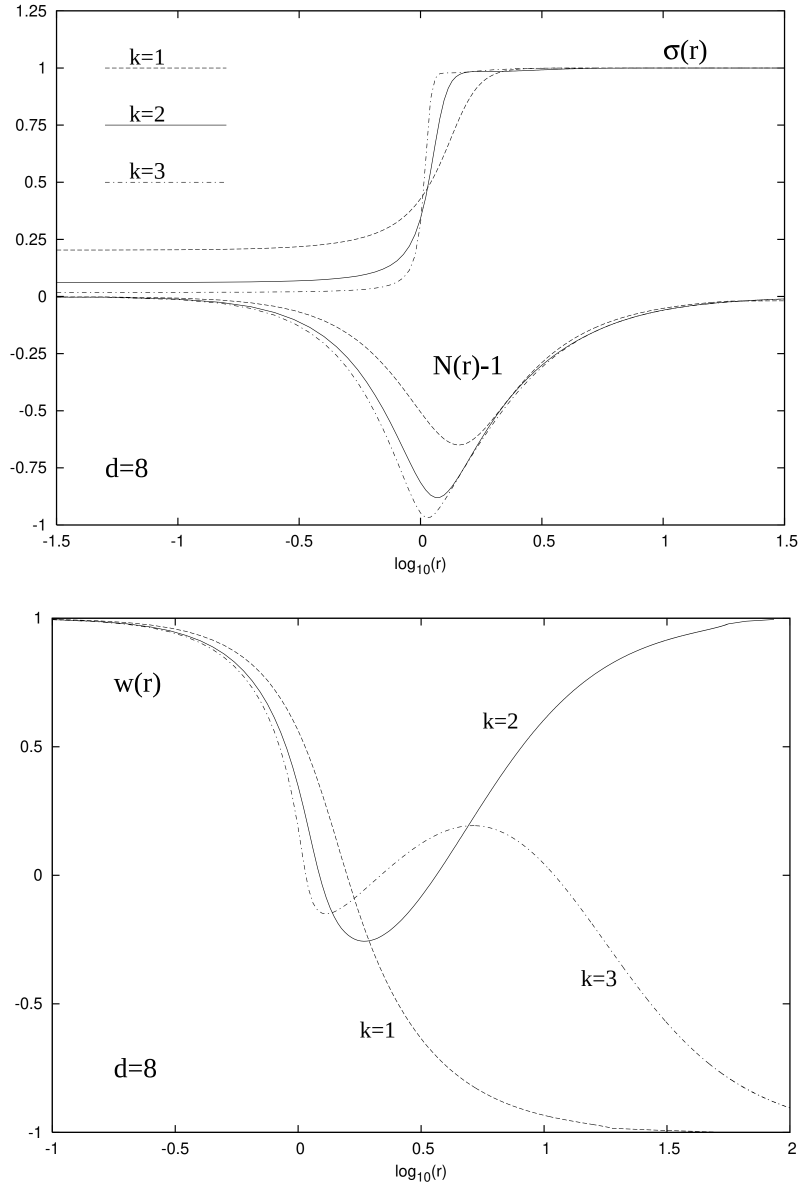


Figure 1. The profiles of the metric functions $N(r)$, $\sigma(r)$ and gauge function $w(r)$ are presented for $k-$node globally regular solutions of the $p=2$ gravity-Yang-Mills model in $d=8$ dimensions.

When looking for numerical solutions, it is convenient to redefine

$$r \rightarrow \left(\frac{2^{2(p-1)}\tau_p}{(2p!)\kappa_p}\right)^{1/(4p-2)} r$$

such that for any $(d,\ p)$ no free parameter appears in the field equations.

The field equations implies the relations

$$\frac{d}{dr}\left[r^{d-2p-1}(1-N)^p\right] = r^{d-2}W^{p-1}\left[2pN\left(\frac{1}{r}\frac{dw}{dr}\right)^2 + (d-2p-1)W\right], \tag{13}$$

$$\frac{d\sigma}{dr} = 2r^{2p-3}\sigma(1-N)^{1-p}W^{p-1}w'^2, \tag{14}$$

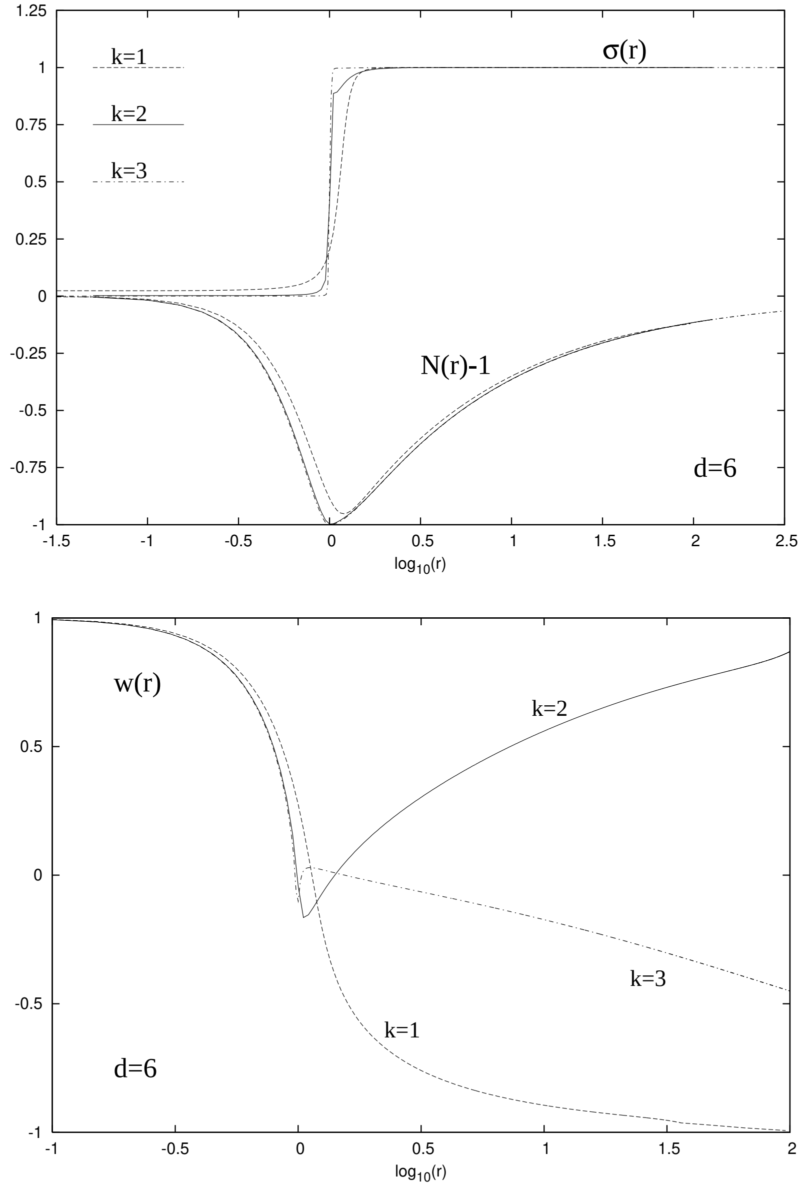


Figure 2. One, two and three nodes solutions of the $p = 2$ gravity-Yang-Mills model in $d = 6$ dimensions.

for the metric function, and

$$\frac{d}{dr}\left(r^{d-4}\sigma W^{p-1} N w'\right) \;=\; \sigma r^{d-6} W^{p-2} w(w^2-1)\left(2(p-1)N\left(\frac{1}{r}\frac{dw}{dr}\right)^2 + (d-2p-1)W\right). \quad (15)$$

for the gauge potential.

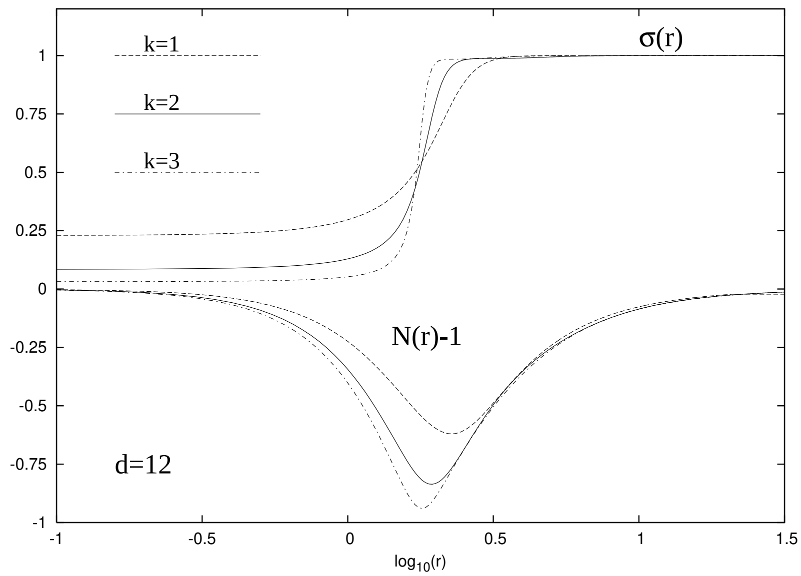
For globally regular solutions, finite energy requirements and regularity of the metric at $r = 0$ give

$$w = 1 - br^2 + O(r^4), \quad N = 1 - 4b^2r^2 + O(r^3), \quad \sigma = \sigma_0(1 + 4b^2r^2) + O(r^4), \quad (16)$$

where $\sigma_0,\ b$ are two positive constant. The analysis of the field equations as $r \to \infty$ gives

$$N = 1 - \frac{M_0^{1/p}}{r^{(d-2p-1)/p}} + \dots, \quad w = \pm 1 + \frac{c}{r^{(2p-d+1)/p}} + \dots, \quad (17)$$

$$\sigma = 1 + \frac{2p(4c^2)^p M_0^{(1-p)/p}}{(d-2p-1)(3p-1) + 2p^2}\frac{1}{r^{2p+(d-2p-1)(3p-1)/p}} + \dots,$$



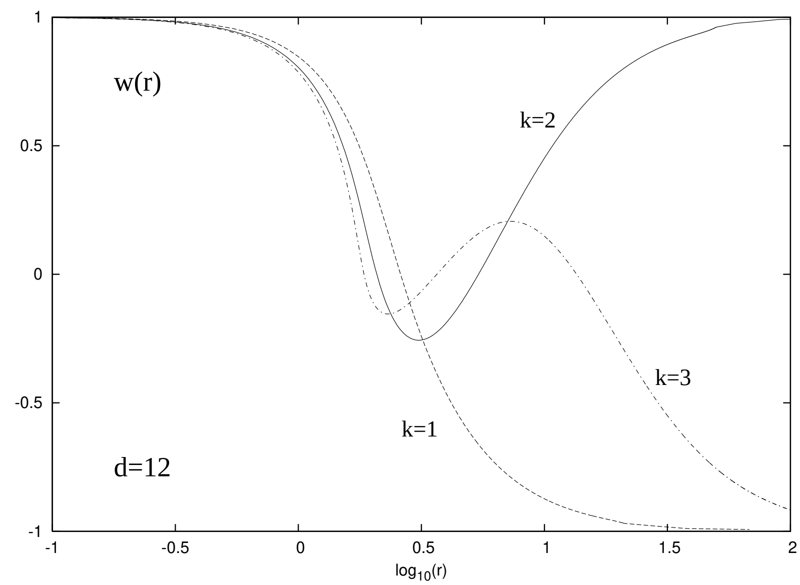


Figure 3. The profiles for the metric functions $N(r)$, $\sigma(r)$ and gauge function $w(r)$ are presented for $k-$node globally regular solutions for the $p=3$ gravity-Yang-Mills system in $d=12$ dimensions.

with M_0, c arbitrary constants. In analogy to the Einstein gravity, it is natural to identify the constant M_0 that enters the asymptotic of N with the mass $\mathbf{M}$ of the solutions, up to a $d-$dependent factor. To see this let us notice that (13) can be written as

$$\frac{d}{dr}[r^{d-2p-1}(1-N)^p] \quad = \quad r^{d-2}T^{(p)t}_{\ \ t} \tag{18}$$

which in the $(p=1, d=4)$ case of standard Einstein gravity leads to the usual definition of mass. Here $T^{(p)t}_{\ \ t}$ is the 'energy density' component of the stress-energy tensor defined in (5). By keeping with this analogy, 'the total energy' inside a $(d-2)$-sphere S_{d-2} with radius r will be given by

$$\int_0^r dr' \int_{S_{d-2}} d\Omega_{(d-2)}\ r'^{d-2}W^{p-1}(r')\left[2pN(r')\left(\frac{1}{r'}\frac{dw}{dr'}\right)^2+(d-2p-1)W(r')\right] = Area(S_{d-2})r^{d-2p-1}(1-N)^p \tag{19}$$

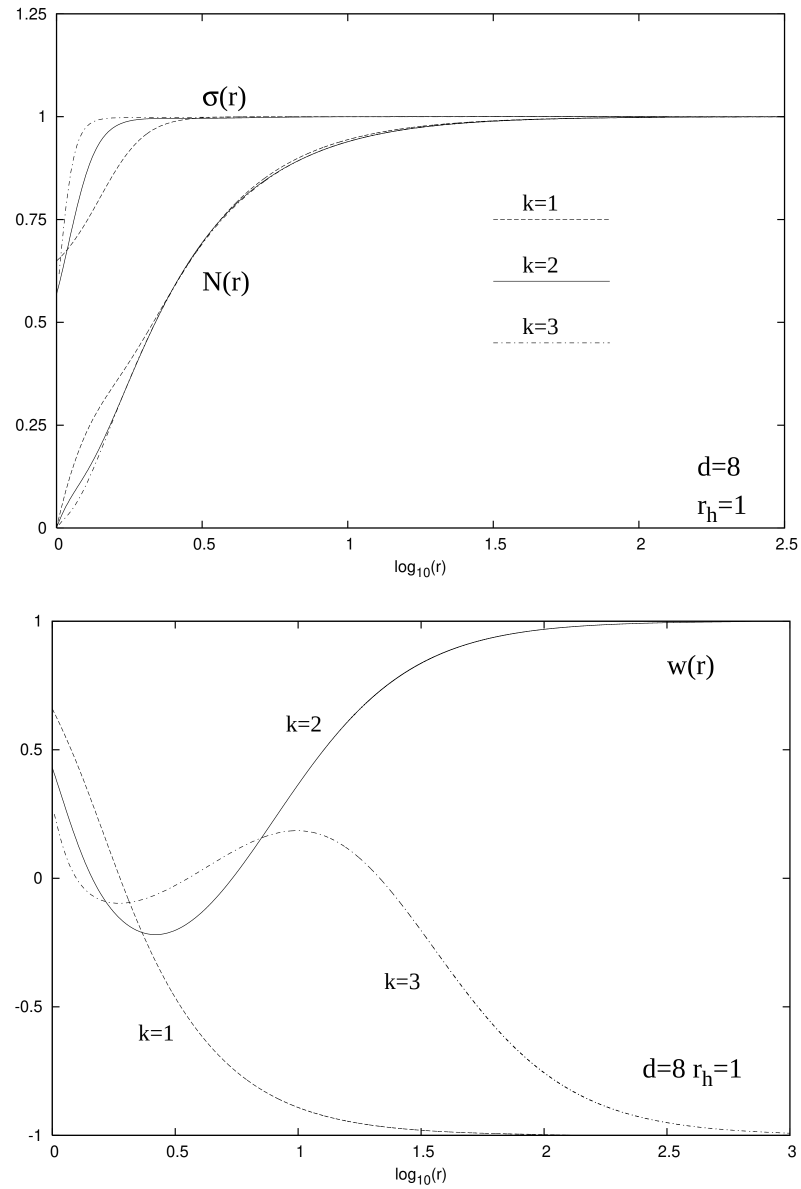


Figure 4. One, two and three nodes black holes solutions of the $p = 2$ gravity-Yang-Mills model in $d = 8$ dimensions.

where we made use of the regularity of the function $N(r)$ at origin $r = 0$ (see (16)). Taking now the limit $r \to \infty$ we find that the total energy is finite and given precisely by $\mathbf{M}$=$Area(S_{d-2})M_0$ as claimed.

The equations of motion (13)-(15) have been solved for $p = 2,\ 3$ and a range of d. Considering first globally regular configurations, we follow the usual approach and, by using a standard ordinary differential equation solver, we evaluate the initial conditions (16) at $r = 10^{-4}$ for global tolerance 10^{-12}, adjusting for fixed shooting parameter b and integrating towards $r \to \infty$.

As expected, these solutions have many features in common with the well-known $d = 4,\ p = 1$ BK solutions. By adjusting the free parameter b appearing in the expansion (16), we "shoot" for global solutions with the right asymptotics. The solutions are indexed by k-the number of nodes of the gauge potential w, which is always bounded within the strip $|w(r)| \le 1$.

For the k-th solution, the function $w(r)$ has k nodes in the interval $0 < r < \infty$, such that $w(\infty) = (-1)^k$. Typical profiles for $p = 2$ solutions are plotted in Figures 1, 2 (for $d = 8,\ 6$) and $p = 3,\ d = 12$ configurations (Figure 3). It is worth noting here that for a p-gravity and p-YM theory, when the spacetime dimension is

$4p$, then the profiles of the functions $N(r)$, $\sigma(r)$ and $w(r)$ are perfectly smooth as seen from Figures 1 (for $d=8$) and Figures 3 (for $d=12$), as is the case for the BK solutions in $d=4$, while for dimensions different from $d=4p$, e.g. for $d=6$ depicted in Figure 2, these profiles are less smooth and the numerical analysis is correspondingly more delicate. This is one the features highlighting the similarity of the solutions to these theories *modulo* $4p$.

The behaviour of the metric functions N and σ is similar for all k's. The metric functions σ increases with growing r from $\sigma(0)=\sigma_0>0$ at the origin to $\sigma(\infty)=1$. As k increases, N develops a more and more deep minimum at some r_m, closely approaching the zero value for large enough values of k, as indicated in Figures 1-3. At the same time, the value at the origin of the metric function σ strongly decreases with k.

The limiting solution with $k=\infty$ can also be investigated [12]. Since the Schwarzschild coordinate system breaks down in this limit, we should use a different parameterization and the equations are formulated as a dynamical system. Similar to the $p=1$ case, this turns out to be non-asymptotically flat.

The system (13)-(15) presents also black hole solutions. The boundary conditions at infinity are still given by (17), and are now supplemented with the requirement that there is a regular event horizon at some $r=r_h>0$, with $N(r)>0$ for $r>r_h$. The local power series in the vicinity of the horizon reads

$$w(r)=w_h+w'_h(r-r_h)+O(r-r_h)^2,\ N(r)=N'_h(r-r_h)+O(r-r_h)^2,\ \sigma(r)=\sigma_h+\sigma'_h(r-r_h)+O(r-r_h)^2, \quad (20)$$

with

$$N'_h=\frac{(d-2p-1)}{r_h}(1-r^{2p}W_h^p),\quad w'_h=w_h(w_h^2-1)(d-2p-1)\frac{1}{r^2N'_h},\quad \sigma'_h=2\sigma_h r_h^{2p-3}W_h^{p-1}w_h'^2, \qquad (21)$$

and $W_h=\frac{(1-w_h^2)^2}{r_h^4}$. Similar to the $d=4,\ p=1$ case, one find a sequence of global solutions in the interval $r_h<r<\infty$, for any value of the event horizon $r_h>0$. These solutions are parameterized again by the node number k of the gauge function w. For any (r_h,k) the behaviour of the functions w,σ is qualitatively similar to that for regular solutions. The gauge function w starts from some value $0<w_h<1$ at the horizon and after k oscillations around zero tends asymptotically to $(-1)^k$. In this case again one can show that $|w|<1$ everywhere outside the horizon. In the limit $r_h\to 0$ the event horizon shrinks to zero and the black hole solutions converges pointwise to the corresponding regular configuration. In Figure 4 we exhibit the one, two and three nodes solutions of the $p=2$ model in eight dimensions; similar solutions have been found for $p=2, d=6$ and $p=3,\ d=9,12$.

The stability of solutions in the usual $p=1,\ d=4$ case has been studied extensively, both perturbatively and at the non-linear level (see the discussion in [2] and the references therein). It turns out that all known regular and black hole solutions in that case are unstable with respect to small spherically symmetric perturbations. The most obvious indication of the instability comes from the absence of a topological charge in the YM sector. This is obvious in the $d=4$ case, but when several members of the YM hierarchy are present like in [3, 4, 8], in some of these theories a Pontryagin charge is defined, leading to stable solutions. The stability question in such models has been studied in [13]. In the present work however we have restricted to a single (the p-th) member of the YM hierarchy, so that no topological charge can be accommodated in the case of static solutions. Indeed, the stability analysis in [13] leading to the conclusion that the solutions in dimensions in which a Pontryagin charge is not defined are unstable (sphalerons) applies to the models studied in the present Section, and is not repeated here.

Another aspect of the stability of gravitating YM fields is the interesting case when a negative cosmological constant is introduced, in the usual EYM model with $p=1,\ d=4$ [14, 15]. In that case, one can see from the asymptotic analysis that the value of the gauge field function $w(r)$ at infinity is not restricted to ± 1, and in particular $\lim_{r\to\infty} w(r)=0$ is allowed. This is the asymptotic value for a static monopole, *i.e.* the particular solution in question is a finite energy lump with (topological) monopole charge, rendering it stable. Unfortunately this property does not persist [2] in the $p\geq 2$ models with negative cosmological constant, implying that these would be exclusively sphalerons. As a result we have eschewed an analysis of these.

[2]This can be seen from the large r asymptotic analysis in the case of $p\geq 2$ models with negative cosmological constant, which leads exclusively to $\lim_{r\to\infty} w(r)=\pm 1$.

We close this Section with several comments on the thermodynamic properties of the black hole solutions. The Hawking temperature of these configurations can easily be found by using the standard Euclidean method. For the line element (9), if we treat t as complex, then its imaginary part is a coordinate for a non-singular Euclidean submanifold iff it is periodic with period

$$\beta = \frac{4\pi}{N'(r_h)\sigma(r_h)}. \tag{22}$$

Then continuous Euclidean Green functions must have this period, so by standard arguments the Hawking temperature is $T = 1/\beta$. For any (p, d) this is found to decrease with the node number k. To compute the entropy of these solutions, one may use the standard relationship between the temperature and the entropy, $S = \int d\mathbf{M}/T + S_0$, with S_0 a mass-independent constant. Although further study is necessary, we expect this expression to differ from the standard one quarter event horizon area value, which holds for $p = 1$ EH gravity only [16, 17].

4 Exact gravity-Yang-Mills solutions in $d = 2p + 1$

In the particular dimensions $d = 2p + 1$, the analogues of (6),

$$\mathcal{L}_{(2p+1)} = e\left(\frac{\kappa_p}{2p}\ R_{(p,1)} + \frac{\tau_p}{2(2p)!}\ \mathrm{Tr}\, F(2p)^2\right)\ . \tag{23}$$

do not support static finite energy solutions, however, their solutions can be constructed in closed form. Since to the best of our knowledge no exact (nontrivial) solution is known in the literature for the coupled gravity-Yang-Mills equations [3], we discuss here the basic properties of these $d = 2p + 1$ configurations. The straightforward generalisation in the presence of a cosmological constant is presented in Appendix B.

After several redefinitions of the theory's constants, the effective Lagrangean of this theory reads

$$L = \sigma\left[\frac{d}{dr}(N-1)^p - cr^{1-2p}N(w^2-1)^{2p-2}\left(\frac{dw}{dr}\right)^2\right], \tag{24}$$

where $c \sim \tau/\kappa$ is a free parameter (in this general case, we do not fix the sign of c). It is also convenient to redefine the gauge potential according to

$$a(w) = \int (w^2-1)^{p-1}dw = (-1)^{p+1}{}_2F_1(\frac{1}{2}, 1-p, \frac{3}{2}, w^2), \tag{25}$$

(${}_2F_1(a, b, c, z)$ being the hypergeometric function), such that the system (24) admits the first integral

$$a' \quad \equiv \quad (w^2-1)^{p-1}w' = \frac{\alpha}{r^{1-2p}N\sigma}, \tag{26}$$

where α is an arbitrary real constant.

The metric variables σ and N satisfy the equations:

$$\frac{dX^p}{dr} \quad = \quad r^{2p-1}\left[\frac{c\alpha^2}{(X+1)Y}\right], \tag{27}$$

$$\frac{dY}{dr} \quad = \quad -\frac{2c\alpha^2}{p}\frac{r^{2p-1}}{(X+1)^2X^{p-1}}, \tag{28}$$

where $X = N - 1$, $Y = \sigma^2$. This implies the relation:

$$\frac{dY}{dX} = -\frac{2Y}{(X+1)}, \tag{29}$$

which gives

$$Y(X+1)^2 = C, \tag{30}$$

[3]See, however, the exact solution with planar symmetry of the $d = 4$ EYM equations with a negative cosmological constant $\Lambda = -3$ presented in [18].

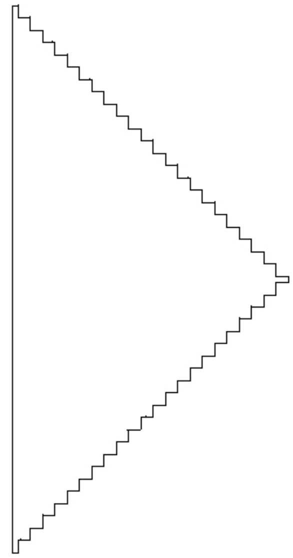


Figure 5. Penrose diagram of the metric (36) for $c > 0$. Wavy lines correspond to curvature singularities.

($i.e.$ $N\sigma = C^{1/2}$) and we can set $C = 1$, without any loss of generality.[4] Replacing this expression into the X-equation one finds the relation:

$$X^p {}_2F_1(p, 1, 1+p, -X) = \frac{c\alpha^2}{2p} r^{2p} + \beta, \tag{31}$$

where β is an integration constant.

Expressed in a form which employs X as coordinate, the general solution reads

$$ds^2 = f_1(X) dX^2 + r^2(X) d\Omega^2_{2p-1} - \frac{dt^2}{X+1}, \tag{32}$$

where

$$f_1(X) = \frac{1}{X+1}\left[\frac{pX^{p-1}}{c\alpha^2(X+1)}\right]^2 r^{2-4p}(X), \tag{33}$$

$$r(X) = \left[\frac{2p}{c\alpha^2}\left(X^p {}_2F_1(p, 1, 1+p, -X) - \beta\right)\right]^{\frac{1}{2p}}. \tag{34}$$

It follows straightforwardly that the expression for the transformed gauge potential a is

$$a(X) = a_0 + \frac{\alpha}{2p} r^{2p}(X), \tag{35}$$

with a_0 a constant of integration, which from (25) gives the expression of the gauge potential. One can see that this general solution is not asymptotically flat (as the matter fields do not vanish at spacelike infinity). The solution expressed in the r-coordinate takes a simple expression for $p = 1$ only

$$\begin{aligned} ds^2 &= f_0 e^{-c\alpha^2 r^2/2}(dr^2 - dt^2) + r^2 d\varphi^2 \\ w(r) &= w_0 + \frac{\alpha}{2} r^2, \end{aligned} \tag{36}$$

where α, f_0 and w_0 are integration constants. In order to eliminate the conical singularities in the (r, φ) sector we require that $f_0 = 1$. It is straightforward to compute the Kretschmann scalar for this metric:

$$K = R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} = \frac{3c^2\alpha^4 e^{c\alpha^2 r^2}}{f_0^2} \tag{37}$$

[4] We assume that the integration constant C is positive, $i.e.$ $\sigma^2 > 0$.

while the Ricci scalar is proportional to $e^{c\alpha^2 r^2/2}$. Notice that if $c > 0$ these curvature scalars blow up in the limit $r \to \infty$, while apparently the geometry is well behaved and becomes regular when $c < 0$. The Penrose diagram of the metric (36) in the $c > 0$ case is presented in Figure 5. Each point in the diagram represents a circle of radius r, as described by the φ coordinate.

While it might be surprising to find null curvature singularities at infinity, to understand their presence notice that defining the following null coordinates:

$$u = t - r, \qquad v = t + r, \tag{38}$$

we can write the metric as:

$$ds^2 = -f_0 e^{-c\alpha^2(v-u)^2/8} du dv + \frac{1}{4}(v-u)^2 d\varphi^2 \tag{39}$$

while K becomes:

$$K = \frac{3c^2\alpha^4 e^{c\alpha^2(v-u)^2/4}}{f_0^2} \tag{40}$$

It is now clear that in the (r, t)-plane null rays are represented as straight lines at 45^o with respect to the coordinate lines and furthermore K diverges as $u \to \infty$ with $v = const.$ as well as for $v \to \infty$ with $u = const.$.

To get a better appreciation of the properties of the $c > 0$ 3-dimensional geometry it is of interest to examine in more detail the behaviour of the timelike and null geodesics. These will illuminate the nature and the effects of the naked curvature singularities at infinity. Since t and φ are cyclical coordinates we can write down directly the following constants of motion:

$$e^{-c\alpha^2 r^2/2}\dot{t} = E, \qquad r^2\dot{\varphi} = L \tag{41}$$

while the radial coordinate r satisfies the equation:

$$\frac{1}{2}\dot{r}^2 + V(r) = 0 \tag{42}$$

which is the equation of a material point with unit mass and zero energy, moving in an effective potential given by:

$$V(r) = -\frac{E^2 e^{c\alpha^2 r^2}}{2} + \frac{L^2 e^{c\alpha^2 r^2/2}}{2r^2} - \frac{\epsilon e^{c\alpha^2 r^2/2}}{2} \tag{43}$$

where $\epsilon = -1$ for timelike geodesics, $\epsilon = 0$ for null geodesics and $\epsilon = 1$ for spacelike geodesics. Consider first the timelike geodesics, *i.e.* $\epsilon = -1$. For non-radial geodesics $L \neq 0$ and for $E \neq 0$ we have $V(r) \to \infty$ and the region near origin is classically forbidden, while in the limit $r \to \infty$ we have $V(r) \to -\infty$, which means that the particles accelerate so that $|\dot{r}| \to \infty$ as $r \to \infty$. For radial geodesics $L = 0$ and the potential reaches a finite value as $r \to 0$, while again $V(r) \to -\infty$ as $r \to \infty$. These properties continue to hold for null geodesics. Let us notice that null and even radial timelike geodesics can reach spatial infinity in finite intervals of the affine parameter. Thence the $c > 0$ geometry is pathological.

By contrast, if $c < 0$, generically $V(r) \to 0$ when $r \to \infty$ and therefore the region $r \to \infty$ is indeed at infinity in terms of proper distance and null right rays cannot reach spatial infinity in finite intervals of the affine parameter. Then for $c < 0$ the geometry is indeed free of any curvature singularities. That this is indeed the case can also be confirmed by examining the components of the Riemann tensor components in an orthonormal frame in the large r limit.

For either sign of c, near origin the spacetime geometry is flat and regular if $f_0 = 1$. It is interesting to note that one can also have the case in which $f_0 < 0$, in which case the geometry is time-dependent. However, one can easily check that the geometry is pathological if $c > 0$ and regular for $c < 0$.

One should also notice that the 3-dimensional metric is directly written in a Weyl-Papapetrou form. When $c < 0$ this will allow us to make an interesting connection with a specific time-dependent axially symmetric geometry of the form:

$$ds_4^2 \;=\; e^{-\sqrt{-2c\alpha^2}t}dz^2 + e^{\sqrt{-2c\alpha^2}t}\left[f_0 e^{-c\alpha^2 r^2/2}(dr^2 - dt^2) + r^2 d\varphi^2\right], \tag{44}$$

which is a solution of the vacuum EH field equations in 4 dimensions, $R_{ik} = 0$. Upon dimensional reduction along the z coordinate we obtain precisely the 3 dimensional geometry (36) that is now supported by a time-dependent scalar field $\phi = \sqrt{-2c\alpha^2}t$, which is a solution of the equations of motion derived from the Lagrangian:

$$\mathcal{L}_3 \;=\; eR_{(1,1)} - \frac{1}{2}e(\partial\phi)^2 \tag{45}$$

One can now perform the dualisation of the scalar field to obtain a magnetic 2-form field strength and it is easy to check that the final solution corresponds to the magnetic version of the Reissner-Nordstrøm solution in 3 dimensions [19]-[21].

The 4 dimensional geometry is also free of curvature singularities. If we perform the analytical continuations $t \to iz$, $z \to it$ and also $c \to -c$ we obtain the Euclidean version of the 3 geometry (36) where now $c > 0$. Its Lorentzian version is obtained if we further analytically continue $z \to it$ with the final result that (36) with $c > 0$ is now supported by a time dependent scalar field whose kinetic term has the wrong sign. This can be regarded as an indirect confirmation of our previous result that the geometry (36) for $c > 0$ has pathological properties.

5 Conclusions

The general aim of this work was to study the properties of gravitating gauge field systems whose gravitational part consists of higher order gravitational curvature terms, e.g. Gauss-Bonnet terms, *i.e.* necessarily in higher dimensions. In particular, our scope is limited to static finite energy solutions, which are also spherically symmetric and asymptotically flat. As we know from previous work that the presence of the usual Einstein-Hilbert (EH, $p = 1$ gravity) term masks the effects of all higher order ($p \geq 2$) terms, we have chosen to study models featuring only one gravitational term with $p \geq 2$, whose specific value is chosen according to the dimensionality d of the spacetime. This still leaves open the choice of the YM terms.

A dual aim in this work was to bring out some general features of gravitating YM solutions, seen in the original BK [1] case with $d = 4$, EH $p = 1$ gravity, and usual $p = 1$ YM term. This has led us to study models with p-th order gravity and $2p$-th order YM in $d = 4p$ dimensions. This is the content of results in Section 3, where we have verified that all qualitative properties of the BK solutions are repeated in dimensions *modulo* $4p$ (although nontrivial finite mass solutions exists also for $2p + 2 \leq d \leq 4p$). Like the BK solution, these are constructed numerically and are likewise all (unstable) sphalerons.

As a result of our general considerations we realised that a somewhat different hierarchy of models, namely those again with p-th order gravity and $2p$-th order YM but now in $d = 2p+1$ dimensions, actually supported exact solutions in closed form. These configurations were discussed in Section 4. Their properties depend essentially on the sign of the coupling constant $c \sim \tau/k$. While we expect that the physically meaningful configurations be described by negative values for c, in Section 4 we studied in more details the $p = 1$ case. We found that indeed $c > 0$ leads to a pathological geometry, while for $c < 0$ the solution is perfectly regular and we anticipate these properties to hold for higher values of p. Led on by the work in Section 4, featuring solutions in closed form, we branched out to introduce a cosmological constant to the models studied in Section 4. This also led to solutions in closed form, presented in the Appendices A and B. In Appendix A we considered the gauge decoupled versions of these models since this turns out to yield a hierarchy of solutions the first ($p = 1$) member of which is the static BTZ solution [9]. Unfortunately the remarkable geometric features of the $p = 1$ solution [9] are not repeated in the $p \geq 2$ cases, and a fairly extensive analysis of this is given in Appendix A. We found that for generic values of the parameters these solutions are pathological in that they exhibit naked curvature singularities (that can be hidden inside cosmological type horizons), however, for special values of the parameters regular $(a)dS/flat$ backgrounds are obtained. In Appendix B, the closed form solutions of the models in Section 4 with a non-vanishing Λ are presented. The general solution has two branches and in the $\Lambda \to 0$ limit only one of them will survive to give the solution discussed in Section 4. The general form of these solutions is very complicated and this impedes a general analysis of their properties. We do expect however that since they have no horizons, they will generically exhibit naked curvature singularities.

While one might question the physical relevance of the new exact solutions found in this paper since the form of our gravitational and matter Lagrangians is non-standard, we take the point of view that given the scarcity of known non-trivial exact solutions of the YM system coupled to gravity, any new exact solutions that can be found in closed analytical form might shed some light on the properties of such complicated systems. Moreover, in literature there have been studied modifications of the standard Einstein-Hilbert gravity by considering invariant quantities constructed from the curvature scalar and/or the Riemann tensor. In general it is known that for a gravitational Lagrangian constructed out of the metric and the Ricci tensor it is possible to perform a conformal transformation (or more generally a Legendre transformation) to a metric expressed in the Einstein frame, solution of the standard Einstein-Hilbert gravity coupled with (exotic) matter fields (see for instance [22, 23] and references therein). The question if a similar reasoning can be applied to the gravitational lagrangians considered in this paper remains an interesting topic for further research.

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A A BTZ hierarchy in $(2p+1)$-dimensions

It is well known that while gravity in $d = 3$ spacetime dimensions in the absence of matter is dynamically trivial, in the presence of a cosmological constant it supports highly nontrivial asymptotically anti-deSitter (AdS) solutions (BTZ) [9].

Our purpose here is to present a hierarchy of gravitational models with non-vanishing cosmological constant, which support exact solutions in closed form, proposing a generalisation of the static $d = 3$ BTZ solution. It turns out that these models are defined in spacetime dimensions $d = 2p + 1$ and are described by (6) with the YM terms suppressed. This can be written by augmenting (6), with YM fields suppressed, by a cosmological constant,

$$\mathcal{L}_\Lambda = \mathcal{L}_{(2p+1)}\big|_{F=0} - (2p+1)!\, e\, \Lambda\,, \tag{A.1}$$

whose $p = 1$ member supports the familiar BTZ solution [9].

The exact solutions in question are the static spherically symmetric field configurations resulting from the imposition of spherical symmetry, given by the metric Ansatz (9) in Schwarzschild coordinates, where here r is the $2p$ dimensional radial coordinate.

Here, we introduce the cosmological constant, *i.e.* we adopt the system (A.1). Subjecting the latter to the Ansatz (9), after a suitable rescaling we have the reduced Lagrangian [5]

$$L^{\Lambda}_{(p,1)} = \frac{1}{2^{2p-1}}\frac{(d-2)!}{(d-2p-1)!}\,\sigma\,\left[\frac{d}{dr}(N-1)^p - r^{2p-1}\,\Lambda\right]. \tag{A.2}$$

We immediately find the following solutions

$$(N-1)^p = \frac{1}{2p}r^{2p}\,\Lambda + \text{const.} \qquad , \qquad p(N-1)^{p-1}\frac{d\sigma}{dr} = 0 \;\; => \;\; \sigma = \text{const.} \tag{A.3}$$

[5]Note that Λ has here the opposite sign as compared to the standard choice in literature

Notice that for even values of p we have two branches in our solutions. We will discuss the global structure of these metrics separately according to even or odd values of p. Also, we will denote the constant that appears in (A.3) by M, as this integration constant will be proportional in some cases to the mass of a black hole for black hole type spacetimes. We will also define $\Lambda = \pm\frac{2p}{\ell^{2p}}$ to simplify notations. For $\Lambda > 0$ we obtain then in general:

$$N(r) \quad = \quad 1 \pm \left(\frac{r^{2p}}{\ell^{2p}} + M\right)^{\frac{1}{p}}, \tag{A.4}$$

where the minus sign defines a second branch of solutions for even values of p only. To analyse the case of a negative cosmological constant $\Lambda < 0$ one simply analytically continues $\ell \to i\ell$, while the case $\Lambda = 0$ is obtained in the limit $\ell \to \infty$.

Several particular cases of interest are $p = 1$ and $p = 2$. The former solution corresponds to the celebrated BTZ solution [9] for $\Lambda = \frac{2}{\ell^2} > 0$:

$$\begin{aligned} ds^2 &= -N(r)dt^2 + N^{-1}(r)dr^2 + r^2 d\theta^2 \\ N(r) &= \frac{r^2}{\ell^2} - M, \end{aligned} \tag{A.5}$$

while $\Lambda < 0$ corresponds to 3-dimensional dS space. If $\Lambda = 0$ the spacetime is flat and, unless $M = 0$, it contains a conical singularity at origin. For $p = 2$ the metric is 5-dimensional and solves the pure Gauss-Bonnet equations with a cosmological constant:

$$\begin{aligned} ds^2 &= -N(r)dt^2 + N^{-1}(r)dr^2 + r^2 d\Omega_3^2 \\ N(r) &= 1 \pm \left(\pm\frac{r^4}{\ell^4} + M\right)^{\frac{1}{2}} \end{aligned} \tag{A.6}$$

In general, we have two branches of our solutions, which correspond to the choice of the sign in front of the radical. Inside the radical the choice of sign is dictated by the sign of the cosmological constant. For a negative cosmological constant we have to consider strictly positive values for M and, in this case, the radial coordinate r will take values only in a finite interval. We find, however, that there are curvature singularities at both end points of this interval, separated by a black hole type horizon in between. For a positive cosmological constant, both positive and negative values of the parameter M are allowed as long as the expression under the radical is positive. We find that there is a naked curvature singularity located at $r = 0$, which is hidden inside a cosmological type horizon. When $M = 0$ we obtain AdS spacetime as the positive branch solution, respectively dS as the negative branch solution. If $\Lambda = 0$ we obtain a solution containing a naked curvature singularity at origin and having a deficit of solid angle. As we shall see in the followings, the general solutions in higher dimensions exhibit similar properties as the lower-dimensional cases.

The detailed analysis of the global structure of these spacetimes involves a discussion of the singularities, horizons and the asymptotic structure. We will be mainly interested in curvature singularities and these will be identified using the Kretschmann invariant. For the spherical symmetric ansatz (9) the Kretschman scalar can be written as:

$$\begin{aligned} K &= R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} \\ &= (N'')^2 + \frac{2(d-2)}{r^2}(N')^2 + \frac{2(d-1)(d-2)}{r^4}(1-N)^2 \end{aligned} \tag{A.7}$$

From the general form of $N(r)$ we can see that there is a curvature singularity located at $r = 0$ and also at points where $\frac{r^{2p}}{2p}\Lambda + M = 0$. Around these curvature singularities the Kretschmann invariant behaves as $K \sim O\left(\frac{M^{\frac{2}{p}}}{r^4}\right)$. Notice that these metrics are regular only if $M = 0$.

The asymptotic behaviour is controlled by the dominant term in $N(r)$ as $r \to \infty$. In general, for odd values of p one can have asymptotically $(a)dS$ spaces, however, for even values of p we find that we have to restrict the range of the radial coordinate such that $\pm\frac{r^{2p}}{\ell^{2p}} + M \geq 0$. For $\Lambda = 0$ we find that the asymptotic structure is controlled by the values of the parameter M. If $M = 0$ we obtain the flat spacetime. If $M \neq 0$ then in general we obtain spaces with deficits or surfeits of solid angle.

To characterise the horizons we use the following definitions [24]: a horizon located at $r = r_h$ is a null hypersurface with finite curvature, such that $N(r_h) = 0$. A black hole horizon is defined by the condition $N'(r_h) > 0$; a horizon for which $N'(r_h) < 0$ and r_h is the largest root of $N(r)$ will be called a cosmological horizon. If $N'(r_h) < 0$ and r_h is not the largest root then $r = r_h$ will define an inner horizon. If $N'(r_h) = 0$ then $r = r_h$ would correspond to an extreme horizon. Notice however that in our solutions this can happen only if $r_h = 0$ and since $r = 0$ is the location of a curvature singularity the spacetime will be singular.

Consider first spacetime geometries corresponding to odd values of p. We will examine the global structure for each value of Λ and M.

- $\Lambda > 0$ For any value of M the spacetimes are asymptotically adS. If $M > 0$, there is a curvature singularity at $r = 0$ and since $N(r)$ does not vanish for any value of r there are no horizons, hence the spacetime contains a globally naked singularity. If $M = 0$ then the spacetime is adS. For $M < 0$ we find that $N(r)$ can have a zero at $r = r_h$ such that $\frac{r_h^{2p}}{\ell^{2p}} + M = -1$. However, since $\frac{r^{2p}}{\ell^{2p}} + M = 0$ for a value $r_s > r_h$, then $r = r_s$ defines a curvature singularity that is not covered by a horizon and therefore it corresponds to a globally naked singularity.

- $\Lambda = 0$ In this case $N(r) = 1 + M^{\frac{1}{p}}$ and there is a curvature singularity at $r = 0$. We exclude from our discussion the value $M = -1$ for which $N(r) \equiv 0$. If $M > 0$ by rescaling the coordinates we can bring the metric in the following form:

$$ds^2 = -dt^2 + dr^2 + (1 + M^{\frac{1}{p}})r^2 d\Omega_{d-2}^2 \tag{A.8}$$

  The spacetime has a surfeit of solid angle as one can see by computing the surface area of a sphere with radius r and comparing its value with the one calculated in flat spacetime. If $M = 0$ we obtain the flat spacetime. For negative values of M such that $1 + M^{\frac{1}{p}} > 0$ the spacetime has a deficit of solid angle. If $1 + M^{\frac{1}{p}} < 0$ then r is a timelike coordinate and the spacetime corresponds to a Milne-type spacetime having a deficit (or surfeit) of solid angle for $M < -2^p$ (respectively $M > -2^p$).

- $\Lambda < 0$ If $M < 0$ then $N(r) = 0$ when $\frac{r^{2p}}{\ell^{2p}} - M = 1$ and the curvature singularity at $r = 0$ is hidden behind a horizon located at $r_h = l(1 + M)^{\frac{1}{2p}}$. Notice that in order to obtain real values for r_h we have to restrict the values of M such that $M > -1$. Since $N'(r_h) < 0$ this corresponds to a cosmological horizon and the spacetime is asymptotically dS. For $M = 0$ the spacetime becomes dS while for $M > 0$ the spacetime contains a globally naked singularity located at $r_s = lM^{\frac{1}{2p}}$.

Let us consider next the geometries corresponding to even values of p. As we have previously mentioned, there are two branches of solutions and we shall discuss each branch separately. The negative branch has:

$$N(r) = 1 - \left(\pm\frac{r^{2p}}{\ell^{2p}} + M\right)^{\frac{1}{p}} \tag{A.9}$$

Here the upper sign corresponds to a positive cosmological constant $\Lambda > 0$, the lower sign corresponds to $\Lambda < 0$, while the case $\Lambda = 0$ is obtained in the limit $\ell \to \infty$. Notice that we have to restrict the values of the parameters M, l and of the radial coordinate r such that $\pm\frac{r^{2p}}{\ell^{2p}} + M \geq 0$.

- $\Lambda > 0$ For $M > 0$ the spacetime is asymptotically dS and it has a curvature singularity at $r = 0$, hidden inside a cosmological horizon located at $r_h = l(1 - M)^{\frac{1}{2p}}$. Notice that in order to obtain real values for r_h we must restrict the values for M such that $M < 1$. For $M = 1$ then $r_h = 0$ and the spacetime contains a globally naked singularity. If $M > 1$ there is a globally naked singularity located at $r = 0$. If $M = 0$ we obtain dS. For $M < 0$ we find a singularity at $r = 0$ inside of a cosmological horizon located at $r_h = \ell(1 - M)^{\frac{1}{2p}}$.

- $\Lambda = 0$ In order to avoid complex values for $N(r) = 1 - M^{\frac{1}{2p}}$ we must require that $M \geq 0$. If $M = 0$ we obtain Minkowski spacetime. If $M > 0$ the spacetime contains a globally naked singularity at $r = 0$ and it has a deficit of solid angle.

- $\Lambda < 0$ If $M \leq 0$ we find that $f(r)$ takes complex values. If $M > 0$ we must also restrict the values of the radial coordinate to a finite interval such that $M - \frac{r^{2p}}{\ell^{2p}} \geq 0$. There exists a black hole type horizon at $r_h = \ell(M-1)^{\frac{1}{2p}}$ only if $M > 1$. However, at both endpoints of the radial coordinate there are curvature singularities. For $M = 1$ there is a naked curvature singularity at $r = 0$ and also another one at $r = \ell$.

Consider next the positive branch solutions. In this case we have:

$$N(r) = 1 + \left(\pm\frac{r^{2p}}{\ell^{2p}} + M\right)^{\frac{1}{p}} \tag{A.10}$$

The analysis is very similar with the one performed for odd values of p. There exists however an extra restriction that $\pm\frac{r^{2p}}{l^{2p}} + M \geq 0$. The equality sign corresponds to a curvature singularity location. We find then that these spacetimes are generically singular unless $M = 0$ and in the cases where there are regular horizons, these correspond to cosmological horizons such that the spacetimes still contain globally naked singularities.

B $D = 2p + 1$ gravity-Yang-Mills solutions with $\Lambda \neq 0$

For $p = 1,\ d = 4$, the EYM equations with a cosmological term present solutions with very different properties as compared to the $\Lambda = 0$ case [14, 15]. Their $d > 4$ generalisations in Einstein gravity with higher terms in the Yang–Mills hierarchy have been discussed recently in [5]. It is therefore natural to consider solutions of the p-th gravity-Yang-Mills system (23) in the presence of a cosmological constant, in $d = 2p+1$ dimensions.

Using the same notations as in Section 4, the reduced Lagrangean of this system reads

$$L = \sigma\left[\frac{d}{dr}(N-1)^p - cr^{1-2p}N(w^2-1)^{2p-2}\left(\frac{dw}{dr}\right)^2 - \Lambda r^{2p-1}\right], \tag{B.1}$$

the resulting field equations being solved by using the same techniques as in the $\Lambda = 0$ case.

Employing the same variables, $X = N - 1$ and $Y = \sigma^2$, we find the same σ-equation, while the function N satisfy the equation

$$\frac{dX^p}{dr} = r^{2p-1}\left[\frac{c\alpha^2}{(X+1)Y} + \Lambda\right], \tag{B.2}$$

the gauge field equations still admitting the first integral (26).

Therefore (29) is generalized to

$$\frac{dY}{dX} = -\frac{2Y}{(X+1)\big(1 + \lambda Y(X+1)\big)}, \tag{B.3}$$

where $\lambda = \Lambda/c\alpha^2$. If we define a new function $Z = (X+1)Y$, we find that (B.3) can be written as:

$$\frac{dZ}{dX} = \frac{Z}{X+1}\frac{\lambda Z - 1}{\lambda Z + 1} \tag{B.4}$$

and upon integration we obtain:

$$\frac{(\lambda Z - 1)^2}{Z} = \frac{X+1}{C} \tag{B.5}$$

where C is an integration constant. This equation has now two solutions, denoted by $Z_\pm$ and given by:

$$\lambda Z_\pm = \frac{\sqrt{X+c_1}\mp\sqrt{X+1}}{\sqrt{X+c_1}\pm\sqrt{X+1}}, \tag{B.6}$$

where $c_1 = 4\lambda C + 1$. In consequence, we obtain two branches for our solution:

$$Y_\pm = \frac{c\alpha^2}{\Lambda(1+X)}\frac{\sqrt{X+c_1}\pm\sqrt{X+1}}{\sqrt{X+c_1}\mp\sqrt{X+1}}, \tag{B.7}$$

Notice that when taking the limit $\Lambda \to 0$ only the negative branch will survive and we obtain precisely the solution given in (30). The positive branch solution has no asymptotically flat limit. Replacing these expressions in (27), we arrive at a relation similar to (31):

$$\frac{1}{2p}r^{2p} + \beta = \frac{p}{2\Lambda}F_\pm(X), \tag{B.8}$$

where

$$F_\pm(X) = \int X^{p-1}\left(1\pm\sqrt{\frac{X+1}{X+c_1}}\right)dX. \tag{B.9}$$

It appears that it is not possible to find a general expression of this integral. Several particular cases of potential interest in which we can integrate (B.9) are:

$$\begin{aligned} F_\pm(X) &= X \pm \sqrt{(X+1)(X+c_1)} \mp (c_1-1)\log(\sqrt{X+1}+\sqrt{X+c_1}), \quad \text{for} \quad p=1. \\ F_\pm(X) &= \frac{1}{8}(4X^2 \pm 2\sqrt{(X+1)(X+c_1)}(1-3c_1+2X) \\ &\quad \pm(c_1-1)(3c_1+1)\log(1+c_1+2X+2\sqrt{(X+1)(X+c_1)})) \quad \text{for} \quad p=2. \end{aligned} \tag{B.10}$$

The general solution takes a simpler form when expressed using X as coordinate with

$$r(X) = \left(\frac{p^2}{\Lambda}F_\pm(X) - 2p\beta\right)^{\frac{1}{2p}}. \tag{B.11}$$

Therefore, the general metric of the $\Lambda \neq 0$ solution is given by

$$ds^2 = g_1(X)dX^2 + r^2(X)d\Omega_{d-2}^2 - g_2(X)dt^2 \tag{B.12}$$

with

$$\begin{aligned} g_1(X) &= \left[\frac{pr^{-2p-1}(X)}{2\Lambda}X^{p-1}\left(1\pm\sqrt{\frac{X+1}{X+c_1}}\right)\right]^2\frac{1}{X+1}, \\ g_2(X) &= \frac{c\alpha^2}{\Lambda}\frac{\sqrt{X+c_1}\pm\sqrt{X+1}}{\sqrt{X+c_1}\mp\sqrt{X+1}}. \end{aligned} \tag{B.13}$$

From the above form of the general solution, notice that we have to restrict the X coordinate such that $X \geq -1$ and that there are no horizons. However $g_1(X)$ will blow up as $X \to -1$ and therefore we conclude that in general such spaces will have pathological features.

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